Theoretical Notes

Note 370

June 18, 2023

Gauge Theory in Classical Electrodynamics

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Abstract— This note is a revised version of a paper the authors have published [1]. A significantly condensed version of this paper may also be seen in [2]. Jackson [3] states, "It seems necessary from time to time to show that the electric and magnetic fields are independent of the choice of gauge for the potentials." We have demonstrated the veracity of this statement in classical electrodynamics with an example. Vector and scalar potentials and an arbitrary gauge relationship are used for convenience in solving boundary value problems involving electromagnetic (EM) fields. Two of the most common gauges are the Lorenz and Coulomb gauges. However, they are not actually needed if one assumes that currents and charges are continuous functions of space and time. The EM fields, in this case, can be obtained directly without the intermediate step of calculating the potentials and taking various derivatives to obtain the fields. It is less commonly known that the Lorenz and Coulomb gauges are two special cases of a generalized gauge called the velocity gauge (or *v*-gauge). This generalized gauge is usually not taught to the students of electrodynamics. In this paper, we review the properties of various gauges and address the more general issue of gauge invariance in classical electromagnetic theory with an example.

I. INTRODUCTION

Vector and scalar potentials and their gauges are a convenient way of solving boundary value problems where the goal is to derive measurable electromagnetic fields. They are commonly introduced in textbooks to facilitate the computation of fields from specified current and charge densities, $\vec{J}(\vec{r},t)$ and $\rho(\vec{r},t)$. However, potentials are not strictly needed in classical electrodynamics if one accepts that the currents and charges are continuous functions of space and time. The first author of this paper was a student of the iconic Prof. R. W. P. King [4], who taught him, "Who needs potentials?" if the electromagnetic fields $\vec{E}(\vec{r},t)$ and $\vec{B}(\vec{r},t)$ can be solved directly from Maxwell's equations.

$$\nabla \times \vec{E}(\vec{r},t) = -\frac{\partial \vec{B}(\vec{r},t)}{\partial t} \qquad (1a)$$

$$\nabla \times \vec{H}(\vec{r},t) = \vec{J}(\vec{r},t) + \frac{\partial \vec{D}(\vec{r},t)}{\partial t} \qquad (1b)$$

$$\nabla \cdot \vec{B}(\vec{r},t) = 0 \qquad (1c)$$

$$\nabla \cdot \vec{D}(\vec{r},t) = \rho(\vec{r},t) \qquad (1d)$$

where $\vec{B} = \mu_0 \vec{H}$ and $\vec{D} = \varepsilon_0 \vec{E}$ in free space.

Potentials are presented to students in electromagnetics as a vehicle for static and dynamic computing fields. Many students are familiar with Lorenz and Coulomb gauges. It is somewhat less known that these two gauges are special cases of a generalized velocity gauge (v-gauge) [5-8], which is characterized by a non-physical parameter v that has the units of a velocity and can vary between c (the speed of light) and infinity. The two special cases of this gauge, named for Lorenz and Coulomb, result from setting $v = c$ and $v = \infty$, respectively, in the v-gauge relationship between scalar and vector potentials. The use of gauges prompts several truly relevant questions. If one uses the different gauges provided by v, are the electromagnetic fields the same? In other words, is gauge invariance a sacred principle in classical electrodynamics? In addition, is the set of gauges that can be used to compute an electromagnetic field discrete, or is it continuous, as the generalized velocity gauge suggests?

There has been some controversy regarding gauge invariance and uniqueness. For example, Engelhardt [9] has claimed that Maxwell's equations have non-unique solutions. It is also interesting to note that Wu and Wu [10] have shown that gauge invariance is violable in Quantum Physics by considering the decay of Higgs Boson into two photons! Given these considerations from published works, reviewing gauge theory and establishing its fundamental precepts is worthwhile.

This note shows that gauge invariance is <u>not</u> violated in classical electrodynamics for the continuous set of potentials encompassed in the v-gauge. Furthermore, we will see that the parameter v need not be limited to the range of c to infinity, but it can take on any value in the range $-\infty < v < \infty$ except $v = 0$. It also can be extended to any number in the complex v-plane, excluding the origin. Thus, the v-gauge provides a doubly infinite set of gauge functions for determining the EM fields.

This note is organized as follows. In section II, we describe the *v*-gauge using SI units. Much of the mathematical approach in this paper is based on the CGS unit formulation in Jackson's technical report [3]. Section III presents an example calculation of fields radiated by a time harmonic Hertzian dipole using the velocity gauge potentials. As demonstrated, the scalar potential will have an apparent velocity determined by the selection of *v* in the velocity gauge. The vector potential will have a contribution that propagates with velocity *c* and another with apparent velocity *v*. Although the potentials will appear to be non-causal in the general case (unless the Lorenz gauge is selected), the electric and magnetic fields computed using these potentials will remain causal and will be invariant to the gauge's apparent velocity *v*. Some summarizing comments are offered in section IV, followed by acknowledgments and a list of references.

II. GAUGE THEORY IN CLASSICAL ELECTRODYNAMICS

Gauge theory has been introduced in classical electrodynamics for convenience and simplicity rather than as a necessity. Considering a volumetric distribution of current and charge, as shown in Fig.1, we are interested in determining the vector electromagnetic fields $\vec{E}$ and $\vec{B}$ at a location (r,θ,ϕ) outside the source region.

For this problem, it is possible to solve Maxwell's equations directly to yield integral formulas for fields in terms of sources, as shown in [11] and numerous other references.

When the right side of (1a) is zero or negligible for quasi-static and electrostatic problems, the E-field may be represented using the gradient of a scalar potential, the same function used to describe voltage in circuits. A single scalar potential function representing three components of the vector E-field simplifies the solution.

Extending the use of potentials to solve the time-dependent Maxwell's equations in (1a) to (1d), or to their frequency-domain versions, originates with (1c): $\nabla\cdot\vec{B}(\vec{r},t)=0$. This infers the absence of magnetic monopoles, which means that magnetic flux lines close on themselves instead of terminating on magnetic charges. Since this divergence of $\vec{B}(\vec{r},t)$ vanishes, one can write

$$\vec{B}(\vec{r},t)=\nabla\times\vec{A}(\vec{r},t) \quad . \qquad (2)$$

The consequence of (2) is that one is free to choose the divergence of the vector potential $\nabla\cdot\vec{A}(\vec{r},t)$ to be any convenient scalar function we wish, and there will be no effect on the B-field.

Following Jackson's formulation in [3], we substitute (2) into (1a), and we note that the curl of the gradient of a scalar function, namely the scalar potential, is zero. This allows us to represent the electric field using both the scalar and vector potentials $\phi(\vec{r},t)$ and $\vec{A}(\vec{r},t)$ as

$$\vec{E}(\vec{r},t) = -\nabla\phi(\vec{r},t) - \frac{\partial\vec{A}(\vec{r},t)}{\partial t} \quad . \qquad (3)$$

If one performs the curl operation on both sides of (3), we will recover (1a). The derivation of (3) also indicates that there are an infinite number of pairs of scalar and vector potentials that generate the electric and magnetic fields using (3) and (2). To show this, we recall that the curl of the gradient of a scalar function is zero $\nabla\times(\nabla\chi(\vec{r},t)) = 0$ and

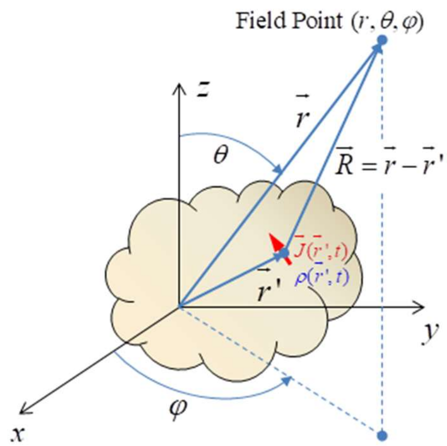


Fig. 1. A current and charge distribution producing EM fields at (r,θ,φ).

