THEORETICAL NOTES

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SOLUTION OF TWO DIMENSIONAL-TWO REGION ELECTROMAGNETIC GROUND RESPONSE*

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ABSTRACT

A solution for the Two Dimensional-Two Region electromagnetic ground response has been developed which relates the surface components of the electric field to the surface components of the magnetic field. This has been accomplished by deriving a universal functional form for a dimensionless Green's Function. The Green's Function provides increasingly more accurate approximations to the response for each successive reflection from the second layer. This result would appear to provide simplification and reduced computer running time in the numerical modelling of the HABEMP when the ground response is coupled to finite-difference methods for solving the atmospheric part of the problem.

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1. INTRODUCTION

In a previous study[1] we showed that for an N-Layer earth model it is possible to express the three components of the electric field and the vertical component of the magnetic field on the surface of the earth as a space-time integration of the two horizontal components of the magnetic field. In particular, it was shown that if $\vec{r_s}$ is a point on the surface (x, y *plane*), and if Y_i is a member of the set:

$$Y_i(\vec{r_s}, t) = \{H_z, E_x, E_y, E_z\}, \tag{1.1}$$

then every member of the set Y_i on the surface of a finite-conducting earth can be related to the horizontal components of the surface magnetic field through the equations:

$$Y_1(\vec{r_s}, t) = \int_0^t \int_{\vec{r'_s}} G_{ix}(\vec{r_s} - \vec{r'_s}, t - t') H_x(\vec{r'_s}, t') dx' dy' dt'$$
$$+ \int_0^t \int_{\vec{r'_s}} G_{iy}(\vec{r_s} - \vec{r'_s}, t - t') H_y(\vec{r'_s}, t') dx' dy' dt' \tag{1.2}$$

The functions G_{ix} and G_{iy} are Green's Functions which are determined from the solution of the N-Region ground model shown in Figure 1.

The result given by Eq. (1.2) can possibly provide considerable simplification in the numerical modelling of the HABEMP when the ground response is coupled to finite-difference methods for solving the atmospheric part of the problem. When this approach can be used it obviates the necessity of developing a numerical representation of the ground, which then reduces the number of variables in the problem and hence the computer running time (and cost). On the other hand, the reduction of machine variables must be weighed against the speed of the numerical computation for the integral boundary conditions arising from the Green's Function formalism. This question is as yet unresolved; in any event it will depend on the number of ground layers being considered, the values of the electrical parameters, and the time range of interest.

Eq. (1.2) is derived by solving the N-Region three dimensional problem in the Fourier (space) and Laplace (time) domains and ultimately performing inverse Fourier/Laplace transforms. (The reader is referred to reference 1 for details.) Although the formalism and mathematical procedure are general, the analytical Fourier/Laplace transform inversion may not always be possible. For the purposes of assessing the feasibility of using the theory of reference 1 four special-case solutions were developed. These included the following models:

(a) One Layer/One Dimension
(b) One Layer/Two Dimensions
(c) One Layer/Three Dimensions
(d) Two Layers/One Dimension

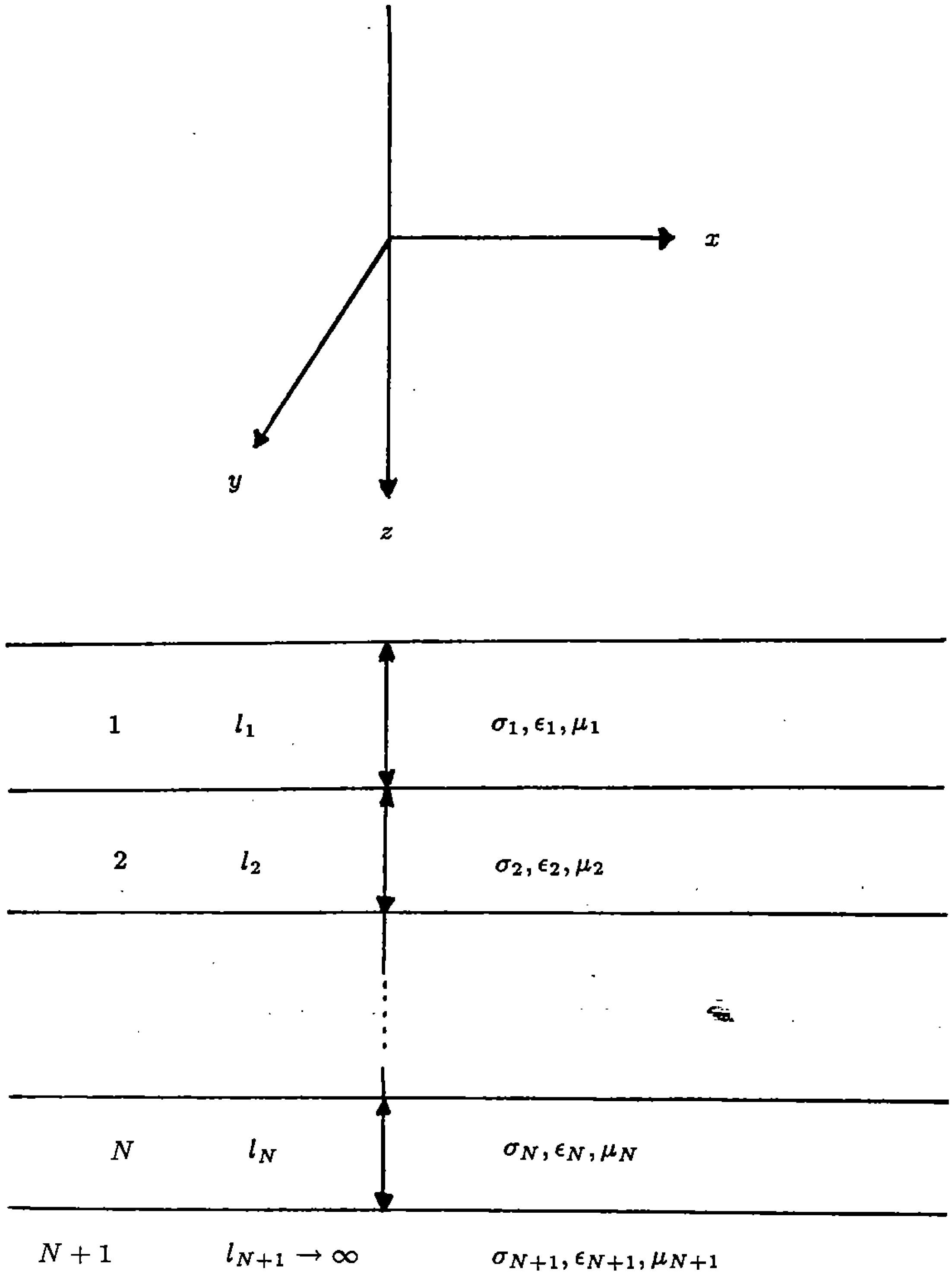

FIG 1. (N + 1)-Region Model

In this study we extend the number of solutions to now include the Two Layer/Two Dimensional case. The geometry for this case is shown in Figure 2. Following the analysis of reference 1 we consider the case where the displacement current in the ground is less than the conduction current. We assume that spatial variations in the y-direction are neglected. Our principal concern is to establish the space-time relationships between E_x, E_y and the driving functions, H_x, H_y.

2. MATHEMATICAL NOTATION

As will become evident in Sections (3) and (4) of this report there are several mathematical operations which are performed on the four surface values of the fields used in this analysis; namely $E_x(x,t), E_y(x,t), H_x(x,t), H_y(x,t)$. The purpose of this section is to define these operations and develop a shorthand notation for dealing with them. If $F(x,t)$ represents any one of the aforementioned four functions the Fourier and Laplace transforms are defined by the following operations:

$$\hat{F}(k,t) = \underline{M}F(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}F(x,t)dx \tag{2.1}$$

$$\tilde{F}(x,s) = \underline{L}F(x,t) = \int_{0}^{\infty} e^{-st}F(x,t)dt \tag{2.2}$$

$$\underline{M} = \text{Fourier Transform operator} \tag{2.3}$$

$$\underline{L} = \text{Laplace Transform operator} \tag{2.4}$$

The double transformed function, $\bar{F}(k,s)$, is given by:

$$\begin{aligned} \bar{F}(k,s) &= \underline{M}\,\underline{L}F(x,t) = \underline{L}\,\underline{M}F(x,t) \\ &= \frac{1}{2\pi}\int_{0}^{\infty} e^{-st}\int_{-\infty}^{\infty} e^{-ikx}F(x,t)dxdt \end{aligned} \tag{2.5}$$

As indicated in Eq. (2.5), the order in which the transforms are taken is immaterial. The inverse Fourier and Laplace operators are defined by operations:

$$\underline{M}^{-1}\hat{F}(k,t) = \int_{-\infty}^{\infty} e^{ikx}\hat{F}(k,t)dk = F(x,t) \tag{2.6}$$

$$\underline{L}^{-1}\tilde{F}(x,s) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\tilde{F}(x,s)ds = F(x,t) \tag{2.7}$$

