THEORETICAL NOTES

NOTE 361

Fractured Solution Method (FSM) For Solving Maxwell's Equations

by

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CONTENTS

1 INTRODUCTION

1.1 Analytical solutions of the wave equation for impulsive sources frequently give rise to expressions of the form

$$\delta(t) \rightarrow S\delta\left(t-\frac{z}{c}\right) + Au\left(t-\frac{z}{c}\right) \tag{1}$$

where an impulsive source gives rise to a travelling impulse of variable strength together with a modulated travelling step wave. Examples of this form are to be found in the solution for current flow in an infinite cylinder in free space when subjected to an impulsive wave [L1], the reflection of impulsive waves from conducting dielectrics [L2] and the passage of waves through both lossless and lossy plasma [L3].

1.2 Adopting this form of solution *ab initio* leads to a set of first order equations together with the original equation and is accordingly over-determined. However, the equations only have to have consistent solutions at the wave front and imposing this condition allows the determination of S and the amplitude of A at $t=z/c$. Analytical solutions are thus obtained at the wavefront, the very region of space-time that is most difficult for a numerical attack.

1.3 A trial substitution of either component of the formal solution separately, rapidly demonstrates that neither is a possible solution of the original equation. This observation gives rise to the name of the method - the **Method of Fractured Solutions.** The power of the method lies in its ability to cope equally with spatio-temporal variations in material parameters, the most general variations of which do not allow the reduction of Maxwell's equations to a wave equation in one field component. The approach here is to impose the formal solution directly on the field components in Maxwell's equations resulting in analytical solutions, up to an integral, for each of the field components.

1.4 An obvious application for these methods is in the transmission of electromagnetic waves through time and space dependent plasmas, such as in the EMP source region and in stratified conductors such as the ground.

1.5 It is to be noted that this functional form is not an approximation but constitutes the complete solution. The information that can be extracted however is only partial, in that the solutions readily obtained relate only to the wavefront. The equation that determines A is the original equation. The value of the method is that it produces analytical results at the wavefront enabling late-time (low-frequency) approximations to be matched at the wavefront, producing approximations valid over all space-time, and provides analytical data for validating numerical solutions.

2 MAXWELL'S EQUATIONS

2.1 Maxwell's equations for an isotropic medium can be written

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{2}$$

$$\nabla \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J} \tag{3}$$

$$\nabla . \boldsymbol{B} = 0 \tag{4}$$

$$\nabla . \boldsymbol{D} = \rho \tag{5}$$

Combining these with the constitutive relations

$$\boldsymbol{D} = \varepsilon \boldsymbol{E} \tag{6}$$

$$\boldsymbol{B} = \mu \boldsymbol{H} \tag{7}$$

$$\boldsymbol{J} = \sigma \boldsymbol{E} \tag{8}$$

they become

$$\nabla \times \boldsymbol{E} = -\frac{\partial [\mu \boldsymbol{H}]}{\partial t} \tag{9}$$

$$\nabla \times \boldsymbol{H} = \frac{\partial [\varepsilon \boldsymbol{E}]}{\partial t} + \sigma \boldsymbol{E} \tag{10}$$

Making use of the vector relation

$$\nabla \times (\nabla \times \boldsymbol{X}) = \nabla (\nabla . \boldsymbol{X}) - \nabla^2 \boldsymbol{X} \tag{11}$$

and observing that within a conductor we may set

$$\rho = 0 \tag{12}$$

we obtain

$$\nabla^2 \boldsymbol{E} = \frac{\partial}{\partial t} [\nabla \times (\mu \boldsymbol{H})] \tag{13}$$

$$-\nabla^2 \boldsymbol{H} = \frac{\partial}{\partial t} [\nabla \times (\varepsilon \boldsymbol{E})] + \nabla \times (\sigma \boldsymbol{E}) \tag{14}$$

3 UNIFORM COLLISIONLESS PLASMA

3.1 If we now consider the case of constant parameters, these equations reduce to

$$\nabla^2 \boldsymbol{E} = \mu \frac{\partial}{\partial t} (\nabla \times \boldsymbol{H}) \tag{15}$$

$$\nabla^2 \boldsymbol{H} = -\varepsilon \frac{\partial}{\partial t} (\nabla \times \boldsymbol{E}) - \sigma \nabla \times \boldsymbol{E} \tag{16}$$

Eliminating $\boldsymbol{H}$ and $\boldsymbol{E}$ in turn yields

$$\nabla^2 \boldsymbol{E} = \mu \varepsilon \frac{\partial^2 \boldsymbol{E}}{\partial t^2} + \mu \sigma \frac{\partial \boldsymbol{E}}{\partial t} \tag{17}$$

$$\nabla^2 \boldsymbol{H} = \mu \varepsilon \frac{\partial^2 \boldsymbol{H}}{\partial t^2} + \mu \sigma \frac{\partial \boldsymbol{H}}{\partial t} \tag{18}$$

which is of course the standard separation of the fields obtaining the wave equation for each component. We now restrict our attention to the one dimensional case, obtaining

$$\frac{\partial^2 E}{\partial z^2} = \mu \varepsilon \frac{\partial^2 E}{\partial t^2} + \mu \sigma \frac{\partial E}{\partial t} \tag{19}$$

$$\frac{\partial^2 H}{\partial z^2} = \mu \varepsilon \frac{\partial^2 H}{\partial t^2} + \mu \sigma \frac{\partial H}{\partial t} \tag{20}$$

3.2 We now determine the impulsive response by imposing the fractured form of the solution,

$$E = S_o e^{-\alpha z} \delta\left(t - \frac{z}{c}\right) + E_T u\left(t - \frac{z}{c}\right) \tag{21}$$

this particular form being chosen for algebraic convenience.

