Theoretical Notes

Note 359

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Representations of Reflection and Transmission Functions of Canonical Electromagnetic Waveforms at a Conducting Half-Space

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Abstract

This paper presents analytical expressions for reflected and transmitted electromagnetic fields if a Delta function pulse impinges on a conductive half-space. These impulse response functions were obtained by analytic continuation of the Fresnel coefficients into the whole complex ω-plane and their Fourier transform back to time domain. From a mathematical point of view, the Fresnel coefficients are double-valued with a branch cut, and it is important for all further considerations to remain on the "physical" Riemann surface. Besides the branch cut no further singularities (e. g. poles) appear on this Riemann surface.

All response functions for reflection $R(\tau)$ and transmission $T(\tau)$ can be represented by sums of two terms. The first term stems from the path integral encircling the branch cut. The second term corresponds to the original wave function reduced by a factor identical with the asymptotic value (i.e. $\omega \to \infty$) of the Fresnel coefficient.

Other representations of the response function are also given (e. g. in terms of modified Bessel functions, and series expansions), and some examples for the convolution with specific canonical waveforms (unit step function, exponential decay, reciprocal sum of exponentials) in the time domain are presented.

Contents

1. Introduction

As was shown in Ref. 1, the problem of scattering of an electromagnetic pulse incident on a conductive half-space can be solved analytically by means of Fourier transform techniques. By analytical continuation, the frequency-dependent Fresnel coefficients for reflection and transmission are defined in the whole complex frequency plane, then multiplied by the Fourier transform of the pulse shape of the electric field, and transformed back to the time domain by contour integration around the singularities. In particular, it was shown that the Fresnel coefficients have only branch cuts along the interval $(0, -is_0)$ of the negative imaginary axis and are analytical functions otherwise on the 'physical' Riemann surface. Therefore, in the time domain, there appear integrals along both sides of the cut which in general cannot be performed in closed form. Results were given for both states of polarization of the incident electromagnetic wave. A brief review of this technique is presented in Appendix A.

More recently, two papers were presented at the 1990 Nuclear Electromagnetic Pulse Meeting (Ref. 2 and 3) choosing a somewhat different approach. They apply inverse Laplace transforms to the Fresnel coefficients which corresponds to the problem of finding the time domain response function of an incident Delta function pulse.

The inverse Laplace transform is then accomplished with the aid of Laplace transform tables. Response functions for any other pulse shapes can be calculated by application of the convolution theorem in time domain.

There are some advantages of these approaches in comparison with that of Ref. 1. The response function has to be calculated just once, i. e. for the Delta function. A Fourier or Laplace transform of the various incident pulse shapes is not required, because their scattered and transmitted fields can be calculated just by convolution in time domain. Moreover, the integrals remaining in the response functions contain the time variable only as the upper limit of integration. Therefore, from a numerical point of view, the time dependence can be computed consecutively.

On the other hand, the performance of convolution integrals can compensate the numerical advantages unless the wave form is an exponential function or sums of them (e. g., the "Bell Laboratory waveform"). One of the standard wave forms, the inverse sum of of two exponentials, cannot as easily be dealt with.

The intention of the present paper is to throw a bridge between the different approaches, in particular, to prove that the analytical results are identical. Both states of polarization will be considered.

2. Reflected Fields in the Case of Horizontal Polarization

In Ref. 1 it was shown that the reflected field can be determined by inverse Fourier transformation according to

$$E_2(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} E_0(\omega) R(\omega)\, e^{-i\omega t} d\omega \tag{1}$$

where $E_0(\omega)$ is the Fourier transform of the incident pulse and $R(\omega)$ denotes the analytic continuation of the Fresnel coefficient on the 'physical' Riemann surface.

For a Dirac Delta function, the Fourier transform is given by

$$E_0(t) = \delta(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\omega t} d\omega \tag{2}$$

Thus,

$$\delta(\omega) = 1. \tag{3}$$

Hence, the product $E_0(\omega)R(\omega)$ has a branch cut along $(0, -is_0)$ and is an analytical function otherwise.

However, it has to observed that for the case of an impinging impulse function $E_0(\omega)\cdot R(\omega) \equiv R(\omega)$ does not vanish at infinity. It rather takes constant values R_∞. Therefore, Jordan's theorem does not apply when evaluating the integral Eq. (1) by means of contour integration, and the contributions from the half circles at infinity must be added to the contributions of the branch cut (see Appendix B).

According to Appendices A and B we therefore have

$$R_h(t) \equiv E_2(t) = -\frac{2}{\pi}\cos\theta\; u(t) \int_0^{s_o} \frac{W(-is)}{\sigma/\epsilon_o - s(\epsilon-1)}\, e^{-st} ds + R_{h,\infty}\,\delta(t) \qquad (4)$$

where

$$W(-is) = \sqrt{s\sigma/\epsilon_o - s^2(\epsilon - \sin^2\theta)} \qquad (5)$$

$$s_o = \frac{\sigma}{\epsilon_o(\epsilon - \sin^2\theta)} \qquad (6)$$

$$R_{h,\infty} = \frac{\cos\theta - \sqrt{\epsilon - \sin^2\theta}}{\cos\theta + \sqrt{\epsilon - \sin^2\theta}}$$

and

$$u(t) = \int_{-\infty}^{t} \delta(t')dt' = \begin{cases} 0 & \text{if } t<0 \\ 1 & \text{if } t>0 \end{cases}$$

In Ref. 1, these expressions were evaluated by numerical integration. In the present paper, Eq. (4) will be rewritten in such a way that it can be compared with the corresponding expressions in Ref. 2.

To begin with we introduce dimensionless times

$$\tau = \frac{1}{2} s_o t \geq 0 \qquad (7)$$

and substitute in Eq. (4) s by

$$\rho = 2s/s_o - 1 \qquad (8)$$

to arrive at

$$R_h(\tau) = \frac{\sigma \cos\theta \, e^{-\tau} \, u(\tau)}{\pi \epsilon_o (\epsilon - 1)\sqrt{\epsilon - \sin^2\theta}} \int_{-1}^{+1} \frac{\sqrt{1-\rho^2}}{\rho - \rho_o} e^{-\rho\tau} \, d\rho + \frac{s_o}{2} R_{h,\infty} \, \delta(t) \tag{9}$$

with

$$\rho_o = \frac{\epsilon + \cos 2\theta}{\epsilon - 1} \tag{10}$$

Using the identity

$$\frac{e^{-\rho\tau}}{\rho - \rho_o} = e^{-\rho_o \tau} \left[\frac{1}{\rho - \rho_o} - \int_0^{\tau} e^{-(\rho - \rho_o)\tau'} d\tau' \right] \tag{11}$$

we obtain

$$R_h(\tau) = - \frac{\sigma}{\epsilon_o} \frac{\cos\theta \, u(\tau)}{(\epsilon - 1)\sqrt{\epsilon - \sin^2\theta}} \quad e^{-(1+\rho_o)\tau} \cdot \left[\int_0^{\tau} e^{\rho_o \tau'} I_1(\tau') \frac{d\tau'}{\tau'} - \frac{1}{\pi} \int_{-1}^{+1} \frac{\sqrt{1-\rho^2}}{\rho - \rho_o} d\rho \right] +$$

$$+ \frac{s_o}{2} R_{h,\infty} \, \delta(t) \tag{12}$$

where I_1 is the modified Bessel function with the integral representation

$$I_1(\tau) = \frac{\tau}{\pi} \int_{-1}^{+1} \sqrt{1-\rho^2} \; e^{\mp\rho\tau} d\rho \tag{13}$$

and the second integral in Eq. (12) can be evaluated for example by means of contour integration in the complex plane:

$$\frac{1}{\pi} \int_{-1}^{+1} \frac{\sqrt{1-\rho^2}}{\rho - \rho_o} d\rho = \sqrt{\rho_o^2 - 1} - \rho_o \tag{14}$$

If in Eq. (11) the lower limit of integration would have been extended from 0 to $-\infty$, Eq. (12) will simplify to

$$R_h(\tau) = -\frac{\sigma}{\epsilon_o} \frac{\cos\theta\ u(\tau)}{(\epsilon-1)\sqrt{\epsilon-\sin^2\theta}}\ e^{-(1+\rho_o)\tau} \int_{-\infty}^{\tau} e^{\rho_o \tau'}\ I_1(\tau')\ \frac{d\tau'}{\tau} + \frac{s_o}{2} R_{h,\infty}\delta(t) \qquad (12')$$

from which as a by-product we obtain the identity

$$\int_0^{\infty} e^{-\rho_o \tau'}\ I_1(\tau')\ \frac{d\tau'}{\tau'} = \rho_o - \sqrt{\rho_o^2-1}$$

Numerical Results

To reduce the computational efforts, ρ_o as given by Eq. (10) will be introduced as an independent variable. Therefore, the integral in Eq. (9) has to be evaluated only for a one-parametric instead of a two-parametric set of variables $\{\epsilon,\theta\}$.

Since

$$\epsilon-1 = \frac{2\cos^2\theta}{\rho_o-1}$$

$$\epsilon-\sin^2\theta = \frac{\rho_o+1}{\rho_o-1}\cos^2\theta$$

Eq. (9) writes as

$$R_h(\tau) = \frac{s_o}{2\pi}\, u(\tau) \sqrt{\rho_o^2-1} \int_{-1}^{1} \frac{\sqrt{1-\rho^2}}{\rho-\rho_o}\, e^{-(1+\rho)\tau} d\rho + \frac{s_o}{2}\left[\sqrt{\rho_o^2-1} - \rho_o\right]\delta(\tau) \qquad (15)$$

Thus, actually any two of the three parameters $\{\sigma,\ \epsilon,\ \theta\}$ are independent and can be condensed to the set $\{s_o,\ \rho_o\}$. Some results for the numerical integration in Eq. (15) are presented in Fig. 1 and Fig. 2. The contribution of the Delta function term is not shown. Therefore, strictly speaking the plots apply only for $\tau>0$.