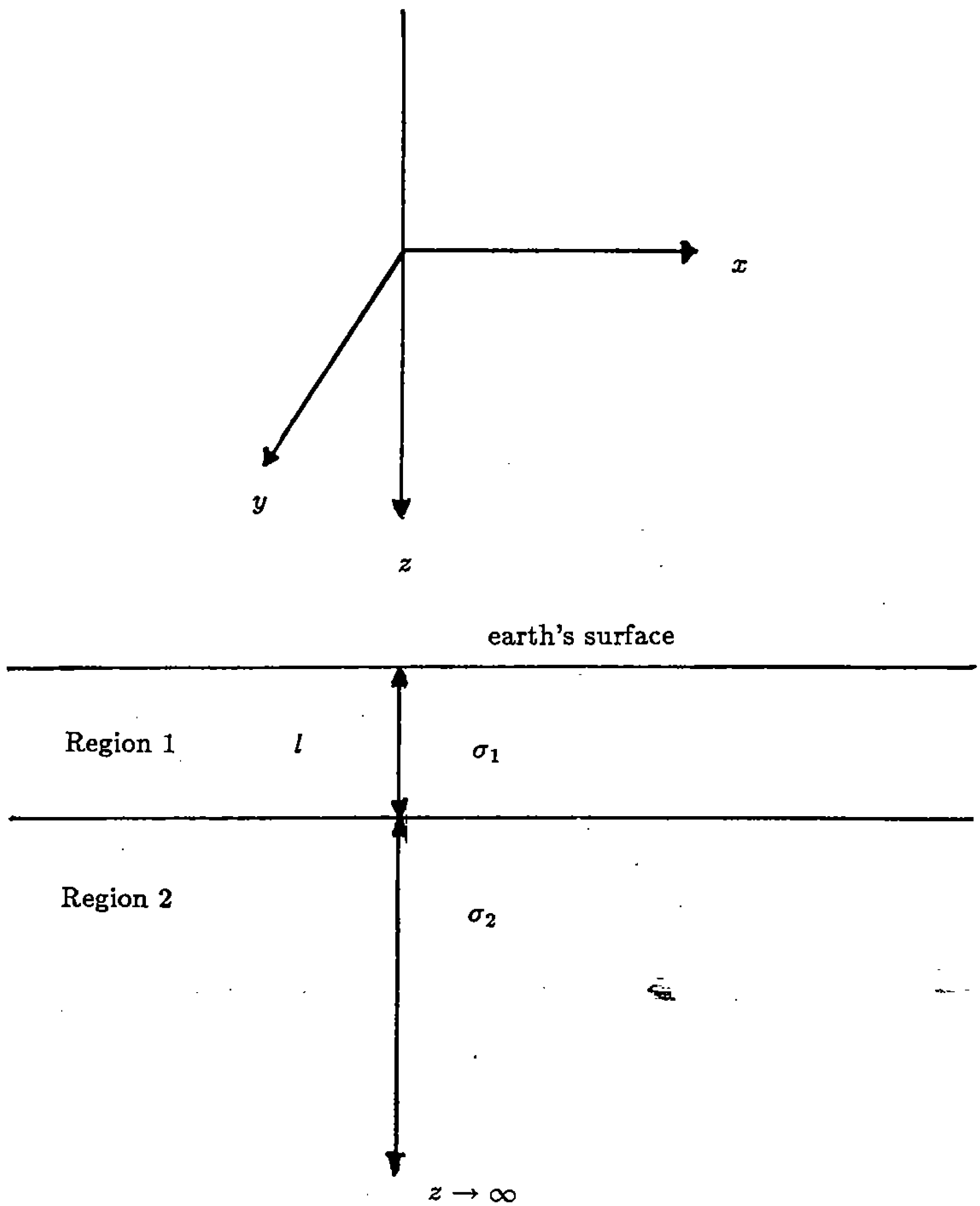

FIG 2. Geometric Considerations

It also follows that:

$$\underline{M}^{-1}\bar{F}(k,s) = \tilde{F}(x,s) \tag{2.8}$$

$$\underline{L}^{-1}\bar{F}(k,s) = \hat{F}(k,t) \tag{2.9}$$

Using the Faltung Theorem we can write for any functions $\bar{f}_1(k,s)$ and $\bar{f}_2(k,s)$

$$\begin{aligned}\underline{M}^{-1}\left(\bar{f}_1(k,s)\bar{f}_2(k,s)\right) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}_1(x-x',s)\tilde{f}_2(x',s)dx' \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}_2(x-x',s)\tilde{f}_1(x',s)dx' \end{aligned} \tag{2.10}$$

The shorthand notation for the operation of Eq. (2.10) is

$$\underline{M}^{-1}\left(\bar{f}_1(k,s)\bar{f}_2(k,s)\right) = \tilde{f}_1(x,s) \otimes \tilde{f}_2(x,s), \tag{2.11}$$

where the "$\otimes$" stands for the space integration of Eq. (2.10). It also readily follows that

$$\underline{M}^{-1}\left(\hat{f}_1(k,t)\hat{f}_2(k,t)\right) = f_1(x,t) \otimes f_2(x,t) \tag{2.12}$$

The Convolution Theorem for the Laplace Transform yields the following result:

$$\begin{aligned}\underline{L}^{-1}\left(\bar{f}_1(k,s)\bar{f}_2(k,s)\right) &= \int_0^t \hat{f}_1(k,t-t')\hat{f}_2(k,t')dt' \\ &= \int_0^t \hat{f}_2(k,t-t')\hat{f}_1(k,t')dt' \end{aligned} \tag{2.13}$$

The shorthand notation for the convolution integration in Eq. (2.13) is

$$\underline{L}^{-1}\left(\bar{f}_1(k,s)\bar{f}_2(k,s)\right) = \hat{f}_1(k,t) * \hat{f}_2(k,t) \tag{2.14}$$

where "$*$" denotes the convolution operation. We also have

$$\underline{L}^{-1}\left(\tilde{f}_1(x,s)\tilde{f}_2(x,s)\right) = f_1(x,t) * f_2(x,t) \tag{2.15}$$

The mathematical formalism and notation developed through Eq. (2.15) provides a compact way of identifying the Green's Function which relates the surface values of the electric field to the magnetic field components. For example, as shown in Section (3) we have the relationship

$$\bar{E}_y(k,s) = \bar{G}(k,s)\bar{H}_x(k,s) \tag{2.16}$$

Using Eqs. (2.8) and (2.9) we can write:

$$\underline{M}^{-1}\,\underline{L}^{-1}\bar{E}_y(k,s) = \underline{M}^{-1}\hat{E}_y(k,t) = E_y(x,t) \tag{2.17}$$

If we identify $\bar{G}(k,s)$ as $\bar{f}_1(k,s)$ and $\bar{H}_x(k,s)$ as $\bar{f}_2(k,s)$ we deduce the relationship:

$$\begin{aligned}\underline{M}^{-1}\,\underline{L}^{-1}\bar{E}_y(k,s) &= \underline{M}^{-1}\,\underline{L}^{-1}\left(\bar{G}(k,s)\bar{H}_x(k,s)\right)\\ \bar{E}_y(x,t) &= G(x,t) \otimes * H_x(x,t)\\ \bar{E}_y(x,t) &= \int_{-\infty}^{\infty}\int_{0}^{t} G(x-x',t-t')H_x(x',t')dx'dt' \end{aligned} \tag{2.18}$$

where

$$\begin{aligned}G(x,t) &= \underline{M}^{-1}\,\underline{L}^{-1}\bar{G}(k,s) = \underline{L}^{-1}\,\underline{M}^{-1}\bar{G}(k,s)\\ &= \text{Green's Function}\end{aligned} \tag{2.19}$$

Employing the techniques of reference 1 we are able to derive an expression for $\bar{G}(k,s)$. The thrust of this investigation is to develop techniques for determining $G(x,t)$ from the inverse Fourier/Laplace transforms.

3. EQUATIONS AT SURFACE OF EARTH

It is shown in Eq. (3.21) of reference 1 that for the nth region of an N-Region ground model the equations for $\bar{E}_{x,n}(z,k_x,k_y,s)$ and $\bar{E}_{y,n}(z,k_x,k_y,s)$ in terms of the magnetic field components, $\bar{H}_{x,n}(z,k_x,k_y,s), \bar{H}_{y,n}(z,k_x,k_y,s)$ are given by:

$$\bar{E}_{x,n} = \left(\frac{\gamma_n}{\sigma_n}\right)\left[-\bar{H}_{y,n} + \left(\frac{1}{\gamma_n^2}\right)(k_y k_x \bar{H}_{x,n} + k_y^2 \bar{H}_{y,n})\right] \tag{3.1}$$

$$\bar{E}_{y,n} = \left(\frac{\gamma_n}{\sigma_n}\right)\left[\bar{H}_{x,n} - \left(\frac{1}{\gamma_n^2}\right)(k_x^2 \hat{H}_{x,n} + k_x k_y \hat{H}_{y,n})\right] \tag{3.2}$$

where

$$\begin{aligned}\sigma_n &= \text{conductivity} \quad (a)\\ \gamma_n &= \pm\lambda_n (+ \text{ used for upward wave; - used for downward wave}) \quad (b)\\ \lambda_n &= \sqrt{s\mu_n\sigma_n + k_x^2 + k_y^2} \quad (c)\end{aligned} \tag{3.3}$$

We immediately note from Eqs. (3.1) and (3.2) that if either k_y or k_x is equal to zero(this corresponds to neglecting spatial variations in the y and x respectively) the equations are decoupled into two independent pairs. For example, setting $k_y = 0$ gives;

$$\bar{E}_{x,n} = -\left(\frac{\gamma_n}{\sigma_n}\right)\bar{H}_{y,n} \tag{3.4}$$

$$\bar{E}_{y,n} = \left(\frac{\gamma_n}{\sigma_n}\right)\left(1 - \frac{k_x^2}{\gamma_n^2}\right)\bar{H}_{x,n}, \tag{3.5}$$

with γ_n now being given by

$$\gamma_n = \pm\sqrt{s\mu_n\sigma_n + k_x^2} \tag{3.6}$$

When Eqs. (3.4) - (3.6) are used in the solution of the two-region problem of Figure (2) it can be shown that the surface field equations are given by:

$$\bar{E}_x = r_1\left[\bar{H}_y + 2\sum_{n=1}^{\infty}\bar{\Psi}_r^n\bar{H}_y\right] \tag{3.7}$$

$$\bar{E}_y = -w_1\left[\bar{H}_x + 2\sum_{n=1}^{\infty}\bar{\Psi}_w^n\bar{H}_x\right], \tag{3.8}$$

where

$$r_1 = \frac{\lambda_1}{\sigma_1}\ (a), r_2 = \frac{\lambda_2}{\sigma_2}\ (b)$$

$$w_1 = \frac{s\mu}{\lambda_1}\ (c), w_2 = \frac{s\mu}{\lambda_2}\ (d)$$

$$\lambda_1 = \sqrt{s\mu\sigma_1 + k^2}\ (e), \lambda_2 = \sqrt{s\mu\sigma_2 + k^2}\ (f)$$

$$\bar{\Psi}_r = \left(\frac{r_2 - r_1}{r_2 + r_1}\right)\exp(-2\lambda_1 l)\ (g)$$

$$\bar{\Psi}_w = \left(\frac{w_2 - w_1}{w_2 + w_1}\right)\exp(-2\lambda_1 l)\ (h) \tag{3.9}$$

For brevity we have replaced k_x by k, it being understood that we only considering spatial variations in the x direction.

