EM Implosion Memos

Memo 2

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Analytic Calculation of Fields on z- Axis Near Second Focus

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Abstract

This paper continues the development of the field focused at the second focus of a prolate- spheroidal reflector. Here we explore the behavior of the impulse term along the z-axis as one passes through this focus.

Section 1. Introduction

This paper is an extension of two previous papers [1,2] .

In [1] some analytic approximations were developed for the focal waveform produced at the second focus of a prolate-spheroidal reflector due to a TEM wave launched from the first focus. This is extended to consider the spot size of the peak field near the second focus.
In [2] parameter study of the focal waveform was produced at the second focus of a prolate-spheroidal reflector due to a TEM wave launched from the first focus

In this paper we find the analytic calculation of fields on z- axis near second focus.

In Section .2 we show that the impulsive part of the waveform at the second focus can be described by a delta-like pulse forming for $z < z_0$ and in the limit as $z \to z_0$ gives the required true delta function. This is a physical example of the formation of a delta function.
In Section .3 ,the aperture integral gives the same result (at early time) as the exact incident wave before truncation. This gives confidence in the aperture integration.
In Section .4, we can see that the area of the $\delta -$like pulse is same for both after and before z_0.
In Section .5 we illustrate these results with a graphical example.

Section 2. Exact Solution for $z < z_0$ up to Aperture Truncation of Signal

First of all, we will find the exact E_2 for $z < z_0$ time after pulse arrival when solution no longer goes to 0.

We can write (3.13 and 3.10 in [1])

$$\begin{aligned}\vec{E}_2 &= E_0 \frac{a+z_0}{r_2}\left[\sec^2\left(\frac{\theta_2}{2}\right)\cos(\theta_2)\,\vec{1}_{\theta_2} - 2\tan\left(\frac{\theta_2}{2}\right)\frac{\sin(\phi_2)}{\sin(\theta_2)}\vec{1}_{\phi_2}\right]u\left(t + \frac{r_2}{c} - 2\frac{a}{c}\right) \\ &= E_0 \frac{a+z_0}{r_2}\left[\frac{2\cos(\phi_2)}{1+\cos(\theta_2)}\vec{1}_{\theta_2} - \frac{2\sin(\phi_2)}{1+\cos(\theta_2)}\vec{1}_{\theta_2}\right]u\left(t + \frac{r_2}{c} - 2\frac{a}{c}\right)\end{aligned} \tag{2.1}$$

On the z-axis $\theta_2 = 0$, ϕ_2 is arbitrary so lets take $\phi_2 = 0$, giving

$$r_2 = z_0 - z \tag{2.2}$$

$$E_2 = E_0 \frac{a+z_0}{z_0 - z} u\left(t + \frac{r_2}{c} - 2\frac{a}{c}\right) \text{ (oriented in the x direction)}$$

Substitute E_0 and $\cot\left(\frac{\theta_c}{2}\right)$ (2.16 in [2]) in $\vec{E}_2$

$$E_0 = \frac{V_0}{\pi f_g}\frac{1}{a - z_0}\cot\left(\frac{\theta_c}{2}\right) \tag{2.3}$$

$$\cot\left(\frac{\theta_c}{2}\right) = \left[\frac{a - z_p}{a + z_p}\right]^{1/2}\frac{b}{a + z_0}$$

Then we have,

$$\begin{aligned}E_2 &= \frac{V_0}{\pi f_g}\frac{1}{a - z_0}\frac{a+z_0}{z_0 - z}\left[\frac{a - z_p}{a + z_p}\right]^{1/2}\frac{b}{a + z_0}u\left(t + \frac{z_0 - z}{c} - 2\frac{a}{c}\right) \\ &= \frac{V_0}{\pi f_g}\frac{1}{z_0 - z}\left[\frac{a - z_p}{a + z_p}\right]^{1/2}\frac{b}{a - z_0}u\left(t + \frac{z_0 - z}{c} - 2\frac{a}{c}\right)\end{aligned} \tag{2.4}$$