using this, we can develop different vector and scalar potentials $\vec{A}'(\vec{r},t)$ and $\phi'(\vec{r},t)$ that are then used to compute EM fields,

$$\vec{A}'(\vec{r},t) = \vec{A}(\vec{r},t) + \nabla\chi(\vec{r},t) \qquad (4a)$$

$$\phi'(\vec{r},t) = \phi(\vec{r},t) - \partial\chi(\vec{r},t)/\partial t \quad . \qquad (4b)$$

giving

$$\begin{aligned}\vec{B}'(\vec{r},t) &= \nabla\times\vec{A}'(\vec{r},t) = \nabla\times\vec{A}(\vec{r},t) + \nabla\times(\nabla\chi(\vec{r},t)) \\ &= \nabla\times\vec{A}(\vec{r},t) = \vec{B}(\vec{r},t)\end{aligned} \qquad (4c)$$

and

$$\begin{aligned}\vec{E}'(\vec{r},t) &= -\nabla\phi'(\vec{r},t) - \frac{\partial\vec{A}'(\vec{r},t)}{\partial t} = -\nabla\left(\phi(\vec{r},t) - \frac{\partial\chi(\vec{r},t)}{\partial t}\right) - \frac{\partial}{\partial t}\left(\vec{A}(\vec{r},t) + \nabla\chi(\vec{r},t)\right) \\ &= -\nabla\phi(\vec{r},t) - \frac{\partial\nabla\chi(\vec{r},t)}{\partial t} - \frac{\partial\vec{A}(\vec{r},t)}{\partial t} + \frac{\partial\nabla\chi(\vec{r},t)}{\partial t} = -\nabla\phi(\vec{r},t) - \frac{\partial\vec{A}(\vec{r},t)}{\partial t} = \vec{E}(\vec{r},t) \;.\end{aligned} \qquad (4d)$$

The scalar function $\chi(\vec{r},t)$ is termed the "gauge function" for the potentials and has units of Tesla-m^2. This function is used to convert pairs of vector and scalar potentials from one gauge to another via (4a) and (4b) while keeping the same electric and magnetic fields [3].

The coupled equations for the scalar and vector potentials are found as follows:

$$\nabla^2 \phi(\vec{r},t) + \frac{\partial}{\partial t}(\nabla \cdot \vec{A}(\vec{r},t)) = -\frac{\rho(\vec{r},t)}{\varepsilon_o} \qquad (5a)$$

$$\nabla^2 \vec{A}(\vec{r},t) - \frac{1}{c^2}\left(\frac{\partial^2 \vec{A}(\vec{r},t)}{\partial t^2}\right) - \nabla\left(\nabla \cdot \vec{A}(\vec{r},t) + \frac{1}{c^2}\frac{\partial \phi(\vec{r},t)}{\partial t}\right) = -\mu_o \vec{J}(\vec{r},t). \qquad (5b)$$

We observe that (5a) and (5b) are coupled partial differential equations for the scalar and vector potentials.

*A. **Lorenz and Coulomb Gauges Explained***

Let us examine the third term on the left side of (5b), recalling that we have yet to set the value of $\nabla \cdot \vec{A}(\vec{r},t)$. The following choices for this quantity lead to the two well-known gauges:

$$\nabla \cdot \vec{A}(\vec{r},t) = -\frac{1}{c^2}\frac{\partial \phi(\vec{r},t)}{\partial t} \qquad (\textit{Lorenz gauge}) \qquad (6)$$

$$\nabla \cdot \vec{A}(\vec{r},t) = 0 \qquad (\textit{Coulomb gauge}). \ (7)$$

We can now solve for the potentials for these two gauges by substituting $\nabla \cdot \vec{A}(\vec{r},t)$ into the coupled equations of (5a) and (5b). We will delineate the potentials for the two gauges by subscripts *L* and *C*, respectively, for the Lorenz and Coulomb gauges.

For the *Lorenz* gauge, (5a) and (5b) become,

$$\nabla^2 \phi_L(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2 \phi_L(\vec{r},t)}{\partial t^2} = -\frac{\rho(\vec{r},t)}{\varepsilon_o} \qquad (8a)$$

$$\nabla^2 \vec{A}_L(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2 \vec{A}_L(\vec{r},t)}{\partial t^2} = -\mu_o \vec{J}(\vec{r},t)\ . \qquad (8b)$$

We observe that the two potential equations decouple completely for the Lorenz gauge.

The potentials in (8), resulting from the Lorenz gauge, can be solved [3] to yield

$$\phi_L(\vec{r},t) = \frac{1}{4\pi\varepsilon_o}\iiint_V \frac{1}{R}\rho\left(\vec{r}\,', t - R/c\right)dV' \qquad (9a)$$

and

$$\vec{A}_L(\vec{r},t) = \frac{\mu_o}{4\pi}\iiint_V \frac{1}{R}\vec{J}\left(\vec{r}\,',\, t-R/c\right)dV' \quad , \qquad \text{(9b)}$$

where the spatial integrations of the sources at $\vec{r}\,'$ in volume V produce contributions to the potentials at the location $\vec{r}$. These contributions are causally retarded by a delay of *R/c,* where $\vec{R} = \vec{r} - \vec{r}\,'$; $R = \left|\vec{r} - \vec{r}\,'\right|$.

For the *Coulomb* gauge, (5a) and (5b) become

$$\nabla^2 \phi_C(\vec{r},t) = -\frac{\rho(\vec{r},t)}{\varepsilon_o} \qquad \text{(10a)}$$

$$\nabla^2 \vec{A}_C(\vec{r},t) - \frac{1}{c^2}\left(\frac{\partial^2 \vec{A}_C(\vec{r},t)}{\partial t^2}\right) = -\mu_o\, \vec{J}(\vec{r},t) + \frac{1}{c^2}\nabla\frac{\partial \phi_C(\vec{r},t)}{\partial t} \quad . \qquad \text{(10b)}$$

Observe that for the Coulomb gauge, the two potential equations decouple in a unique way. Equation (10a) has only the scalar potential, and (10b) has both scalar and vector potentials. One way to solve this set of equations is to first solve (10a) for the scalar potential, as shown in (11a) below, and then substitute this result into (10b) to determine the solution for the vector potential. The development will involve a local integration over $\vec{J}(\vec{r},t)$ and a full space integration over the gradient term $\nabla\, \partial\phi_C(\vec{r},t)/\partial t$ computed from the solution to (10a). Alternatively, one can use a gauge function, as discussed next in Section B, to obtain the vector potential in terms of localized integrations over the charge and current densities.

Carrying out the first analysis approach allows us to write the scalar and vector Coulomb potentials corresponding to (3.2), (3.10), and (3.16) of [3] as,

$$\phi_C(\vec{r},t) = \frac{1}{4\pi\varepsilon_o}\iiint_V \frac{1}{R}\rho(\vec{r}\,',\, t)\,dV' \qquad \text{(11a)}$$

$$\begin{aligned}\vec{A}_C(\vec{r},t) = &\frac{\mu_o}{4\pi}\iiint_V \frac{1}{R}\left[\vec{J}\left(\vec{r}\,',t-R/c\right) - c\hat{R}\rho\left(\vec{r}\,',t-R/c\right)\right]dV' \\ &+ \frac{1}{4\pi\varepsilon_o}\iiint_V \frac{\hat{R}}{R^2}\int_0^{R/c} d\tau\, \rho(\vec{r}\,',t-\tau)\,dV'\end{aligned} \qquad \text{(11b)}$$

or alternatively, by

$$\begin{aligned}\vec{A}_C(\vec{r},t) = &\frac{\mu_o}{4\pi}\iiint_V \frac{1}{R}\left[\vec{J}\left(\vec{r}\,',t-R/c\right) - \hat{R}\left(\hat{R}\cdot\vec{J}\left(\vec{r}\,',t-R/c\right)\right)\right]dV' \\ &+ \frac{1}{4\pi\varepsilon_o}\iiint_V \frac{1}{R^3}\int_0^{R/c} \tau\, d\tau\left[3\hat{R}\left(\hat{R}\cdot\vec{J}(\vec{r}\,',t-\tau)\right) - \vec{J}(\vec{r}\,',t-\tau)\right]dV' \; .\end{aligned} \qquad \text{(11c)}$$

In this last expression $\hat{R} = \vec{R} / R$ denotes the unit vector in the direction of $\vec{R}$.