Eqs. (3.7) and (3.8) reduce to the One Layer-Two Dimensional case considered in reference 1 when $\sigma_2 = \sigma_1$. In this situation $\bar{\Psi}_r$ and $\bar{\Psi}_w$ are both equal to zero, and we obtain the solution:

$$\bar{E}_{x,0} = r_1\bar{H}_y \tag{3.10}$$

$$\bar{E}_{y,0} = -w_1 \bar{H}_x \tag{3.11}$$

The space-time behavior of $E_{y,0}(x,t)$ is given by[1]:

$$E_{y,0} = \underline{M}^{-1}\, \underline{L}^{-1} \bar{E}_{y,0}$$

$$E_{y,0} = -\frac{\mu}{2\pi} \int_0^t \int_{-\infty}^{\infty} \frac{1}{(t-t')} \exp\left[-\frac{(x-x')^2 \mu \sigma_1}{4(t-t')}\right] \left(\frac{\partial}{\partial t'} H_x(x',t')\right) dx'dt' \tag{3.12}$$

For the One Layer-Two Dimensional case Eq. (3.12) represents a means of establishing the boundary condition on top of the earth. This equation can be written in finite-difference form and thus can be used in conjunction with the respective HABEMP equations above the earth's surface to provide a self consistent representation of the overall physical model. The key question is "Under what conditions is the method more effecient than representing the ground in a finite-difference approximation ?" The answer to this will be forthcoming in the near future.

In the case of the Two Layer-Two Dimensional case being considered here the deduction of $E_y(x,t)$ in terms of $H_x(x,t)$ is more complicated than that of Eq. (3.12) because of the complexity of $\bar{\Psi}_w(k,s)$. A similar statement can be made for the E_x, H_y pair; however, for brevity this is not being considered since the analysis is similar to the E_y, H_x case.

It is possible to express the solution of Eq. (3.8) in several different ways, leading in turn to different algorithms for completing the calculation of $E_y(x,t)$. For example, performing the $\underline{L}^{-1}\, \underline{M}^{-1}$ operation directly on Eq. (3.8) leads to the result:

$$E_y = E_{y,0} + 2\sum_{n=1}^{\infty} E_{y,n}, \tag{3.13}$$

where

$$E_{y,n} = G_n(x,t) \otimes * H_x(x,t) \quad : n \geq 1 \tag{3.14}$$

$$G_n(x,t) = \underline{L}^{-1}\, \underline{M}^{-1} \left(-\bar{\Psi}_w^n w_1\right) \tag{3.15}$$

An alternate, and mathematically equivalent formalism is based on the utilization of the knowledge of $E_{y,0}$ deduced from Eq. (3.12). We can write Eq. (3.8) in the form:

$$\bar{E}_y = \bar{E}_{y,0} + 2\sum_{n=1}^{\infty} \bar{\Psi}_w^n \bar{E}_{y,0}, \tag{3.16}$$

which then leads to the relationship

$$E_y = E_{y,0} + 2\sum_{n=1}^{\infty} E_{y,n} \tag{3.17}$$

$E_{y,n}$ is expressed as:

$$E_{y,n} = G_n(x,t) \otimes *E_{y,0}(x,t), \tag{3.18}$$

with $G_n(x,t)$ now given by:

$$G_n(x,t) = \underline{L}^{-1}\, \underline{M}^{-1}\left(\bar{\Psi}_w^n\right) \tag{3.19}$$

The analysis of this investigation is concerned with the determination of $G_n(x,t)$ from Eq. (3.19).

There is however, one additional method which can be used, which is related to Eq. (3.19). We include this for completeness. Consider, for example, the sequence of functions

$$\begin{aligned}
&\bar{E}_{y,1} = \bar{\Psi}_w \bar{E}_{y,0} \quad (a)\\
&\bar{E}_{y,2} = \bar{\Psi}_w^2 \bar{E}_{y,0} = \bar{\Psi}_w \bar{E}_{y,0} \quad (b)\\
&\cdot\\
&\cdot\\
&\cdot\\
&\bar{E}_{y,n} = \bar{\Psi}_w \bar{E}_{y,n-1} \quad (c)
\end{aligned} \tag{3.20}$$

Using Eq. (3.20) we can write:

$$E_y = E_{y,0} + 2\sum_{n=1}^{\infty} E_{y,n}, \tag{3.21}$$

$E_{y,n}$ is now given by:

$$E_{y,n} = G_1 \otimes *E_{y,n-1} \tag{3.22}$$

with

$$G_1(x,t) = \Psi_w(x,t) = \underline{L}^{-1}\, \underline{M}^{-1}\left(\bar{\Psi}_w(k_x, s)\right) \tag{3.23}$$

It is also observed by comparing Eqs. (3.19) and (3.23) that $G_n(x,t)$ is the $(n-1)^{st}$ space-time convolution of $\Psi_w(x,t)$. That is,

$$\begin{aligned}
&G_2 = \underline{L}^{-1}\, \underline{M}^{-1}\left(\bar{\Psi}_w^2\right) = \Psi_w \otimes *\Psi_w \quad (a)\\
&G_3 = \Psi_w \otimes *\Psi_w \otimes *\Psi_w \quad (b)\\
&\cdot\\
&\cdot\\
&\cdot\\
&G_n = \Psi_w \quad \overbrace{\hspace{3cm}}^{(n-1)\text{convolutions}} \quad \Psi_w \quad (c)
\end{aligned} \tag{3.24}$$

In summary, it is clear that the determination of $\Psi_w(x,t)$, as given by Eq. (3.23) is the basic building block of the calculation. This is the focus of the effort of the next section.

4. MATHEMATICAL STRUCTURE OF $G_n(x,t)$

From Eq. (3.9) we can write

$$\bar{\Psi}_w^n = (g(k,s)v(k,s))^n = \bar{g}_n \bar{v}_n \tag{4.1}$$

where

$$\bar{g}_n(k,s) = \left(\frac{w_2 - w_1}{w_2 + w_1}\right)^n \tag{4.2}$$

$$\bar{v}_n(k,s) = \exp(-2n\lambda_1 l) \tag{4.3}$$

Examination of Eq. (3.16) shows that the electric field at the surface can be considered as a sum of terms involving multiple round-trip reflections from the second layer. This can be seen for example by first examining the inverse Laplace Transform of $\bar{v}_n(k,s)$.

We have

$$\begin{aligned} \hat{v}_n(k,t) &= \underline{L}^{-1}\bar{v}_n(k,s) \\ &= \underline{L}^{-1}\left(\exp(-2nl\sqrt{s\mu\sigma_1 + k^2})\right) \\ &= \underline{L}^{-1}\left(\exp(-2nl\sqrt{\mu\sigma_1}\sqrt{s+\alpha_1})\right) \end{aligned} \tag{4.4}$$

where

$$\alpha_1 = \frac{k^2}{(\mu\sigma_1)} \tag{4.5}$$

Using the formula

$$\underline{L}^{-1} f(s+\alpha_1) = e^{-\alpha_1 t}\underline{L}^{-1} f(s) \tag{4.6}$$

we obtain

$$\hat{v}_n(k,t) = e^{-\alpha_1 t}\theta_n(t) \tag{4.7}$$

where

$$\theta_n(t) = \frac{n}{2\sqrt{\pi}}\frac{\sqrt{T_R}}{t^{\frac{3}{2}}}\exp\left(\frac{-n^2 T_R}{4t}\right) \tag{4.8}$$

and

$$\begin{aligned} T_R = L^2\mu\sigma_1 \quad &= \quad \text{two-way diffusion time to second layer} \quad (a) \\ L = 2l = \text{round-trip distance} \quad &(b) \end{aligned} \tag{4.9}$$

As observed from Eq. (4.8), for

$$\left(\frac{T_R}{t}\right) > 1 \tag{4.10}$$

the damping will become severe and perhaps only one term in the expansion will be necessary. Moreover, θ_n is in general a rapidly decreasing function of n as can be seen by examining its maximum value, $\theta_{n,\,max}$. This is determined by soving the equation

$$\frac{d\theta_n}{dt} = 0 \tag{4.11}$$

for the time at which the maximum occurs; this time is given by:

$$t_m = \frac{n^2 T_R}{6} \tag{4.12}$$

and the corresponding value of θ_n is

$$\theta_{n,\,max} = \frac{6^{\frac{3}{2}}}{2\sqrt{\pi}} \frac{1}{T_R} \frac{1}{n^2} e^{-\frac{3}{2}} \tag{4.13}$$

The foregoing equation supports the conjecture that higher-order terms provide diminishing contributions to the overall solution.

The main difficulty in determining

$$G_n(x,t) = \underline{L}^{-1}\, \underline{M}^{-1} \bar{\Psi}_w^n = \underline{M}^{-1} \underline{L}^{-1} \bar{\Psi}_w^n \tag{4.14}$$

is attributed to performing the inverse Laplace Transform of $\bar{g}_n(k,s)$. Using Eqs. (3.9c) and (3.9d) in Eq. (4.2) yields:

$$\bar{g}_n(k,s) = \left(\frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2}\right)^n = \bar{g}_1^n \tag{4.15}$$

where

$$\bar{g}_1 = \left(\frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2}\right) \tag{4.16}$$

and λ_1, λ_2 are given by Eqs. (3.9e) and (3.9f) respectively. Let us first consider some of the mathematical properties of $\bar{g}_1^n$. We write g_1 in the form:

$$\bar{g}_1 = \left(\frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2}\right)\left(\frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_2}\right) = \frac{(\lambda_1 - \lambda_2)^2}{(\lambda_1^2 - \lambda_2^2)} \tag{4.17}$$

Using Eqs. (3.9e) and (3.9f) we have:

$$\lambda_1^2 - \lambda_2^2 = s\mu(\sigma_1 - \sigma_2) \tag{4.18}$$

Substituting Eqs. (4.17) and (4.18) into Eq. (4.15), and subsequently using the binomial expansion for $(\lambda_1 - \lambda_2)^{2n}$ gives:

$$\bar{g}_n = \left(\frac{1}{s\mu(\sigma_1 - \sigma_2)}\right)^n \sum_{m=0}^{2n} b_{2n,m}\lambda_1^{2n-m}(-\lambda_2)^m \tag{4.19}$$

where $b_{2n,m}$ is the binomial coefficient

$$b_{2n,m} = \frac{(2n)!}{(2n-m)!m!} \tag{4.20}$$

Let us now consider the structure of the terms Eq. (4.19). For the even terms, characterized by

$$m = 2m'; \quad (0 \leq m' \leq n) \tag{4.21}$$

we have

$$\lambda_1^{2n-2m'}(-\lambda_2)^{2m'} = (s\mu\sigma_1 + k^2)^{n-m'}(s\mu\sigma_2 + k^2)^{m'} \tag{4.22}$$

For the odd numbered terms, characterized by

$$m = 2m'' + 1; \quad 0 \leq m'' \leq n - 1 \tag{4.23}$$

we can write

$$\lambda_1^{2n-m''}(-\lambda_2)^{m''} = -(s\mu\sigma_1 + k^2)^{n-m''-1}(s\mu\sigma_2 + k^2)^{m''}\,(s\bar{\Gamma})$$

where

$$\bar{\Gamma} = \frac{\sqrt{s\mu\sigma_1 + k^2}\sqrt{s\mu\sigma_2 + k^2}}{s} \tag{4.25}$$

By expressing $(s\mu\sigma_1 + k^2)^{n-m'}$, $(s\mu\sigma_2 + k^2)^{m'}$, $(s\mu\sigma_1 + k^2)^{m''}$, and$(s\mu\sigma_2 + k^2)^{m''}$ as a binomial expansion, and substituting the results into Eq. (4.19) it can be shown by combining the summations that $\bar{g}_n$ is of the form:

$$\bar{g}_n = \bar{g}_{n1} + \bar{g}_{n2}$$

where

$$\bar{g}_{n1} = \frac{1}{s^n(\mu(\sigma_1 - \sigma_2))^n}\left(\sum_{r=0}^{n} A_r s^r k^{2p}\right) \tag{4.26}$$

$$\bar{g}_{n2} = \frac{1}{s^n(\mu(\sigma_1 - \sigma_2))^n}\left(\sum_{r=0}^{n} B_r s^r k^{2q}\right)(s\bar{\Gamma}) \tag{4.27}$$

In the foregoing expressions A_r and B_r are constants which depend on σ_1 , σ_2 and the index r, and the integers p and q are positive linear functions of the summation index, r.

