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Modeling a Lens with Dielectric Constant Inversely Proportional to Radius Squared

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Abstract

A lens design procedure, with constant wavelength to cross section ratio as ε_r increases, is used to obtain better focusing at the second focal point of a prolate-spheroidal IRA. It is shown that the lens behavior can be analyzed by a correspondence to a lens behaving like an exponential transmission line.

1 Introduction

In [3,4] the analytical, numerical and experimental focal waveforms of a prolate spheroidal IRA are presented. Early papers [1,2] discussed addition of a lens at the second focal point of a prolate spheroidal IRA for better concentrating at the second focal point. Figure 1 shows this lens geometry and z is the transmission-line coordinate.

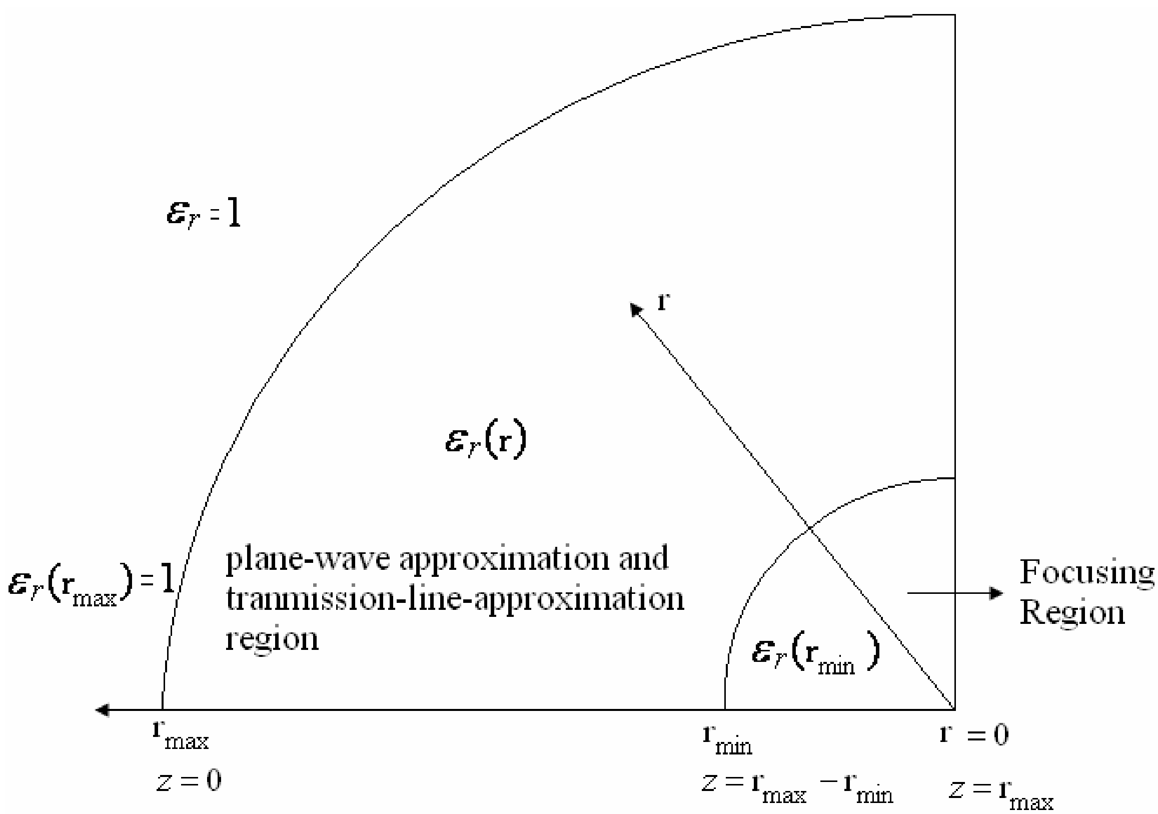


Figure 1. Lens Geometry

$\varepsilon_r(\mathrm{r})$ is the dielectric constant of the tapered lens [5,2]. For $r_{min} < r < r_{max}$ region we use the transmission line results to estimate peak and droop of wave reaching r_{min}. Inside r_{min} the dielectric constant can be defined as

$$\varepsilon_r = \varepsilon_r(r_{min}) \qquad 0 < r < r_{min}, \tag{1}$$

And this is a focusing region for which the transmission-line results do not apply however the spot size analysis in [3] can be applied.