It is noteworthy that the scalar potential at the observer location given by (11a) for the Coulomb gauge is not retarded but is instantaneous. In contrast, the vector potential has source contributions retarded by *R/c*. The Coulomb gauge scalar potential is fully non-causal, reaching the observer instantaneously from the source as if it propagated with infinite speed!

What is interesting is the underlying fact that these potentials are non-unique. Although we are now able to determine the electromagnetic fields from these non-unique potentials, we defer this calculation to introduce the *v*-gauge [5-8]. We will obtain the same electromagnetic fields, regardless of the gauge used, as implied by (4c) and (4d).

*B. **Generalized v-Gauge***

The *v*-gauge is akin to the Lorenz gauge, and it is obtained by replacing the speed of light *c* with a *generalized* speed *v* in our choice of $\nabla \cdot \vec{A}(\vec{r},t)$. Drury [6] has an excellent explanation of this generalized *v*-gauge.

In this gauge, we let

$$\nabla \cdot \vec{A}_V(\vec{r},t) + \frac{1}{v^2}\frac{\partial \phi_V(\vec{r},t)}{\partial t} = 0 \ . \tag{12}$$

Following the formulation of Jackson [3] but using SI units, we arrive at the expression for the scalar potential, as

$$\phi_V(\vec{r},t) = \frac{1}{4\pi\varepsilon_o}\iiint_V \frac{1}{R}\,\rho\left(\vec{r}\,',\, t - R/v\right) dV' . \tag{13}$$

The above solution for ϕ_V in the *v*-gauge reduces to the corresponding solutions for Lorenz gauge in (9a) and Coulomb gauge in (11a) when $v = c$ and $v \to \infty$.

Next, we need to determine the vector potential for the *v*-gauge. Once again, we follow Jackson's formulation for finding the vector potential. We begin using the scalar gauge function $\chi(\vec{r},t)$ appearing in (4b) to define differences of scalar potentials between the Lorenz and *v*-gauges,

$$\frac{\partial \chi_V}{\partial t} = \phi_L(\vec{r},t) - \phi_V(\vec{r},t) = \frac{1}{4\pi\varepsilon_o}\iiint_V \frac{1}{R}\left[\rho\left(\vec{r}\,',t - R/c\right) - \rho\left(\vec{r}\,',t - R/v\right)\right] dV'. \tag{14}$$

The bracketed term in (14) can be written as

$$\left[\rho\left(\vec{r}\,',t-R/c\right)-\rho\left(\vec{r}\,',t-R/v\right)\right]=\int_{R/v}^{R/c}d\tau\frac{\partial}{\partial t}\rho(\vec{r}\,',t-\tau) \quad . \quad (15)$$

Substituting (15) into (14), we obtain

$$\frac{\partial\chi_V}{\partial t}=\frac{1}{4\pi\varepsilon_o}\iiint_V\frac{1}{R}\int_{R/v}^{R/c}d\tau\frac{\partial}{\partial t}\rho(\vec{r}\,',t-\tau)\,dV' \quad . \qquad (16)$$

Then integrating with respect to *t,* we find

$$\chi_V(\vec{r},t)=\frac{-1}{4\pi\varepsilon_o}\iiint_V\frac{1}{R}\int_{R/v}^{R/c}d\tau\,\rho(\vec{r}\,',t-\tau)\,dV' \,. \qquad (17)$$

Ultimately, we will need the gradient of (17) to evaluate the vector potential with the *v*-gauge. To compute the gradient, we differentiate only the spatial terms involving *R*, as indicated by the brackets in the equation below,

$$\nabla\chi_V(\vec{r},t)=-\frac{1}{4\pi\varepsilon_o}\iiint_V\nabla\left(\frac{1}{R}\int_{R/v}^{R/c}\right)d\tau\,\rho(\vec{r}\,',t-\tau)\,dV' \quad . \quad (18)$$

Using the product rule and Leibniz's rule together with $\nabla\vec{R}=\hat{R}$ and $\nabla\left(1/R\right)=-\hat{R}/R^2$ we obtain

$$\nabla\chi_v(\vec{r},t)=\frac{1}{4\pi\varepsilon_o}\iiint_V\left[-\frac{\hat{R}}{R^2}\tau\rho(\vec{r}\,',t-\tau)\Big|_{R/v}^{R/c}+\frac{\hat{R}}{R^2}\int_{R/v}^{R/c}d\tau\,\rho(\vec{r}\,',t-\tau)\right]dV'. \qquad (19)$$

Both terms of (19) can be more compactly written using integration by parts, resulting in

$$\nabla\chi_V(\vec{r},t)=\frac{-1}{4\pi\varepsilon_o}\iiint_V\frac{\hat{R}}{R^2}\left[\int_{R/v}^{R/c}\tau\frac{\partial\rho(\vec{r}\,',t-\tau)}{\partial\tau}d\tau\right]dV' \,. \quad (20)$$

With the continuity relation $\nabla'\cdot\vec{J}(\vec{r},t-\tau)=-\partial\rho(\vec{r}\,',t-\tau)/\partial t=+\partial\rho(\vec{r}\,',t-\tau)/\partial\tau$, we can write (20) as

$$\nabla\chi_V(\vec{r},t)=\frac{-1}{4\pi\varepsilon_o}\iiint_V\frac{\hat{R}}{R^2}\int_{R/v}^{R/c}\tau\,d\tau\,\nabla'\cdot\vec{J}(\vec{r}\,',t-\tau)\,dV'. \qquad (21)$$

This expression is the SI unit version of Jackson's (7.6) in [3]. With the gradient relationship $\nabla' f(\vec{R},t)=-\nabla f(\vec{R},t)$ and integration by parts, we obtain the k^{th} Cartesian component of the gradient,

$$\left(\nabla \chi_V(\vec{r},t)\right)_k = \frac{-1}{4\pi\varepsilon_o}\iiint_V \frac{\partial}{\partial x_k}\left[\frac{\hat{R}}{R^2}\int_{R/v}^{R/c} d\tau\, \tau\, J_k(\vec{r}\,',t-\tau)\right] dV\,'. \tag{22}$$

Several further steps lead to the v-gauge vector potential reported by Jackson in his (7.7) as

$$\begin{aligned}\vec{A}_V(\vec{r},t) &= \frac{\mu_o}{4\pi}\iiint_V \frac{1}{R}\left[\vec{J}\left(\vec{r}\,',t-R/c\right) - \hat{R}\left(\hat{R}\cdot\vec{J}\left(\vec{r}\,',t-R/c\right)\right) + \frac{c^2}{v^2}\hat{R}\left(\hat{R}\cdot\vec{J}\left(\vec{r}\,',t-R/c\right)\right)\right] dV\,' \\ &+ \frac{1}{4\pi\varepsilon_o}\iiint_V \frac{1}{R^3}\int_{R/v}^{R/c} \tau\, d\tau\left[3\hat{R}\left(\hat{R}\cdot\vec{J}(\vec{r}\,',t-\tau)\right) - \vec{J}(\vec{r}\,',t-\tau)\right] dV\,'.\end{aligned} \tag{23a}$$

By using the continuity equation in developing the equation above, we can express the vector potential entirely in terms of the current density. However, if one were to retain the charge density and compute the vector potential using integrations over both current and charge sources, the vector potential takes the alternate form given by Jackson in his (7.8), which in SI units, is

$$\vec{A}_V(\vec{r},t) = \frac{\mu_o}{4\pi}\iiint_V \frac{1}{R}\left[\vec{J}\left(\vec{r}\,',t-R/c\right) - \hat{R}\,c\rho\left(\vec{r}\,',t-R/c\right) + \hat{R}\frac{c^2}{v}\rho\left(\vec{r}\,',t-R/v\right) + \frac{\hat{R}}{R}c^2\int_{R/v}^{R/c} d\tau\ \rho(\vec{r}\,',t-\tau)\right] dV\,'. \tag{23b}$$

As a check of our previous statement, if we set $v = c$ in (23a) or (23b), we obtain the Lorenz gauge expression of (9b). Likewise, if we let $v \to \infty$ these expressions reduce to the Coulomb gauge expressions of (11c) and (11b). Equations (13) and (23a or b) complete the formulation of the scalar and vector potentials for the v-gauge.