It is not necessary to go into the tedious details to establish the implications of Eqs. (4.26) and (4.27) regarding the computation of $G_n(x,t)$ from Eq. (4.14).

Using Eqs. (4.1) and (4.3) in Eqs. (4.26) and (4.27) we have:

$$\bar{\Psi}^n_w = \bar{\Psi}^n_{w,1} + \bar{\Psi}^n_{w,2}$$

where

$$\bar{\Psi}^n_{w,1} = \frac{1}{(\mu(\sigma_1-\sigma_2))^n}\sum_r^n A_r\left(\frac{k^{2p}}{s^{n-r}}\bar{v}_n(k,s)\right) \tag{4.28}$$

$$\bar{\Psi}^n_{w,2} = \frac{1}{(\mu(\sigma_1-\sigma_2))^n}\sum_r^{n-1} B_r\left(\frac{k^{2q}}{s^{n-r}}\bar{v}_n(k,s)s\bar{\Gamma}\right) \tag{4.29}$$

From Eq. (4.14) we deduce

$$G_n(x,t) = G_{n1}(x,t) + G_{n2}(x,t) \tag{4.30}$$

where

$$G_{n1} = \frac{1}{(\mu(\sigma_1-\sigma_2))^n}\sum_r^n A_r Y_r(x,t) \tag{4.31}$$

$$G_{n2} = \frac{1}{(\mu(\sigma_1-\sigma_2))^n}\sum_r^{n-1} B_r Z_r(x,t) \tag{4.32}$$

$$Y_r(x,t) = \underline{M}^{-1}\,\underline{L}^{-1}\left(\frac{k^{2p}}{s^{n-r}}\bar{v}_n(k,s)\right) \tag{4.33}$$

$$Z_r(x,t) = \underline{M}^{-1}\,\underline{L}^{-1}\left(\frac{k^{2q}}{s^{n-r}}\bar{v}_n(k,s)s\bar{\Gamma}\right) \tag{4.34}$$

If we now let

$$v_n(x,t) = \underline{M}^{-1}\,\underline{L}^{-1}\bar{v}_n(k,s) \tag{4.35}$$

and

$$\Phi_n(x,t) = \underline{M}^{-1}\,\underline{L}^{-1}(\bar{v}_n(k,s)s\bar{\Gamma}(k,s)) \tag{4.36}$$

it then follows Eqs. (2.6) and (2.7) that

$$Y_r(x,t) = \int\cdots\int dt_1 dt_2\cdots dt_{n-r}\left(-\frac{\partial^2}{\partial x^2}\right)^p v_n(x,t_1) \tag{4.37}$$

$$Z_r(x,t) = \int\cdots\int dt_1 dt_2\cdots dt_{n-r}\left(-\frac{\partial^2}{\partial x^2}\right)^q \Phi_n(x,t_1) \tag{4.38}$$

In summary, we have shown that if one can determine $v_n(x,t)$ and $\Phi_n(x,t)$ it is possible to determine the Green's Function $G_n(x,t)$ by space derivates on v_n and Φ_n followed by repeated time integrations. The utility of this approach depends on the simplicity and speed of performing the time integration.

In Eq. (4.7) we showed

$$\hat{v}_n(k,t) = \underline{L}^{-1}\bar{v}_n(k,s) = \exp\left(-\left(\frac{k^2}{\mu\sigma_1}\right)t\right)\theta_n(t) \tag{4.39}$$

We now have

$$v_n(x,t) = \underline{M}^{-1}\,\underline{L}^{-1}\bar{v}_n(k,s) = \theta_n(t)R(x,t), \tag{4.40}$$

where

$$\begin{aligned} R(x,t) &= \int_{-\infty}^{\infty} e^{ikx}e^{-\frac{k^2t}{\mu\sigma_1}}\,dk \\ &= \frac{\sqrt{\pi\mu\sigma_1}}{\sqrt{t}}\exp\left(-\frac{x^2\mu\sigma_1}{4t}\right) \end{aligned} \tag{4.41}$$

It is observed that $v_n(x,t)$ can be expressed in closed-form which facilitates the computation of Eq. (4.37).

Now let us consider the deduction of $\Phi_n(x,t)$. We have

$$\begin{aligned} \Phi_n(x,t) &= \underline{M}^{-1}\,\underline{L}^{-1}\left(s\bar{v}_n(k,s)\bar{\Gamma}(k,s)\right) \\ &= \frac{\partial}{\partial t}\left(\underline{M}^{-1}\,\underline{L}^{-1}\bar{P}_n(k,s)\right) \end{aligned} \tag{4.42}$$

where

$$\bar{P}_n(k,s) = \bar{v}_n(k,s)\bar{\Gamma}(k,s) \tag{4.43}$$

By examining Eqs. (4.7), (4.8) and (4.25), which are the constitutents of Eq. (4.43), we notice that the only difference between $\bar{P}_n$ and $\bar{P}_1(k,s)$ is the replacement of T_R by n^2T_R. Thus, if we can determine

$$P_1(x,t) = \underline{M}^{-1}\,\underline{L}^{-1}\bar{P}_1(k,s) \tag{4.44}$$

we can determine $P_n(x,t)$ by substituting n^2T_R for T_R in the resultant expression.

For the purposes of this investigation we are limiting the calculation to the evaluation of

$$G_1(x,t) = \underline{M}^{-1}\,\underline{L}^{-1}\left(\bar{g}_1(k,s)\bar{v}_1(k,s)\right) \tag{4.45}$$

This provides the contribution to the surface value of the electric field from the first round trip reflection. In addition, it will be seen in Section (5) that the calculation of $G_1(x,t)$ includes as one of its components, the computation of $\bar{P}_1(x,t)$.

5. CALCULATION OF $G_1(x,t)$

Starting from Eq. (4.45) we have:

$$
\begin{aligned}
G_1(x,t) &= \underline{M}^{-1}\,\underline{L}^{-1}\left(\bar{g}_1(k,s)\bar{v}_1(k,s)\right) \\
&= \underline{M}^{-1}\int_0^t \hat{g}_1(k,t')\hat{v}_1(k,t-t')dt' \\
&= \int_{-\infty}^{\infty} e^{ikx}\int_0^t \hat{g}_1(k,t')\hat{v}_1(k,t-t')dt'\;dk
\end{aligned} \tag{5.1}
$$

Using Eq. (4.17) we can write:

$$\bar{g}_1 = \left(\frac{1}{s\mu(\sigma_1-\sigma_2)}\right)\left(\lambda_1^2+\lambda_2^2-2\lambda_1\lambda_2\right) \tag{5.2}$$

$$\bar{g}_1 = \frac{s\mu(\sigma_1+\sigma_2)+2k^2-2\mu\sqrt{\sigma_1\sigma_2}\sqrt{s+\alpha_1}\sqrt{s+\alpha_2}}{s\mu(\sigma_1-\sigma_2)} \tag{5.3}$$

where

$$
\begin{aligned}
\alpha_1 &= \frac{k^2}{\mu\sigma_1} \quad (a) \\
\alpha_2 &= \frac{k^2}{\mu\sigma_2} \quad (b)
\end{aligned} \tag{5.4}
$$

We also have:

$$\hat{g}_1(k,t) = \left(\frac{\sigma_1+\sigma_2}{\sigma_1-\sigma_2}\right)\delta(t) + \frac{2k^2}{\mu(\sigma_1-\sigma_2)}H(t) - \frac{2\sqrt{\sigma_1\sigma_2}}{(\sigma_1-\sigma_2)}\underline{L}^{-1}\bar{Q} \tag{5.5}$$

where

$$
\begin{aligned}
\bar{Q} &= \left(\frac{\sqrt{s+\alpha_1}\sqrt{s+\alpha_2}}{s}\right) = \frac{1}{\mu\sqrt{\sigma_1\sigma_2}}\bar{\Gamma} \quad (a) \\
\delta(t) &= \text{Delta Function} \quad (b) \\
H(t) &= \text{Step Function} \quad (c)
\end{aligned} \tag{5.6}
$$

From Eq. (5.5) we can write

$$\hat{g}_1(k,t) = \hat{g}_{11} + \hat{g}_{12} + \hat{g}_{13}, \tag{5.7}$$

where

$$
\begin{aligned}
\hat{g}_{11} &= \left(\frac{\sigma_1+\sigma_2}{\sigma_1-\sigma_2}\right)\delta(t) \quad (a)\\
\hat{g}_{12} &= \frac{2k^2}{\mu(\sigma_1-\sigma_2)}H(t) \quad (b)\\
\hat{g}_{13} &= \frac{-2\sqrt{\sigma_1\sigma_2}}{(\sigma_1-\sigma_2)}\hat{Q}(k,t) \quad (c)\\
\hat{Q}(k,t) &= \underline{L}^{-1}\left(\frac{\sqrt{s+\alpha_1}\sqrt{s+\alpha_2}}{s}\right) \quad (d)
\end{aligned}
\tag{5.8}
$$

As shown in Appendix A, $\hat{Q}(k,t)$ is given by:

$$
\begin{aligned}
\hat{Q}(k,t) &= e^{-\alpha t}\left[\beta I_1(\beta t)+\alpha I_o(\beta t)\right]\\
&+(\alpha^2-\beta^2)\int_0^t e^{-\alpha u}I_o(\beta u)du+\delta(t)
\end{aligned}
\tag{5.9}
$$

where

$$
\begin{aligned}
\alpha &= \frac{1}{2\mu}\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}\right)k^2 \quad (a)\\
\beta &= \frac{1}{2\mu}\left(\frac{1}{\sigma_1}-\frac{1}{\sigma_2}\right)k^2 \quad (b)
\end{aligned}
\tag{5.10}
$$