We can normalize E_2 as

$$e_2 = \frac{\pi f_g \ell}{V_0}E_2 = \frac{\ell}{z_0 - z}\left[\frac{a - z_p}{a + z_p}\right]^{1/2}\frac{b}{a - z_0}u\left(t + \frac{z_0 - z}{c} - 2\frac{a}{c}\right) = e_\delta' u\left(t + \frac{z_0 - z}{c} - 2\frac{a}{c}\right) \tag{2.5}$$

$$e_\delta' = \frac{\ell}{z_0 - z}\left[\frac{a - z_p}{a + z_p}\right]^{1/2}\frac{b}{a - z_0} \tag{2.6}$$

This result applies for a time up until the signal from the truncation of the aperture is seen.

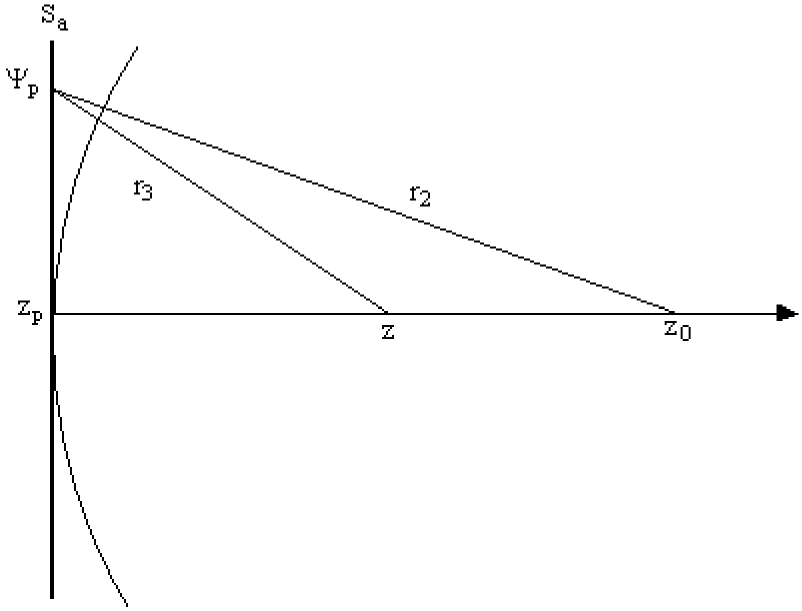

Figure 2.1 z values for $z < z_0$

For convenience we define retarded time t_r such that $t_r = 0$ is the time of arrive of the direct ray along the z-axis. The field from Ψ_p then arrives at the observer on the z-axis at a retarded time

$$\begin{aligned} ct_r &= -r_2 + r_3 + [z_0 - z] \equiv ct_{r\ end} \\ &= -\left[\Psi_p{}^2 + [z_0 - z_p]^2\right]^{1/2} + \left[\Psi_p{}^2 + [z - z_p]^2\right]^{1/2} + [z_0 - z] \end{aligned} \tag{2.7}$$

For z near z_0 this is approximately

$$\begin{aligned} r_3 - r_2 &= \left[\Psi_p{}^2 + [z - z_p]^2\right]^{1/2} + \left[\Psi_p{}^2 + [z_0 - z_p]^2\right]^{1/2} \\ &= \left[\Psi_p{}^2 + [z_0 - z_p + [z - z_0]]^2\right]^{1/2} + \left[\Psi_p{}^2 + [z_0 - z_p]^2\right]^{1/2} \text{ let's find} \end{aligned}$$

$$\begin{aligned} &\left[\Psi_p{}^2 + [z_0 - z_p + [z - z_0]]^2\right]^{1/2} = \left[\Psi_p{}^2 + [z_0 - z_p]^2 + 2[z_0 - z_p][z - z_0] + [z - z_0]^2\right]^{1/2} \\ &= \left[\Psi_p{}^2 + [z_0 - z_p]^2\right]^{1/2}\left[1 + \frac{2[z_0 - z_p][z - z_0] + [z - z_0]^2}{\Psi_p{}^2 + [z_0 - z_p]^2}\right]^{1/2} \\ &= \left[\Psi_p{}^2 + [z_0 - z_p]^2\right]^{1/2} + \frac{[z_0 - z_p][z - z_0]}{\left[\Psi_p{}^2 + [z_0 - z_p]^2\right]^{1/2}} + O\left([z - z_0]^2\right) \end{aligned} \tag{2.8}$$