2 Analytical Calculations for $r_{min} < r < r_{max}$ Region

Let's assume $\lambda \propto r$ in the $\varepsilon_r(\mathrm{r})$ medium in which we have a converging wave toward $r = 0$. One can define λ as the wavelength for some characteristic frequency f in the pulse,giving

$$\begin{aligned} &f\lambda = \mathrm{v}, \quad \mathrm{v} = c\,\varepsilon_r^{-1/2}(r), \\ &f \propto 1/\Delta t, \ \text{a constant} \end{aligned} \tag{2}$$

where v is the speed of propagation of the lens. c is the speed of light in the free space and Δt is the pulse width (or time of interest after the initial pulse "step" rise). One uses these approximations

$$\begin{aligned} &\lambda \propto r, \\ &\mathrm{v} \propto \lambda / \Delta t \propto r / \Delta t. \end{aligned} \tag{3}$$

Then v can be defined as

$$\mathrm{v} = c\,\varepsilon_r^{-1/2}(r) = Cr\,. \tag{4}$$

From (4) $\varepsilon_r(\mathrm{r})$ can be found as

$$\begin{aligned} &\varepsilon_r^{1/2}(\mathrm{r}) = \frac{c}{Cr}, \\ &\varepsilon_r(\mathrm{r}) = \frac{D}{r^2} = \left[\frac{r_{max}}{r}\right]^2. \end{aligned} \tag{5}$$

The speed of propagation is

$$\mathrm{v}(r) = \mathrm{c}/\sqrt{\varepsilon_{\mathrm{r}}(r)} = cr / r_{max}, \tag{6}$$

the ratio can be found as

$$c / \mathrm{v} = r_{max} / r\,. \tag{7}$$

3 Changing the Radial Coordinate

We want to create a new coordinate ς in which we want to make the wave propagation speed equal to the speed of light c locally as

$$\frac{d\varsigma}{dt} = c \qquad \text{for } r_{min} < r < r_{max}\ . \tag{8}$$

Integrating (8) gives us

$$\varsigma = ct\,, \tag{9}$$

where ς is the equivalent spatial coordinate or equivalent transmission-line coordinate. t is the time for the wave to reach ς we have in our ε_r varying medium

$$\frac{dr}{dt} = \mathrm{v}\,. \tag{10}$$

By substituting (10) in (8) and integrating it one can obtain

$$\varsigma = \int_r^{r_{max}} \frac{c}{\mathrm{v}} dr \ > \left[r_{max} - r\right], \tag{11}$$

slower velocity implies $\varsigma > \left[r_{max} - r\right]$ except as $\varsigma \to 0,\ \mathrm{v} \to c$.
One can substitute (7) in (10) as

$$\varsigma = \int_r^{r_{max}} \frac{r_{max}}{r'} dr' = r_{max}\, ln(r')\Big|_r^{r_{max}} = r_{max}\, ln(\frac{r_{max}}{\mathrm{r}}) \tag{12}$$

(12) shows that as $r \to 0,\ \varsigma \to \infty$ therefore our lens is spatially limited as in [2].
The ratio of r / r_{max} can be found as

$$\frac{r}{r_{max}} = e^{-\varsigma / r_{max}}\,. \tag{13}$$

Note that r / r_{max} has an exponential behavior as $\varsigma \to \infty, r \to 0$.
Let us change the transmission-line coordinate as

$$z = r_{max} - r$$
$$\frac{z}{r_{max}} = 1 - \frac{r}{r_{max}} \tag{14}$$

as $\varepsilon_r \to \infty$ r_{max} is like the length of the transmission line. One can find z as a function of ς and r_{max} as

$$z = r_{max}(1 - e^{-\varsigma / r_{max}}) \tag{15}$$

4. Relation to the Telegrapher Equations

The wave propagation can be described by the source-free telegrapher equations [5(2.3)] as

$$\begin{aligned} \frac{d}{dz}\tilde{V}(z,\mathrm{s}) &= -\tilde{Z}'(z,s)\tilde{I}(z,s) \\ \frac{d}{dz}\tilde{I}(z,\mathrm{s}) &= -\tilde{Y}'(z,s)\tilde{V}(z,s) \end{aligned}, \tag{16}$$