Using Eqs. (4.7) and (4.8) we can write:

$$
\hat{v}_1(k,t) = e^{-\frac{k^2 t}{\mu\sigma_1}}\left(\frac{\sqrt{T_R}}{2\sqrt{\pi}t^{\frac{3}{2}}}\right)\exp\left(-\frac{T_R}{4t}\right) \tag{5.11}
$$

If we insert Eqs. (5.7) - (5.11) into Eq. (5.1), and note that we can combine the $\delta(t)$ of Eq. (5.9) with that of Eq. (5.8) we can write:

$$
G_1(x,t) = G_{1,1}+G_{1,2}+G_{1,3} \tag{5.12}
$$

where

$$
G_{1,j} = \int_{-\infty}^{\infty} e^{ikx}\int_0^t \hat{\phi}_j(k,t')\hat{v}_1(k,t-t')dt'dk \;;\; j=1,2,3 \tag{5.13}
$$

$$
\hat{\phi}_1(k,t) = \left(\frac{\sqrt{\sigma_1}-\sqrt{\sigma_2}}{\sqrt{\sigma_1}+\sqrt{\sigma_2}}\right)\delta(t) \tag{5.14}
$$

$$\hat{\phi}_2(k,t) = \frac{2k^2}{\mu(\sigma_1-\sigma_2)}H(t) \tag{5.15}$$

$$\hat{\phi}_3(k,t) = \frac{-2\sqrt{\sigma_1\sigma_2}}{(\sigma_1-\sigma_2)}\left[\hat{L}_1(k,t)+\hat{L}_2(k,t)+\hat{L}_3(k,t)\right] \tag{5.16}$$

$$\hat{L}_1(k,t) = \beta e^{-\alpha t} I_1(\beta t) \tag{5.17}$$

$$\hat{L}_2(k,t) = \alpha e^{-\alpha t} I_o(\beta t) \tag{5.18}$$

$$\hat{L}_3(k,t) = (\alpha^2-\beta^2)\int_0^t e^{-\alpha u} I_o(\beta u)\,du \tag{5.19}$$

It should be noted that although $\sigma_1-\sigma_2$ appears in the denominator of Eqs. (5.15) and (5.16), the sum of $\phi_2+\phi_3$ must equal zero in the limit of $\sigma_1=\sigma_2$ (note that ϕ_1 is already zero in this case). This follows from the observation that there can be no reflections in this situation. This is easily seen by using approximations to $\hat{L}_1, \hat{L}_2, \text{and} \hat{L}_3$ in the limit of $\beta \to 0$.

In the next subsections we shall perform the calculations of $G_{1,j}$.

5.1 CALCULATION OF $G_{1,1}(x,t)$

From Eqs. (5.13) and (5.14) we have:

$$\begin{aligned}\int_0^t \hat{\phi}_1(k,t)v_1(k,t-t')dt' &= \left(\frac{\sqrt{\sigma_1}-\sqrt{\sigma_2}}{\sqrt{\sigma_1}+\sqrt{\sigma_2}}\right)\int_0^t \delta(t')\hat{v}_1(k,t-t')dt' \\ &= \left(\frac{\sqrt{\sigma_1}-\sqrt{\sigma_2}}{\sqrt{\sigma_1}+\sqrt{\sigma_2}}\right)e^{-\frac{k^2 t}{\mu\sigma_1}}f(t)\end{aligned} \tag{5.20}$$

where

$$f(t) = \frac{\sqrt{T_R}}{2\sqrt{\pi}t^{\frac{3}{2}}}\exp\left(\frac{-T_R}{4t}\right) \tag{5.21}$$

Performing the integration over k space yields:

$$G_{1,1} = \left(\frac{\sqrt{\sigma_1}-\sqrt{\sigma_2}}{\sqrt{\sigma_1}+\sqrt{\sigma_2}}\right)f(t)\int_{-\infty}^{\infty} e^{ikx}e^{-\frac{k^t}{\mu\sigma_1}}dk$$

$$G_{1,1} = \left(\frac{\sqrt{\sigma_1}-\sqrt{\sigma_2}}{\sqrt{\sigma_1}+\sqrt{\sigma_2}}\right)f(t)\exp\left(-\frac{x^2\mu\sigma_1}{4t}\right)\frac{\sqrt{\pi\mu\sigma_1}}{\sqrt{t}}$$

$$G_{1,1} = \left(\frac{\sqrt{\sigma_1}-\sqrt{\sigma_2}}{\sqrt{\sigma_1}+\sqrt{\sigma_2}}\right)\left(\frac{1}{2}\right)\frac{T_R}{t^2}\frac{1}{L}\exp\left[-\frac{T_R}{4t}\left(1+\frac{x^2}{L^2}\right)\right] \tag{5.22}$$

5.2 Calculation of $G_{1,2}$

From Eqs. (5.13) and (5.15) we have:

$$G_{1,2} = \int_{-\infty}^{\infty} e^{ikx} \int_{0}^{t} \frac{2k^2}{\mu(\sigma_1 - \sigma_2)} e^{-\frac{k^2 t'}{\mu\sigma_1}} f(t') dt' dk \tag{5.23}$$

Replacing k^2 by $-\frac{\partial^2}{\partial x^2}$ we obtain

$$G_{1,2} = -\frac{2}{\mu(\sigma_1 - \sigma_2)} \frac{\partial^2}{\partial x^2} \int_{0}^{t} f(t'dt' \int_{-\infty}^{\infty} e^{ikx} e^{\frac{k^2 t'}{\mu\sigma_1}} dk \tag{5.24}$$

The integral in Eq. (5.24) is the same as that in Eq. (5.22), so that we can write

$$G_{1,2} = -\frac{2}{\mu(\sigma_1 - \sigma_2)} \frac{\partial^2}{\partial x^2} \int_{0}^{t} \frac{1}{2} \frac{T_R}{(t')^2} \frac{1}{L} \exp - \left(\frac{1}{4} \frac{T_R (1 + \frac{x^2}{L^2})}{t'} \right) dt' \tag{5.25}$$

Making the substitution

$$u' = \frac{1}{4} \frac{T_R}{t'} \left(1 + \frac{x^2}{L^2} \right) = \frac{r}{t'} \tag{5.26}$$

permits the integration to be performed through the relationship:

$$du' = -\frac{r}{(t')^2} dt' \tag{5.27}$$

We obtain:

$$G_{1,2}(x,t) = -\frac{4}{\mu L} \left(\frac{1}{\sigma_1 - \sigma_2} \right) \frac{\partial^2}{\partial x^2} (U(x,t)) \tag{5.28}$$

where

$$U(x,t) = \left(\frac{1}{1 + \frac{x^2}{L^2}} \right) e^{-\frac{T_R}{4} \frac{(1 + \frac{x^2}{L^2})}{t}} \tag{5.29}$$

Using Eq. (3.18), the contribution from the first reflection will be given by:

$$E_{y,1} = G_1(x,t) \otimes *E_{y,0}(x,t), \tag{5.30}$$

and in particular the contribution from $G_{1,2}$ is:

$$E_{y,1,2} = \frac{-4}{\pi} \frac{1}{L(\sigma_1 - \sigma_2)} \int_{-\infty}^{\infty} \int_{0}^{t} \left(\frac{\partial^2}{\partial x'^2} U(x',t') \right) E_{y,0}(x - x', t - t') dx' dt' \tag{5.31}$$

Based on application of the Faltung theorem discussed in Appendix B the foregoing integration can be converted to the form:

$$E_{y,1,2} = \frac{-4}{\pi L} \frac{1}{(\sigma_1 - \sigma_2)} \int_{-\infty}^{\infty} \int_{0}^{t} U(x', t') \left(\frac{\partial^2 E_{y,0}(z, t - t')}{\partial z^2} \right)_{z=x-x'} dx' dt'; \tag{5.32}$$

It may turn out that for computational purposes that the form given by Eq. (5.32) is easier to evaluate.

5.3 Calculation of $G_{1,3}$

Inserting Eq. (5.16) into Eq. (5.13) and rearranging gives the following expression for $G_{1,3}$:

$$G_{1,3}(x,t) = -\frac{2\sqrt{\sigma_1 \sigma_2}}{(\sigma_1 - \sigma_2)} \int_{0}^{t} dt' f(t - t') \left(N_1(x,t') + N_2(x,t') + N_3(x,t') \right), \tag{5.33}$$

where

$$N_1 = \int_{-\infty}^{\infty} \beta e^{ikx} e^{-\alpha t'} I_1(\beta t') e^{-\frac{k^2}{\mu \sigma_1}(t-t')} dk \tag{5.34}$$

$$N_2 = \int_{-\infty}^{\infty} \alpha e^{ikx} e^{-\alpha t'} I_o(\beta t') e^{-\frac{k^2}{\mu \sigma_1}(t-t')} dk \tag{5.35}$$

$$N_3 = \int_{-\infty}^{\infty} (\alpha^2 - \beta^2) e^{ikx} \int_{0}^{t'} e^{-\alpha u} I_o(\beta u) du \; e^{-\frac{k^2}{\mu \sigma_1}(t-t')} dk \tag{5.36}$$

Eqs. (5.34) – (5.36) can be simplified using the formulas

$$I_o(z) = \frac{1}{\pi} \int_{0}^{\pi} e^{z \cos\theta} d\theta \tag{5.37}$$

$$I_1(z) = \frac{1}{\pi} \int_{0}^{\pi} e^{z \cos\theta} \cos\theta d\theta, \tag{5.38}$$

and writing α and β in the form:

$$\alpha = \left(\frac{1}{\mu \sigma_a} \right) k^2 \tag{5.39}$$

$$\beta = \left(\frac{1}{\mu\sigma_b}\right) k^2, \tag{5.40}$$

where

$$\frac{1}{\sigma_a} = \frac{1}{2}\left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2}\right); \sigma_a = \frac{2(\sigma_1\sigma_2)}{(\sigma_2 + \sigma_1)} \tag{5.41}$$

$$\frac{1}{\sigma_b} = \frac{1}{2}\left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2}\right); \sigma_b = \frac{2(\sigma_1\sigma_2)}{(\sigma_2 - \sigma_1)} \tag{5.42}$$