So we can find

$$r_3 - r_2 = \frac{[z_0 - z_p][z - z_0]}{\left[\Psi_p^{\ 2} + [z_0 - z_p]^2\right]^{1/2}}$$

$$ct_{r\ end} = [z_0 - z]\left[1 - \frac{[z_0 - z_p]}{\left[\Psi_p^{\ 2} + [z_0 - z_p]^2\right]^{1/2}}\right] = [z_0 - z]\left[\frac{r_2 - z_0 + z_p}{r_2}\right] \tag{2.9}$$

We notive that e_2 is proportional to $[z - z_0]^{-1}$ and ct_r is proportional to $[z - z_0]$. The product gives the “area “ under the pulse as

$$\begin{aligned} E_{2\,mag}\ t_{r\ end} &= \frac{V_0}{\pi f_g}\frac{1}{z_0 - z}\left[\frac{a - z_p}{a + z_p}\right]^{1/2}\frac{b}{a - z_0}\frac{[z_0 - z]}{c}\left[1 - \frac{[z_0 - z_p]}{\left[\Psi_p^{\ 2} + [z_0 - z_p]^2\right]^{1/2}}\right] \\ &= \frac{V_0}{\pi f_g c}\left[\frac{a - z_p}{a + z_p}\right]^{1/2}\frac{b}{a - z_0}\left[1 - \frac{[z_0 - z_p]}{\left[\Psi_p^{\ 2} + [z_0 - z_p]^2\right]^{1/2}}\right] \\ &= \frac{V_0}{\pi f_g c}\left[\frac{a - z_p}{a + z_p}\right]^{1/2}\frac{b}{a - z_0}\left[1 - \left[1 + \left[\frac{\Psi_p}{z_0 - z_p}\right]^2\right]^{-1/2}\right] \end{aligned} \tag{2.10}$$

This is like an impulse , going to zero width as $z \to z_0$. Let us compare this with (5.1) in [509]. They are exactly same! This shows that the impulsive part of the waveform at the second focus can be described by a delta-like pulse forming for $z < z_0$ and in the limit as $z \to z_0$ gives the required true delta function. This is a physical example of the formation of a delta function.

We can find the normalized value of the (2.10) from (2.5) and (2.9) so

$$\begin{aligned} E_{2\,mag}\ t_{r\ end} &= \frac{\ell}{z_0 - z}\left[\frac{a - z_p}{a + z_p}\right]^{1/2}\frac{b}{a - z_0}\frac{[z_0 - z]}{c}\left[\frac{r_2 - z_0 + z_p}{r_2}\right] \\ &= \frac{b\ell}{c(a - z_0)}\left[\frac{a - z_p}{a + z_p}\right]^{1/2}\left[\frac{r_2 - z_0 + z_p}{r_2}\right] \end{aligned} \tag{2.11}$$

Section 3. Approximate Solution for $z < z_0$ by Aperture Integration for Early Time with z Near z_0