Forming the various derivatives

$$\frac{\partial E}{\partial z} = -\frac{S_o e^{-\alpha z}}{c}\delta' - \left(\alpha S_o e^{-\alpha z} + \frac{E_T}{c}\right)\delta + \frac{\partial E_T}{\partial z}u \tag{22}$$

$$\frac{\partial^2 E}{\partial z^2} = \frac{S_o e^{-\alpha z}}{c^2}\delta'' + \frac{1}{c}\left(2\alpha S_o e^{-\alpha z} + \frac{E_T}{c}\right)\delta' + \left(\alpha^2 S_o e^{-\alpha z} - \frac{2}{c}\frac{\partial E_T}{\partial z}\right)\delta + \frac{\partial^2 E_T}{\partial z^2}u \tag{23}$$

$$\frac{\partial E}{\partial t} = S_o e^{-\alpha z}\delta' + E_T\delta + \frac{\partial E_T}{\partial t}u \tag{24}$$

$$\frac{\partial^2 E}{\partial t^2} = S_o e^{-\alpha z}\delta'' + E_T\delta' + 2\frac{\partial E_T}{\partial t}\delta + \frac{\partial^2 E_T}{\partial t^2}u \tag{25}$$

We note that the constructs u, δ, δ', δ'' are not functions but distributions, and that we are using the extended meaning of differentiation due to Schwartz [L4]. The prime is used throughout to denote differentiation with respect to the argument.

3.3 Substituting these derivatives into the wave equations we obtain two further equations involving all of these distributions. However the distributions are linearly independent and accordingly the coefficients of the various distributions must vanish identically. This process results in the following set of equations, before cancellations

$$(\delta'') \qquad \frac{S_o e^{-\alpha z}}{c^2} = \mu\varepsilon\, S_o e^{-\alpha z} \tag{26}$$

$$(\delta') \qquad \frac{1}{c}\left(2\alpha S_o e^{-\alpha z} + \frac{E_T}{c}\right) = \mu\varepsilon E_T + \mu\sigma S_o e^{-\alpha z} \tag{27}$$

$$(\delta) \qquad \alpha^2 S_o e^{-\alpha z} - \frac{2}{c}\frac{\partial E_T}{\partial z} = 2\mu\varepsilon\frac{\partial E_T}{\partial t} + \mu\sigma E_T \tag{28}$$

$$(u) \qquad \frac{\partial^2 E_T}{\partial z^2} = \mu\varepsilon\frac{\partial^2 E_T}{\partial t^2} + \mu\sigma\frac{\partial E_T}{\partial t} \tag{29}$$

Cancelling common factors, the first of these reduces to

$$\mu\,\varepsilon = \frac{1}{c^2} \tag{30}$$

a not unexpected result.

Making use of this result the second equation yields

$$\alpha = \frac{\mu\sigma c}{2} \tag{31}$$

and the third equation gives

$$\frac{\partial E_T}{\partial z} + \frac{1}{c}\frac{\partial E_T}{\partial t} + \frac{\mu\sigma c}{2}E_T = \frac{c}{2}\left(\frac{\mu\sigma c}{2}\right)^2 S_o e^{\frac{-\mu\sigma c}{2}z} \tag{32}$$

If we now set

$$E_T = e^{-\alpha z}G \tag{33}$$

the equation reduces to

$$\frac{\partial G}{\partial z} + \frac{1}{c}\frac{\partial G}{\partial t} = \frac{c}{2}\alpha^2 S_o \tag{34}$$

By inspection, this has a solution

$$G = g\left(t-\frac{z}{c}\right) + \frac{c}{2}\alpha^2 S_o z \tag{35}$$

where g is an arbitrary function. Therefore we have

$$E_T = e^{-\alpha z}\left[g\left(t-\frac{z}{c}\right) + \frac{c}{2}\alpha^2 S_o z\right] \tag{36}$$

3.4 It is readily checked that this solution is incompatible with equation (29). However, this solution results from the δ - function whose distribution only exists at the wavefront. Accordingly we only impose this solution at the wavefront, and so determine a boundary condition for the fourth equation. At the wavefront

$$[E_T]_{z=ct} = e^{-\alpha z}\left[g(0) + \frac{c}{2}\alpha^2 S_o z\right] \tag{37}$$

When $z = 0$, the amplitude of the tail, E_T, must be zero, and so we have $g(0) = 0$ which gives

$$[E_T]_{z=ct} = \frac{c}{2}\left(\frac{\sigma}{2\varepsilon c}\right)^2 S_o z\, e^{-\frac{\sigma}{2\varepsilon c}z} \tag{38}$$

Equation **(29)** is, of course, the original equation. The advantage of the above procedure is that we now know the amplitude of the tail function at the wavefront, enabling late-time approximations to be matched to this result and so obtain reasonable approximations over all space-time.

3.5 In this particular instance the problem has an analytical solution [L5], and we can validate the above procedure. Taking Laplace transforms of each term of the E-field equation, and denoting transformations by a tilde, there results

$$\frac{\partial^2 \tilde{E}}{\partial z^2} = [s\mu\sigma + s^2\mu\varepsilon]\,\tilde{E} \tag{39}$$

This has a general solution

$$\tilde{E} = A\, e^{\sqrt{s\mu\sigma + s^2\mu\varepsilon}\, z} + B\, e^{-\sqrt{s\mu\sigma + s^2\mu\varepsilon}\, z} \tag{40}$$

The boundary conditions are

$$z=0,\ E=S_o\delta(t) \Rightarrow \tilde{E}\Rightarrow S_o \ \therefore\ B = S_o \tag{41}$$

$$z\rightarrow\infty, \quad E\rightarrow 0 \quad \Rightarrow \tilde{E}\rightarrow 0 \ \therefore\ A=0 \tag{42}$$

Noting the transform pair [L6]

$$\frac{e^{-k\sqrt{s(s+a)}}}{\sqrt{s(s+a)}} \Rightarrow e^{-at/2}\, I_o\left(\frac{a\sqrt{t^2-k^2}}{2}\right) u(t-k) \tag{43}$$

We write, on making use of

$$c^2 = \frac{1}{\mu\varepsilon} \tag{44}$$

$$\tilde{E} = -\frac{\partial}{\partial\left(\frac{z}{c}\right)}\left[\frac{S_o \exp\left(-\sqrt{s\left(s+\frac{\sigma}{\varepsilon}\right)}\ \frac{z}{c}\right)}{\sqrt{s\left(s+\frac{\sigma}{\varepsilon}\right)}}\right] \tag{45}$$

$$\Rightarrow E = -\frac{\partial}{\partial\left(\frac{z}{c}\right)}\left[S_o e^{\frac{\sigma}{2\varepsilon}t} I_o\left(\frac{\sigma}{2\varepsilon}\sqrt{t^2-\frac{z^2}{c^2}}\right) u\left(t-\frac{z}{c}\right)\right] \tag{46}$$