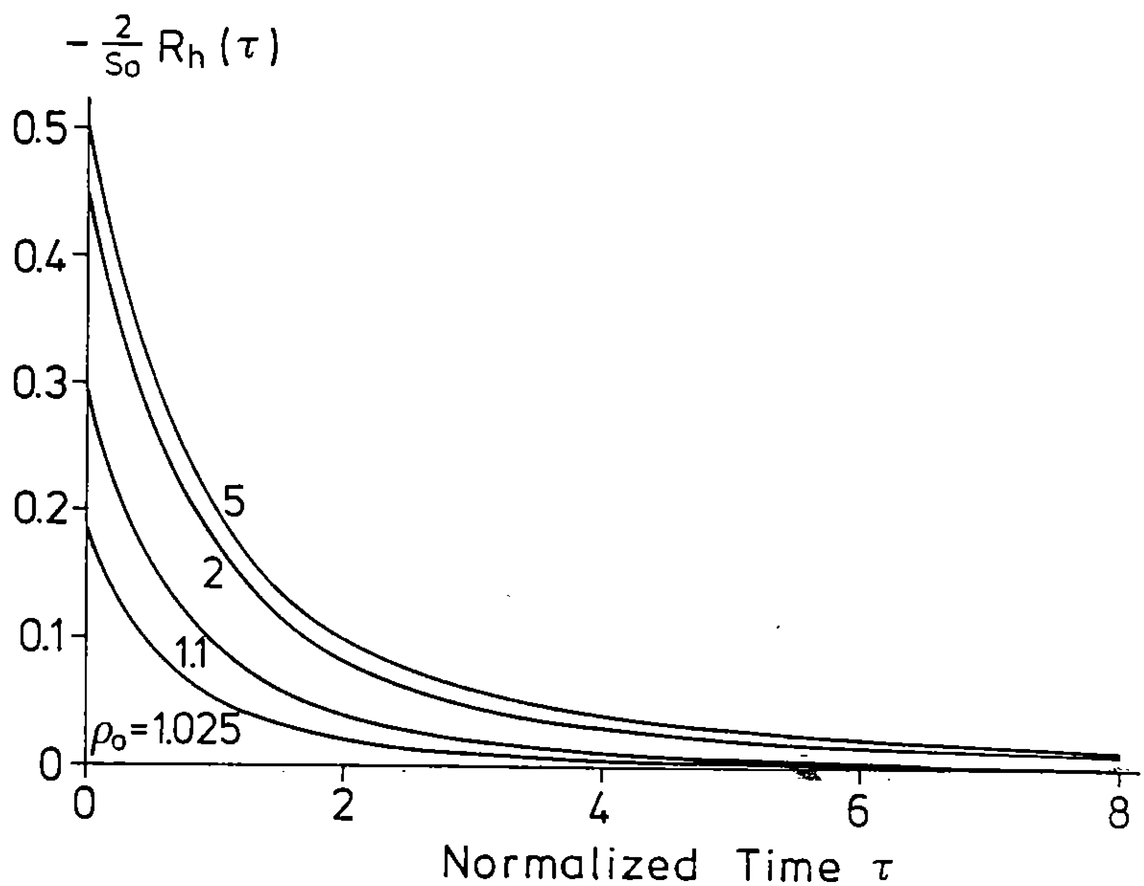

Fig. 1: Reflected delta function response in case of horizontal polarization

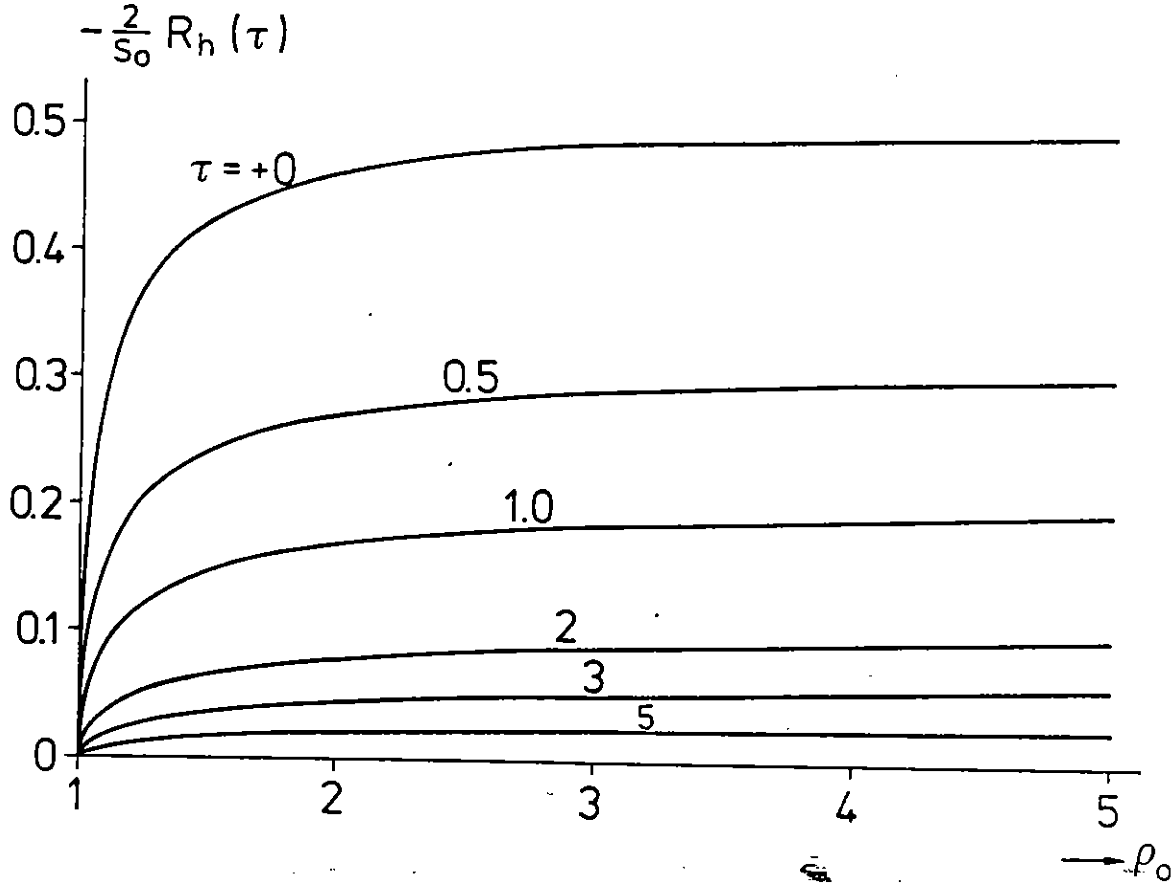

Fig. 2: Reflected delta function response as a function of ρ_0 in case of horizontal polarization

3. Reflected Fields in the Case of Vertical Polarization

According to Appendices A and B, the reflected pulse for an incident Delta function pulse is given by

$$R_V(t) = \frac{2}{\pi} u(t) \cos\theta \int_0^{s_o} \frac{(s\epsilon-\sigma/\epsilon_o)\, W(-is)\, e^{-st}}{\cos^2\theta(s\epsilon-\sigma/\epsilon_o)^2+W^2(-is)} ds + R_{V,\infty}\, \delta(t) \tag{16}$$

After some tedious calculation, the denominator D(s) can be factorized as follows

$$D(s) = \left[\epsilon^2\cos^2\theta + \sin^2\theta - \epsilon\right](s-s_1)\ (s-s_2) \tag{17}$$

where s_1 and s_2 are the two zeros of the denominator:

$$s_1 = \frac{\sigma}{\epsilon_o} \frac{1}{\epsilon-1} \tag{18a}$$

$$s_2 = \frac{\sigma}{\epsilon_o} \frac{1}{\epsilon-tg^2\theta} \tag{18b}$$

As emphasized in Ref. 1, these zeros correspond to poles on the imaginary axis of he "unphysical" Riemann surface, they do not appear on the "physical" sheet of the two-fold Riemann surface. Therefore, they do not contribute to the inverse Fourier transform.

It is noticed that $|s_{1,2}|>s_o> 0$. Hence, there is no singularity within the range of integration. Change of variable according to Eq. (8) and partial fraction decomposition of 1/D(s) yields

$$R_V(\tau) = \frac{1}{\pi}\frac{\sigma}{\epsilon_o}\frac{\cos\theta}{\cos2\theta}\frac{u(\tau)}{(\epsilon-\sin^2\theta)^{3/2}}$$

$$\cdot\ e^{-\tau}\sum_{i=1}^{2}(-1)^i e^{-\tau}\int_{-1}^{+1}\left[(1+\rho)\epsilon-2(\epsilon-\sin^2\theta)\right]\frac{\sqrt{1-\rho^2}}{\rho-\rho_i}\ e^{-\rho\tau}d\rho +$$

$$+\ \frac{s_o}{2} R_{V,\infty}\, \delta(\tau)$$

$$= - \frac{s_o}{2\pi} \frac{u(\tau)}{\cos 2\theta} \sqrt{\rho_1^2 - 1} \cdot e^{-\tau} \left[\int_{-1}^{+1} \frac{\sqrt{1-\rho^2}}{\rho - \rho_1} e^{-\rho\tau} d\rho \; - \right.$$

$$\left. - \left(1 - \epsilon \frac{\rho_1 - \rho_2}{\rho_1 + 1}\right) \int_{-1}^{+1} \frac{\sqrt{1-\rho^2}}{\rho - \rho_2} e^{-\rho\tau} d\rho \right] + \frac{s_o}{2} R_{V,\infty} \delta(\tau) \tag{19}$$

with

$$\rho_1 = 2s_1/s_o - 1 = \frac{\epsilon + \cos 2\theta}{\epsilon - 1} \tag{20a}$$

$$\rho_2 = 2s_2/s_o - 1 = \frac{\epsilon - tg^2\theta \cdot \cos 2\theta}{\epsilon - tg^2\theta} \tag{20b}$$

As in Section 2, we introduce the identity

$$\frac{e^{-\rho\tau}}{\rho - \rho_i} = e^{-\rho_i \tau} \left[\frac{1}{\rho - \rho_i} - \int_o^{\tau} e^{-(\rho - \rho_i)\tau'} d\tau' \right], \quad i = 1,2 \tag{21}$$

by which

$$\frac{1}{\pi} \int_{-1}^{+1} \frac{\sqrt{1-\rho^2}}{\rho - \rho_i} e^{-\rho\tau} d\rho = e^{-\rho_i \tau} \left[\sqrt{\rho_i^2 - 1} - \rho_i - \int_o^{\tau} e^{\rho_i \tau'} I_1(\tau') \frac{d\tau'}{\tau'} \right] \tag{22}$$

where I_1 is the modified Bessel function of order 1 as given by Eq. (13). Comparison with the result for horizontal polarization (Eq. (12)) shows that there is complete analogy in time dependence for the different zeros ρ_o and ρ_i (i=1,2), respectively.

The special case $\theta = 45^o$ requires some caution because of degenerate roots $\rho_1 = \rho_2 \equiv \rho_o$. We have

$$\lim_{\theta \to \frac{\pi}{4}} \frac{\rho_1 - \rho_2}{\cos 2\theta} = -2 \frac{2\epsilon - 1}{(\epsilon-1)^2} \qquad (23)$$

and therefore

$$R_V(\tau) = -\frac{2}{\pi} \frac{u(\tau)e^{-\tau}}{(\epsilon-1)^2\sqrt{2\epsilon-1}} \int_{-1}^{+1} \frac{\rho\epsilon-(\epsilon-1)}{(\rho-\rho_o)^2} \sqrt{1-\rho^2}\, e^{-\rho\tau} d\rho + \frac{s_o}{2} R_{V,\infty}\delta(\tau) \qquad (24)$$

$$\text{where } \rho_o = \frac{\epsilon}{\epsilon-1} \quad \text{and } R_{V,\infty} = \frac{\epsilon - \sqrt{2\epsilon-1}}{\epsilon + \sqrt{2\epsilon-1}} \qquad (25)$$

Figs. 3 and 4 show some numerical results for different parameters ϵ and θ. In contrast to the case of horizontal polarization, the set of parameters cannot again be grouped together into a single quantity.

For vertical incidence ($\theta=0$) the reflected E field should be indentical for both states of polarization.

This can easily be verified since for $\theta=0$

$$\rho_1 = \frac{\epsilon+1}{\epsilon-1} = \rho_o \qquad \rho_2 = 1$$

$$1-\epsilon \frac{\rho_1-\rho_2}{\rho_1+1} = 0$$

and therefore Eq.(19) and Eq.(15) turn out to be identical (except for the opposite sign which is due to the fact that R_h and R_V are defined as response functions for the electric and magnetic field vectors, respectively).

Eq. (19) shows that the integrand changes sign for

$$\rho = 1 - \frac{2}{\epsilon}\sin^2\theta.$$

Therefore, for sufficiently large relative dielectric constants ϵ and/or sufficiently small angles of incidence R_V remains positive.

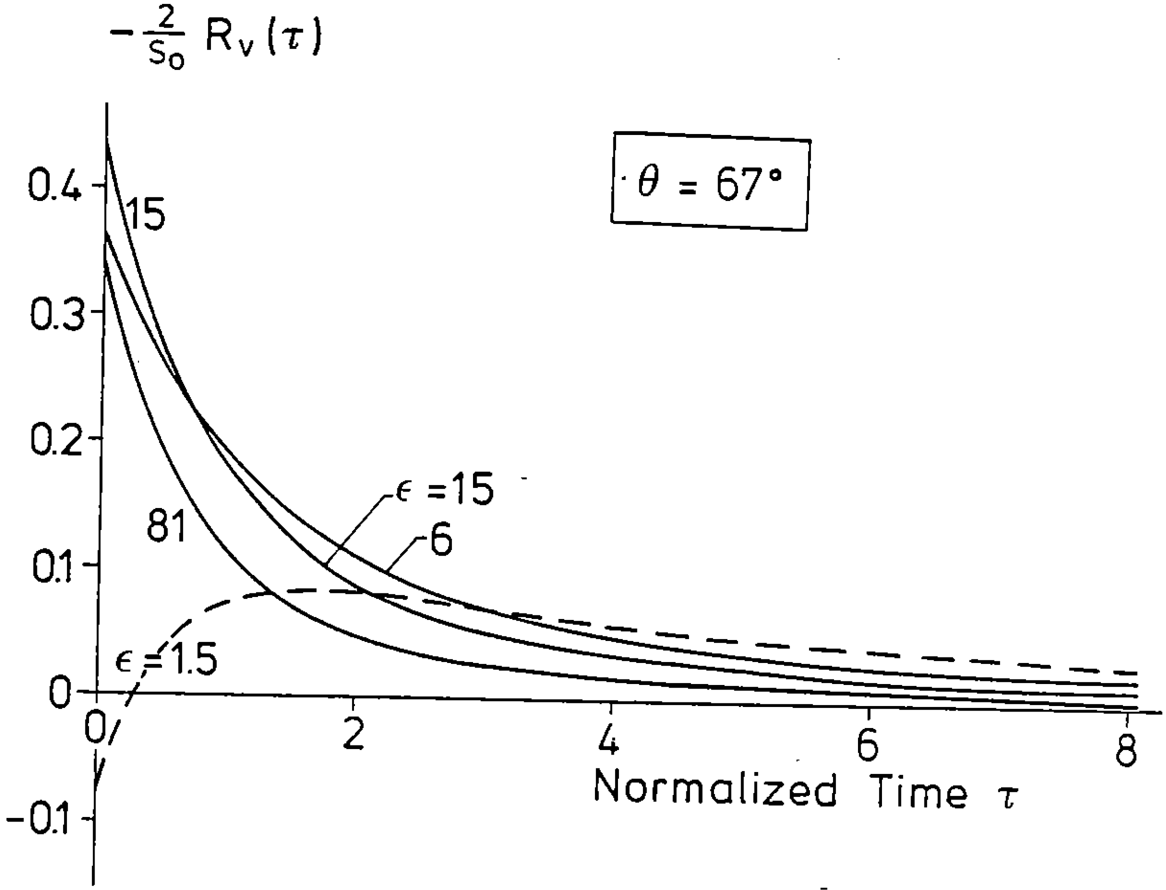

Fig. 3: Reflected delta function response in case of vertical polarization. Angle of incidence $\theta = 67°$