One of the reviewers pointed out that under the Coulomb gauge where the vector potential has zero divergence, as shown in (7) and in (12) as $v \to \infty$, its contribution to both the electric and magnetic fields, via (2) and (3), are solenoidal (also termed "transversal" in the context of 3D spatial Fourier Transforms). The scalar potential, whose contribution via the gradient in (3), provides the electric field with an irrotational component (termed "longitudinal" in the realm of 3D Fourier Transforms). When using the v-gauge with finite velocity, so the divergence of the vector potential is not zero using (12), the vector potential has both solenoidal (transversal) and irrotational (longitudinal) portions. This can be seen directly the change of gauge formula shown in (4a) when applied to the cases of Coulomb and finite velocity, $\vec{A}_V(\vec{r},t) = \vec{A}_C(\vec{r},t) + \nabla\chi(\vec{r},$ Since $\vec{A}_C$ is solenoidal and the gradient of the gauge function is irrotational it follows that the finite velocity v-gauge vector potential will contribute both categories of vector function to the electric field while still contributing only a solenoidal part to the magnetic field due to the curl operation in (2). This can also be seen explicitly in the gauge transformation steps depicted in (4c) and (4d).

*C. **Evaluation of the E and B Fields***

The last step in this process is to derive the electromagnetic fields from the potentials in the generalized *v*-gauge, using (2) for *B* and (3) for *E*.

We calculate the two terms in (3) separately for the E-field. Using (13), we calculate the first term as

$$-\nabla\phi_V(\vec{r},t)=\frac{1}{4\pi\varepsilon_o}\iiint_V \frac{1}{R}\left[\frac{\hat{R}}{R}\rho(\vec{r}\,',t-R/v)+\hat{R}\frac{1}{v}\frac{\partial}{\partial t}\rho(\vec{r}\,',t-R/v)\right]dV' \qquad (24)$$

The second term of (3) is found using (23b), and this is

$$-\frac{\partial\vec{A}_V(\vec{r},t)}{\partial t}=\frac{\mu_o}{4\pi}\iiint_V \frac{1}{R}\left\{\left[-\frac{\partial\vec{J}(\vec{r}\,',t')}{\partial t}+\hat{R}c\frac{\partial\rho(\vec{r}\,',t')}{\partial t}\right]_{ret\,c}-\hat{R}\,\frac{c^2}{v}\left[\frac{\partial\rho(\vec{r}\,',t')}{\partial t}\right]_{ret\,v}-\frac{\hat{R}}{R}c^2\int_{R/v}^{R/c}d\tau\frac{\partial\rho(\vec{r}\,',t-\tau)}{\partial t}\right\}dV' \qquad (25a)$$

The term $\partial\rho/\partial t$ in the integral of (25a) is equal to $-\partial\rho/\partial\tau$, and we can evaluate the integral to yield

$$-\frac{\partial\vec{A}_V(\vec{r},t)}{\partial t}=\frac{\mu_o}{4\pi}\iiint_V \frac{1}{R}\left\{\left[-\frac{\partial\vec{J}(\vec{r}\,',t')}{\partial t}+c\hat{R}\frac{\partial\rho(\vec{r}\,',t')}{\partial t}+\frac{\hat{R}}{R}c^2\rho(\vec{r}\,',t')\right]_{ret\,c}-\left[\frac{\hat{R}}{R}c^2\rho(\vec{r}\,',t')+\hat{R}\frac{c^2}{v}\frac{\partial\rho(\vec{r}\,',t')}{\partial t}\right]_{ret\,v}\right\}dV'. \qquad (25b)$$

By adding (24) and (25b), we obtain the v-gauge electric field as

$$\begin{aligned}\vec{E}_V(\vec{r},t)=&\frac{1}{4\pi\varepsilon_o}\iiint_V \frac{1}{R}\left[\frac{\hat{R}}{R}\rho(\vec{r}\,',t')+\hat{R}\frac{1}{v}\frac{\partial\rho(\vec{r}\,',t')}{\partial t}\right]_{ret\,v}dV'\\&+\frac{\mu_o}{4\pi}\iiint_V \frac{1}{R}\left\{\left[-\frac{\partial\vec{J}(\vec{r}\,',t')}{\partial t}+c\hat{R}\frac{\partial\rho(\vec{r}\,',t')}{\partial t}+c^2\frac{\hat{R}}{R}\rho(\vec{r}\,',t')\right]_{ret\,c}-\left[c^2\frac{\hat{R}}{R}\rho(\vec{r}\,',t')+\frac{c^2}{v}\hat{R}\frac{\partial\rho(\vec{r}\,',t')}{\partial t}\right]_{ret\,v}\right\}dV'\end{aligned} \qquad (26)$$

In this expression $c^2=1/(\mu_o\varepsilon_o)$ and we find that the retarded terms $t'=t-(R/v)$ from the scalar and vector potentials cancel, and this gives us a final expression for the electric field as

$$\vec{E}_V(\vec{r},t)=\vec{E}_L(\vec{r},t)=\frac{\mu_o}{4\pi}\iiint_V \frac{1}{R}\left[-\frac{\partial\vec{J}(\vec{r}\,',t')}{\partial t}+c\hat{R}\frac{\partial\rho(\vec{r}\,',t')}{\partial t}+c^2\frac{\hat{R}}{R}\rho(\vec{r}\,',t')\right]_{ret\,c}dV' \qquad (27)$$

This expression is the same as would be found by substituting the Lorenz gauge potentials of (9a) and (9b) into (4d) to produce the E-field. It is emphasized that the resultant electric field found using the *v*-

gauge potentials is precisely the same as using $v = c,$ and it does not depend on the selected v at all. This result means that the electric field is gauge invariant.

We can derive the corresponding magnetic flux density by taking the curl of the v-gauge vector potential from Eq(23b) as follows:

$$\vec{B}_V(\vec{r},t) = \nabla\times\vec{A}_V(\vec{r},t)$$
$$= \frac{\mu_o}{4\pi}\nabla\times\iiint_V \frac{1}{R}\left[\vec{J}(\vec{r}\,',t-R/c) - \hat{R}\,c\,\rho\big(\vec{r}\,',t-R/c\big) + \hat{R}\frac{c^2}{v}\rho\big(\vec{r}\,',t-R/v\big) + \frac{\hat{R}}{R}c^2\int_{R/v}^{R/c} d\tau\ \rho(\vec{r}\,',t-\tau)\right]dV\,'. \tag{28}$$

Since $\nabla\times\left[\hat{R}\,\text{any scalar function}\right] = 0$ the terms containing the charge density produce no contribution. This shows that the B field from the v-gauge depends only on the current density and is given by

$$\vec{B}_V(\vec{r}\,,t) = \frac{\mu_o}{4\pi}\nabla\times\left[\iiint_V \frac{1}{R}\vec{J}(\vec{r}\,',t-R/c)\,dV\,'\right]. \tag{29}$$

As expected, the same result is obtained by substituting the Lorenz gauge vector potential from (9b) into (4b) to generate the B-field. By taking the field point curl inside of the integral, we can use the product rule for curls, $\nabla\times\left[\vec{J}\left(\vec{r}\,',\tau\right)/R\right] = \left(1/R\right)\nabla\times\vec{J}\left(\vec{r}\,',\tau\right) + \nabla\left(1/R\right)\times\vec{J}\left(\vec{r}\,',\tau\right)$
to rewrite (29) as

$$\vec{B}_V(\vec{r}\,',t) = \frac{\mu_o}{4\pi}\iiint_V\left[\frac{1}{R}\ \nabla\times\vec{J}(\vec{r}\,',t') - \frac{\hat{R}}{R^2}\times\vec{J}(\vec{r}\,',t')\right]_{ret} dV\,'. \tag{30}$$