Carrying out the differentiation

(47)

which is precisely the fractured form of the solution. Observing that [7]

$$f(z)\,\delta(a-z) = f(a)\,\delta(a-z) \tag{48}$$

the solution reduces to

$$E = S_o e^{-\frac{\sigma}{2\varepsilon}t}\left[\delta\left(t-\frac{z}{c}\right) + \frac{\sigma z}{2\varepsilon c}\frac{I_1\left(\frac{\sigma}{2\varepsilon}\sqrt{t^2-\frac{z^2}{c^2}}\right)}{\sqrt{t^2-\frac{z^2}{c^2}}}\, u\left(t-\frac{z}{c}\right)\right] \tag{49}$$

To obtain the amplitude of E_T at the wavefront, we allow $t \to z/c$ in the second term, and there results

$$[E_T]_{z=ct} = S_o\frac{c}{2}\left(\frac{\sigma}{2\varepsilon c}\right)^2 z\, e^{-\frac{\sigma z}{2\varepsilon c}} \tag{50}$$

which is the previous formula (38).

4 LOSSY PLASMA WITH VARIABLE CONDUCTIVITY

4.1 Having illustrated the method of fractured solutions on an elementary problem we now consider the more difficult problem of a wave travelling in a medium with a spatial dependency of the conductivity. We must now write

$$\nabla^2 \boldsymbol{E} = \mu \frac{\partial}{\partial t} (\nabla \times \boldsymbol{H}) \tag{51}$$

$$\nabla^2 \boldsymbol{H} = -\varepsilon \frac{\partial}{\partial t} (\nabla \times \boldsymbol{E}) - \nabla \times (\sigma \boldsymbol{E}) \tag{52}$$

where

$$\sigma = \sigma(r) \tag{53}$$

Eliminating $\boldsymbol{H}$ from the first of these

$$\nabla^2 \boldsymbol{E} = \mu \varepsilon \frac{\partial^2}{\partial t^2} \boldsymbol{E} + \mu \sigma(r) \frac{\partial \boldsymbol{E}}{\partial t} \tag{54}$$

Making use of the vector identity

$$\nabla \times (\phi \boldsymbol{a}) = \phi \nabla \times \boldsymbol{a} + (\nabla \phi) \times \boldsymbol{a} \tag{55}$$

the equation for the magnetic field becomes

$$\nabla^2 \boldsymbol{H} = \mu \varepsilon \frac{\partial^2 \boldsymbol{H}}{\partial t^2} + \mu \sigma(r) \frac{\partial \boldsymbol{H}}{\partial t} + (\nabla \sigma(r)) \times \boldsymbol{E} \tag{56}$$

Restricting our attention to the one dimensional case the equations reduce to

$$\frac{\partial^2 E}{\partial z^2} = \mu \varepsilon \frac{\partial^2 E}{\partial t^2} + \mu \sigma(z) \frac{\partial E}{\partial t} \tag{57}$$

$$\frac{\partial^2 H}{\partial z^2} = \mu \varepsilon \frac{\partial^2 H}{\partial t^2} + \mu \sigma(z) \frac{\partial H}{\partial t} + \frac{d\sigma(z)}{dz} E \tag{58}$$

We can eliminate E from this latter equation, but at the expense of some complexity. The result is

(59)

4.2 We first consider the E-field equation. We proceed in the same manner as before, that is we introduce the fractured form of solution as a trial solution and attempt to determine the coefficients of the distributions.

The input is

$$E = S_o \delta(t) \tag{60}$$

and the output, or the response of the medium, is taken to be

$$E = S\,\delta\left(t - \frac{z}{c}\right) + E_T\,u\left(t - \frac{z}{c}\right) \tag{61}$$

where S is the strength of the travelling impulse, and can be regarded as depending on z, it only existing when $z = ct$; E_T is the 'tail' function which depends on both z and t. Forming the various derivatives

$$\frac{\partial E}{\partial z} = \frac{\partial E_T}{\partial z} u + \left[\frac{\partial S}{\partial z} - \frac{E_T}{c}\right]\delta - \frac{S}{c}\delta' \tag{62}$$

$$\frac{\partial^2 E}{\partial z^2} = \frac{\partial^2 E_T}{\partial z^2} u + \left[\frac{\partial^2 S}{\partial z^2} - \frac{2}{c}\frac{\partial E_T}{\partial z}\right]\delta - \frac{1}{c}\left[2\frac{\partial S}{\partial z} - \frac{E_T}{c}\right]\delta' + \frac{S}{c^2}\delta'' \tag{63}$$

$$\frac{\partial E}{\partial t} = \frac{\partial E_T}{\partial t} u + \left[\frac{\partial S}{\partial t} + E_T\right]\delta + S\,\delta' \tag{64}$$

$$\frac{\partial^2 E}{\partial t^2} = \frac{\partial^2 E_T}{\partial t^2} u + \left[\frac{\partial^2 S}{\partial t^2} + 2\frac{\partial E_T}{\partial t}\right]\delta + \left[2\frac{\partial S}{\partial t} + E_T\right]\delta' + S\,\delta'' \tag{65}$$

The various distributions are linearly independent, and so we may equate coefficients, and we obtain the following set of equations

$$(\delta'') \qquad \frac{S}{c^2} = \mu\varepsilon S \tag{66}$$

$$(\delta') \qquad \left[\frac{E_T}{c^2} - \frac{2}{c}\frac{\partial S}{\partial z}\right] = \mu\varepsilon\left[2\,\frac{\partial S}{\partial t} + E_T\right] + \mu\sigma(z)\,S \tag{67}$$

$$(\delta) \qquad \left[\frac{\partial^2 S}{\partial z^2} - \frac{2}{c}\frac{\partial E_T}{\partial z}\right] = \mu\varepsilon\left[\frac{\partial^2 S}{\partial t^2} + 2\,\frac{\partial E_T}{\partial t}\right] + \mu\,\sigma(z)\left[\frac{\partial S}{\partial t} + E_T\right] \tag{68}$$

$$(u) \qquad \frac{\partial^2 E_T}{\partial z^2} = \mu\varepsilon\,\frac{\partial^2 E_T}{\partial t^2} + \mu\sigma(z)\,\frac{\partial E_T}{\partial t} \tag{69}$$