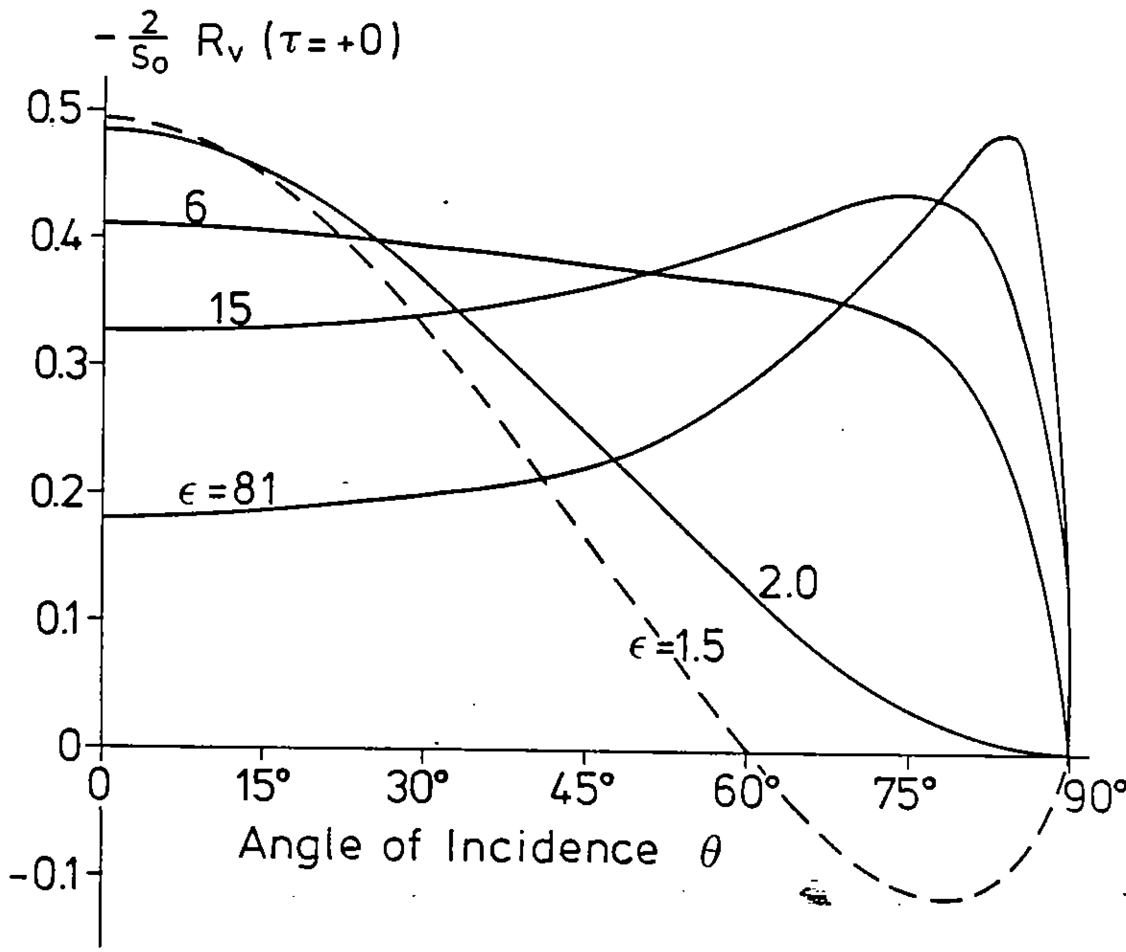

Fig. 4: Reflected delta function response as a function of the angle of incidence in case of vertical polarization

4. Transmitted Fields in the Case of Horizontal Polarization

For the transmitted electric field of a horizontally polarized Delta function pulse impinging on the surface of a conducting medium the following expression was obtained in Ref. 1 (see also Appendix A).

$$T_h(x,\ t) = \frac{i}{2\pi} u(t-t_o) \int_0^{s_o} (T^+(-is)-T^-(-is))\ e^{-st} ds + T_{h,\infty}(0)\ e^{-\frac{s_o}{2} t_o}\ \delta(t-t_o) \quad (26)$$

where

$$T^+(-is)-T^-(-is) = 4i\ \cos\theta\ \frac{s\ \cos\theta\ \sin\left[\frac{x}{c} W(-is)\right] - W(-is)\ \cos\left[\frac{x}{c} W(-is)\right]}{\sigma/\epsilon_o - s(\epsilon-1)} \quad (27)$$

$$T_{h,\infty}(0) = \sqrt{\rho_o^2-1} - \rho_o + 1 \qquad \text{(see Appendix B)}$$

and $t_o = \frac{x}{c}\sqrt{\epsilon-\sin^2\theta} > 0.$

Introducing dimensionless times (see Eq. (7))

$$\tau = \frac{1}{2} s_o t \quad (28)$$

and penetration depths

$$\xi = \frac{1}{2} s_o \sqrt{\epsilon-\sin^2\theta}\ \frac{x}{c} \geq 0 \quad (29)$$

we have

$$T_h(\xi,\tau) = \frac{s_\rho}{2\pi}\, u(\tau-\xi)\, \sqrt{\rho_o^2-1}\cdot e^{-\tau}\int_{-1} \frac{d\rho}{\rho-\rho_o}\left\{\sqrt{\frac{\rho_o-1}{\rho_o+1}}\,(1+\rho)\,\sin\left[\xi\sqrt{1-\rho^2}\right] - \sqrt{1-\rho^2}\,\cos\left[\xi\sqrt{1-\rho^2}\right]\right\} e^{-\rho\tau} + \frac{s_\rho}{2} T_{h,\infty}(0)\, e^{-\xi}\,\delta(\tau-\xi)$$

$$= \frac{s_\rho}{2\pi}\, u(\tau-\xi)\,\sqrt{\rho_o^2-1}\; e^{-\tau}\left\{\frac{\sqrt{\rho_o-1}}{\sqrt{\rho_o+1}}\int_{-1}^{+1}\sin\left[\xi\sqrt{1-\rho^2}\right] e^{-\rho\tau}\,d\rho + \left[\sqrt{\rho_o^2-1} - \frac{\partial}{\partial\xi}\right]\int_{-1}^{+1}\frac{\sin\left[\xi\sqrt{1-\rho^2}\right]}{\rho-\rho_o}\, e^{-\rho\tau}\,d\rho\right\} + \frac{s_\rho}{2} T_{h,\infty}(0)\, e^{-\xi}\,\delta(\tau-\xi) \qquad (30)$$

Figs. 5 and 6 show transmitted response functions calculated on the basis of Eq. (30) for $\tau > \xi$. These functions can change sign in a certain depth which is due to the fact that for a monochromatic wave the phase lags increasingly with deeper penetration until it reaches 180° at the skin depth.

Another representation of the transmitted response function will be derived in the following.

First, we consider the integral

$$F(\xi,\tau) = \int_{-1}^{+1} \sin\left[\xi\sqrt{1-\rho^2}\right] e^{-\rho\tau}\,d\rho$$

$$= 2\frac{\partial}{\partial\xi}\int_0^1 \frac{d\rho}{\sqrt{1-\rho^2}}\cos\left[\xi\sqrt{1-\rho^2}\right]\cosh(\rho\tau)$$

$$= 2\frac{\partial}{\partial\xi}\int_0^1 \frac{d\rho}{\sqrt{1-\rho^2}}\cos(\xi\rho)\cosh\left(\tau\sqrt{1-\rho^2}\right)$$

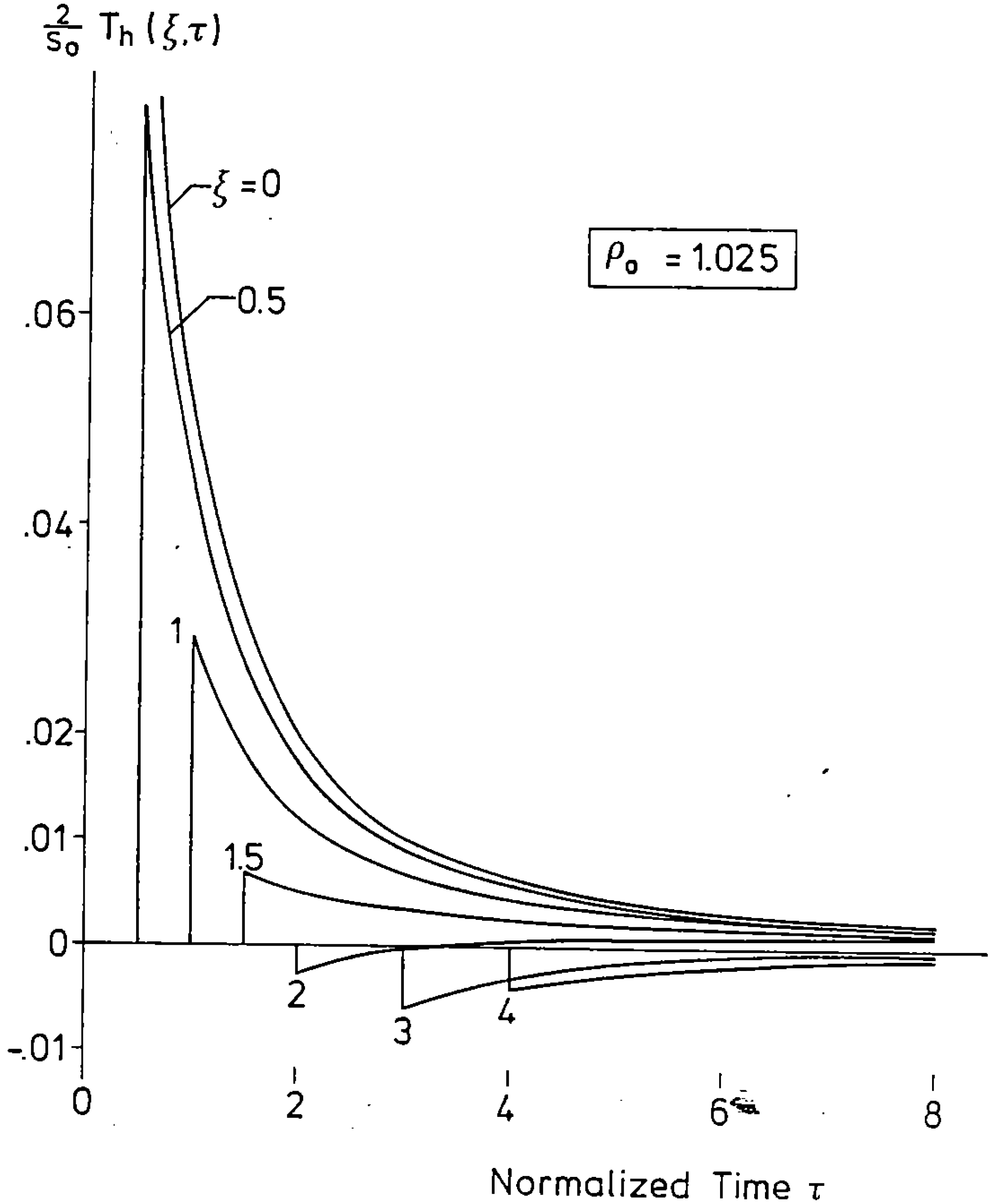

Fig. 5: Transmitted delta function response in penetration depths ξ in case of horizontal polarization (ρ_o = 1.025)