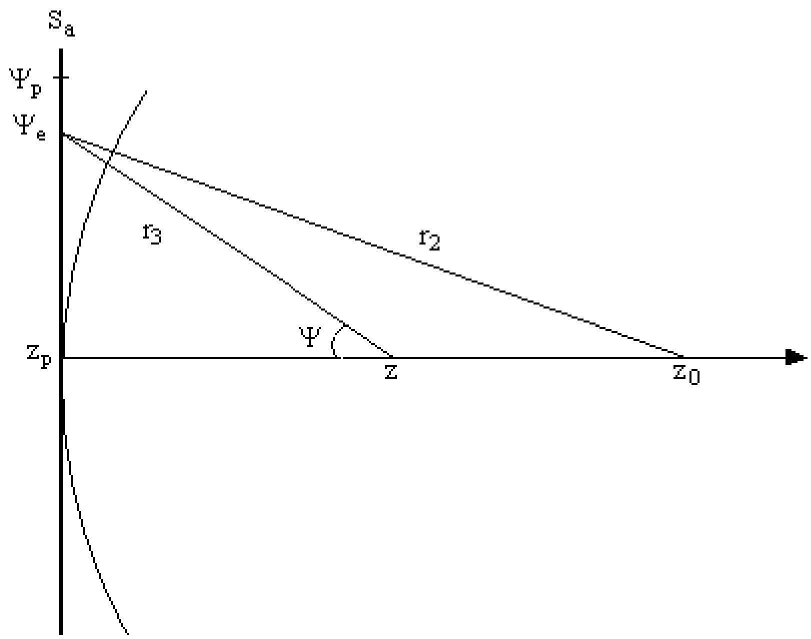

Figure 3.1 z values for $z < z_0$

The field seen at observer at $z\,\vec{1}_z$ is seen at a time later

$$ct_r = \left[r_3 - \left[z - z_p\right]\right] - \left[r_2 - \left[z_0 - z_p\right]\right]$$

$$r_3 = \left[\Psi^2 + \left[z - z_p\right]^2\right]^{1/2} \quad \text{and} \quad r_2 = \left[\Psi^2 + \left[z_0 - z_p\right]^2\right]^{1/2} \quad \text{so} \tag{3.1}$$

$$ct_r = \left[\left[\Psi^2 + \left[z - z_p\right]^2\right]^{1/2} - \left[z - z_p\right]\right] - \left[\left[\Psi^2 + \left[z_0 - z_p\right]^2\right]^{1/2} - \left[z_0 - z_p\right]\right]$$

For small Ψ we have

$$ct_r = \frac{1}{2}\Psi_e{}^2\left[\frac{1}{\left[z - z_p\right]} - \frac{1}{\left[z_0 - z_p\right]}\right] \quad \text{so}$$

$$\Psi_e = \left[\frac{2ct_r}{\dfrac{1}{\left[z - z_p\right]} - \dfrac{1}{\left[z_0 - z_p\right]}}\right]^{1/2} \tag{3.2}$$

If $|z_0 - z|$ is small

$$
\begin{aligned}
ct_r &= [r_3 - r_2] + z_0 - z \\
ct_r - [z_0 - z] &= r_3 - r_2 \\
r_3 - r_2 &= \left[\Psi_e^2 + z^2\right]^{1/2} - \left[\Psi_e^2 + z_0^2\right]^{1/2} \\
&= \left[\Psi_e^2 + z_0^2 + z^2 - z_0^2\right]^{1/2} - \left[\Psi_e^2 + z_0^2\right]^{1/2}\left[1 - \frac{1}{2}\frac{z_0^2 - z^2}{\left[\Psi_e^2 + z_0^2\right]}\right] \\
&= \frac{1}{2}\frac{z_0^2 - z^2}{\Psi_e^2 + z_0^2} = \frac{[z_0 + z]}{2}\frac{[z_0 - z]}{\Psi_e^2 + z_0^2}
\end{aligned}
\tag{3.3}
$$

So we have

$$
\begin{aligned}
ct_r - [z_0 - z] &\approx + \frac{[z_0 + z]}{2}\frac{[z_0 - z]}{\Psi_e^2 + z_0^2} \\
\Psi_e^2 &= z_0^2 + \frac{1}{2}\left[\frac{z_0 + z}{\frac{ct_r}{z_0 - z} - 1}\right]
\end{aligned}
\tag{3.4}
$$

We want to take the surface integral of (4.2 in [1]) to find a new form for E_δ. It does not involve a step-function from S_a. It is now dispersed such that the integration limits can be functions of time.

