Theoretical Notes
Note 311
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A NOTE ON EMP PROPAGATION OVER IMPERFECTLY CONDUCTING GROUND

K.S.H. Lee
The Dikewood Corporation
Santa Monica, CA 90405

The integral equation formulated by Longmire for the outgoing field over an imperfectly conducting ground beyond the source region is solved exactly by the method of Srivastav. From the exact solution various asymptotic forms of the solution are found, from which a simplified representation of the exact solution is constructed in the frequency as well as the time domains.

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I. INTRODUCTION

The AFWL code SCX in its present form treats the ground perfectly conducting in the region beyond the outer boundary (Fig. 1a), where the outgoing radiation field is expanded in multipoles. Recently a method has been proposed to account for the ground impedance in the outer region (Ref. 1). The method is essentially as follows. The region of interest is above the ground, near the air-ground interface, and beyond the outer boundary (Fig. 1a). In this region the outgoing wave F, $F = r(B_\phi + E_\theta/c)$, satisfies the diffusion equation and impedance boundary condition at the interface, as shown in Figure 1b. At x = 1 (outer boundary), F is taken to be unity and can be scaled to any function of the complex frequency s. Let the surface value of F be denoted by F_1, i.e., $F_1(x) \equiv F(x \geq 1,0)$. Then, with the help of the Green's function, the following integral equation for F_1 results

$$F_1(x) = 1 - \frac{\beta}{\sqrt{2\pi}} \int_1^x \frac{F_1(x')}{\sqrt{x'(1-x'/x)}}\, dx' \tag{1}$$

where $\beta = \sqrt{sr_o/c}\, Z_s/Z_o$, r_o = radius of outer boundary, $Z_s = \sqrt{\mu_o s/(s\varepsilon + \sigma)}$ = surface impedance of the ground in MKS units, and $Z_o = \sqrt{\mu_o/\varepsilon_o}$ = impedance of free space, c = vacuum speed of light. In Reference 1 the method of power-series expansion was first tried to solve Equation 1, but met with no success. The original differential equation (Fig. 1b) was finally adopted to generate numerical results.

The purpose of this note is to show that the integral Equation 1 can be solved exactly without any approximation. Asymptotic solutions for (i) $x \to \infty$, $x \to 1$, and (ii) $|\beta| \to \infty$, $|\beta| \to 0$ can be easily obtained from the exact solution. From the asymptotic solutions a simple expression can be constructed to approximate the exact solution. In the time domain this approximate expression corresponds to the simple form $\tau \exp(-\alpha\tau^2)$.