4.3 The first of these equations yields, as before

$$c^2 = \frac{1}{\varepsilon\mu} \tag{70}$$

Making use of this result, the tail component cancels from the second equation which reduces to

$$-\frac{2}{c}\frac{\partial S}{\partial z} = \frac{2}{c^2}\frac{\partial S}{\partial t} + \mu\sigma(z)\,S \tag{71}$$

We have chosen to display the variable conductivity as a function of z, and so for the impulse propagation it is appropriate to express S as a function of z only. Accordingly this equation reduces to

$$\frac{\partial S}{\partial z} = \frac{-c\mu\sigma(z)}{2}\,S \tag{72}$$

which has the solution

$$S = S_o \exp\left[-\int_o^z \frac{\sigma(z')}{2\varepsilon c}\,dz'\right] \tag{73}$$

The same solution is obtained if we consider S to be a function of t only, and regard

$$\sigma = \sigma(ct) \tag{74}$$

4.4 Utilising the solution of the first two equations, the third equation reduces to

$$\frac{\partial E_T}{\partial z} + \frac{1}{c}\frac{\partial E_T}{\partial t} + \frac{\sigma(z)}{2\varepsilon c}E_T = \frac{c}{2}\left[\left(\frac{\sigma(z)}{2\varepsilon c}\right)^2 - \frac{\sigma'(z)}{2\varepsilon c}\right]S \qquad (75)$$

which we note is a first order equation, and again the solution must only be imposed at the wavefront. Accordingly we may regard E_T as depending on z only, and we may delete the term in $\partial E_T/\partial t$. Note that we do not replace

$$\frac{1}{c}\frac{\partial E_T}{\partial t} \rightarrow \frac{\partial E_T}{\partial z} \qquad (76)$$

as this would be taking the spatial variation into account twice. The equation then becomes an ordinary differential equation, and the solution is

$$[E_T]_{z=ct} = \frac{c}{2}S_o \exp\left[-\int_o^z \frac{\sigma(z')}{2\varepsilon c}\,dz'\right] \int_o^z \left[\left(\frac{\sigma(z')}{2\varepsilon c}\right)^2 - \frac{\sigma'(z')}{2\varepsilon c}\right]dz' \qquad (77)$$

4.5 We now turn to the magnetic field equation, where we make the Fractured Solution substitution

$$H = S_H\delta\left(t - \frac{z}{c}\right) + H_T u\left(t - \frac{z}{c}\right) \qquad (78)$$

The distribution coefficient equations can be written down by inspection, the only equation with an additional term being derived from the coefficients of δ. Accordingly we have

$$(\delta'') \qquad \frac{S_H}{c^2} = \mu\varepsilon S_H \qquad (79)$$

$$(\delta') \qquad H_T - \frac{2}{c}\frac{\partial S_H}{\partial z} = \mu\varepsilon\left[2\frac{\partial S_H}{\partial t} + H_T\right] + \mu\sigma(z)S_H \qquad (80)$$

$$(\delta) \qquad \frac{\partial^2 S_H}{\partial z^2} - \frac{2}{c}\frac{\partial H_t}{\partial z} = \mu\varepsilon\left[\frac{\partial^2 S_H}{\partial t^2} + 2\frac{\partial H_t}{\partial t}\right] + \mu\sigma(z)H_t + \sigma'(z)S_E \qquad (81)$$

$$(u) \qquad \frac{\partial^2 H_T}{\partial z^2} = \mu\varepsilon\frac{\partial^2 H_T}{\partial t^2} + \mu\sigma(z)\frac{\partial H_T}{\partial t} + \sigma'(z)E_T \qquad (82)$$

From (δ'), by analogy with the E-field equation, we have immediately

$$S_H = \sqrt{\frac{\varepsilon}{\mu}}\, S_o \exp\left[-\int_o^z \frac{\sigma(z')}{2\varepsilon c}\, dz'\right] \tag{83}$$

The (δ) equation reduces to

$$\frac{\partial H_T}{\partial z} + \frac{1}{c}\frac{\partial H_T}{\partial t} + \frac{\sigma(z)}{2\varepsilon c} H_T = \frac{c}{2}\left[\left(\frac{\sigma(z)}{2\varepsilon c}\right)^2 - \frac{\sigma'(z)}{2\varepsilon c} - \sqrt{\frac{\mu}{\varepsilon}}\,\sigma'(z)\right] S_H \tag{84}$$

and this has a solution

$$[H_T]_{z=ct} = \frac{c}{2}\sqrt{\frac{\varepsilon}{\mu}}\, S_o \exp\left[-\int_o^z \frac{\sigma(z')\, dz'}{2\varepsilon c}\right]\left[\int_o^z\left(\frac{\sigma(z')}{2\varepsilon c}\right)^2 dz - \frac{3\sigma'(z')}{2\varepsilon c}\right] \tag{85}$$

4.6 The space-derivatives of the tail function at the wavefront are obtainable directly from the amplitude result. In particular, for the constant conductivity case we have

$$[E_T]_{z=ct} = \frac{c\alpha^2}{2} S_o z e^{-\alpha z} \tag{86}$$

Differentiating this result we have

$$\left[\frac{\partial E_T}{\partial z}\right]_{z=ct} = \frac{c\alpha^2}{2} S_o\, e^{-\alpha z}\,(1 - \alpha z) \tag{87}$$

To confirm this result we extract the tail function from the complete solution

$$E_T = \alpha\, S_o\, z\, e^{-\alpha z}\, \frac{I_1\left[\frac{\sigma}{2\varepsilon}\sqrt{t^2 - \frac{z^2}{c^2}}\right]}{\sqrt{t^2 - \frac{z^2}{c^2}}} \tag{88}$$