$$
\begin{aligned}
E_\delta &= \frac{E_0}{\pi c}\frac{d}{dt}\int_0^{\Psi_e}\int_0^{2\pi}\frac{z - z_p}{r_3^2}\frac{a + z_0}{r_2 + z_0 + z_p}\left[\frac{z_0 - z_p}{r_2}\cos^2(\phi) + \sin^2(\phi)\right]\Psi d\theta d\Psi \\
&= \frac{E_0}{c}\frac{d}{dt}\int_0^{\Psi_e}\frac{z - z_p}{r_3^2}\frac{a + z_0}{r_2}\Psi d\Psi
\end{aligned}
\tag{3.5}
$$

Lets take the time derivative of the integral

$$
E_\delta = \frac{d\Psi_e}{dt}\frac{E_0}{c}\frac{z - z_p}{r_3^2}\frac{a + z_0}{r_2}\Psi \tag{3.6}
$$

We can find $\frac{d\Psi_e}{dt}$ from (3.2)

$$c\frac{dt_r}{d\Psi_e} = \Psi_e\left(\frac{1}{[z-z_p]} - \frac{1}{[z_0-z_p]}\right)$$

$$\frac{d\Psi_e}{c\,dt_r} = \Psi_e^{-1}\left(\frac{1}{[z-z_p]} - \frac{1}{[z_0-z_p]}\right)^{-1} \qquad (3.7)$$

So (3.5) becomes

$$E_\delta = \Psi_e^{-1}\left(\frac{1}{[z-z_p]} - \frac{1}{[z_0-z_p]}\right)^{-1} \frac{E_0}{c}\frac{z-z_p}{r_3^2}\frac{a+z_0}{r_2}\Psi_e \qquad (3.8)$$

$$E_\delta = \left(\frac{1}{[z-z_p]} - \frac{1}{[z_0-z_p]}\right)^{-1} E_0 \frac{z-z_p}{\Psi_e^2+(z-z_p)^2}\frac{a+z_0}{\left(\Psi_e^2+(z_o-z_p)^2\right)^{1/2}}$$

$$= E_0\left(\frac{1}{[z-z_p]} - \frac{1}{[z_0-z_p]}\right)^{-1}\frac{1}{(z-z_p)}\frac{a+z_0}{(z_o-z_p)} \quad \text{as } \Psi_e \to 0 \qquad (3.9)$$

$$= \frac{V_0}{\pi f_g}\frac{1}{z_0-z}\left[\frac{a-z_p}{a+z_p}\right]^{1/2}\frac{b}{a-z_0}$$

As we can see it is same as E_2 in (2.4) . This shows that the aperture integral gives the same result (at early time) as the exact incident wave before truncation. This gives confidence in the aperture integration. The reader can note that since the above gives a pulse of width greater than zero, one can add a correction term (zero at zero retarded time) from E_s, also dispersed like E_δ.

Section 4. Approximate Solution for $z > z_0$ by Aperture Integration for Early Time with z Near z_0

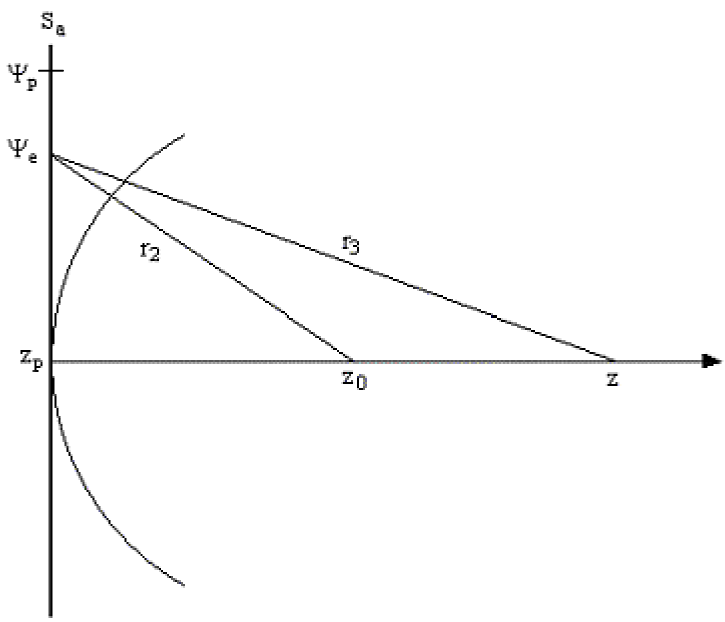

Figure 4.1 z values for $z > z_0$

Imagine Ψ_e close to Ψ_p for which

$$r_3 = \left[\Psi_e{}^2 + \left[z - z_p\right]^2\right]^{1/2} \ , \ r_2 = \left[\Psi_e{}^2 + \left[z_0 - z_p\right]^2\right]^{1/2} \tag{4.1}$$