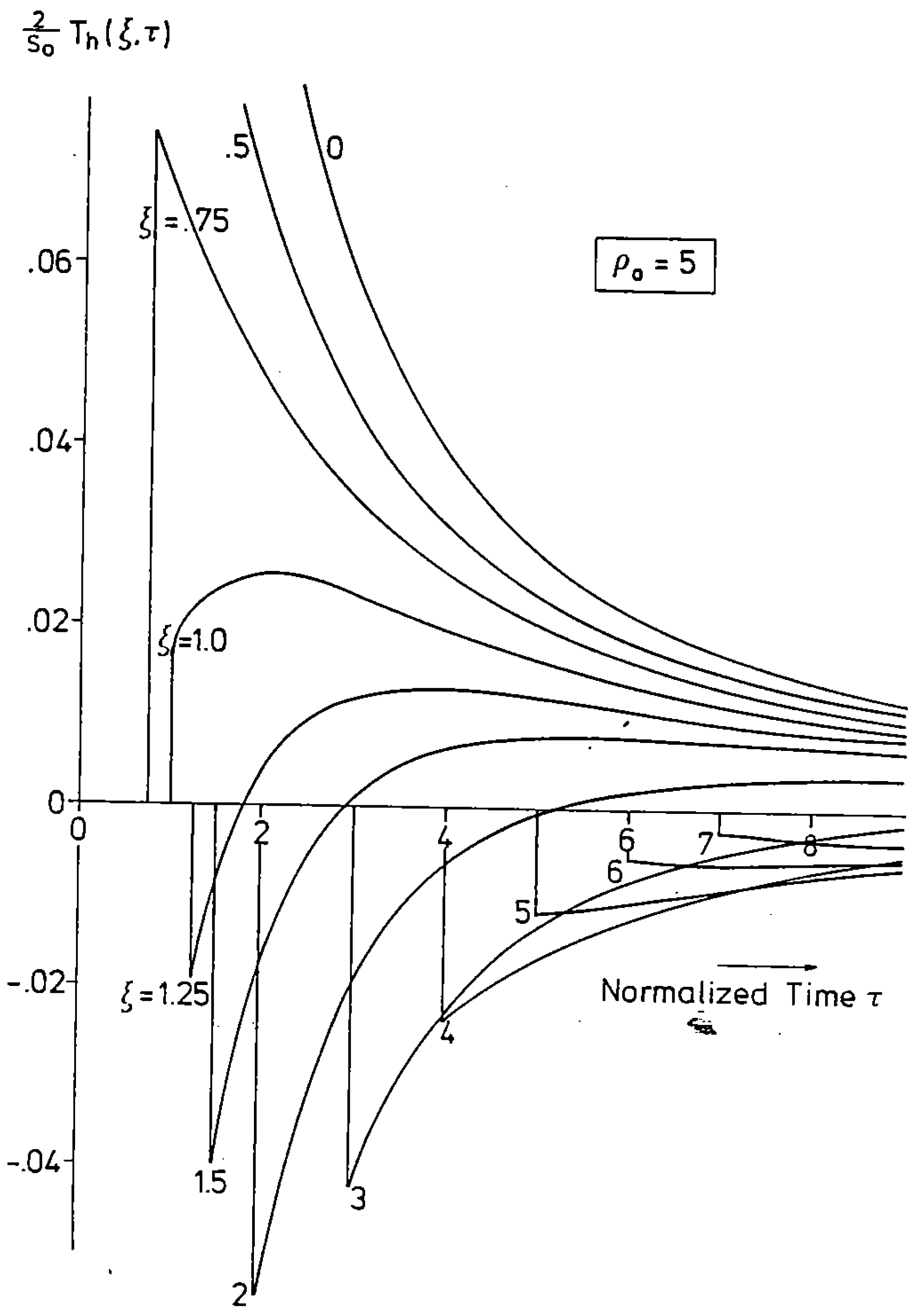

Fig. 6: Transmitted delta function response in penetration depths ξ in case of horizontal polarization ($\rho_o = 5$)

$$= -\pi \frac{\partial}{\partial \xi} \; I_0 \left[\sqrt{\tau^2 - \xi^2}\right] \qquad \text{(Ref. 5, p. 518)}$$

$$= \pi \frac{\xi}{\sqrt{\tau^2 - \xi^2}} \; I_1 \left(\sqrt{\tau^2 - \xi^2}\right) \tag{31}$$

Second, we observe that

$$G(\xi, \tau) = \int_{-1}^{+1} \frac{\sin\left(\xi\sqrt{1-\rho^2}\right)}{\rho - \rho_0} e^{-\rho\tau} \, d\rho \tag{32}$$

satisfies a second-order differential equation

$$\frac{\partial^2 G}{\partial \xi^2} - (\rho_0^2 - 1)\; G(\xi, \tau) = (\rho_0 - \frac{\partial}{\partial \tau})\; F(\xi, \tau) \tag{33}$$

With the boundary condition $G(0,\tau) = 0$ the solution of Eq. (33) becomes

$$G(\xi,\tau) = \frac{1}{\sqrt{\rho_0^2 - 1}} \int_0^{\xi} \sinh\left[(\xi - \xi')\sqrt{\rho_0^2 - 1}\right] \cdot F(\xi',\tau)\, d\xi' \; +$$

$$+ \frac{1}{\sqrt{\rho_0^2 - 1}} \sinh\left(\xi\sqrt{\rho_0^2 - 1}\right) \; \frac{\partial G}{\partial \xi}\bigg|_{\xi = 0} \tag{34}$$

where (see Eqs. (9) to (14))

$$\frac{\partial G}{\partial \xi}\bigg|_{\xi = 0} = \int_{-1}^{+1} \frac{\sqrt{1-\rho^2}}{\rho - \rho_0} \; e^{-\rho\tau} \, d\rho$$

$$= \pi \, e^{-\rho_0 \tau} \left[\sqrt{\rho_0^2 - 1} - \rho_0 \cdot \int_0^{\tau} e^{\rho_0 \tau'} I_1(\tau') \frac{d\tau'}{\tau'}\right] \tag{35}$$

A different representation of $G(\xi,\tau)$ is derived in Appendix D. Eq. (34) can further be simplified by means of integration by parts and Eq. (33)

$$G(\xi,\tau) = -\frac{\pi}{\rho_o^2-1}\left(\rho_o - \frac{\partial}{\partial\tau}\right)\int_0^{\xi} \cosh\left[(\xi-\xi')\sqrt{\rho_o^2-1}\right] I_0\left(\sqrt{\tau^2-\xi'^2}\right)\cdot d\xi'$$

$$+\frac{1}{\sqrt{\rho_o^2-1}}\sinh\left(\xi\sqrt{\rho_o^2-1}\right)\frac{\partial G}{\partial\xi}\Big|_{\xi=o} \tag{36}$$

With this we arrive at the final expression for the transmitted response function

$$T_h(\xi,\tau) = \frac{s_o}{2}u(\tau-\xi)\,e^{-\tau}\left\{(\rho_o-1)\,\xi\,\frac{I_1\left(\sqrt{\tau^2-\xi^2}\right)}{\sqrt{\tau^2-\xi^2}} - \right.$$

$$\left. -\frac{1}{\sqrt{\rho_o^2-1}}\int_0^{\xi} e^{-(\xi-\xi')\sqrt{\rho_o^2-1}}\left[\rho_o I_0\left(\sqrt{\tau^2-\xi'^2}\right) - \tau\,\frac{I_1\left(\sqrt{\tau^2-\xi'^2}\right)}{\sqrt{\tau^2-\xi'^2}}\right]d\xi'\right.$$

$$+\,e^{-\xi\sqrt{\rho_o^2-1}}\,\frac{\partial G(\xi,\tau)}{\partial\xi}\Big|_{\xi=o} + \frac{s_o}{2}T_{h,\infty}(0)e^{-\xi}\delta(\tau-\xi) \tag{37}$$

However, from a computational point of view, the equivalent expression as given by Eq. (30) seems to be far easier to handle.

5. Series Expansions

Since in Ref. 3 approximate analytical solutions are presented in the form of series expansions in terms of modified Bessel functions I_κ, we also attempt to find series expansions of the exact solutions Eq. (9) or (19).

Without loss of generality, only the case of horizontal polarization will be considered. In Eq. (9) we encounter the integral

$$L(\tau) = \frac{1}{\pi} \int_{-1}^{+1} \frac{\sqrt{1-\rho^2}}{\rho-\rho_o} e^{-\rho\tau} d\rho \tag{39}$$

The Taylor's series expansion of $L(\tau)$ about $\tau = 0$ is

$$L(\tau) = \sum_{\kappa=0}^{\infty} \frac{\tau^\kappa}{\kappa!} \frac{d^\kappa L(\tau)}{d\tau^\kappa} \Bigg|_{\tau=o}$$

$$= \frac{1}{\pi} \sum_{\kappa=o}^{\infty} (-1)^\kappa \frac{\tau^\kappa}{\kappa!} \int_{-1}^{+1} \rho^\kappa \frac{\sqrt{1-\rho^2}}{\rho-\rho_o} d\rho \tag{40}$$

For $\kappa = 0$ and 1

$$L_0(\rho_o) = \frac{1}{\pi} \int_{-1}^{+1} \frac{\sqrt{1-\rho^2}}{\rho-\rho_o} d\rho = \sqrt{\rho_o^2 -1} - \rho_o \tag{41}$$

$$L_1(\rho_o) = \frac{1}{\pi} \int_{-1}^{+1} \rho \frac{\sqrt{1-\rho^2}}{\rho-\rho_o} d\rho = \frac{1}{\pi} \int_{-1}^{+1} \sqrt{1-\rho^2}\, d\rho + \rho_o L_0(\rho_o) \tag{42}$$

$$= \frac{1}{2} - \rho_o^2 - \rho_o \sqrt{\rho_o^2 - 1}$$

were already evaluated in the previous sections. For $\kappa > 2$ we have in analogy to Eq. (42)

$$L_{\kappa+1}(\rho_o) = L_{\kappa+1}(0) + \rho_o L_\kappa(\rho_o) \tag{43}$$

We observe that

$$L_{\kappa+1}(0) = \frac{1}{\pi} \int_{-1}^{+1} \rho^\kappa \sqrt{1-\rho^2}\, d\rho \tag{44}$$

is zero for uneven values of κ.

For even values of κ, the integral is evaluated by contour integration in the complex plane. The complex function

$$F_\kappa(z) = z^\kappa \sqrt{z^2-1} \tag{45}$$

has a branch cut along the interval [-1, +1]. At infinity, as seen from the expansion

$$\lim_{z\to\infty} F(z) = z^{\kappa+1} \sum_{j=0}^{\infty} \binom{1/2}{j} (-z^2)^j \tag{46}$$

there is a pole in case of $j = (\kappa+2)/2$. Thus, by determining the residue at infinity, we have

$$\mathcal{L}_{\kappa+1}(0) = \begin{cases} \left| \binom{1/2}{\kappa/2+1} \right| & \kappa \text{ even} \\ 0 & \kappa \text{ uneven} \end{cases} \tag{47}$$

with the binomial coefficients

$$\binom{x}{n} = \begin{cases} \frac{1}{n!} x(x-1)\ldots(x-n+1) & n>0 \\ 1 & n=0 \\ 0 & n<0 \end{cases}$$

As a result, all $\mathcal{L}_\kappa(\rho_o)$ can be determined recursively while increasing the order of the expansion (40).

An alternate series expansion is obtained by expanding the denominator of Eq. (39) with regard to ρ about ρ_o

$$\mathcal{L}(\tau) = \frac{-1}{\pi\rho_o} \sum_{\kappa=o}^{\infty} (-1)^\kappa \int_{-1}^{+1} \left(\frac{\rho}{\rho_o}\right)^\kappa \sqrt{1-\rho^2}\, e^{+\rho\tau}\, d\rho \tag{48}$$

This expansion seems to be particularly useful if $\rho_o \gg 1$.

With the definition

$$F_\kappa(\tau) = \frac{\tau}{\pi} \int_{-1}^{+1} \rho^{\kappa-1}\sqrt{1-\rho^2}\; e^{+\rho\tau} d\rho \tag{49}$$

Eq. (48) reads as

$$L(\tau) = \frac{-1}{\rho_0 \tau} \sum_\kappa (-\rho_0)^{-\kappa} F_\kappa(\tau) \tag{50}$$

By definition, F_1 and F_2 are identical with the modified Bessel Functions I_1 and I_2, respectively. However, this correspondence does no longer apply for orders higher than 2 because different recurrence relations are valid for F_κ.