Differentiating

$$\frac{\partial E_T}{\partial z} = \alpha\, S_o e^{-\alpha z}\left[\frac{I_1\left[\frac{\sigma}{2\varepsilon}\sqrt{t^2 - \frac{z^2}{c^2}}\right]}{\sqrt{t^2 - \frac{z^2}{c^2}}}(1 - \alpha z) - \frac{z^2}{c^2}\frac{\sigma}{2\varepsilon}\frac{I_1'\left[\frac{\sigma}{2\varepsilon}\sqrt{t^2 - \frac{z^2}{c^2}}\right]}{t^2 - \frac{z^2}{c^2}} + \frac{z^2}{c^2}\frac{I_1\left[\frac{\sigma}{2\varepsilon}\sqrt{t^2 - \frac{z^2}{c^2}}\right]}{\left(t^2 - \frac{z^2}{c^2}\right)^{\frac{3}{2}}}\right] \tag{89}$$

Letting z-ct

$$\left[\frac{\partial E_T}{\partial z}\right]_{z=ct} = \frac{c\alpha^2}{2} S_o (1-\alpha z)\, e^{-\alpha z} \tag{90}$$

confirming the previous derivation. Generalising the result, we obtain

$$\left[\frac{\partial^n E_T}{\partial z^n}\right]_{z=ct} = (-)^{n-1}\frac{c}{2}\alpha^{n+1}\left(1-(n-1)\alpha - \alpha z\right) e^{-\alpha z} \tag{91}$$

The amplitude of the tail function at the wavefront is independent of t, other than through the relation $z=ct$, and so we have

$$\left[\frac{\partial E_T}{\partial t}\right]_{z=ct} = 0 \tag{92}$$

To confirm this result we return to the complete tail-function

$$E_T = \alpha S_o z e^{-\alpha z} \frac{I_1\left[\frac{\sigma}{2\varepsilon}\sqrt{t^2 - \frac{z^2}{c^2}}\right]}{\sqrt{t^2 - \frac{z^2}{c^2}}} \tag{93}$$

Differentiating this result with respect to t in the neighbourhood of $z=ct$, there results

$$\begin{aligned}\frac{\partial E_T}{\partial t} &= \alpha S_o z e^{-\alpha z}\left[\frac{\frac{\sigma}{4\varepsilon}\, t}{t^2 - \frac{z^2}{c^2}} - \frac{\frac{\sigma}{4\varepsilon}\sqrt{t^2 - \frac{z^2}{c^2}}\; t}{\left(t^2 - \frac{z^2}{c^2}\right)^{\frac{3}{2}}}\right] \\ &= 0\end{aligned} \tag{94}$$

5 COLLISIONLESS UNMAGNETISED PLASMA

5.1 The equation of motion of an electron in an electric field is

$$\frac{\partial^2 x}{\partial t^2} \equiv \ddot{x} = -\frac{e}{m}E \tag{95}$$

while the field equations are

$$\frac{\partial E}{\partial z} + \mu_o \frac{\partial H}{\partial t} = 0 \tag{96}$$

$$\frac{\partial H}{\partial z} + \varepsilon_o \frac{\partial E}{\partial t} = Ne\dot{x} \tag{97}$$

Differentiating **(97)** with respect to t

$$\frac{\partial^2 H}{\partial t \partial z} + \varepsilon_o \frac{\partial^2 E}{\partial t^2} = Ne\ddot{x} \tag{98}$$

Combining with **(95)**

$$\frac{\partial^2 H}{\partial t \partial z} + \varepsilon_o \frac{\partial^2 E}{\partial t^2} = -\frac{Ne^2}{m}E \tag{99}$$

Differentiating **(96)** with respect to z

$$\frac{\partial^2 E}{\partial z^2} + \mu_o \frac{\partial^2 H}{\partial z \partial t} = 0 \tag{100}$$

Combining with **(99)**

$$\frac{\partial^2 E}{\partial z^2} = \mu_o \varepsilon_o \frac{\partial^2 E}{\partial t^2} + \mu_o \frac{Ne^2}{m}E \tag{101}$$

Substituting for the plasma frequency

$$\omega_p^2 = \frac{\mu_o N e^2 c^2}{m} \tag{102}$$

and taking Laplace Transforms

$$\frac{\partial^2 \tilde{E}}{\partial z^2} = \left[\mu_o \varepsilon_o s^2 + \frac{\omega_p^2}{c^2}\right]\tilde{E} \tag{103}$$

which may be re-written as

$$\frac{\partial^2 \tilde{E}}{\partial z^2} = \left[s^2 + \omega_p^2\right] \frac{\tilde{E}}{c^2} \tag{104}$$

Solutions are

$$e^{\pm\sqrt{s^2 + \omega_p^2}\ \frac{z}{c}} \tag{105}$$

5.2 Setting the initial condition to

$$E = S_o \delta(t) \tag{106}$$

$$\tilde{E} = S_o \tag{107}$$

and letting $E \to 0$ as $z \to \infty$, we have the solution

$$\tilde{E} = S_o e^{-\sqrt{s^2 + \omega_p^2}\ \frac{z}{c}} \tag{108}$$

which can also be written as

$$\tilde{E} = -S_o \frac{\partial}{\partial\left(\frac{z}{c}\right)} \frac{e^{-\sqrt{s^2 + \omega_p^2}\ \frac{z}{c}}}{\sqrt{s^2 + \omega_p^2}} \tag{109}$$

The inverse of this transform is

$$E = -S_o \frac{\partial}{\partial\left(\frac{z}{c}\right)} \left[J_o\left(\omega_p \sqrt{t^2 - \left(\frac{z}{c}\right)^2}\right) u\left(t - \frac{z}{c}\right)\right] \tag{110}$$

Carrying out the differentiation

$$E = -S_o \left[-J_o\left[\omega_p \sqrt{t^2 - \left(\frac{z}{c}\right)^2}\right] \delta\left(t - \frac{z}{c}\right) + \frac{\omega_p z}{c\sqrt{t^2 - \left(\frac{z}{c}\right)^2}} J_1\left[\omega_p \sqrt{t^2 - \left(\frac{z}{c}\right)^2}\right] u\left(t - \frac{z}{c}\right)\right] \tag{111}$$