where $\sim$ shows the two-sided Laplace transform over time t and $s = \Omega + j\omega$ is the complex frequency. For transmission line assumed lossless with $\tilde{Z}'(z,s) = sL'(z)$ where $L'(z)$ is the inductance per unit and $\tilde{Y}'(z,s) = sC'(z)$ where $C'(z)$ is the capacitance per unit length. Under the assumption that the tapered transmission line consists of perfect conductors with lossless dielectric one can define $L'(z)$ and $C'(z)$ as

$$L'(z) = \mu\, f_g(z),\; L'(z) = \varepsilon / f_g(z) \tag{17}$$

where $f_g(z)$ is geometric impedance factor and it is

$$f_g(z) = e^{-\varsigma / r_{max}}\,. \tag{18}$$

We have an analogy between

$$V \to E, I \to H\,. \tag{19}$$

Substituting (17) and (18) in (16) one can transform the 1D wave equation to an equivalent ζ space coordinate using (15) as

$$\begin{aligned} \frac{dV(\zeta,s)}{d\zeta} &= -s\mu_0 e^{-\zeta / r_{max}}\; I, \\ \frac{dI(\zeta,s)}{d\zeta} &= -s\varepsilon_0 e^{\zeta / r_{max}}\; V. \end{aligned} \tag{20}$$

This result is similar as the previous spatially limited case [2(4.10)] and both of the results have the same exponential behavior. Therefore we can use these results for a spatially limited lens.

We can use the exact solution of the transfer function neglecting the reflection from the beginning of the lens in [5(A.8)] as

$$\tilde{T} = e^{S+G}\left[cosh\left(\left(S^2 + G^2\right)^{1/2}\right) + \frac{S}{\left(S^2 + G^2\right)^{1/2}} sinh\left(\left(S^2 + G^2\right)^{1/2}\right)\right]^{-1} \tag{21}$$

where S is the normalized complex frequency

$$S = s\,t_{\zeta\ max} = (\Omega + j\omega)t_{\zeta\ max}, \tag{22}$$

where $t_{\zeta\ max}$ is the transit time through lens. The high-frequency gain is defined in [5(3.4)] as

$$\frac{V(r_{min})}{V_0} = g = e^G = \sqrt{Z_2 / Z_1} = \varepsilon_r^{-1/4}(r_{min}) = e^{-\zeta\ max/(2r_{max})} \tag{23}$$

This is actually a decrease which will be overcome by the wave convergence in the lens. One can also define $I(r_{min}) / I_0$ and the wave impedance at $\varsigma = \varsigma_{max}$ or $r = r_{min}$ as

$$\begin{aligned} \frac{I(r_{min})}{I_0} &= 1/g = e^{-G} = \sqrt{Z_1 / Z_2} = \varepsilon_r^{1/4}(r_{min}) = e^{\zeta\ max/(2r_{max})} \\ \frac{V(r_{min})}{I(r_{min})} &= Z_w(r_{min}) = \frac{Z_0}{\varepsilon_r^{1/2}(r_{min})} \end{aligned} \tag{24}$$