It is important to recognize that

$$\left[\nabla\times\vec{J}(\vec{r}\,',t')\right]_{ret} = \left.\frac{\partial\vec{J}(\vec{r}\,',t')}{\partial t'}\right|_{ret}\times\nabla\left(t-\frac{R}{c}\right) = \frac{1}{c}\left.\frac{\partial\vec{J}(\vec{r}\,',t')}{\partial t'}\right|_{ret}\times\hat{R}\ . \tag{31}$$

Substituting (31) into (30) yields

$$\vec{B}_V(\vec{r},t') = \vec{B}_L(\vec{r},t') = \frac{\mu_o}{4\pi}\iiint_V\left[\frac{1}{R^2}\vec{J}(\vec{r}\,',t') + \frac{1}{c\,R}\frac{\partial}{\partial t'}\vec{J}(\vec{r}\,',t')\right]_{ret}\times\hat{R}\,dV\,'\ . \tag{32}$$

Thus, like the electric field, the B field is independent of v and is also gauge invariant. Equations (27) and (32) are given in CGS units in [6] in (5.2) and (5.3) and are termed the “Jefimenko” expressions. They appear in several texts [12-14].

In concluding this section, we emphasize that while the potentials depend on the gauge used, such as the v-gauge, the Lorenz gauge, or the Coulomb gauge, the electromagnetic fields are gauge invariant.

III. EXAMPLE USING THE v-GAUGE

For example, consider the ideal Hertz dipole shown in Fig. 2. We assume a time-harmonic variation of a current element $I(t) = I_o \cos\omega t$ flowing in an infinitesimally thin conductor along the z-axis over a differential length $-d\ell/2 \le z' \le d\ell/2$. The current density may be expressed using Dirac delta functions and Heaviside unit step functions as

$$\vec{J}\left(\vec{r}\,',t\right) = \hat{z}\, I(t)\,\delta(x')\,\delta(y')\left[u\left(z'+d\ell/2\right) - u\left(z'-d\ell/2\right)\right]. \tag{33}$$

Due to charge continuity $I(t) = dq/dt$, there exist equal and opposite time-varying charges $\pm q(t)$ at the ends of the differential filament, as shown in the figure where $q(t) = (I_0/\omega)\sin\omega t$. The resultant charge density is given by

$$\rho(\vec{r}\,',t) = q(t)\,\delta(x')\,\delta(y')\left[\delta\left(z'-d\ell/2\right) - \delta\left(z'+d\ell/2\right)\right]. \tag{34}$$

The Lorenz gauge scalar potential given by (9a) is evaluated below. Integration of the delta functions sifts out the point values of the charges $\pm q(t)$ at the ends of the filament. Contributions of these charges to the potential are delayed by propagation over the distances R_1 and R_2 shown in the figure.

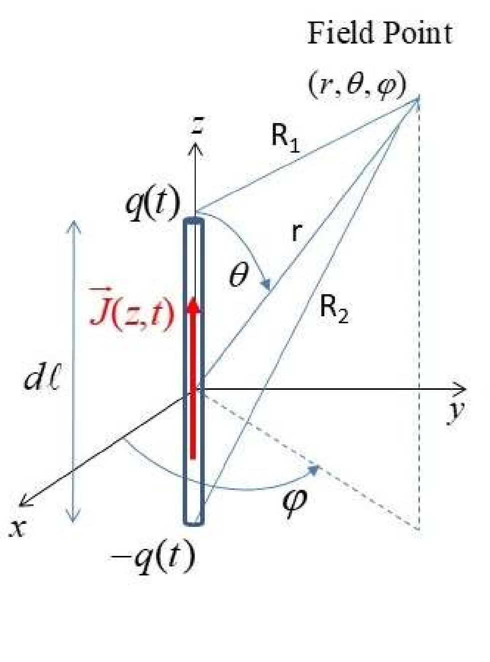


Fig. 2. Hertzian dipole in a spherical coordinate system.

$$\phi_L(r,\theta,t)=\frac{1}{4\pi\varepsilon_o}\iiint_V \frac{1}{R}\rho\left(\vec{r}\,',t-R/c\right)dV'$$
$$=\frac{1}{4\pi\varepsilon_o}\left[\frac{1}{R_1}q\left(t-R_1/c\right)-\frac{1}{R_2}q\left(t-R_2/c\right)\right]=\frac{I_o}{4\pi\varepsilon_o\omega}\left[\frac{1}{R_1}\sin\omega\left(t-R_1/c\right)-\frac{1}{R_2}\sin\omega\left(t-R_2/c\right)\right] \quad (35)$$
$$\approx\frac{I_o}{4\pi\varepsilon_o\omega}\left[\frac{\sin\omega\left(t-\frac{r}{c}+\frac{d\ell}{2c}\cos\theta\right)}{r-\frac{d\ell}{2}\cos\theta}-\frac{\sin\omega\left(t-\frac{r}{c}-\frac{d\ell}{2c}\cos\theta\right)}{r+\frac{d\ell}{2}\cos\theta}\right]=\frac{I_o d\ell\cos\theta}{4\pi\varepsilon_o}\left[\frac{1}{cr}\cos\omega\left(t-r/c\right)+\frac{1}{\omega r^2}\sin\omega\left(t-r/c\right)\right].$$

The first approximation of (35) (on line 3) assumes that $r >> dl$. The final version of (35) results from using the relationship $\sin(\alpha\pm\beta)=\sin\alpha\cos\beta\pm\cos\alpha\sin\beta$.

The Lorenz gauge vector potential may be computed using (9b). Since $r >> dl$, the term $1/R(z')\doteq 1/r$ and we can write the resulting Lorenz vector potential as

$$\vec{A}_L(\vec{r},t)=\frac{\mu_o}{4\pi}\iiint_V \frac{1}{R}\vec{J}\left(\vec{r}\,',t-R/c\right)dV'$$
$$=\frac{\mu_o\hat{z}}{4\pi}\iiint_V\left\{\frac{1}{R}I\left(t-R/c\right)\delta(x')\,\delta(y')\ \left[u\left(z'+d\ell/2\right)-u\left(z'-d\ell/2\right)\right]\right\}dx'dy'dz' \quad (36)$$
$$=\frac{\mu_o\hat{z}}{4\pi r}\ I\left(t-r/c\right)=\hat{z}\frac{\mu_o I_0\,d\ell}{4\pi r}\ \cos\omega\left(t-r/c\right)\ .$$

Notice that the Lorenz potentials propagate away from the dipole with velocity c. The vector potential $\vec{A}_L$ follows the time variation of the current while ϕ_v has portions that separately track the time variation of current and charges on the dipole.

The v-gauge scalar potential can be computed by comparing (13) to (35), where we replace c with v:

$$\phi_V(r,\theta,t)=\frac{1}{4\pi\varepsilon_o}\iiint_V\frac{1}{R}\rho\left(\vec{r}\,',t-R/v\right)dV'\approx\frac{I_0 d\ell\cos\theta}{4\pi\varepsilon_o}\left[\frac{1}{vr}\cos\omega\left(t-r/v\right)+\frac{1}{\omega r^2}\sin\omega\left(t-r/v\right)\right]. \quad (37)$$

This potential function propagates away from the source at velocity v.

For the Coulomb gauge with $v\to\infty$ (37) gives

$$\phi_C(r,\theta,t)=\frac{I_0 d\ell\cos\theta}{4\pi\varepsilon_o\,\omega r^2}\sin\left(\omega t\right) \quad (38)$$

where the portion involving the current vanishes, leaving a result that is identical to the scalar potential for a static dipole. This potential tracks the charge variation in real-time at any distance and without a time delay.