We have the equivalence

$$f(t)\,\delta(t) \equiv f(0)\,\delta(t) \qquad (112)$$

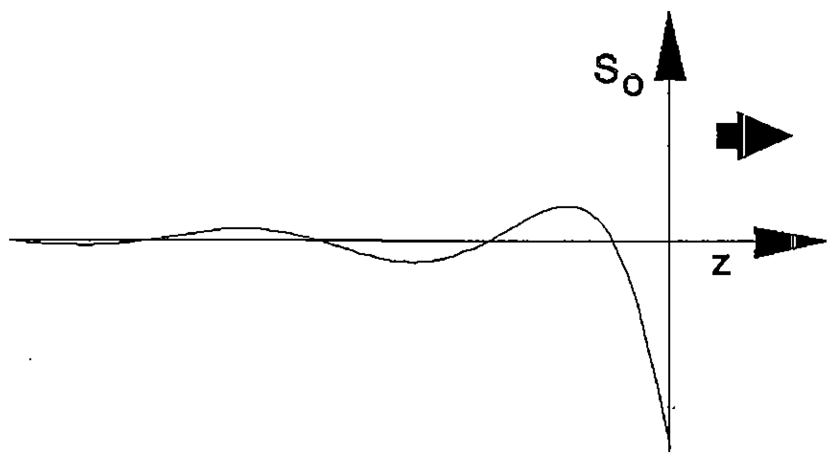
and the solution reduces to

$$E = S_o\left[\delta\left(t-\frac{z}{c}\right) - \frac{\omega_p z}{c\sqrt{t^2-\left(\frac{z}{c}\right)^2}}\,J_1\left[\omega_p\sqrt{t^2-\left(\frac{z}{c}\right)^2}\right]u\left(t-\frac{z}{c}\right)\right] \qquad (113)$$

which is the fractured solution form. From this result we obtain

$$[E_T]_{z=ct} = -\frac{\omega_p^2}{2}\frac{z}{c}S_o \qquad (114)$$

We see from **(113)** that the δ-function propagates into the plasma with a growing tail function, and a qualitative graph is indicated below



The amplitude of the tail at the wavefront tends to $-\infty$ with z, while the distance of the first zero behind the wavefront tends to zero with $1/z$.

5.3 If we conduct a fractured solution analysis, we make the substitution

$$E = S\delta\left(t - \frac{z}{c}\right) + E_T u\left(t - \frac{z}{c}\right) \tag{115}$$

and form the relevant derivatives

$$\frac{\partial E}{\partial z} = -\frac{S}{c}\delta' + \left[\frac{\partial S}{\partial z} - \frac{E_T}{c}\right]\delta + \frac{\partial E}{\partial z}u \tag{116}$$

$$\frac{\partial^2 E}{\partial z^2} = \frac{S}{c^2}\delta'' - \left[\frac{2}{c}\frac{\partial S}{\partial z} - \frac{E_T}{c^2}\right]\delta' + \left[\frac{\partial^2 S}{\partial z^2} - \frac{2}{c}\frac{\partial E_T}{\partial z}\right]\delta + \frac{\partial^2 E_T}{\partial z^2}u \tag{117}$$

$$\frac{\partial E}{\partial t} = S\delta' + E_T\delta + \frac{\partial E_T}{\partial t}u \tag{118}$$

$$\frac{\partial^2 E}{\partial t^2} = S\delta'' + E_T\delta' + 2\frac{\partial E_T}{\partial t}\delta + \frac{\partial^2 E_T}{\partial t^2}u \tag{119}$$

Substituting into the equation for the fields and equating the coefficients to zero

$$(\delta'') \quad \frac{S}{c^2} = \mu_o\varepsilon_o S \tag{120}$$

$$(\delta') \quad -\left[\frac{2}{c}\frac{\partial S}{\partial z} - \frac{E_T}{c^2}\right] = \mu_o\varepsilon_o E_T \tag{121}$$

$$(\delta) \quad \frac{\partial^2 S}{\partial z^2} - \frac{2}{c}\frac{\partial E_T}{\partial z} = 2\mu_o\varepsilon_o\frac{\partial E_T}{\partial t} + \frac{\mu_o N e^2}{m}S \tag{122}$$

$$(u) \quad \frac{\partial^2 E_T}{\partial z^2} = \mu_o\varepsilon_o\frac{\partial^2 E_T}{\partial t^2} + \frac{\mu_o N e^2}{m}E_T \tag{123}$$

From these we obtain

$$(\delta'') \quad c = \frac{1}{\sqrt{\mu_o\varepsilon_o}} \tag{124}$$

$$\therefore S = f(t) \tag{125}$$

Therefore S is a function of t only.

At $t=0$,

$$E = S_o \delta(t) \tag{126}$$

$$\therefore S = S_o \tag{127}$$

$$(\delta) \quad -\frac{2}{c}\frac{\partial E_T}{\partial z} = 2\mu_o \varepsilon_o \frac{\partial E_T}{\partial t} + \frac{\mu_o N e^2}{m} S_o \tag{128}$$

This equation is only imposed at the wavefront and so we can delete the $\partial E_T/\partial t$ term, and we find

$$\left[E_T\right]_{z=ct} = -\frac{\mu_o N e^2 c}{2m} z S_o \tag{129}$$

which may be re-written as

$$\left[E_T\right]_{z=ct} = -\frac{\omega_p^2}{2}\frac{z}{c} S_o \tag{130}$$

in agreement with the complete solution.

6 COLLISIONLESS MAGNETISED PLASMA

6.1 The propagation of arbitrary pulses through the ionosphere is a problem of some complexity, involving electron motion in the Earth's magnetic field, with this motion coupled to the radiation fields. In addition, the electron density and magnetic field depend on altitude. This note is a preliminary study of impulse propagation through a uniform plasma with a uniform imposed magnetic field, the method of attack being that of the Method of Fractured Solutions. Though this method is not essential to the solution of this particular problem, it has significant advantages when the medium parameters have a spatial dependency.

A further simplification is to consider a collisionless plasma; that is, the mean free path of the electrons is taken to be so large that dissipation can be ignored.