On S_a fields arrive at the time

$$ct_{r\,a} = \left[z_0 - z_p\right] - \left[\Psi_e{}^2 + \left[z_0 - z_p\right]^2\right]^{1/2} \tag{4.2}$$

which sets $ct_{r\,a} = 0$ on the aperture center . Fields arrive at z at the time

$$ct_{r\,a} + r_3 \tag{4.3}$$

The first fields at z are from $\Psi = \Psi_p$, at the time

$$ct_{r\,\mathrm{first}} = \left[z_0 - z_p\right] - \left[\Psi_p{}^2 + \left[z_0 - z_p\right]^2\right]^{1/2} + \left[\Psi_p{}^2 + \left[z - z_p\right]^2\right]^{1/2} \tag{4.4}$$

The last fields come along the z axis at

$$ct_{r\,last} = \left[z_0 - z_p\right] - \left[z_0 - z_p\right] + z - z_p = z - z_p \tag{4.5}$$

So define retarded time by subtracting $z - z_p$ so the pulse stops at 0 but begins at

$$\begin{aligned} ct_{r\,begin} &= \left[z_0 - z_p\right] - \left[\Psi_p{}^2 + \left[z_0 - z_p\right]^2\right]^{1/2} + \left[\Psi_p{}^2 + \left[z - z_p\right]^2\right] - \left[z - z_p\right] \\ &= \left[z - z_p\right] - \left[\Psi_p{}^2 + \left[z_0 - z_p\right]^2\right]^{1/2} + \left[\Psi_p{}^2 + \left[z - z_p\right]^2\right] < 0 \end{aligned} \tag{4.6}$$

From an arbitrary point on S_a

$$ct_r = \left[z_0 - z\right] - \left[\Psi^2 + \left[z_0 - z_p\right]^2\right]^{1/2} + \left[\Psi^2 + \left[z - z_p\right]^2\right]^{1/2} \tag{4.7}$$

We can take the derivative of (4.7)

$$c\frac{dt}{d\Psi_e} = -\frac{\Psi_e}{\left[\Psi_e{}^2 + \left[z_0 - z_p\right]^2\right]^{1/2}} + \frac{\Psi_e}{\left[\Psi_e{}^2 + \left[z - z_p\right]^2\right]^{1/2}} \tag{4.8}$$

$$c\frac{dt}{d\Psi_e}\bigg|_{\Psi_e = \Psi_p} = -\frac{\Psi_p}{\left[\Psi_p{}^2 + \left[z_0 - z_p\right]^2\right]^{1/2}} + \frac{\Psi_p}{\left[\Psi_p{}^2 + \left[z - z_p\right]^2\right]^{1/2}} \tag{4.9}$$

So E_δ can be found as,

$$\begin{aligned} E_\delta &= \frac{E_0}{c}\frac{d\Psi}{dt}\bigg|_{\Psi_p} \frac{z - z_p}{r_3{}^2}\frac{a + z_0}{r_2}\Psi_p \\ &= \frac{V_0\ell}{\pi f_g \ell}\frac{1}{a - z_0}\cot\left(\frac{\theta_c}{2}\right)\frac{d\Psi}{dt}\bigg|_{\Psi_p} \frac{z - z_p}{r_3{}^2}\frac{a + z_0}{r_2}\Psi_p \end{aligned} \tag{4.10}$$