To compute the v-gauge vector potential, we will use the gauge function and (4a),

$$\vec{A}_V(r,\theta,t) = \vec{A}_L(r,\theta,t) + \nabla\chi_v(r,\theta,t) \quad . \tag{39}$$

Then applying (4c) gives

$$\frac{\partial\chi_V(r,\theta,t)}{\partial t} = \phi_L(r,\theta,t) - \phi_V(r,\theta,t)$$
$$= \frac{I_o d\ell\cos\theta}{4\pi\varepsilon_o}\left[\frac{1}{cr}\cos\omega(t-r/c) + \frac{1}{\omega r^2}\sin\omega(t-r/c) - \frac{1}{vr}\cos\omega(t-r/v) - \frac{1}{\omega r^2}\sin\omega(t-r/v).\right] \tag{40}$$

By integrating (40) over t, with zero integration constant, we obtain

$$\chi_V(r,\theta,t) = \frac{I_0 d\ell\cos\theta}{4\pi\varepsilon_o\,\omega}\left[\frac{1}{cr}\sin\omega(t-r/c) - \frac{1}{\omega r^2}\cos\omega(t-r/c) - \frac{1}{vr}\sin\omega(t-r/v) + \frac{1}{\omega r^2}\cos\omega(t-r/v)\right]. \tag{41}$$

The gradient operation is performed in spherical coordinates, noting no φ -dependence to yield

$$\nabla\chi_V(r,\theta,t) = \hat{r}\frac{\partial\chi_v}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial\chi_v}{\partial\theta}$$
$$= \hat{r}\frac{I_o d\ell\cos\theta}{4\pi\varepsilon_o\,\omega}\begin{bmatrix} -\frac{2}{cr^2}\sin\omega(t-r/c) - \frac{\omega}{c^2 r}\cos\omega(t-r/c) \\ +\frac{2}{\omega r^3}\cos\omega(t-r/c) + \frac{2}{vr^2}\sin\omega(t-r/v) + \frac{\omega}{v^2 r}\cos\omega(t-r/v) - \frac{2}{\omega r^3}\cos\omega(t-r/v)\end{bmatrix}$$
$$-\hat{\theta}\frac{I_o d\ell\sin\theta}{4\pi\varepsilon_o\,\omega}\left[\frac{1}{cr^2}\sin\omega(t-r/c) - \frac{1}{\omega r^3}\cos\omega(t-r/c) - \frac{1}{vr^2}\sin\omega(t-r/v) + \frac{1}{\omega r^3}\cos\omega(t-r/v)\right] \tag{42}$$

Adding (42) to the spherical components of $\vec{A}_L(\vec{r},t)$ in (36) with vector components $A_r = A_z\cos\theta$ and $A_\theta = -A_z\sin\theta$ yields the final expression for the v-gauge vector potential.

$$\vec{A}_V(r,\theta,t)=\vec{A}_C(r,\theta,t)+\nabla\chi_V(r,\theta,t)=$$

$$=\hat{r}\,\frac{I_o d\ell\cos\theta}{4\pi\varepsilon_o\,\omega}\left[\begin{array}{l}-\dfrac{2}{c r^2}\sin\omega\big(t-r/c\big)+\dfrac{2}{\omega r^3}\cos\omega\big(t-r/c\big)\\[2ex] +\dfrac{2}{v r^2}\sin\omega\big(t-r/v\big)+\dfrac{\omega}{v^2 r}\cos\omega\big(t-r/v\big)-\dfrac{2}{\omega r^3}\cos\omega\big(t-r/v\big)\end{array}\right] \tag{43}$$

$$-\hat{\theta}\,\frac{I_o d\ell\sin\theta}{4\pi\varepsilon_o\,\omega}\left[\begin{array}{l}\dfrac{\omega}{c^2 r}\cos\omega\big(t-r/c\big)+\dfrac{1}{c r^2}\sin\omega\big(t-r/c\big)\\[2ex] -\dfrac{1}{\omega r^3}\cos\omega\big(t-r/c\big)-\dfrac{1}{v r^2}\sin\omega\big(t-r/v\big)+\dfrac{1}{\omega r^3}\cos\omega\big(t-r/v\big)\end{array}\right].$$

This result can be obtained more directly by substituting (33) and (34) into (23b). However, this does not give the insight gained using the gauge function starting with (39). Interestingly, the v-gauge vector potential in (43) has some parts due to the Lorenz gauge portion propagating at c while other portions propagate with $v \neq c$. Note also that if we select $v = c$ in the v-gauge terms of (41) and (42), simplifications can be made, and (43) gives the Lorenz gauge result in (36).

For the v-gauge, we have introduced a dimensionless scalar potential function $\Phi(kr,v)$ and two dimensionless vector potential functions $\Xi_r(kr,v)$ and $\Xi_\theta(kr,v)$ as shown in (44).

$$\phi_v(r,\theta,\omega)=\frac{I_o Z_0\, kd\ell\cos\theta}{4\pi}\Phi(kr,v)\quad\text{where}\quad \Phi(kr,v)=\frac{1}{kr}\left[\frac{c}{v}+\frac{1}{jk\,r}\right]e^{-jk_v r} \tag{44a}$$

$$\vec{A}_v\big(r,\theta,\omega\big)=\frac{I_o Z_o\, kd\ell}{4\pi\, c}(\hat{r}\,\Xi_r+\hat{\theta}\,\Xi_\theta) \tag{44b}$$

where

$$\Xi_r(kr,v)=\frac{\cos\theta}{kr}\left[\left(-\frac{2}{jk\,r}+\frac{2}{(kr)^2}\right)e^{-jkr}+\left(\frac{c}{v}\right)^2\left(1+\frac{2}{jk_v\,r}-\frac{2}{(k_v r)^2}\right)e^{-jk_v r}\right] \tag{44c}$$

$$\Xi_\theta(kr,v)=-\frac{\sin\theta}{kr}\left[\left(1+\frac{1}{jkr}-\frac{1}{(kr)^2}\right)e^{-jkr}+\left(\frac{c}{v}\right)^2\left(-\frac{1}{jk_v\,r}+\frac{1}{(k_v r)^2}\right)e^{-jk_v r}\right] \tag{44d}$$

Magnitudes of these potentials are plotted in Fig.3a as a function of the dimensionless parameter kr to demonstrate their dependence on v.

Oscillations for the non-Lorenz cases that appear in the vector potential plots of Figs. 3b and 3c are caused by rapidly changing complex phasor interference between portions of the potentials having spatial variation $e^{jk_v r}$ and $e^{-jk_v r}$. The effect appears most strongly at radial distances where the magnitudes of these two phasor portions are comparable.

We may summarize the scalar and vector potentials for the Hertzian dipole for various gauges in Table 1.