The new recurrence relations are derived by applying integration by parts to

$$F_{\kappa-1} - F_{\kappa+1} = \frac{\tau}{\pi} \int_{-1}^{+1} d\rho\; \rho^{\kappa-2} \sqrt{1-\rho^2}^{\,3}\; e^{\rho\tau}$$

$$= -\frac{1}{\pi} \int_{-1}^{+1} d\rho \left[(1-\rho^2)(\kappa-2)\rho^{\kappa-3} - 3\rho^{\kappa-1}\right] \sqrt{1-\rho^2}\; e^{\rho\tau}$$

$$= -\frac{1}{\pi} \int_{-1}^{+1} d\rho \left[(\kappa-2)\rho^{\kappa-3} - (\kappa+1)\rho^{\kappa-1}\right] \sqrt{1-\rho^2}\; e^{\rho\tau} \tag{51}$$

Hence, for $\kappa > 2$ we have

$$F_{\kappa+1} = F_{\kappa-1} - \frac{1}{\tau} \left[(\kappa+1) F_\kappa - (\kappa-2) F_{\kappa-2}\right] \tag{52}$$

and $F_{\kappa+1}$ can therefore be determined recursively from the modified Bessel functions I_0 and I_1 for which very accurate polynomial approximations are available (e.g. Ref. 4) in case of a numerical evaluation of the series expansion Eq.(50).

6. The Case of General Incident Pulse Shapes

Once having solved the reflection problem for an incident Delta function pulse, the reflected fields for general incident pulses $E_i(t)$ can easily be calculated by convolution

$$E_r(t) = \int_{-\infty}^{t} E_i(t') \, R(t-t') \, dt'$$

$$= \int_{0}^{\infty} E_i(t-t'') \, R(t'') \, dt'' \qquad (53)$$

The convolution becomes particularly simple if the incident field can be expressed as sums/differences of decaying exponentials (dimensionless times τ will be used in what follows)

$$E_i(\tau) = E_o \, e^{-\alpha\tau} u(\tau) \qquad (54)$$

where $u(\tau) = 1$ for $\tau > 0$ and $u(\tau) = 0$ for $\tau < 0$.

With Eqs. (15) and (54) we have after performing the integration over τ''

$$E_{r,h}(\tau) = \frac{1}{\pi} E_o \sqrt{\rho_o^2 - 1} \int_{-1}^{+1} d\rho \, \frac{\sqrt{1-\rho^2} \left[e^{-\alpha\tau} - e^{-(1+\rho)\tau}\right]}{(\rho-\rho_o)(\rho+1-\alpha)} + R_{h,\infty} e^{-\alpha\tau}, \tau \geq 0 \qquad (55)$$

which can be cast into the standard form by partial fraction decomposition of the denominator. It is worthwhile noting that any possible zero of the denominator is cancelled out by the numerator. If desirable, Eq. (55) can be written in terms of modified Bessel functions analogous to Eq. (12).

Compared to the approach chosen in Ref. 1, Eq. (55) has the advantage that the problem associated with poles of $E_i(\omega)$ located on the branch cut of the Fresnel coefficients is circumvented by performing the convolution in the time domain instead of dealing with products in the frequency domain.

As a special case of Eq. (54) we consider the unit step function

$$E_i(t) = u(t)$$

for which Eq. (55) reads for $\tau \geq 0$ and $E_o = 1$.

$$E_{r,h}(\tau) = \frac{1}{\pi}\sqrt{\rho_o^2-1} \int_{-1}^{+1} d\rho \frac{\sqrt{1-\rho^2}}{(\rho-\rho_o)(\rho+1)} \left[1-e^{-(1+\rho)\tau}\right] + R_{h,\infty} \tag{56}$$

On partial fraction decomposition we obtain ($\tau \geq 0$)

$$E_{r,h}(\tau) = \frac{1}{\pi}\sqrt{\frac{\rho_o-1}{\rho_o+1}} \left\{\sqrt{\rho_o^2-1} - \rho_o + 1 - \right. \tag{57}$$

$$\left. - \int_{-1}^{+1} d\rho \sqrt{1-\rho^2} \left[\frac{1}{\rho-\rho_o} - \frac{1}{\rho+1}\right] e^{-(1+\rho)\tau}\right\} + \sqrt{\rho_o^2-1} - \rho_o$$

$$= -1 - \sqrt{\frac{\rho_o-1}{\rho_o+1}} \left\{e^{-\tau}\left[I_o(\tau) + I_1(\tau)\right] + e^{-(1+\rho)\tau} \int_{-\infty}^{\tau} e^{\rho_o\tau} I_1(\tau') \frac{d\tau'}{\tau'}\right\}$$

The last line was obtained with the aid of Ref. 4:

$$\int_{0}^{\tau} e^{-\tau} I_1(\tau') \frac{d\tau'}{\tau'} = - e^{-\tau} \left[I_o(\tau) + I_1(\tau)\right] + 1 \tag{58}$$

For comparison, the contour integration approach of Ref. 1 is applied to the unit step function in Appendix C.

The transmitted unit step function response is calculated similarly. Figs. 7 to 11 show some numerical results for reflection and transmission.

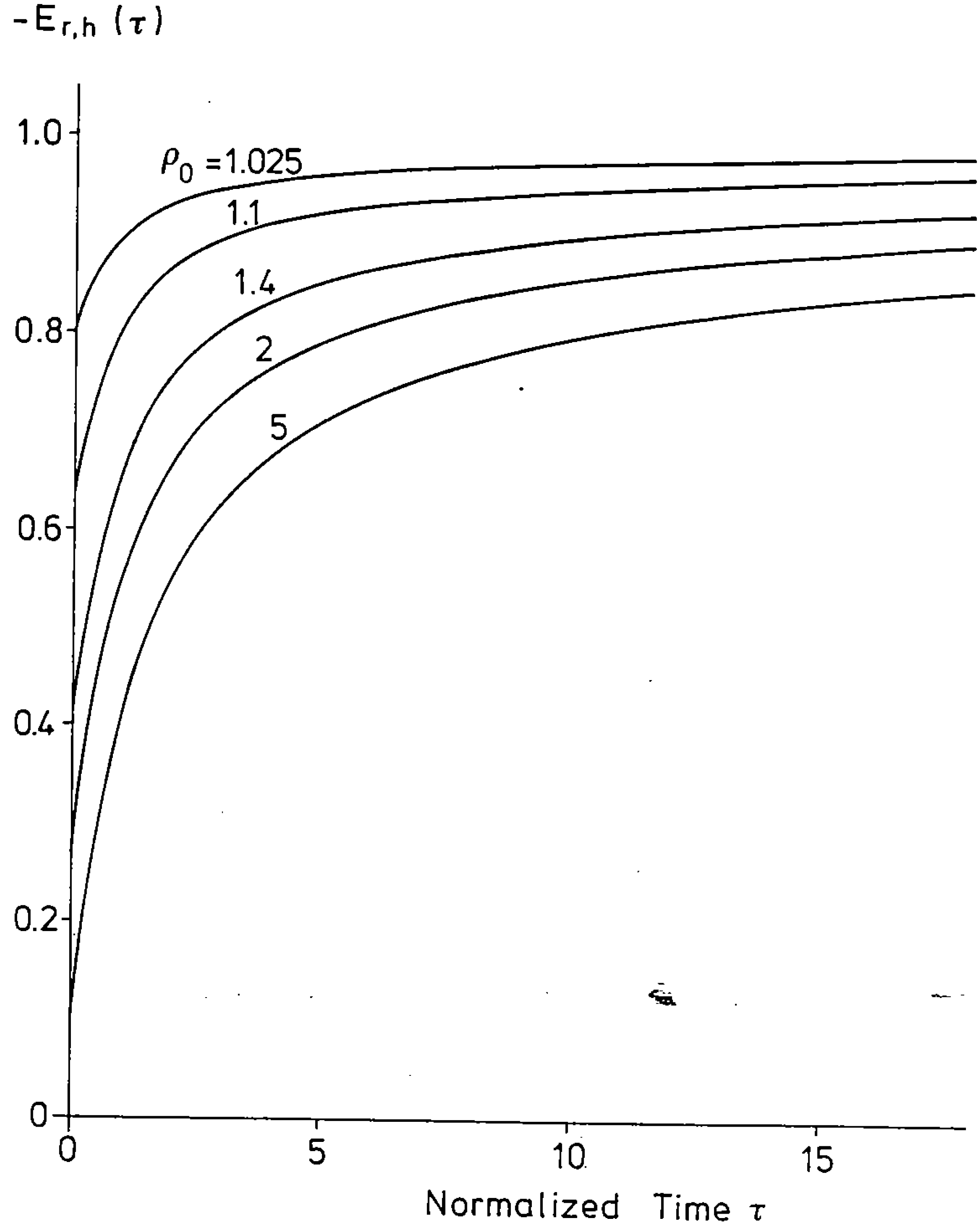

Fig. 7: Reflected unit step function response in case of horizontal polarization

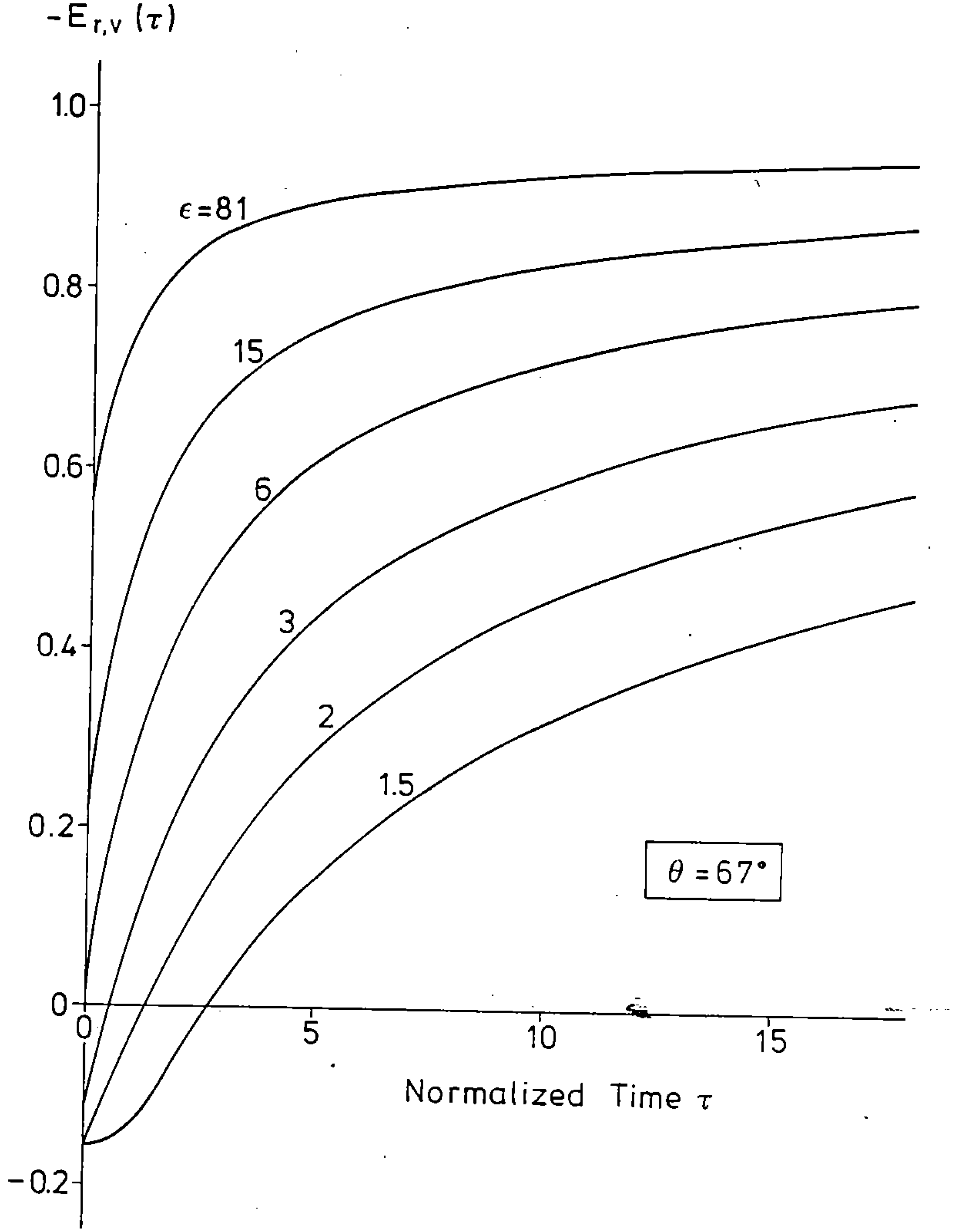

Fig. 8: Reflected unit step function response in case of vertical polarization. Angle of incidence $\theta = 67°$