E_δ can be normalized as,

$$\begin{aligned} e_\delta{}' &= \frac{\pi f_g \ell}{V_0}E_\delta = \frac{\ell}{a - z_0}\cot\left(\frac{\theta_c}{2}\right)\frac{z - z_p}{r_3{}^2}\frac{\Psi_p}{r_2}\frac{1}{\Psi_p}\left[-\frac{1}{r_2} + \frac{1}{r_3}\right]^{-1} \\ &= \frac{\ell b}{a - z_0}\left[\frac{a - z_p}{a + z_p}\right]\frac{z - z_p}{r_3}\left[r_3 - r_2\right]^{-1} \end{aligned} \tag{4.11}$$

We need to expand $\left[r_3 - r_2\right]$ for small $\left[z - z_0\right]$

$$\begin{aligned} r_3 &= \left[\Psi_p{}^2 + \left[\left[z - z_0\right] + \left[z_0 - z_p\right]\right]^2\right]^{1/2} \\ &= \left[\Psi_p{}^2 + \left[z_0 - z_p\right]^2\right]^{1/2}\left[1 + \frac{2\left[z - z_0\right]\left[z_0 - z_p\right] + \left[z - z_0\right]}{\left[\Psi_p{}^2 + \left[z_0 - z_p\right]^2\right]} + O\left(\left[z - z_0\right]\right)^2\right]^{1/2} \end{aligned} \tag{4.12}$$

$$r_2 = \left[\Psi_p{}^2 + [z_0 - z_p]^2\right]^{1/2} \tag{4.13}$$

$$r_3 - r_2 = \frac{[z - z_0]\,[z - z_p]}{r_2} \quad \text{as} \quad \Psi_e \to \Psi_p$$

$$\frac{r_2}{r_3} = 1 + O([z - z_0]) \cong 1$$

So the normalized field is;

$$e_\delta{}' = \frac{\ell b}{a - z_0}\left[\frac{a - z_p}{a + z_p}\right]^{1/2} \frac{1}{z - z_0} \tag{4.14}$$

It is same as (2.5)

The asymptotic form of $ct_{\,begin}$ for small $[z - z_0]$ is

$$\begin{aligned} ct_{\,r\,begin} &= [z_0 - z] - r_2 + \left[r_2{}^2 + 2[z - z_0][z_0 - z_p] + [z - z_0]^2\right]^{1/2} \\ &= [z_0 - z]\left[1 - \frac{z_0 - z_p}{r_2}\right] \\ &= [z_0 - z]\left[\frac{r_2 - z_0 + z_p}{r_2}\right] \text{ (negative)} \\ &= -ct_{r\,end} \quad (\text{as in in (2.8)}) \end{aligned} \tag{4.15}$$

The integral (or area) of the pulse is just

$$\begin{aligned} E_\delta \; t_{r\,begin} &= \frac{\ell b}{a - z_0}\left[\frac{a - z_p}{a + z_p}\right]^{1/2} \frac{1}{z - z_0}\frac{r_2}{r_3}\frac{1}{c}[z_0 - z]\left[\frac{r_2 - z_0 + z_p}{r_2}\right] \\ &= \frac{1}{c}\frac{\ell b}{a - z_0}\left[\frac{a - z_p}{a + z_p}\right]^{1/2}\left[\frac{-r_2 + z_0 - z_p}{r_3}\right] \end{aligned} \tag{4.16}$$

One can see with comparing (4.16) and (2.9) the area of E_δ is same both after and before z_0.

Section.5 Concluding Remarks

In order to illustrate what our results have shown, let us make a graph showing the normalized pulse shape for various $z - z_0$ as one goes from negative values through the second focus to positive values. For negative values the pulse follows after zero retarded time. For positive values the pulse precedes zero retarded time. For our example we take the simple case from [1,2] for which,

$$z_p = 0,\ \frac{\Psi_0}{\ell} = .5,\ \frac{a}{\ell} = \frac{5}{8},\ \frac{b}{\ell} = \frac{4}{8},\ \frac{z_0}{\ell} = \frac{3}{8},\ r_2 = a \quad \text{so}$$

$$e_\delta{}' = \frac{b\ell}{a - z_0}\frac{1}{|z - z_0|} \tag{5.1}$$

and which is related to the 3,4,5 right triangle.