6.2 The Equations

We consider an ionised medium containing N electrons per unit volume, with an imposed magnetic field H_o. We now direct a plane impulsive EM wave in the direction of H_o, which we will call the z-direction. This EM wave causes the electrons to move and the resulting currents modify the fields. The equations of motion for a charge in an arbitrary EM field are L3:

$$\ddot{x} = -\frac{e}{m}E_x - \frac{e\mu_o}{m}(\dot{y}H_o - \dot{z}H_y) \tag{131}$$

$$\ddot{y} = -\frac{e}{m}E_y - \frac{e\mu_o}{m}(\dot{z}H_x - \dot{x}H_o) \tag{132}$$

$$\ddot{z} = -\frac{e\mu_o}{m}(\dot{x}H_y - \dot{y}H_x) \tag{133}$$

The ratio of the force exerted by the magnetic vector of the travelling wave to the force exerted by the electric vector is equal to v/c, where v is the velocity. Accordingly we may ignore terms involving H_x and H_y, and the set of equations reduces to:

$$\ddot{x} = -\frac{e}{m}E_x - \frac{e\mu_o H_o}{m}\dot{y} \tag{134}$$

$$\ddot{y} = -\frac{e}{m}E_y + \frac{e\mu_o H_o}{m}\dot{x} \tag{135}$$

The motion takes place in the x-y plane and the wave vectors are so restricted. This allows us to take advantage of the complex notation by writing

$$\ddot{z} = 0 \tag{136}$$

$$v = x + iy \tag{137}$$

$$E = E_x + iE_y \tag{138}$$

$$H = H_x + iH_y \tag{139}$$

equation of motion

$$\ddot{v} - i\frac{e\mu_o H_o}{m}\dot{v} = -\frac{e}{m}E \tag{140}$$

The field equations are:

$$\frac{\partial E_x}{\partial z} + \mu_o\frac{\partial H_y}{\partial t} = 0 \tag{141}$$

$$-\frac{\partial E_y}{\partial z} + \mu_o\frac{\partial H_x}{\partial t} = 0 \tag{142}$$

$$\frac{\partial H_y}{\partial z} + \varepsilon_o\frac{\partial E_x}{\partial t} = Ne\dot{x} \tag{143}$$

$$\frac{\partial H_x}{\partial z} - \varepsilon_o\frac{\partial E_y}{\partial t} = -Ne\dot{y} \tag{144}$$

Multiplying **(142)** by *-i* and adding the result to **(141)**, we obtain

$$\frac{\partial E}{\partial z} - i\mu_o\frac{\partial H_y}{\partial t} = 0 \tag{145}$$

and in a similar fashion, from **(143)** and **(144)**

$$\frac{\partial H}{\partial z} + i\varepsilon_o\frac{\partial E}{\partial t} = iNe\dot{v} \tag{146}$$

Equations **(140)**, **(145)** and **(146)** constitute the set of equations to be solved. Differentiating **(146)** with respect to time, we obtain

$$\frac{\partial^2 H}{\partial t \partial z} + i\varepsilon_o \frac{\partial^2 E}{\partial t^2} = iNe\ddot{v} \qquad (147)$$

Making use of **(146)** and **(140)**, this equation becomes

$$\frac{\partial^2 H}{\partial t \partial z} + i\varepsilon_o \frac{\partial^2 E}{\partial t^2} = i\frac{Ne^2 E}{m} + i\frac{e\mu_o H_o}{m}\left[\frac{\partial H}{\partial z} + i\varepsilon_o \frac{\partial E}{\partial t}\right] \qquad (148)$$

Differentiating **(145)** with respect to z

$$\frac{\partial^2 E}{\partial z^2} - i\mu_o \frac{\partial^2 H}{\partial z \partial t} = 0 \qquad (149)$$

Substituting for $\partial^2 H/\partial t \partial z$ from **(149)** into **(148)**, we obtain

$$\frac{\partial^2 E}{\partial z^2} = \mu_o \varepsilon_o \frac{\partial^2 E}{\partial t^2} + \frac{Ne^2 \mu_o}{m} E - \frac{e\mu_o^2 H_o}{m}\left[\frac{\partial H}{\partial z} + i\varepsilon_o \frac{\partial E}{\partial t}\right] \qquad (150)$$

and so we have succeeded in eliminating v and reducing the equations to be solved to **(145)** and **(150)**. It is possible to eliminate H from **(150)**, but no advantage is gained. Noting that the plasma frequency, ω_p, is given by

$$\omega_p = \sqrt{\frac{Ne^2 \mu_o c^2}{m}} \qquad (151)$$

and making the substitution

$$\frac{\eta}{c^2} = \frac{e\mu_o^2 H_o}{m} \qquad (152)$$

we have

$$\frac{\partial^2 E}{\partial z^2} = \mu_o \varepsilon_o \frac{\partial^2 E}{\partial t^2} + \frac{\omega_p^2}{c^2} E - \frac{\eta}{c^2}\left[\frac{\partial H}{\partial z} + i\varepsilon_o \frac{\partial E}{\partial t}\right] \qquad (153)$$

The pair of equations may be tackled by the Fractured Solution method directly.

6.3 The Fractured Solution Equations

We impose the Fractured Solution form

$$S_o \delta(t) \rightarrow S\,\delta\!\left(t-\frac{z}{c}\right) + T\,u\!\left(t-\frac{z}{c}\right) \tag{154}$$

where S, the strength of the delta-function, and T, the amplitude of the tail function may be functions of z and t. Generally neither term satisfies the differential equation on its own, the sum being required to obtain a solution.

To apply the method, we first develop the various derivatives of the Fractured Solution, where we are making use of the Schwartz definition of derivatives for distributions. For the E-field we write

$$E = S_E\,\delta\!\left(t-\frac{z}{c}\right) + E_T\,u\!\left(t-\frac{z}{c}\right) \tag{155}$$

and we obtain

$$\frac{\partial E}{\partial t} = S_E\,\delta' + E_T\,\delta + \frac{\partial E_T}{\partial t}\,u \tag{156}$$

We have chosen to let S_E depend on z only as the δ-function only exists at $z=ct$. Primes indicate differentiation with respect to the argument. Continuing,

$$\frac{\partial^2 E}{\partial t^2} = S_E\,\delta'' + E_T\,\delta' + 2\frac{\partial E_T}{\partial t}\,\delta + \frac{\partial^2 E_T}{\partial t^2}\,u \tag{157}$$

$$\frac{\partial E}{\partial z} = -\frac{S_E}{c}\,\delta' + \left[\frac{\partial S_E}{\partial z} - \frac{E_T}{c}\right]\delta + \frac{\partial E_T}{\partial z}\,u \tag{158}$$

$$\frac{\partial^2 E}{\partial z^2} = \frac{S_E}{c^2}\,\delta'' + \left[-\frac{2}{c}\frac{\partial S_E}{\partial z} + \frac{E_T}{c^2}\right]\delta' + \left[\frac{\partial^2 S_E}{\partial z^2} - \frac{2}{c}\frac{\partial E_T}{\partial z}\right]\delta + \frac{\partial^2 E_T}{\partial z^2}\,u \tag{159}$$