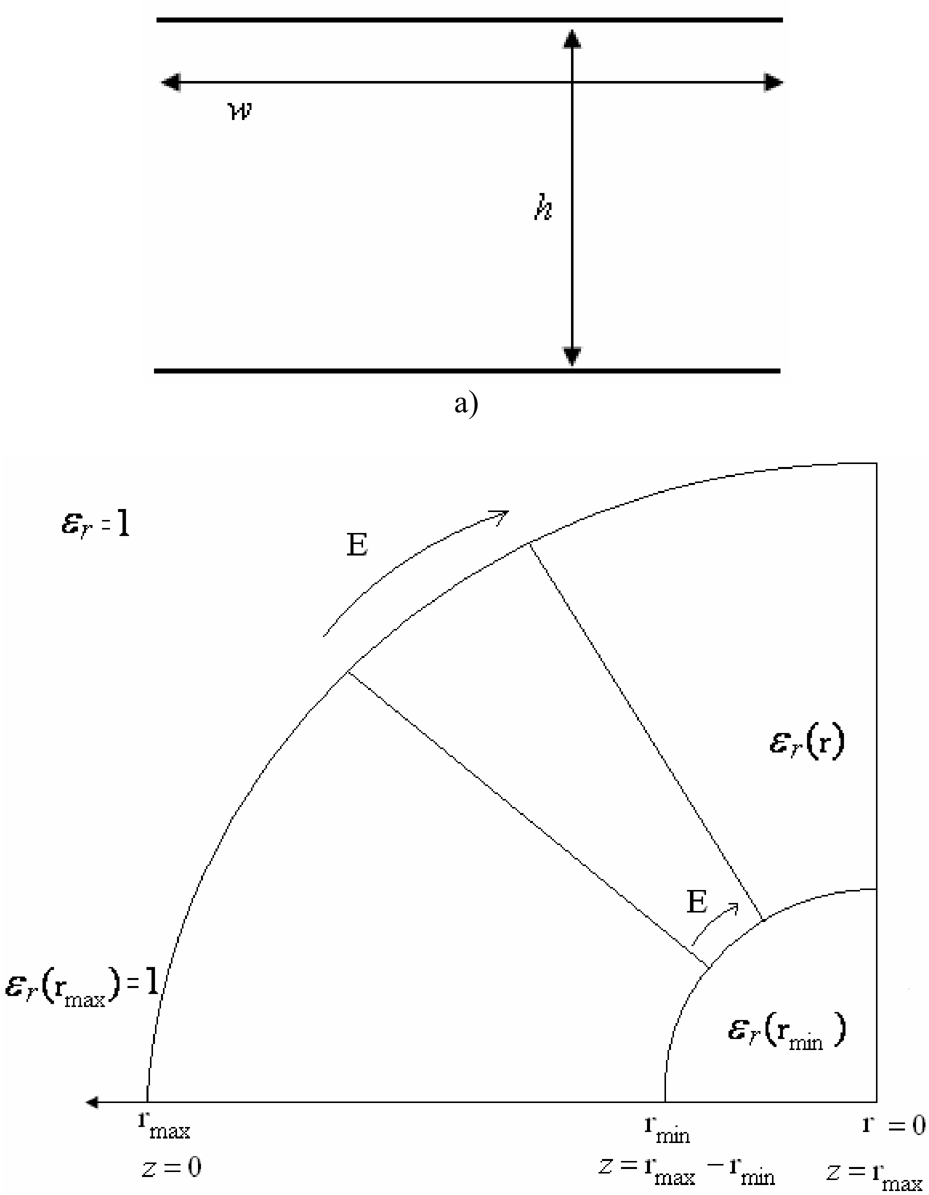
Note that as one goes down in frequency this also neglects any reflection (small) at $r = r_{min}$ where the lens goes into a constant $\varepsilon_r(r_{min})$ region.

5. Relation the Telegrapher Equations to the Fields

Figure 2a and 2b show the transmission-line representation of the fields in the lens. In this case as the wave is focused toward $r = 0$, the height h and the width w of a transmission line representing an incremental portion in a spherical cross section of the lens are decreasing toward $r = 0$ as

$$\begin{aligned} h &\propto r / r_{max}, \\ w &\propto r / r_{max}, \end{aligned} \tag{25}$$

where h / w is independent of r.



b)
Figure 2. a)transmission-line parameters: h and w b)E-Field Focusing

The fields in the transmission line are thus proportional to V and I as

$$E \propto V \frac{r_{max}}{r}, H \propto I \frac{r_{max}}{r} \quad . \tag{26}$$

So E and H are increasing relative to V and I as the wave propagates through the lens.

At the beginning of the lens, at early time, we have

$$\frac{E_0}{H_0} = Z_0 ,$$

In the $r_{min} < r < r_{max}$ region we have, at early time,

$$\frac{V}{V_0} = \frac{E}{E_0}\frac{r}{r_{max}}, \ \frac{I}{I_0} = \frac{H}{H_0}\frac{r}{r_{max}} . \tag{27}$$

On the wave front the wave impedance is

$$Z_w(r) = \frac{V(r)}{I(r)} = \frac{Z_0}{\varepsilon_r^{1/2}(r)} . \tag{28}$$

Power in the transmission-line, $V_0 I_0$, and on the wavefront, VI, are, therefore, equal because of the conservation of energy with no high-frequency reflection, giving

$$VI = V_0 I_0 , \tag{29}$$

The power in an equivalent transmission line is converging toward the focus. The power density, at early times, in the lens is

$$VI\left[\frac{r_{max}}{r}\right]^2 , \tag{30}$$

consistent with (27).