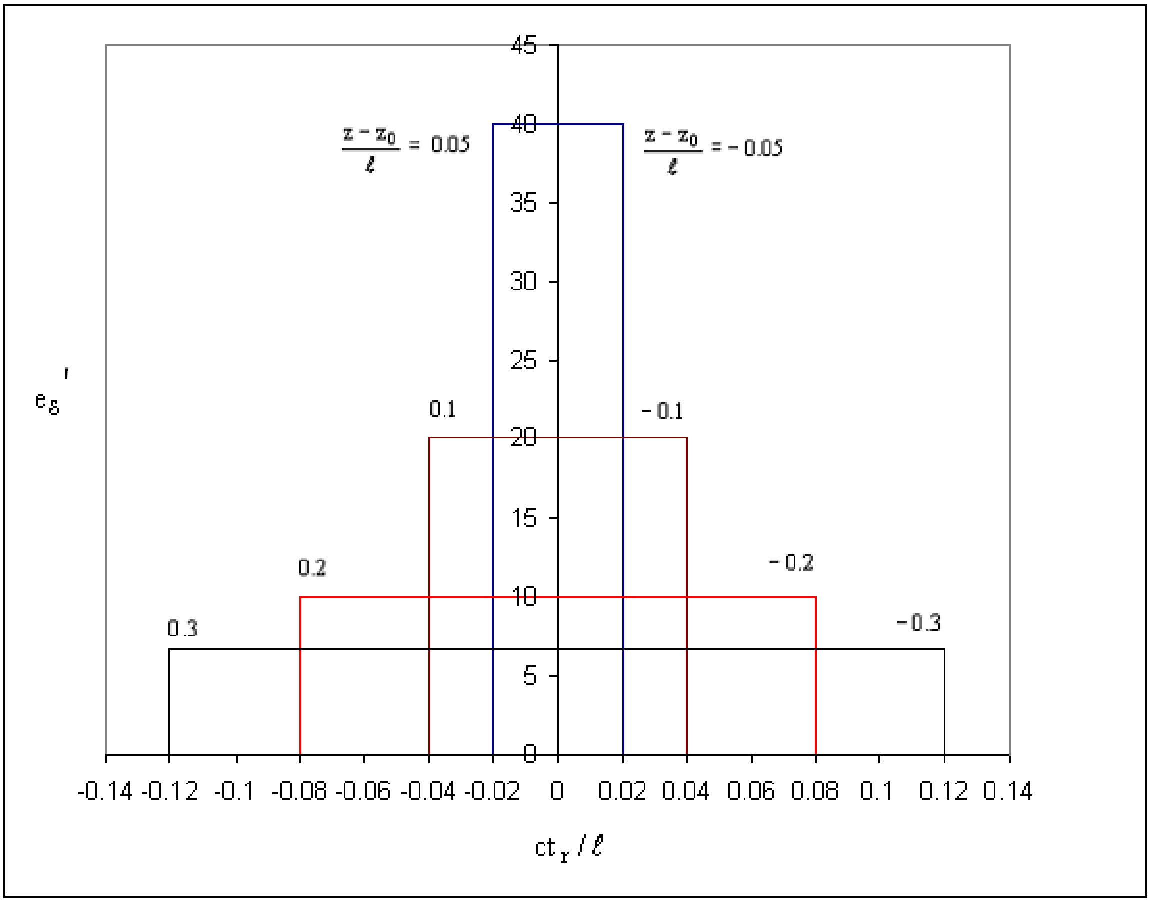

Figure 5.1 Normalizes Pulse Shape for the various $\frac{z - z_0}{\ell}$

One can see from Figure 5.1 the compression of the pulse as $z \to z_0$ and expansion of the pulse for $z > z_0$ as z increases away from z_0.

References

1. C. E. Baum, “Focal Waveform of a Prolate-Spheroidal IRA”, Sensor and Simulation Note 509, February 2006.
2. S.Altunc and C.E.Baum. “Parameter Study for Prolate-Spheroidal IRA”,EM Implosion Memos, Memo 1, May 2006.