Using Eqs. (5.37) – (5.42) in Eqs. (5.34) and (5.35) yields:

$$\begin{aligned} N_1 &= -\frac{\partial^2}{\partial x^2}(n_1) \quad (a) \\ N_2 &= -\frac{\partial^2}{\partial x^2}(n_2) \quad (b) \end{aligned} \tag{5.43}$$

where

$$n_1 = \left(\frac{1}{\mu\sigma_b\pi}\right) \int_0^\pi cos\theta \; d\theta \int_{-\infty}^{\infty} e^{ikx} e^{-(\frac{1}{\mu\sigma_a})k^2t'} e^{(\frac{1}{\mu\sigma_b})k^2t' cos\theta} e^{-\frac{k^2}{\mu\sigma_1}(t-t')} dk \tag{5.44}$$

$$n_2 = \left(\frac{1}{\mu\sigma_a\pi}\right) \int_0^\pi d\theta \int_{-\infty}^{\infty} e^{ikx} e^{-(\frac{1}{\mu\sigma_a})k^2t'} e^{(\frac{1}{\mu\sigma_b})k^2t' cos\theta} e^{-\frac{k^2}{\mu\sigma_1}(t-t')} dk \tag{5.45}$$

The expression for N_3 is simplified by first performing the integration over u in Eq. (5.36). We have

$$\begin{aligned} \int_0^{t'} e^{-\alpha u} I_o(\beta u) \; du &= \frac{1}{\pi} \int_0^\pi d\theta \int_0^{t'} e^{-\alpha u} e^{\beta u \; cos\theta} \; du \\ &= \frac{1}{\pi} \int_0^\pi d\theta \left(\frac{1 - e^{-(\alpha - \beta \; cos\theta)t'}}{(\alpha - \beta \; cos \; \theta)} \right) \end{aligned} \tag{5.46}$$

Inserting Eq. (5.46) into Eq.(5.36) yields

$$N_3 = -\frac{\partial^2}{\partial x^2}(n_3) \tag{5.47}$$

where

$$n_3 = \left(\frac{1}{\mu\sigma_a\pi}\right)\int_0^\pi d\theta S(\theta)\int_{-\infty}^{\infty} e^{ikx}e^{-\frac{k^2(t-t')}{\mu\sigma_1}}\left(1-e^{-(\frac{1}{\mu\sigma_a}-\frac{cos\theta}{\mu\sigma_b})k^2t'}\right)dk \tag{5.48}$$

$$S(\theta) = \left(\frac{1-(\frac{\sigma_a}{\sigma_b})^2}{1-(\frac{\sigma_a}{\sigma_b})cos\theta}\right) \tag{5.49}$$

Using the general formula

$$\int_{-\infty}^{\infty} e^{ikx}e^{-\Lambda k^2}dk = \frac{\sqrt{\pi}}{\sqrt{\Lambda}}\exp\left(\frac{-x^2}{4\Lambda}\right) = F(\Lambda) \tag{5.50}$$

with

$$\Lambda_1 = \frac{1}{\mu\sigma_a}\left(1-\frac{\sigma_a}{\sigma_b}cos\theta\right)t' + \frac{1}{\mu\sigma_1}(t-t') \tag{5.51}$$

$$\Lambda_2 = \frac{t-t'}{\mu\sigma_1} \tag{5.52}$$

we deduce:

$$n_3 = \left(\frac{1}{\mu\sigma_a\pi}\right)C'F(\Lambda_2) - \left(\frac{1}{\mu\sigma_a\pi}\right)\int_0^\pi d\theta S(\theta)F(\Lambda_1) \tag{5.53}$$

where

$$C' = \int_0^\pi S(\theta)d\theta \tag{5.54}$$

Noting that

$$\left|\frac{\sigma_a}{\sigma_b}\right| = \left|\frac{\sigma_2-\sigma_1}{\sigma_2+\sigma_1}\right| < 1 \tag{5.55}$$

gives

$$\int_0^\pi \frac{d\theta}{(1-(\frac{\sigma_a}{\sigma_b})cos\theta)} = \frac{\pi}{\sqrt{1-(\frac{\sigma_a}{\sigma_b})^2}} \tag{5.56}$$

and

$$C' = \pi\sqrt{1-(\frac{\sigma_a}{\sigma_b})^2} \tag{5.57}$$

By comparing Eq. (5.51) with the exponential terms of Eqs. (5.44) and (5.45) we obtain

$$n_1 = \left(\frac{1}{\mu\sigma_b\pi}\right)\int_0^{\pi} cos\theta \; d\theta F(\Lambda_1) \tag{5.58}$$

$$n_2 = \left(\frac{1}{\mu\sigma_a\pi}\right)\int_0^{\pi} d\theta F(\Lambda_2) \tag{5.59}$$

From Eqs. (5.53), (5.58), and (5.59) we have:

$$n = n_1 + n_2 + n_3 \tag{5.60}$$

$$n = \left(\frac{\sqrt{1-(\frac{\sigma_a}{\sigma_b})^2}}{\mu\sigma_a}\right)F(\Lambda_2) + \left(\frac{1}{\mu\sigma_a\pi}\right)\int_0^{\pi} W(\theta)F(\Lambda_1)d\theta \tag{5.61}$$

where

$$W(\theta) = \left(\frac{\sigma_a}{\sigma_b}\right) cos\theta + 1 - S(\theta) \quad (a)$$
$$W(\theta) = \frac{(\frac{\sigma_a}{\sigma_b})^2 sin^2\theta}{1-(\frac{\sigma_a}{\sigma_b})cos\theta} \quad (b) \tag{5.62}$$

Combining Eqs. (5.43), (5.47), (5.60), (5.61) and inserting the latter into (5.33) gives:

$$G_{1,3} = \frac{\partial^2}{\partial x^2}\left[ABI_1 + ACI_2\right] \tag{5.63}$$

where

$$
\begin{aligned}
A &= \frac{2\sqrt{\sigma_1\sigma_2}}{(\sigma_1-\sigma_2)} \quad (a)\\
B &= \frac{\sqrt{1-(\frac{\sigma_a}{\sigma_b})^2}}{\mu\sigma_a} = \frac{\sqrt{1-\rho^2}}{\mu\sigma_a} \quad (b)\\
C &= \frac{(\frac{\sigma_a}{\sigma_b})^2}{(\mu\sigma_a\pi)} = \frac{\rho^2}{\mu\sigma_a\pi} \quad (c)\\
I_1 &= \int_0^t dt' f(t-t')F(\Lambda_2) \quad (d)\\
I_2 &= \int_0^t dt' f(t-t')\langle F(\Lambda_1)\rangle \quad (e)\\
\langle F(\Lambda_1)\rangle &= \int_0^\pi \frac{sin^2\theta}{(1-\rho cos\ \theta)}F(\Lambda_1) \quad (f)\\
\rho &= \left(\frac{\sigma_a}{\sigma_b}\right) = \frac{(\sigma_2-\sigma_1)}{(\sigma_2+\sigma_1)} \quad (g)
\end{aligned}
\tag{5.64}
$$

It is easy to see that I_1 reduces to the same integral as that of Eq. (5.25); that is

$$
I_1 = \frac{2}{L}U(x,t) \tag{5.65}
$$

where $U(x,t)$ is given by Eq. (5.29).

UsingEqs. (5.41) and (5.42) we deduce

$$
\frac{\partial^2(ABI_1)}{\partial x^2} = \frac{4}{\mu L(\sigma_1-\sigma_2)}\frac{\partial^2}{\partial x^2}(U(x,t)) = -G_{1,2}(x,t) \tag{5.66}
$$

We observe that the foregoing contribution provides an exact cancellation to $G_{1,2}$.

Thus, the significant contribution to $G_{1,3}$ comes from the term ACI_2. Unfortunately, this does not appear to be easily integrated, and numerical techniques must be used. For simplicity in presentation we shall render the results in dimensionless form. We first compute the constant AC; we have

$$
AC = \left(\frac{1}{\mu\sqrt{\sigma_1\sigma_2}}\right)\left(\frac{\sigma_1-\sigma_2}{\sigma_1+\sigma_2}\right) = \frac{K}{\mu\sigma_1} \tag{5.67}
$$

where

$$
K = \sqrt{\frac{\sigma_1}{\sigma_2}}\left(\frac{\sigma_1-\sigma_2}{\sigma_1+\sigma_2}\right) \tag{5.68}
$$

Let us now introduce the variable

$$y = \left(\frac{t'}{t}\right) \tag{5.69}$$

in Eq. (5.64d). We can write I_2 in terms of y in the following form:

$$I_2 = \left(\frac{1}{2L}\right) \Pi_o(t_d, \phi_d) \tag{5.70}$$

where

$$\begin{aligned}
\Pi_o &= \frac{1}{t_d} \int_0^1 \frac{1}{(1-y)^{\frac{3}{2}}} e^{-\frac{1}{4t_d(1-y)}} \Omega(\phi_d, y) dy \quad (a) \\
\Omega(\phi_d, y) &= \int_0^{\pi} \frac{sin^2\theta d\theta}{(1-\rho cos\theta)} \frac{1}{\sqrt{\gamma_o}} \exp\left(-\frac{\phi_d}{\gamma_o}\right) \quad (b) \\
\gamma_o &= 1 + \left(\frac{1-r}{2r}\right) y(1+cos\theta) \quad (c) \\
\phi_d &= \frac{x_d^2}{4t_d} \quad (d) \\
t_d &= \frac{t}{T_R} \quad (e) \\
x_d &= \frac{x}{L} \quad (f) \\
r &= \frac{\sigma_2}{\sigma_1} \quad (g) \\
\rho &= \frac{(r-1)}{(r+1)} \quad (h)
\end{aligned} \tag{5.71}$$

Using the dimensionless variables introduced in Eq. (5.71) we write for the Green's Function,

$$G_1 = G_{1,1} + \frac{\partial^2}{\partial x^2}(ACI_2) \tag{5.72}$$

$$\begin{aligned}
G_1 = &\frac{1}{2}\frac{(1-\sqrt{r})}{(1+\sqrt{r})}\left(\frac{1}{T_R L}\right)\left(\frac{1}{t_d^2}\right) \exp\left[-\left(\frac{1+x_d^2}{4t_d}\right)\right] \\
&+ \frac{1}{2}\frac{1}{\sqrt{r}}\frac{(1-r)}{(1-r)}\left(\frac{1}{T_R L}\right)\frac{\partial^2}{\partial x_d^2}\Pi_o
\end{aligned} \tag{5.73}$$

The first term in Eq. (5.73) is that of $G_{1,1}$ (cf. Eq. (5.22c)), expressed in terms of T_R, L, and the dimensionless variables r, t_d, and x_d.