Gauge	Potentials	Remarks
Lorenz	$\phi_L(r,\theta,\omega) \approx \dfrac{I_0 Z_0 d\ell}{4\pi r}\cos\theta\left[1+\dfrac{1}{jk_c\, r}\right]e^{-jk_c r}$	Causal and retarded with speed of c
	$\vec{A}_L\left(r,\theta,\omega\right)=\dfrac{\mu_0 I_0\, Z_o\, d\ell}{4\pi\, r\, c}e^{-jk_c r}\ \hat{z}$	Causal and retarded with speed of c
Coulomb	$\phi_C(r,\theta,\omega)=\dfrac{I_0 d\ell\, Z_o}{4\pi\, r}\cos\theta\left[\dfrac{1}{jk_c r}\right]$	*<u>Non-causal and instantaneous</u>*
	$\vec{A}_C\left(r,\theta,\omega\right)=\dfrac{I_0 Z_o\, d\ell}{4\pi\, rc}\begin{Bmatrix}\hat{r}\,\cos\theta\ 2\left[\left(-\dfrac{1}{jk_c\, r}+\dfrac{1}{\left(k_c\, r\right)^2}\right)e^{-jk_c r}-\dfrac{1}{\left(k_c\, r\right)^2}\right]\\ -\hat{\theta}\ \sin\theta\left[\left(1+\dfrac{1}{jk_c\, r}-\dfrac{1}{\left(k_c\, r\right)^2}\right)e^{-jk_c r}+\dfrac{1}{\left(k_c\, r\right)^2}\right]\end{Bmatrix}$	Contains causal and non-causal components
***v*-Gauge**	$\phi_v(r,\theta,\omega) \approx \dfrac{I_0 Z_0\, d\ell\cos\theta}{4\pi\, r}\left(\dfrac{c}{v}\right)\left[1+\dfrac{1}{jk_v\, r}\right]e^{-jk_v r}$	Causal and retarded with speed of *v*
	$\vec{A}_v\left(r,\theta,\omega\right)=\dfrac{I_0 Z_o\, d\ell}{4\pi r\, c}\times$ $\begin{Bmatrix}\hat{r}\cos\theta\left[\left(-\dfrac{2}{jk_c\, r}+\dfrac{2}{\left(k_c r\right)^2}\right)e^{-jk_c r}+\left(\dfrac{c^2}{v^2}\right)\left(1+\dfrac{2}{jk_v\, r}-\dfrac{2}{\left(k_v r\right)^2}\right)e^{-jk_v r}\right]\\ -\hat{\theta}\sin\theta\left[\left(1+\dfrac{1}{jk_c r}-\dfrac{1}{\left(k_c r\right)^2}\right)e^{-jk_c r}+\left(\dfrac{c^2}{v^2}\right)\left(-\dfrac{1}{jk_v\, r}+\dfrac{1}{\left(k_v r\right)^2}\right)e^{-jk_v r}\right]\end{Bmatrix}$	Causal and retarded with speeds of *c* and *v*

<u>**Notes:**</u>

Z_o = Free space wave impedance $=\sqrt{\mu_o/\varepsilon_o}\approx 377\,\Omega$

c = Speed of light in free space $=1/\sqrt{\mu_o\varepsilon_o}\approx 3\times 10^8$ m/sec

v = Velocity gauge speed parameter $k_c = \omega/c$ $k_v = \omega/v$

Table 1. Scalar and Vector potentials for the Hertzian dipole for various gauges

A. Evaluation of the E and B Fields for the v-gauge

The computation of the B-field for the *v*-gauge proceeds by inserting (43) into (2) and performing the curl operation. Since most of the terms (43) arise from the gradient of the gauge function $\nabla\chi_V$, and because the curl operation on a gradient is zero, the *B*-field will be just the curl of $\vec{A}_L$ in (36), given as

$$\begin{aligned}\vec{B}_V(r,\theta,t) &= \nabla\times\vec{A}_L(r,\theta,t)+\nabla\times\nabla\chi_V(r,\theta,t)=\nabla\times\vec{A}_L(r,\theta,t)\\ &=\frac{\mu_o I_o\,d\ell\sin\theta}{4\pi}\,\hat{\varphi}\left[\frac{1}{r^2}\cos\omega\left(t-r/c\right)-\left(\frac{\omega}{c}\right)\frac{1}{r}\sin\omega\left(t-r/c\right)\right].\end{aligned}\tag{45}$$

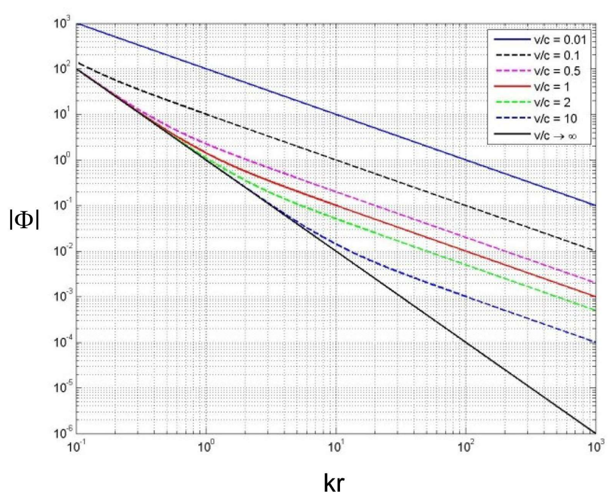
The electric field can be found either by solving (1b),

$$\vec{E}_V(r,\theta,t)=c^2\int\nabla\times\vec{B}_V(r,\theta,t)\,dt \;=c^2\int\nabla\times\vec{B}_L(r,\theta,t)\,dt\tag{46}$$

or by using the potentials via (3). Using (46) sidesteps the need to compute the scalar potential.

We will now use (3) to show that the v-gauge potentials provide the same electric field as the Lorenz gauge. We first obtain the gradient of (37) and the time derivative of (43) as

$$\begin{aligned}\nabla\phi_v(r,\theta,t) &= \hat{r}\frac{\partial\phi_v}{\partial r}+\hat{\theta}\frac{1}{r}\frac{\partial\phi_v}{\partial\theta}\\ &=\hat{r}\frac{I_o d\ell\cos\theta}{4\pi\varepsilon_o}\left[-\frac{2}{vr^2}\cos\omega\left(t-r/v\right)+\frac{\omega}{v^2 r}\sin\omega\left(t-r/v\right)-\frac{2}{\omega r^3}\sin\omega\left(t-r/v\right)\right]\\ &\quad-\hat{\theta}\frac{I_o d\ell\sin\theta}{4\pi\varepsilon_o}\left[\frac{1}{vr^2}\cos\omega\left(t-r/v\right)+\frac{1}{\omega r^3}\sin\omega\left(t-r/v\right)\right]\end{aligned}\tag{47}$$



Fig, 3a. Magnitude of the dimensionless *v*-gauge scalar potential.

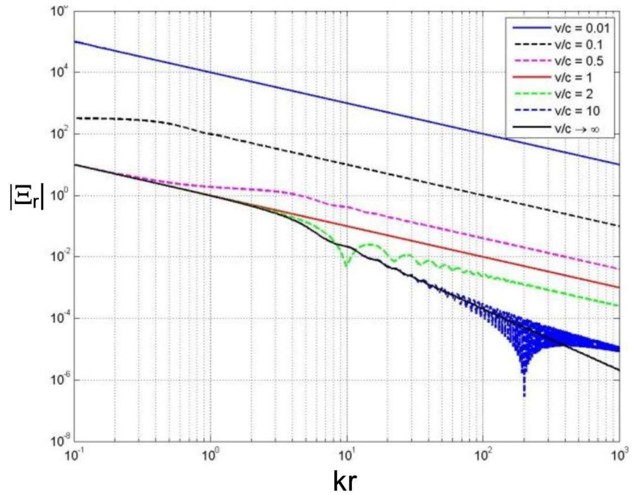


Fig. 3b. Magnitude of the dimensionless *v*-gauge r-component of the vector potential.

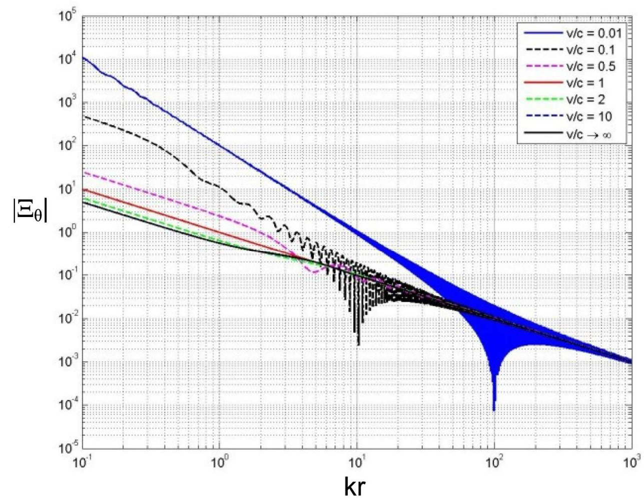


Fig. 3c. Magnitude of the dimensionless v-gauge θ-component of the vector potential.