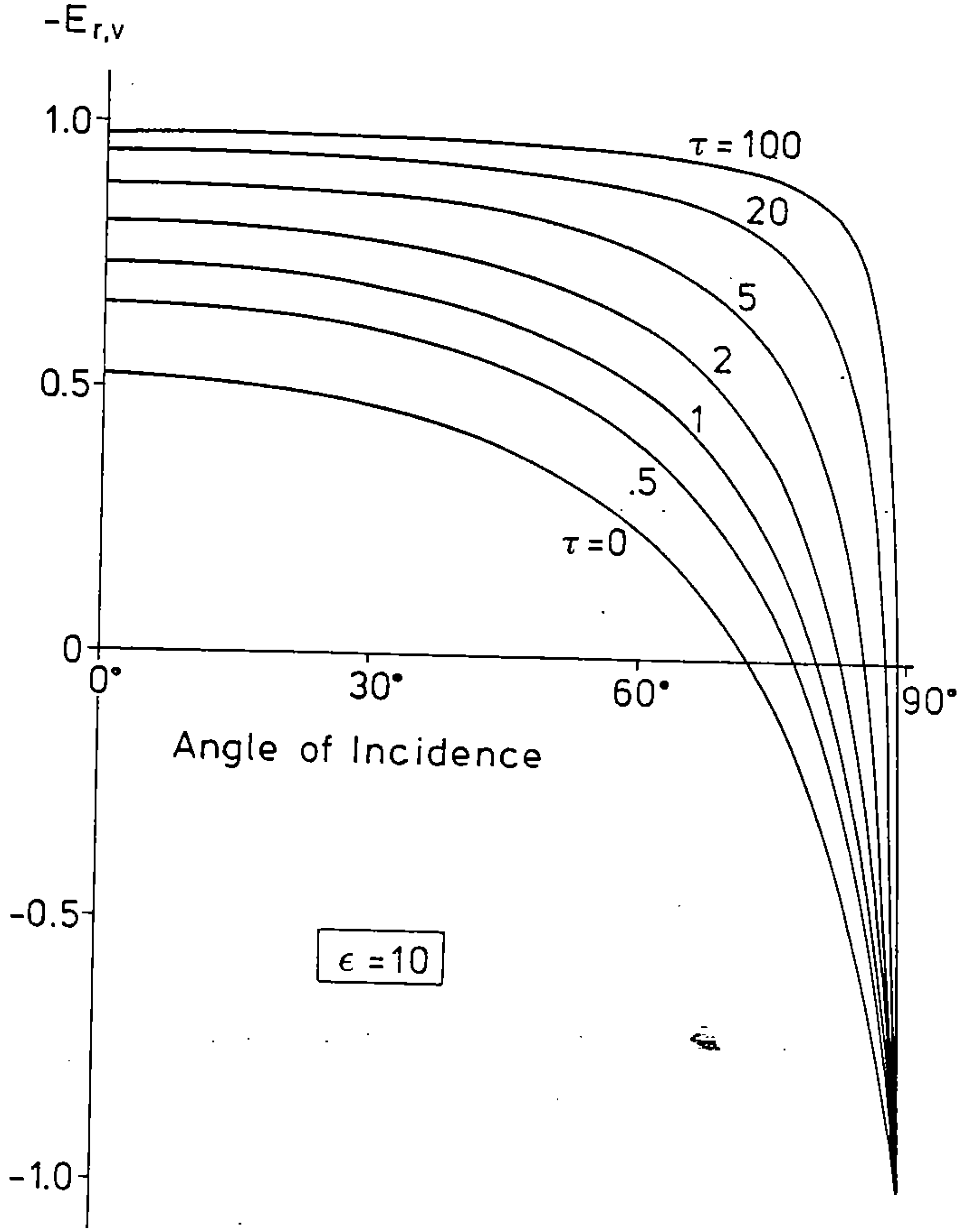

Fig. 9: Reflected unit step function response as a function of angle of incidence in case of vertical polarization

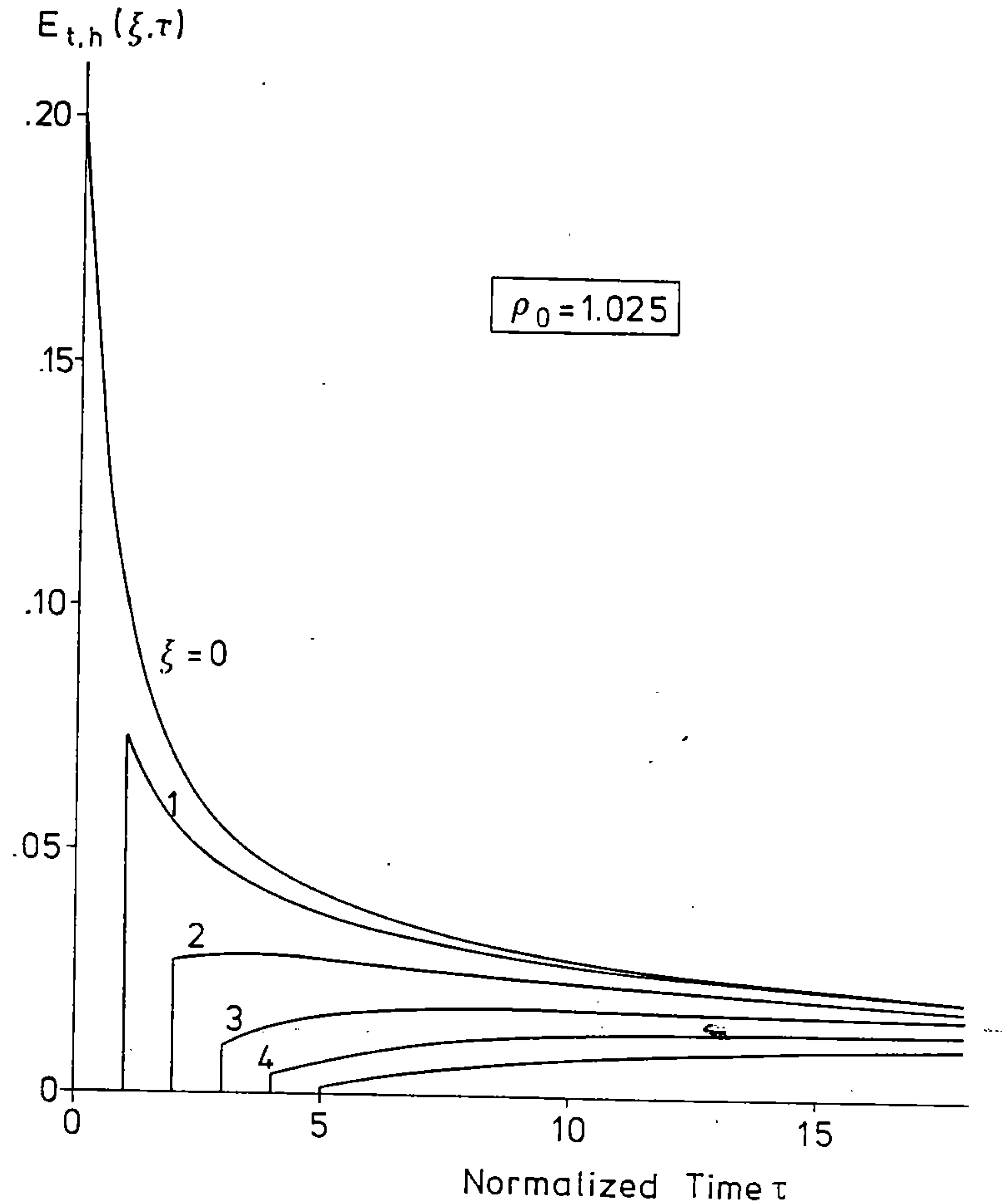

Fig. 10: Transmitted unit step function response in penetration depths ξ in case of horizontal polarization (ρ_o = 1.025)

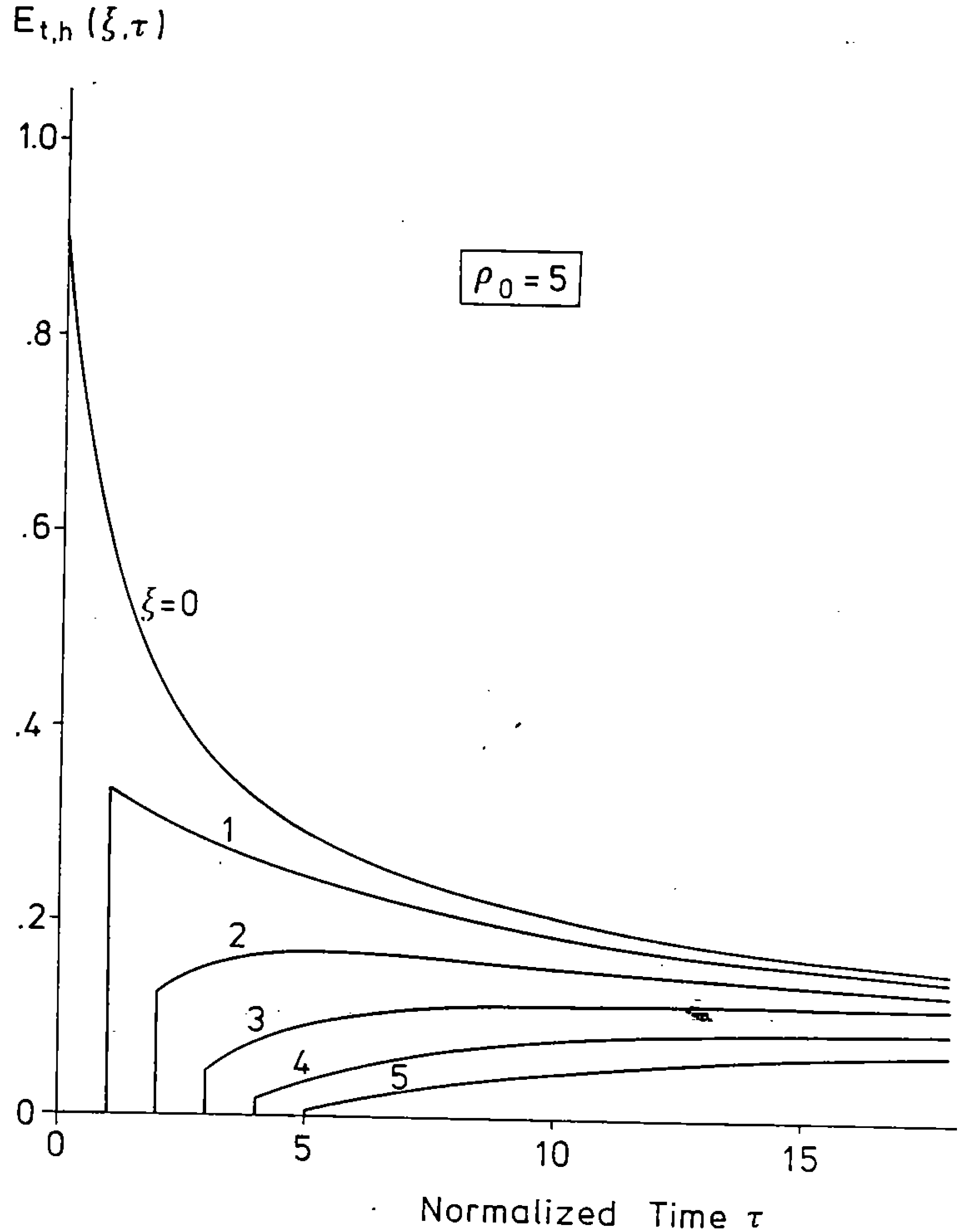

Fig. 11: Transmitted unit step function response in penetration depth ξ in case of horizontal polarization (ρ_o=5)

For other pulse shapes such as the inverse sum of exponentials

$$E_i(\tau) = A \frac{e^{\alpha\tau}}{1+\frac{A}{B}\, e^{(\alpha+\beta)\tau'}} \qquad -\infty<\tau<+\infty \tag{59}$$

the advantages are not so evident because the convolution integral cannot be performed analytically.

In Eq. (59) the time scale can be shifted such that A=B. Unless it is considered to evaluate the convolution integral numerically, a series expansion of Eq. (59), e. g. in terms of exponentials, might be helpful (see Ref. 1).

$$E_i(\tau) = \begin{cases} Ae^{\alpha\tau} \sum\limits_{\kappa=0}^{\infty} (-1)^\kappa e^{\kappa(\alpha+\beta)\tau} & \tau<0 \\ \\ Ae^{\alpha\tau} \sum\limits_{\kappa=1}^{\infty} (-1)^\kappa e^{-\kappa(\alpha+\beta)\tau} & \tau>0 \end{cases} \tag{60}$$

In case of horizontal polarization the reflected fields are given by

<u>(I) $\tau<0$</u>

$$E_{r,H}(\tau) = Ae^{\alpha\tau} \sum_{\kappa=0}^{\infty} (-1)^\kappa e^{\kappa(\alpha+\beta)\tau}\, \frac{1}{\pi}\sqrt{\rho_0^2-1} \int_{-1}^{+1} \frac{d\rho\, \sqrt{1-\rho^2}}{(\rho-\rho_0)(\rho+\rho_\kappa)} + R_{h,\infty}E_i(\tau) \tag{61}$$

where $\rho_\kappa = 1+\alpha+\kappa(\alpha+\beta)$.