In a similar manner we impose the fractured form on the magnetic field and we have the set

$$H = S_H \delta\left(t - \frac{z}{c}\right) + H_T u\left(t - \frac{z}{c}\right) \tag{160}$$

$$\frac{\partial H}{\partial t} = S_H \delta' + H_T \delta + \frac{\partial H_T}{\partial t} u \tag{161}$$

$$\frac{\partial^2 H}{\partial t^2} = S_H \delta'' + H_T \delta' + 2\frac{\partial H_T}{\partial t}\delta + \frac{\partial^2 H_T}{\partial t^2} u \tag{162}$$

$$\frac{\partial H}{\partial z} = -\frac{S_H}{c}\delta' + \left[\frac{\partial S_H}{\partial z} - \frac{H_T}{c}\right]\delta + \frac{\partial H_T}{\partial z} u \tag{163}$$

$$\frac{\partial^2 H}{\partial z^2} = \frac{S_H}{c^2}\delta'' + \left[-\frac{2}{c}\frac{\partial S_H}{\partial z} + \frac{H_T}{c^2}\right]\delta' + \left[\frac{\partial^2 S_H}{\partial z^2} - \frac{2}{c}\frac{\partial H_T}{\partial z}\right]\delta + \frac{\partial^2 H_T}{\partial z^2} u \tag{164}$$

Substitution of these sets into the differential equations would lead to lengthy expressions from which to determine the four field quantities. The distributions are linearly independent and so the coefficients of each of the distributions must be identically zero, yielding a set of three equations from equation **(145)** and a set of four equations from equation **(153)**. Picking out these coefficients, we obtain,

$$(\delta')_1 \qquad S_H = \frac{i}{\mu_o c} S_E \tag{165}$$

$$(\delta)_1 \qquad H_T = \frac{i}{\mu_o c} E_T - \frac{i}{\mu_o}\frac{\partial S_E}{\partial z} \tag{166}$$

$$(u)_1 \qquad \frac{\partial E_T}{\partial z} - i\mu_o \frac{\partial H_T}{\partial t} = 0 \tag{167}$$

$$(\delta'')_2 \quad \frac{S_E}{c^2} = \mu_o \varepsilon_o S_E \tag{168}$$

$$(\delta')_2 \quad -\frac{2}{c}\frac{\partial S_E}{\partial z} + \frac{E_T}{c^2} = \mu_o \varepsilon_o E_T - i\frac{\varepsilon_o \eta}{c^2} S_E + \frac{\eta}{c^3} S_H \tag{169}$$

$$(\delta)_2 \quad \frac{\partial^2 S_E}{\partial z^2} - \frac{2}{c}\frac{\partial E_T}{\partial z} = \mu_o \varepsilon_o 2\frac{\partial E_T}{\partial t} - i\varepsilon_o \frac{\eta}{c^2} E_T + \frac{\omega_p^2}{c^2} S_E - \frac{\eta}{c^2}\left[\frac{\partial S_H}{\partial z} - \frac{H_T}{c}\right] \tag{170}$$

$$(u)_2 \quad \frac{\partial^2 E_T}{\partial z^2} = \mu_o \varepsilon_o \frac{\partial^2 E_T}{\partial t^2} - i\varepsilon_o \frac{\eta}{c^2}\frac{\partial E_T}{\partial t} + \frac{\omega_p^2}{c^2} E_T - \frac{\eta}{c^2}\frac{\partial H_T}{\partial z} \tag{171}$$

We now have seven equations. Equation **(171)** is just the original equation and **(168)** reduces to

$$\mu_o \varepsilon_o = \frac{1}{c^2} \tag{172}$$

Equations **(165)**, **(166)** and **(167)** allow the elimination of S_H and H_T from equations **(169)** and **(170)**. Following this procedure, there results

$$\frac{\partial S_E}{\partial z} = 0 \tag{173}$$

That is, the strength of the delta function does not reduce with distance. Applying this result to **(166)**, **(170)** and **(171)** we have

$$H_T = i\varepsilon_o c\, E_T \tag{174}$$

Hence

$$\frac{\partial H_T}{\partial z} = i\varepsilon_o c \frac{\partial E_T}{\partial z} \tag{175}$$

From **(170)**

$$\omega_p^2 S_E = -\, 2\frac{\partial E_T}{\partial t} - 2c\frac{\partial E_T}{\partial z} \tag{176}$$

6.4 The Fractured Solutions

From equation **(173)**, we have

$$\frac{\partial S_E}{\partial z} = 0 \tag{177}$$

Therefore S_E is a constant as we have excluded dependency on time. The initial condition, and hence the solution is

$$S_E = S_o \tag{178}$$

To obtain the tail amplitude at the wavefront we set

$$\frac{\partial E_T}{\partial t} = 0 \tag{179}$$

and we obtain, from **(176)**

$$\frac{\partial E_T}{\partial z} = -\frac{\omega_p^2}{2c} S_o \tag{180}$$

which gives

$$[E_T]_{z=ct} = -\frac{\omega_p^2}{2c} S_o z \tag{181}$$

which is the same result as for an unmagnetised plasma.

7. SUMMARY

7.1 The method of fractured solutions has been introduced for obtaining partial solutions to Maxwell's equations for media with arbitrary space and time dependent parameters. The solutions are obtained in terms of the impulse response which is separated (fractured) into two components that separately are not solutions to the equations. The partial results are the time/space dependency of the original impulse together with the amplitude at the wavefront of a modified step function (tail function). This latter result is analytical up to an integral and this integral can be evaluated in a number of interesting cases. Further analysis can extract more information as the derivatives of the tail function at the wavefront follow in a simple manner. The results are interesting in their own right, more general solutions can be generated by convolution, and the analytical character provides simple reference data for numerical studies. In addition late-time (low frequency) approximations can be matched to those conditions at the wavefront, so obtaining approximations valid over all space-time.

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