In our case as indicated in Figures 2a and 2b the distance is decreasing. Therefore, we have higher a electric field in a smaller region. This statement also can be made by conservation of energy in the wavefront. The wave impedance in the $r_{min} < r < r_{max}$ region is

$$\frac{E}{H} = \frac{Z_0}{\varepsilon_r^{1/2}} = Z_0 e^{-\varsigma / r_{max}} \tag{31}$$

This lens behavior is the same as that for a spatially limited lens [2(Section 4)]. Therefore, one can use the same equations for this lens.

One can define the transit, normalized and droop time [5(A.11)] parameters as follow

$$
\begin{aligned}
&t_{\zeta\ max} \equiv \zeta_{max} / c && \text{transit time through lens} \\
&\tau \equiv t / t_{\zeta} && \text{normalized time} \\
&\tau_d = 2\ ln^{-2}(g) && \text{normalized droop time} \\
&\quad = t_d / t_{\zeta\ max}
\end{aligned}
\tag{32}
$$

t_d is the droop time. While this is quite accurate for an exponential transmission-line [5], it is only approximate here. This is due to the approximate number of wavelengths across the spherical surface of radius r through which the wave is propagating. However, we are dealing with a pulse for which going to later times involves lower frequencies, and therefore, larger wavelengths, making the approximation less valid.

The electric-field gain, step response quality factor for lens improvement, can be defined as

$$
\begin{aligned}
g_E &= \frac{E_{out}}{E_0} = \varepsilon_r^{1/2} g_V\ e^{-\Delta t / t_d} u(t) \approx \varepsilon_r^{1/2} g_V \left(1 - \Delta t / t_d\right) u(t) \\
&\approx \varepsilon_r^{1/4} \left(1 - \Delta t / t_d\right) u(t),
\end{aligned}
\tag{33}
$$

where $\varepsilon_r^{1/2}$ is the wave convergence factor (defined as enhancement factor F_0 in [1]), Δt is pulse width, $e^{-\Delta t / t_d}$ is the droop, $g_V\ e^{-\Delta t / t_d}$ is the high-frequency transmission-line gain for early times , $g_V = \varepsilon_r^{-1/4}$ and E_0 is the electric field at the focal point when the lens is not there.

One can define t_d from (32) and [(13)7] as

$$
t_d = -\frac{r_{max}}{c} ln\left(\frac{r_{min}}{r_{max}}\right) 2\, ln^{-2}\left(\varepsilon_r^{-1/4}\right)
\tag{34}
$$

In [6], the numerical simulations show us when the outer radii of the lenses are $r_{max} = 15\,\text{cm}$ and $24\,\text{cm}$, we obtain acceptable focusing at the focal point. The outer and inner radii and droop times calculated from (34) for $\varepsilon_r = 9$ and 81 are presented in Table 1.

Table 1 The outer, inner radii, droop times for $\varepsilon_r = 9$ and 81

r_{max} (meters)	0.15		0.24	
ε_r	9	81	9	81
r_{min} (meters)	0.05	0.017	0.08	0.027
t_d (ns)	3.6	1.8	5.8	2.9

One can see from (33) and Table 1 that higher dielectric constant and larger radius give smaller droop times, while obtaining higher electric-field gain.

6. Conclusion

In this paper we start with a lens design with constant wavelength to cross section ratio. ε_r varies [8] and we come up with a spatially limited lens [2(Section 4)]. A lens is designed in which we want to make the wave propagation speed in $r_{min} < r < r_{max}$ region equal to the speed of light c in an equivalent ς coordinate system. The 1D wave equation is transformed to an equivalent ς coordinate and this equation has the same exponential behavior as a spatially limited lens [2(Section 4)]. The region $0 < r < r_{min}$ is a focusing region for which the transmission-line results do not apply however the spot size analysis in [3] can be applied.

Higher dielectric constant and larger radius give smaller droop times, while obtaining higher electric-field gain. Even though t_d is one parameter for lens design one should also consider other parameters such as loss and dispersion. More realistic results can be obtained from experiments.

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