The contribution to the electric field from G_1 is given from Eq. (5.30) where the required integrations can be performed in either $\frac{x_d}{t_d}$ or $\frac{x}{t}$ space. It is also evident from Eqs. (5.31) and (5.32) that the term involving $\frac{\partial^2 \Pi_o}{\partial x_d^2}$ can be converted to an integration of Π_o combined with the second partial derivative of $E_{y,o}$.

Since Π_o cannot be expressed in closed-form, it must be provided in tabular form, either as a function of t_d, x_d or t_d, ϕ_d. This consideration should be reserved when the actual implementation of this result is incorporated in the HABEMP finite-difference model.

Tabular values of $\Pi_o(t_d, \phi_d)$ for selected values of t_d and ϕ_d are rendered in Table I. Plots of Π_o as a function of ϕ_d with t_d as a parameter for $r = 0.5$ are shown in Figure 3, while plots of Π_o as a function of t_d with ϕ_d as a parameter for $r = 0.5$ are shown in Figure 4.

TABLE I

Values Of $\Pi_o(t_d, \phi_d)$ For

$r = 0.3, 0.5, 0.7, 2.0, 5.0$

THE AFOREMENTIONED INFORMATION IS PROVIDED IN THE NEXT FIVE PAGES

values of pi0(tau-d,phi-d) for r = .30

tau-d	phi-d .00	.25	.50	.75	1.00	1.50	2.00	3.00	4.00	5.00
.10	.4189	.3450	.2847	.2354	.1949	.1345	.0935	.0463	.0237	.0126
.30	1.7276	1.4584	1.2341	1.0469	.8902	.6485	.4771	.2657	.1533	.0915
.50	2.0805	1.7740	1.5166	1.2999	1.1171	.8312	.6247	.3630	.2184	.1355
.75	2.1579	1.8532	1.5957	1.3776	1.1925	.9003	.6865	.4104	.2537	.1615
1.00	2.1258	1.8340	1.5865	1.3760	1.1965	.9117	.7015	.4268	.2682	.1734
1.50	1.9976	1.7334	1.5081	1.3156	1.1506	.8867	.6899	.4288	.2750	.1810
2.00	1.8634	1.6227	1.4169	1.2403	1.0886	.8447	.6616	.4165	.2702	.1798
3.00	1.6254	1.4212	1.2459	1.0951	.9649	.7545	.5954	.3801	.2497	.1681
4.00	1.4368	1.2592	1.1065	.9747	.8607	.6760	.5356	.3446	.2280	.1544
5.00	1.2952	1.1370	1.0006	.8828	.7808	.6151	.4888	.3161	.2102	.1430

values of pi0(tau-d,phi-d) for r = .50

tau-d	phi-d .00	.25	.50	.75	1.00	1.50	2.00	3.00	4.00	5.00
.10	.4091	.3340	.2730	.2235	.1832	.1236	.0839	.0393	.0189	.0093
.30	1.7094	1.4274	1.1942	1.0010	.8407	.5965	.4266	.2233	.1205	.0669
.50	2.0696	1.7447	1.4738	1.2476	1.0584	.7664	.5595	.3054	.1718	.0993
.75	2.1547	1.8288	1.5556	1.3260	1.1328	.8319	.6160	.3459	.2000	.1187
1.00	2.1278	1.8140	1.5498	1.3270	1.1387	.8438	.6305	.3603	.2119	.1278
1.50	2.0055	1.7193	1.4772	1.2720	1.0977	.8226	.6215	.3630	.2179	.1340
2.00	1.8744	1.6125	1.3902	1.2012	1.0401	.7848	.5969	.3531	.2146	.1335
3.00	1.6385	1.4151	1.2249	1.0625	.9236	.7022	.5381	.3229	.1989	.1253
4.00	1.4502	1.2553	1.0890	.9467	.8248	.6298	.4846	.2931	.1818	.1153
5.00	1.3084	1.1343	.9856	.8582	.7488	.5735	.4426	.2692	.1679	.106[illegible]

values of pi0(tau-d,phi-d) for r = .70

tau-d	phi-d .00	.25	.50	.75	1.00	1.50	2.00	3.00	4.00	5.00
.10	.4063	.3301	.2685	.2187	.1783	.1189	.0797	.0363	.0169	.0080
.30	1.7093	1.4189	1.1797	.9824	.8195	.5732	.4036	.2042	.1061	.0565
.50	2.0754	1.7385	1.4590	1.2266	1.0331	.7368	.5292	.2789	.1509	.0837
.75	2.1651	1.8257	1.5423	1.3054	1.1069	.8004	.5829	.3158	.1757	.1001
1.00	2.1409	1.8130	1.5382	1.3075	1.1136	.8123	.5968	.3290	.1862	.1078
1.50	2.0213	1.7210	1.4682	1.2549	1.0746	.7925	.5887	.3316	.1915	.1131
2.00	1.8911	1.6155	1.3828	1.1859	1.0190	.7565	.5657	.3227	.1887	.1127
3.00	1.6550	1.4193	1.2196	1.0500	.9056	.6774	.5103	.2953	.1750	.1059
4.00	1.4658	1.2598	1.0849	.9360	.8091	.6078	.4597	.2682	.1601	.0975
5.00	1.3231	1.1389	.9823	.8488	.7348	.5536	.4200	.2463	.1478	.0905

values of pi0(tau-d,phi-d) for r = 2.00

tau-d	phi-d .00	.25	.50	.75	1.00	1.50	2.00	3.00	4.00	5.00
.10	.4258	.3428	.2762	.2227	.1797	.1172	.0767	.0331	.0145	.0064
.30	1.8148	1.4885	1.2223	1.0047	.8268	.5618	.3835	.1811	.0871	.0427
.50	2.2159	1.8324	1.5171	1.2577	1.0439	.7218	.5015	.2457	.1227	.0624
.75	2.3210	1.9308	1.6083	1.3413	1.1202	.7842	.5519	.2774	.1421	.0741
1.00	2.3011	1.9217	1.6070	1.3456	1.1282	.7962	.5649	.2886	.1502	.0796
1.50	2.1798	1.8295	1.5376	1.2940	1.0905	.7775	.5572	.2905	.1543	.0833
2.00	2.0436	1.7205	1.4505	1.2245	1.0351	.7426	.5355	.2826	.1519	.0830
3.00	1.7928	1.5147	1.2816	1.0858	.9211	.6655	.4833	.2585	.1408	.0779
4.00	1.5900	1.3461	1.1412	.9688	.8236	.5974	.4355	.2348	.1288	.0717
5.00	1.4365	1.2180	1.0341	.8791	.7484	.5444	.3980	.2157	.1189	.0665

values of pi0(tau-d,phi-d) for r = 5.00

tau-d	phi-d .00	.25	.50	.75	1.00	1.50	2.00	3.00	4.00	5.00
.10	.4777	.3835	.3080	.2476	.1991	.1289	.0837	.0356	.0153	.0066
.30	2.0449	1.6705	1.3660	1.1179	.9158	.6163	.4162	.1920	.0899	.0427
.50	2.5014	2.0595	1.6974	1.4003	1.1565	.7912	.5435	.2596	.1259	.0620
.75	2.6237	2.1725	1.8008	1.4944	1.2414	.8595	.5975	.2924	.1454	.0734
1.00	2.6034	2.1638	1.8004	1.4997	1.2506	.8725	.6114	.3039	.1535	.0786
1.50	2.4690	2.0620	1.7240	1.4431	1.2093	.8520	.6029	.3056	.1573	.0821
2.00	2.3163	1.9403	1.6272	1.3661	1.1482	.8139	.5793	.2971	.1547	.0817
3.00	2.0336	1.7094	1.4385	1.2119	1.0221	.7295	.5228	.2717	.1433	.0766
4.00	1.8043	1.5197	1.2814	1.0817	.9141	.6548	.4711	.2466	.1310	.0705
.00	1.6307	1.3754	1.1613	.9817	.8307	.5968	.4304	.2265	.1209	.0654

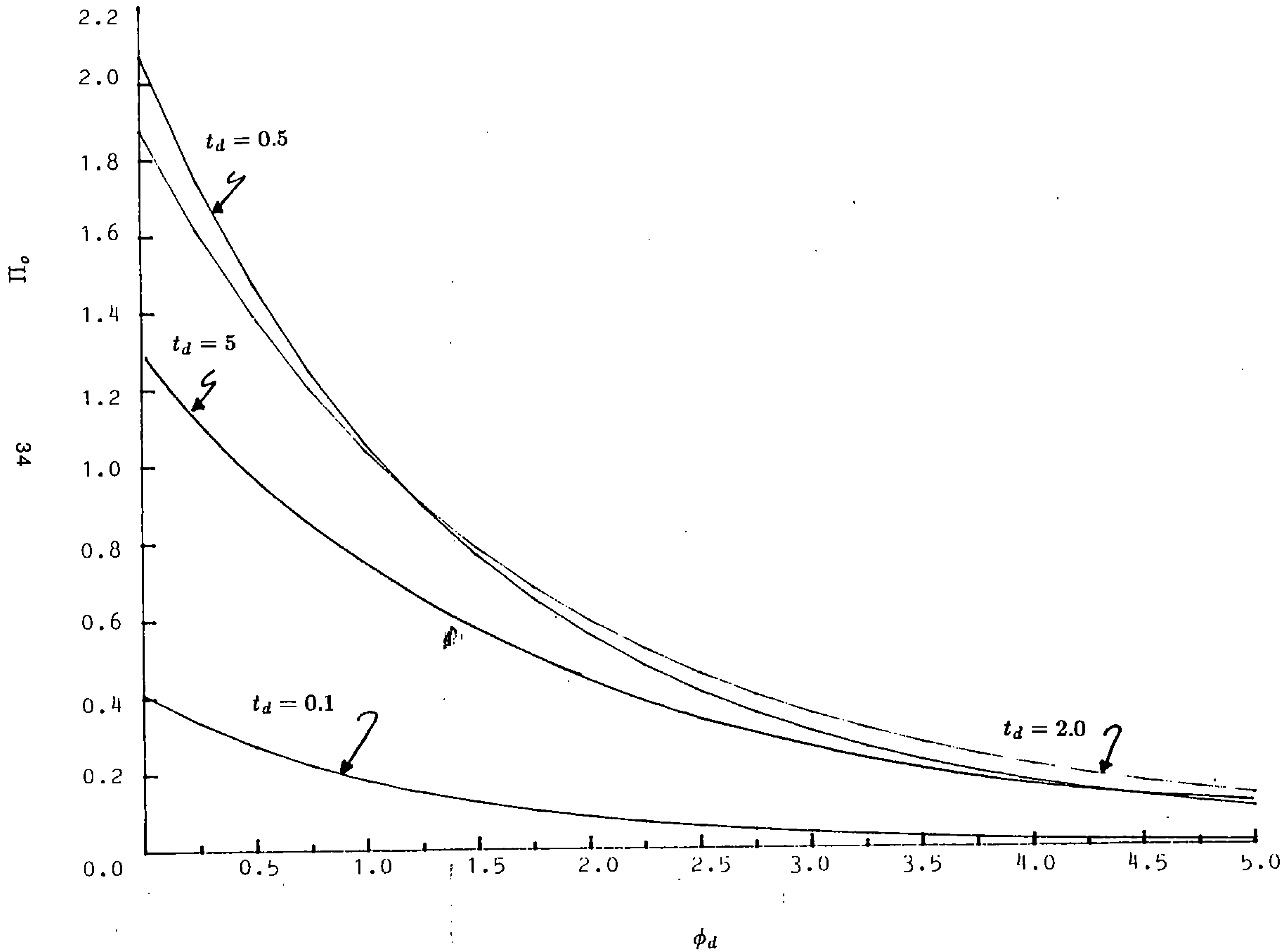

FIG 3. $\Pi_o(t_d, \phi_d)$ As A Function O… ϕ_d With t_d As Parameter For $r = 0.5$