The integrals over ρ can be evaluated immediately upon partial fraction decomposition of the integrand to give

$$E_{r,H}(\tau) = Ae^{\alpha\tau}\sqrt{\rho_0^2-1} \sum_{\kappa=0}^{\infty} (-1)^\kappa \frac{L_0(\rho_0)-L_0(-\rho_\kappa)}{\rho_0+\rho_\kappa}\, e^{\kappa(\alpha+\beta)\tau} + R_{h,\infty}E_i(\tau) \tag{62}$$

$$= Ae^{\alpha\tau} \sum_{\kappa=0}^{\infty} (-1)^\kappa \left[L_0(\rho_0)+\sqrt{\rho_0^2-1}\; \frac{L_0(\rho_0)-L_0(-\rho_\kappa)}{\rho_0+\rho_\kappa}\right] e^{\kappa(\alpha+\beta)\tau}$$

The series converges even for $\tau=0$ since successive terms have alternating sign and go to zero for $\kappa\to\infty$. It can further be shown that Eq. (62) corresponds to the (infinite) sum over Fresnel reflection coefficients evaluated at the poles ω_κ of $E_i\ (\omega)$ in the upper half-plane.

(II) $\tau>0$

$$E_{r,h}(\tau) = -\ Ae^{\alpha\tau} \sum_{\kappa=-1}^{-\infty} (-1)^{\kappa} e^{\kappa(\alpha+\beta)\tau} \frac{1}{\pi}\sqrt{\rho_0^2-1} \int_{-1}^{+1} \frac{d\rho\sqrt{1-\rho^2}}{(\rho-\rho_0)(\rho+\rho_\kappa)} \cdot$$

$$+A\ \frac{1}{\pi}\sqrt{\rho_0^2-1} \int_{-1}^{+1} \frac{d\rho\sqrt{1-\rho^2}}{\rho-\rho_0}\ e^{-(1+\rho)\tau} \sum_{-\infty}^{+\infty} \frac{(-1)^{\kappa}}{\rho+\rho_\kappa} + R_{h,\infty}E_i(\tau)$$

$$= -Ae^{\alpha\tau} \sum_{\kappa=1}^{\infty} (-1)^{\kappa}\left[\mathcal{L}_0(\rho_0)+\sqrt{\rho_0^2-1}\cdot \frac{\mathcal{L}_0(\rho_0)-\mathcal{L}_0(-\rho_\kappa)}{\rho_0+\rho_\kappa}\right] e^{\kappa(\alpha+\beta)} +$$

$$+\ \frac{A}{\alpha+\beta}\sqrt{\rho_0^2-1}\int_{-1}^{+1} \frac{d\rho\sqrt{1-\rho^2}\ e^{-(1+\rho_\kappa)\tau}}{(\rho-\rho_0)\ \sin\left[\frac{\alpha+1+\rho}{\alpha+\beta}\pi\right]} \qquad (63)$$

Again, the infinite sum is associated with the residues of the poles of $E(\omega)$, whereas the integral represents the contributions from the contour integration around the branch cut of the reflection coefficient.

This explicitly exhibits the equivalence of the convolution in time domain and the frequency domain approach chosen in Ref. 1.

APPENDIX A

Applications of Fourier Transform and Contour Integration Techniques to the Evaluation of Reflected and Transmitted Fields

(Summary of Ref. 1)

The complex Fresnel reflection and transmission (i. e. refraction) coefficients for a plane electromagnetic wave incident from the vacuum upon the plane surface of a conducting semi-infinite medium are

$$R_h(\omega) = \frac{\omega\cos\theta - W(\omega)}{\omega\cos\theta + W(\omega)} \qquad \text{(A.1)}$$

$$R_v(\omega) = \frac{(\omega\epsilon + i\sigma/\epsilon_o)\cos\theta - W(\omega)}{(\omega\epsilon + i\sigma/\epsilon_o)\cos\theta + W(\omega)} \qquad \text{(A.2)}$$

$$T_h(\omega) = \frac{2\omega}{\omega\cos\theta + W(\omega)} \, e^{i\frac{x}{c}W(\omega)} \qquad \text{(A.3)}$$

$$T_v(\omega) = \frac{2\cos\theta\sqrt{\omega^2\epsilon + i\omega\sigma/\epsilon_o}}{(\omega\epsilon + i\sigma/\epsilon_o)\cos\theta + W(\omega)} \, e^{i\frac{x}{c}W(\omega)} \qquad \text{(A.4)}$$

where

$$W(\omega) = \sqrt{\omega^2(\epsilon - \sin^2\theta) + i\omega\sigma/\epsilon_o} \qquad \text{(A.5)}$$

and x is the penetration depth in the conducting medium.

Since all these functions are double-valued, the analytic continuation into the complex ω-plane requires some caution in order not to leave the physical Riemann layer.

The Fresnel coefficients have a branch cut on the interval $(0,-is_o)$ of the imaginary axis with

$$s_o = \frac{\sigma}{\epsilon_o(\epsilon - \sin^2\theta)} \tag{A.6}$$

and are analytic otherwise. Their behavior at infinity is discussed in Appendix B.

On the physical Riemann surface, the limiting values of $W(\omega)$ on the branch cut are $(s = i\omega)$

$$w^{\pm}(-is) = \pm \sqrt{s^2(\epsilon - \sin^2\theta) + s\sigma/\epsilon_o}, \qquad s \epsilon (0, s_o) \tag{A.7}$$

when approaching the cut from the right-hand side (+) and left-hand side (-), respectively.

Furthermore,

$$W(is) = \pm i \sqrt{s^2(\epsilon - \sin^2\theta) + s\sigma/\epsilon_o}, \qquad s \epsilon \ (0, s_o) \tag{A.8}$$

on the positive (+) and negative (-) imaginary axis, respectively.

If

$$E_o(\omega) = \int_{-\infty}^{+\infty} E_o(t) e^{i\omega t} dt \tag{A.9}$$

now denotes the Fourier transform of an incident electromagnetic pulse $E_o(t)$ the reflected pulse $E_2(t)$ is calculated according to

$$E_2(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} E_o(\omega) R(\omega) e^{-i\omega t} d\omega \tag{A.10}$$

Table C.1 shows Fourier transforms of a few standard pulses

$E_0(t)$	$E(\omega)$	Remarks
$\delta(t)$	1	Delta function
$u(t)$	$iP\left[\frac{1}{\omega}\right] + \pi\delta(\omega)$	P: principal value u(t): unit step function
$u(t)e^{-\alpha t}$	$\frac{1}{\alpha - i\omega}$	$\alpha>0$
$\frac{1}{e^{-\alpha t} + e^{\beta t}}$	$\frac{\pi}{(\alpha+\beta)\sin\left[\frac{\alpha+i\omega}{\alpha+\beta}\pi\right]}$	$\alpha, \beta>0, -\infty<t<+\infty$ infinite number of poles on the imaginary ω-axis

<u>Table C.1:</u> Fourier Tansforms of Standard Pulses

Dependent on the numerical values of σ, ϵ, and θ, poles of $E(\omega)$ can eventually be located on the branch cut of $W(\omega)$. Only the reciprocal sum of two exponentials shows also poles on the positive imaginary ω-axis.

All Fourier transforms except for the Delta function which requires special treatment (see Appendix B) vanish uniformly at infinity.

To evaluate the integral (A.10), the integration contour is closed by a semi-circle at infinity in the upper half-plane for $\tau<0$. Except for the Delta function, the integrals over these semi-circles do not contribute due to Jordan's Lemma.

If there are no singularities in the upper half-plane, because of Cauchy's theorem,

$$E_2(t) = 0 \text{ for } \tau < 0 \tag{A.11}$$

which, as a consequence of the causality principle, might appear trivial.

In case of the reciprocal sum of exponentials

$$E_2(t) = i \sum_{k=0}^{\infty} \text{Res } E(\omega)\Big|_{\omega=\omega_\kappa} \cdot R(\omega_\kappa)\, e^{-i\omega_\kappa t} \;, \quad t<0 \qquad (A.12)$$

where $\omega_\kappa = -i(\kappa(\alpha+\beta)-\alpha)$ are the poles of $E(\omega)$.

In the lower half-plane ($t>0$), the contribution from the path integral enclosing the branch cut must be added to the residues of $E(\omega)$. We therefore arrive at

$$E_2(t) = \frac{i}{2\pi} \int_0^{s_0} E(-is) \left[R^+(-is)-R^-(-is)\right] e^{-st} ds \; -$$

$$- \; i \sum_\kappa \text{Res } E(\omega)\Big|_{\omega_\kappa} R(\omega_\kappa) e^{-i\omega_\kappa t}, \quad t>0 \qquad (A.13)$$

where $R^\pm$ denote the limiting values of $R(\omega)$ for $W^\pm$.

If there exists a pole ω_j on the interval $(0,-is_0)$ the cut contribution must be rewritten as follows

$$-\frac{i}{2\pi} P \int_0^{s_0} E(-is) \left[R^+(-is) - R^-(-is)\right] e^{-st} ds \quad +$$

$$+ \frac{1}{2} \left[E(\omega)(\omega-\omega_j)\right]\Big|_{\omega=\omega_j} \left[R^+(\omega_j) + R^-(\omega_j)\right] e^{-i\omega_j t} \qquad (A.14)$$

where the first term has to be understood as a Cauchy principal value integral, and the second term originates from the contribution of small semi-circles centered at ω_j on the left (-) and right (+)-hand side of the cut.

What still has to be done is of purely algebraic and numerical nature. Explicit numerical results for different incident EMP shapes were obtained in Ref. 1

APPENDIX B

Fourier Transforms of Fresnel Coefficients

For the evaluation of the Fourier transform

$$R(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} R(\omega) e^{-i\omega t} d\omega \tag{B.1}$$

which is indentical with the Delta function response, it is noticed that $R(\omega)$ has a branch cut along the interval $(0,-is_o)$ of the negative imaginary axis, and takes constant values at infinity

$$R_{h,\infty} = \frac{\cos\theta - \sqrt{\epsilon - \sin^2\theta}}{\cos\theta + \sqrt{\epsilon - \sin^2\theta}}, \tag{B.2}$$

$$R_{v,\infty} = \frac{\epsilon\cos\theta - \sqrt{\epsilon - \sin^2\theta}}{\epsilon\cos\theta + \sqrt{\epsilon - \sin^2\theta}} \tag{B.3}$$

in case of horizontal and vertical polarization, respectively.

If the contour of integration is closed by an infinite half-circle in the lower half-plane, we have by Cauchy's theorem

$$R(t) = \frac{1}{2\pi} \int_{0}^{-is_o} \left[R^{+}(\omega) - R^{-}(\omega)\right] e^{-i\omega t} d\omega + \frac{1}{2\pi} \int_{C_\infty^-} R(\omega) e^{-i\omega t} d\omega \tag{B.5}$$

$$= -\frac{i}{2\pi} \int_{0}^{s_o} \left[R^{+}(-is) - R^{-}(-is)\right] e^{-st} ds + \frac{1}{2\pi} R_\infty \int_{C_\infty^-} e^{-i\omega t} d\omega$$

Note that the integral over C_∞^- does not vanish since the conditions for the application of Jordan's lemma are not satisfied.

This integral can however be rewritten by means of Cauchy's theorem

$$\frac{1}{2\pi} R_\infty \int_{C_\infty^-} e^{-i\omega t} d\omega = \frac{1}{2\pi} R_\infty \int_{-\infty}^{+\infty} e^{-i\omega t} d\omega = R_\infty \, \delta(t) \tag{B.5}$$

since $e^{-i\omega t}$ is an analytical function on the entire complex ω-plane.

For horizontal polarization, Eq. (B.2) can be expressed in terms of ρ_o as defined by Eq. (10)

$$R_{h,\infty} = -\rho_o + \sqrt{\rho_o - 1} \equiv \frac{1}{\pi} \int_{-1}^{+1} \frac{\sqrt{1-\rho}}{\rho - \rho_o} d\rho \tag{B.6}$$

For the transmitted fields, the exponential in the transmission function requires special caution.