$$\frac{\partial \vec{A}_v(r,\theta,t)}{\partial t} = \hat{r}\frac{I_o d\ell\cos\theta}{4\pi\varepsilon_o}\left[\begin{array}{l} -\frac{2}{c r^2}\cos\omega(t-r/c) - \frac{2}{\omega r^3}\sin\omega(t-r/c) \\ +\frac{2}{v r^2}\cos\omega(t-r/v) - \frac{\omega}{v^2 r}\sin\omega(t-r/v) + \frac{2}{\omega r^3}\sin\omega(t-r/v) \end{array}\right]$$
$$+\hat{\theta}\frac{I_o d\ell\sin\theta}{4\pi\varepsilon_o}\left[\begin{array}{l} \frac{\omega}{c^2 r}\sin\omega(t-r/c) - \frac{1}{c r^2}\cos\omega(t-r/c) \\ -\frac{1}{\omega r^3}\sin\omega(t-r/c) + \frac{1}{v r^2}\cos\omega(t-r/v) + \frac{1}{\omega r^3}\sin\omega(t-r/v) \end{array}\right] \quad (48)$$

By substituting (47) and (48) into (3), we can cancel all terms containing the velocity *v,* and the result for the *E*-field will be precisely the same as obtained using the Lorenz gauge:

$$\begin{aligned}\vec{E}_V(\vec{r},t) &= -\nabla\phi_V(\vec{r},t) - \frac{\partial \vec{A}_L(\vec{r},t)}{\partial t} = \vec{E}_L(\vec{r},t) \\ &= \hat{r}\,\frac{I_o d\ell\cos\theta}{4\pi\varepsilon_o}\left[\frac{2}{c\,r^2}\cos\omega\left(t-r/c\right)+\frac{2}{\omega r^3}\sin\omega\left(t-r/c\right)\right] \\ &\quad+\hat{\theta}\,\frac{I_o\,d\ell\sin\theta}{4\pi\varepsilon_o}\left[-\frac{\omega}{c^2 r}\sin\omega\left(t-r/c\right)+\frac{1}{c\,r^2}\cos\omega\left(t-r/c\right)+\frac{1}{\omega r^3}\sin\omega\left(t-r/c\right)\right]\end{aligned} \tag{49}$$

Thus, the parameter *v* can take on any value in a complex *v*-plane (except zero), and the electric field will remain the same.

Alternatively, the frequency-domain expressions for the time-harmonic *B* and *E* fields of (45) and (49) can be written as

$$\begin{aligned}\vec{B}_V(r,\omega) &= \vec{B}_L(r,\omega) \\ &= \hat{\varphi}\,\frac{\mu_o I_o\,d\ell\sin\theta}{4\pi}k^2 e^{-jkr}\left[\frac{1}{(kr)^2}+\frac{j}{kr}\right]\end{aligned} \tag{50}$$

$$\begin{aligned}\vec{E}_V(\vec{r},\omega) &= \vec{E}_L(\vec{r},\omega) \\ &= \frac{I_o d\ell_o}{4\pi}k^2 Z_o e^{-jkr}\left\{\hat{r}\,\cos\theta\left[\frac{2}{(kr)^2}-\frac{j2}{(kr)^3}\right]+\hat{\theta}\sin\theta\left[\frac{j}{kr}+\frac{1}{(kr)^2}-\frac{j}{(kr)^3}\right]\right\}\end{aligned} \tag{51}$$

where the resulting time-harmonic expressions for the fields are given by $\vec{\mathrm{F}}(\vec{r},t) = \mathrm{Re}\left[\vec{F}(\vec{r},\omega)\,e^{j\omega t}\right]$

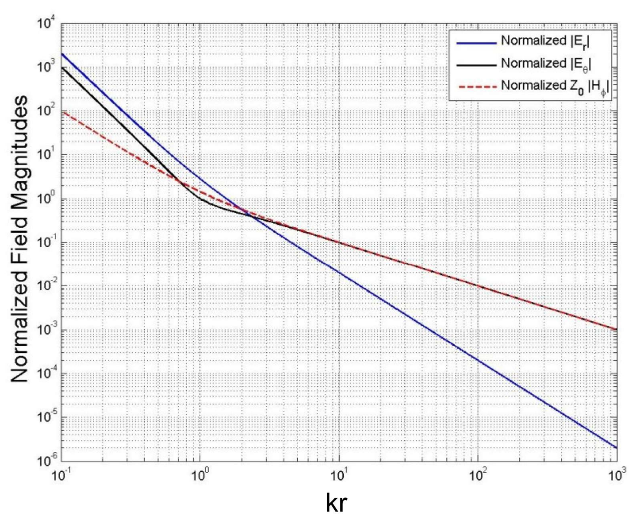


Fig. 4. Normalized EM field magnitudes (independent of *v*) shown as a function of *kr*.

with F denoting either the E or B phasors. The frequency domain expressions above are commonly available in the literature.

The E and B fields produced by the Hertzian dipole have been computed numerically using the frequency domain versions of (2) and (3). This calculation used the *v*-gauge potentials of (37) and (43), which were plotted in Fig. 3. For any selected value of *v,* the resulting E and B fields were identical to those provided in the analytical expressions of (49) and (50).

Figure 4 presents plots of the normalized E and Z_oH field component magnitudes as a function of the parameter *kr*. These are plots of the *kr*-terms within the square brackets in (49) and (50), obtained by removing the common magnitude term $I_o d\ell_o k^2 Z_o e^{-jkr}/4\pi$ and setting the sine and cosine terms to unity. The curves in this plot illustrate the near-field ($kr << 1$) and far-field ($kr >> 1$) behavior of the fields from the dipole source.

IV. SUMMARIZING COMMENTS

In this note, we have reviewed gauge theory in classical electromagnetics. The terms classical electromagnetics and classical electrodynamics are used interchangeably. In classical electromagnetic theory, currents and charges are assumed to be continuous functions of space and time. Under this assumption, it is noted that there is no absolute need to introduce potentials since the electromagnetic fields can be solved and unique solutions obtained directly from Maxwell's equations.

It is noted that most commercial-off-the-shelf (COTS) computational electromagnetic computer codes solve for EM fields directly without using potentials. At least one exception is the COMSOL Magnetic and Electric Fields interface in the quasi-static AC/DC Module, which solves for both scalar and vector potentials.

If we choose for utility or simplicity to introduce the scalar and vector potentials, it is seen that the potentials depend on which gauge is used. However, the electromagnetic fields derived from the non-unique potentials are unique.

An interesting result from using the *v*-gauge formulation is the generally non-causal dependence of the scalar potential on the source charge density unless the Lorenz gauge with $v = c$ is used. The Coulomb gauge with $v = \infty$ provides an instantaneous scalar potential contribution at a distant observation point, while the vector potential has a properly retarded contribution. However, when computed with these potentials, the EM fields are both causal.

V. FUTURE RESEARCH

One of the reviewers commented**:** Is the velocity v a global parameter, i.e., it is the same in all space, or one can introduce space dependence v(x,y,z), maybe, with some compensation term to keep physical values of electrical and magnetic field. Could such dependence make a simpler solution to some EM problem, e.g., coupling with thin-wire structures?

This is an interesting idea, perhaps applied to fields in inhomogeneous materials and can be a new line of research and beyond the scope of this Note.

The same reviewer further asked: The Hamiltonian of the particle in magnetic field H contains vector potential A. Usually, for the quantum-mechanics calculation in magnetic field one use the Landau gauge or symmetrical gauge. It will be interesting try to use this v - gauge, if v coincides with longitudinal velocity of the particle along the direction of the magnetic field. (Especially for the problem of scattering of the particle on the spherically symmetric potential in the quantizing magnetic field).

Certainly, beyond the intended scope of our paper but can also be a topic for future research.

ACKNOWLEDGMENT

The authors are thankful to Dr. Kelvin Lee and Prof. Robert G. Olsen (Emeritus), Washington State University, for their valuable discussions during the preparation of this paper. We are also grateful to Prof. Robert D. Nevels, Texas A&M University, for reading our draft manuscript and suggesting many improvements which have been incorporated. Furthermore, during this Theoretical Note publication, we are thankful to two reviewers who carefully looked through the initial draft and offered valuable suggestions.

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