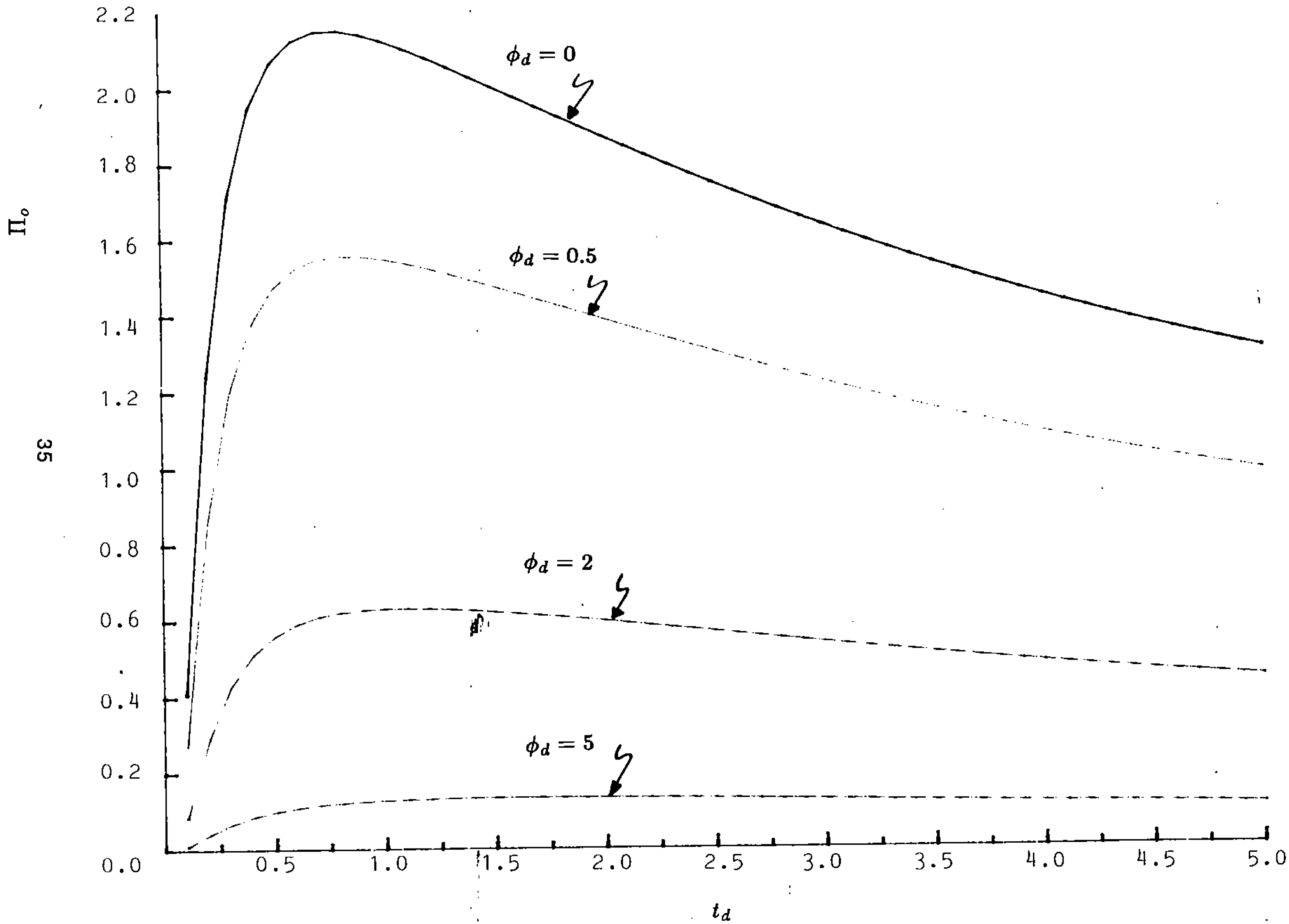

FIG 4. $\Pi_o(T_d, \phi_d)$ As A Function of t_d with ϕ_d As Parameter For $r = 0.5$

6. CONCLUSION

A solution for the Two Region-Two Dimensional electromagnetic ground response has been developed which relates the surface components of the electric field to the surface components of the magnetic field. This has been accomplished by deriving a universal functional form for a dimensionless Green's Function. The Green's Function provides increasingly more accurate approximations to the response for each successive reflection from the second layer. This result would appear to provide simplification and reduced computer running time in the numerical modelling of the HABEMP when the ground response is coupled to finite-difference methods for solving the atmospheric part of the problem.

7. REFERENCES

1. I. Kohlberg, Three-Dimensional Electromagnetic Ground Response for Multi-Layered Earth: Surface Integral Representation with Frequency-Dependent Electrical Parameters, Theoretical Note 355, Aug 88.

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Appendix A: Calculation of $\hat{Q}(k,t)$

The basic building block for the calculation of

$$\hat{Q}(k,t) = \underline{L}^{-1}\left(\frac{\sqrt{s+\alpha_1}\sqrt{s+\alpha_2}}{s}\right) = \underline{L}^{-1}\left(\bar{Q}(k,s)\right) \tag{A.1}$$

is the formula

$$\begin{aligned} \underline{L}^{-1}\left(\bar{R}(s)\right) &= \underline{L}^{-1}\left[\frac{1}{s}\frac{(s+\alpha-\beta)^{\frac{1}{2}}}{(s+\alpha+\beta)^{\frac{1}{2}}}\right] = \hat{R}(t) \\ &= \left[e^{-\alpha t}I_o(\beta t) + (\alpha-\beta)\int_0^t e^{-\alpha u}I_o(\beta u)du\right] H(t) \end{aligned} \tag{A.2}$$

In the foregoing expxression I_o is the modified Bessel Function of zero order, and $H(t)$ is the step function. If we make the identification

$$\begin{aligned} \alpha &= \frac{\alpha_1+\alpha_2}{2} = \frac{1}{2\mu}\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}\right)k^2 \quad (a) \\ \beta &= \frac{\alpha_1-\alpha_2}{2} = \frac{1}{2\mu}\left(\frac{1}{\sigma_1}-\frac{1}{\sigma_2}\right)k^2, \quad (b) \end{aligned} \tag{A.3}$$

we can write (suppressing the k dependence):

$$\bar{Q}(s) = (s+\alpha_1)\bar{R}(s) \tag{A.4}$$

We then have:

$$\begin{aligned} \underline{L}^{-1}\bar{Q}(s) &= \underline{L}^{-1}\left((s+\alpha_1)\bar{R}(s)\right) = \hat{Q}(t) \quad (a) \\ \hat{Q}(t) &= \alpha_1\hat{R}(t) + \frac{d}{dt}\hat{R}(t) \quad (b) \end{aligned} \tag{A.5}$$

Substituting Eq. (A.2) into Eq. (A.5) yields

$$\begin{aligned} \hat{Q}(t) &= e^{-\alpha t}\left[\beta I_1(\beta t) + \alpha I_o(\beta t)\right] \\ &+ (\alpha^2-\beta^2)\int_0^t e^{-\alpha u}I_o(\beta u)du + \delta(t) \end{aligned} \tag{A.6}$$

Appendix B: Application of Faltung Theorem

In the main body of the report we are concerned with calculations of the form

$$f_{12}(x) = \left(\frac{1}{2\pi}\right) \int_{-\infty}^{\infty} f_1(x-y) f_2(y) dy \tag{B.1}$$

where $f_2(y)$ is given by:

$$f_2(y) = \frac{\partial^2}{\partial y^2} \left(g(y)\right) \tag{B.2}$$

For computational purposes it may be desireable to organize the computation of $f_{12}(x)$ so that only the function $g(y)$ appears in the integration, and none of it's derivatives. This is accomplished in the following way: We let

$$g'(y) = \left(\frac{\partial g}{\partial y}\right) \tag{B.3}$$

so that we can write

$$f_2(y) = \frac{\partial}{\partial y} g'(y) \tag{B.4}$$

Substituting Eq. (B.4) in Eq. (B.1) gives:

$$f_{12} = \left(\frac{1}{2\pi}\right) \int_{-\infty}^{\infty} f_1(x-y) \frac{\partial g'}{\partial y}(y) dy \tag{B.5}$$

We now integrate Eq. (B.5) by parts to give

$$\begin{aligned} f_{12} &= \frac{1}{2\pi} \left[g'(y) f_1\,(x-y)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} g'(y) \frac{\partial f_1(x-y)}{\partial y} dy \right] \\ &= -\frac{1}{2\pi} \int_{-\infty}^{\infty} g'(y) \frac{\partial f_1(x-y)}{\partial y} dy \end{aligned} \tag{B.6}$$

Letting

$$z = x - y \tag{B.7}$$

we have

$$\frac{\partial}{\partial y} = \frac{\partial f_1}{\partial z} \frac{\partial z}{\partial y} = -\left(\frac{\partial f_1}{\partial z}\right) \tag{B.8}$$

There results:

$$f_{12}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g'(y) \left(\frac{\partial f_1(z)}{\partial z} \right)_{z=x-y} dy \tag{B.9}$$

By executing the same procedure once more we obtain the desired result

$$f_{12}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(y) \left(\frac{\partial^2 f_1(z)}{\partial^2 z} \right)_{z=x-y} dy \tag{B.10}$$