Series expansion about $\omega = \infty$ yields

$$\lim_{\omega \to \infty} e^{i \frac{x}{c} \sqrt{\omega^2 (\epsilon - \sin^2\theta) + i\omega\sigma/\epsilon_o}} =$$

$$= e^{i \frac{x}{c} \sqrt{\epsilon - \sin^2\theta} \left(\omega + \frac{i\sigma}{2\epsilon_o(\epsilon - \sin^2\theta)} - \ldots\ldots\right)}$$

$$= e^{i\omega t_o - \xi} \tag{B.7}$$

where $\quad t_o = \frac{x}{c} \sqrt{\epsilon - \sin^2\theta}$

and $\quad \xi = \frac{1}{2} \frac{\sigma}{\epsilon_o \sqrt{\epsilon - \sin^2\theta}} \frac{x}{c}$

Hence we have

$$T_{h,\infty}(x) = \frac{2\cos\theta}{\cos\theta + \sqrt{\epsilon - \sin^2\theta}}\, e^{i\omega t_o - \xi} \tag{B.8}$$

$$T_{v,\infty}(x) = \frac{2\sqrt{\epsilon}\cos\theta}{\epsilon\cos\theta + \sqrt{\epsilon - \sin^2\theta}}\, e^{i\omega t_o - \xi} \tag{B.9}$$

In correspondence with Eq. (B.5)

$$\frac{1}{2\pi}\int_{C_\infty^-} T_\infty(x) e^{-i\omega t} d\omega = \frac{1}{2\pi} T_\infty(0)\ e^{-\xi} \int_{-\infty}^{+\infty} e^{-i\omega(t-t_o)} d\omega$$

$$= T_\infty(0)\ e^{-\xi}\ \delta(t-t_o) \tag{B.10}$$

In case of horizontal polarization, $T_\infty(0)$ can again be expressed in terms of ρ_o

$$T_{h,\infty}(0) = 1 - \rho_o + \sqrt{\rho_o^2 - 1} \equiv 1 + R_{h,\infty} \tag{B.11}$$

APPENDIX C

Reflected Electric Fields for an Incident Unit Step Function u(t)

According to Eq. (1), we have

$$E_r(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} u(\omega) R(\omega) e^{-i\omega t} d\omega \tag{C.1}$$

The Fourier transform of u(t) is given by the generalized function (distribution)

$$u(\omega) = - P\left(\frac{1}{i\omega}\right) + \pi\, \delta(\omega) \tag{C.2}$$

where P stands for "Cauchy principal value" when performing the integration over ω

$$E_r(t) = \frac{1}{2} R(0) - \frac{1}{2\pi i} P \int_{-\infty}^{+\infty} \frac{1}{\omega} R(\omega) e^{-i\omega t}\, d\omega \tag{C.3}$$

The same result would have been obtained for an integration contour along the real axis where the pole at $\omega=0$ is encircled by a small half-circle C_ϵ^+ with radius $\epsilon \to 0$ in the upper half-plane (Fig. C.1)

$$\frac{1}{2\pi i} \int_{C_\epsilon^+} \frac{1}{\omega} R(\omega) e^{-i\omega t} d\omega = \frac{1}{2} \tag{C.4}$$

since $R_h(0)=-1$ if we consider horizontal polarization only.

To evaluate Eq. (C.3) for $t>0$ by contour integration, the integration path is closed by an infinitely large half-circle C_∞^- in the lower half-plane. Since the integrand tends to zero at infinity, the integral over C_∞^- vanishes, too, according to Jordan's lemma.

Due to Cauchy's theorem the contour of integration can now be distorted to the closed contour around the branch cut $(0,-is_o)$ and the pole at $\omega=0$ as shown in Fig. C.1

$$E_{r,h}(t) = \frac{1}{2\pi i} \int_{C_\epsilon} \frac{d\omega}{\omega} R_h(\omega)e^{-i\omega t} + \frac{i}{2\pi} \int_o^{s_o} \frac{1}{s} \left[R_h^+(-is)-R_h^-(-is)\right] e^{-st} ds \qquad \text{(C.5)}$$

Note that the integrand does not diverge for $s=0$ because $R_h^+(0)-R_h^-(0) = 0$, and

$$\frac{1}{2\pi i} \int_{C_\epsilon} \frac{d\omega}{\omega} R_h(\omega)e^{-i\omega t} = 1. \qquad \text{(C.6)}$$

Change of the integration variable s to ρ according to Eq. (8) and introduction of normalized times τ then reveals equivalence with Eq. (57) obtained by convolution in the time domain:

$$E_{r,h}(\tau) = -1 - \frac{1}{\pi} \sqrt{\rho_o^2 -1} \int_{-1}^{+1} d\rho \frac{\sqrt{1-\rho^2}}{(\rho-\rho_o)(\rho+1)} e^{-(1+\rho)\tau} \quad , \tau \geqslant 0 \qquad \text{(C.7)}$$

A similar consideration leads to $E_r(\tau) = 0$ for $\tau<0$, as also expected from causality, because there are no singularities in the upper half-plane.

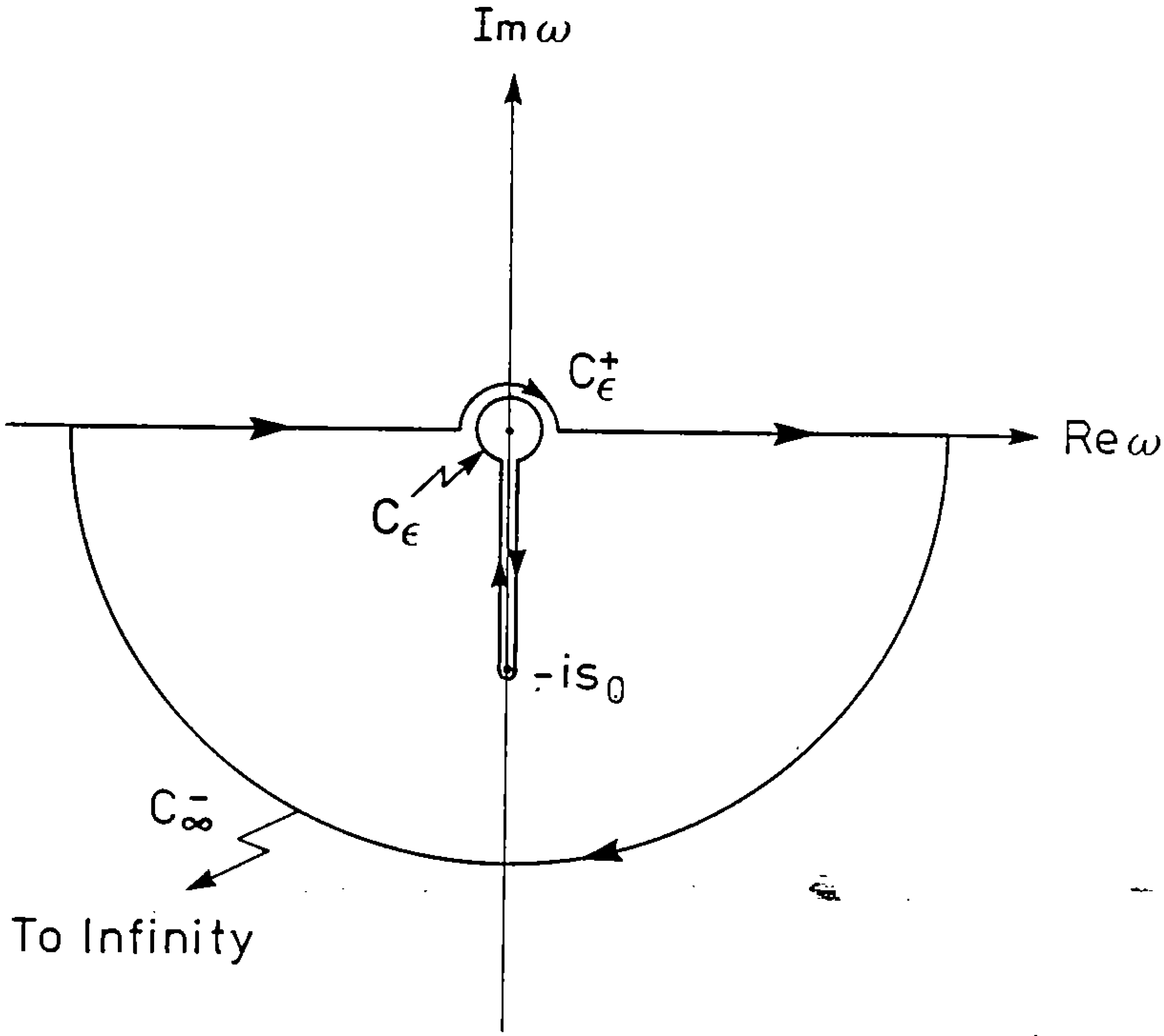

Fig. C.1: Integration contours in the complex ω-plane for an incident unit step function

APPENDIX D

Another Representation of the Transmission Function

Instead of Eq. (34), we use the following representation of $G(\xi,\tau)$ and its derivative

$$G(\xi,\tau) = \int_{-1}^{+1} \frac{\sin\left[\xi\sqrt{1-\rho^2}\right]}{\rho-\rho_o} e^{-\rho\tau}\, d\rho$$

$$= -\, e^{-\rho_o\tau} \int_{-\infty}^{\tau} e^{\rho_o\tau'} F(\xi,\tau')\, d\tau'$$

$$= -\,\pi\xi\, e^{-\rho_o\tau} \int_{-\infty}^{\tau} e^{\rho_o\tau'} \frac{I_1(\sqrt{\tau'^2-\xi^2})}{\sqrt{\tau'^2-\xi^2}}\, d\tau' \qquad \text{(C.1)}$$

$$\frac{\partial}{\partial\xi} G(\xi,\tau) = -e^{-\rho_o\tau} \int_{-\infty}^{\tau} e^{\rho_o\tau'} \frac{\partial}{\partial\xi} F(\xi,\tau')\, d\tau' \qquad \text{(C.2)}$$

where

$$\frac{\partial}{\partial\xi} F(\xi,\tau) = -\,\pi \frac{\partial^2}{\partial\xi^2} I_o\left[\sqrt{\tau^2-\xi^2}\right]$$

$$= \frac{\pi}{\tau^2-\xi^2}\left\{\frac{\tau^2+\xi^2}{\sqrt{\tau^2-\xi^2}}\, I_1\left[\sqrt{\tau^2-\xi^2}\right] - \xi^2 I_o\left[\sqrt{\tau^2-\xi^2}\right]\right\}$$

$$= \frac{\pi}{2}\left\{I_o\left[\sqrt{\tau^2-\xi^2}\right] - \frac{\tau^2+\xi^2}{\tau^2-\xi^2} I_2\left[\sqrt{\tau^2-\xi^2}\right]\right\} \qquad \text{(C.3)}$$

Introducing these expressions into Eq. (30), another representation of the transmission function in terms of modified Bessel functions can be obtained.

APPENDIX E

Asymptotic Behavior of the Response Function for Long Times

For $\tau \gg 1$, Eq. (15) can be approximated as follows

$$R_h(\tau) = \frac{s_o}{2\pi}\sqrt{\rho_o^2-1}\int_{-1}^{+1}\frac{\sqrt{1-\rho^2}}{\rho-\rho_o}\,e^{-(1+\rho)\tau}\,d\rho \qquad (E.1)$$

$$\approx -\frac{s_o}{2\pi}\,e^{-\tau}\frac{\sqrt{\rho_o-1}}{\sqrt{\rho_o+1}}\int_{-1}^{+1}\sqrt{1-\rho^2}\,e^{-\rho\tau}\,d\rho =$$

$$= -\frac{s_o}{2}\frac{\sqrt{\rho_o-1}}{\sqrt{\rho_o+1}}\frac{e^{-\tau}}{\tau}\,I_1(\tau) \qquad (E.2)$$

$$\approx -\frac{s_o}{2}\frac{\sqrt{\rho_o-1}}{\sqrt{\rho_o+1}}\frac{1}{\sqrt{2\pi}}\,\tau^{-3/2} \qquad (E.3)$$

Some numerical examples are compiled in Table E.1 to demonstrate the quality of the approximation

τ	Exact Eq. (E.1)	Eq. (E.2)	Eq. (E.3)
1	-.1697	-.1200	-.2303
2	-.0793	-.0621	-.0814
5	-.02108	-.01893	-.02060
10	-.00739	-.00700	-.00728
20	-.00260	-.00252	-.00257

Table E.1: Comparison of Different Approximations for $2R_h(\tau)/s_o$ $(\rho_o=2)$

Hence, in particular approximation (E.3) seems to be reasonably accurate for $\tau \gtrsim 2$, whereas for short times ($\tau \ll 1$) the Taylor series expansion as given in Section 5 could be used.

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