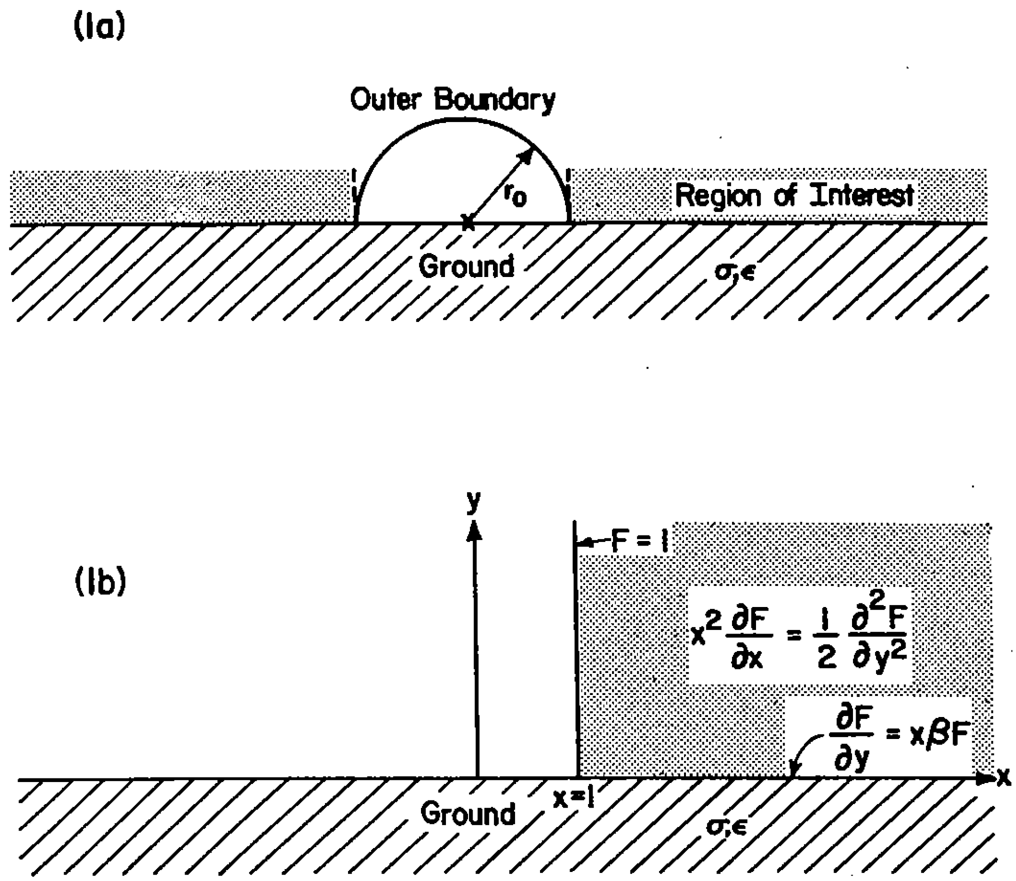

Figure 1. Geometry of the problem

II. EXACT SOLUTION OF INTEGRAL EQUATION

Equation 1 is a special form of the integral equation of Abel type (Ref. 2). Let $f(x) = F_1(x)/\sqrt{x}$. Then Equation 1 can be written as

$$f(x) = \frac{1}{\sqrt{x}} - \frac{\beta}{\sqrt{2\pi}} \int_1^x \frac{f(x')dx'}{\sqrt{x - x'}} \tag{2}$$

Multiplying both sides by $(\xi - x)^{-\frac{1}{2}}$ and integrating the resulting equation over x from 1 to ξ one obtains

$$\int_1^\xi \frac{f(x)dx}{\sqrt{\xi - x}} = \pi - 2 \sin^{-1}\sqrt{1/\xi} - \frac{\beta}{\sqrt{2\pi}} \int_1^\xi \frac{dx}{\sqrt{\xi - x}} \int_1^x \frac{f(x')dx'}{\sqrt{x - x'}} \tag{3}$$

An interchange of the order of integrations gives

$$\int_1^\xi \frac{dx}{\sqrt{\xi - x}} \int_1^x \frac{f(x')dx'}{\sqrt{x - x'}} = \int_1^\xi f(x')dx' \int_{x'}^\xi \frac{dx}{\sqrt{(\xi - x)(x - x')}} = \pi \int_1^\xi f(x')dx' \tag{4}$$

where the inner integral in the second step can be expressed in terms of the beta function $B(\frac{1}{2},\frac{1}{2})$, which has the value π (Ref. 3). Using Equations 2 and 4 in Equation 3 one gets

$$f(\xi) - \frac{\beta^2}{2} \int_1^\xi f(x')dx' = \frac{1}{\sqrt{\xi}} + \beta \sqrt{\frac{2}{\pi}} \sin^{-1}\sqrt{\frac{1}{\xi}} - \beta \sqrt{\frac{\pi}{2}} \tag{5}$$

Differentiation with respect to ξ yields

$$\frac{d}{d\xi} f(\xi) - \frac{\beta^2}{2} f(\xi) = -\frac{1}{2} \xi^{-3/2} - \frac{\beta}{\sqrt{2\pi}} \xi^{-1}(\xi - 1)^{-\frac{1}{2}} \tag{6}$$

The boundary condition for f at $\xi = 1$ can be obtained from Equation 2 and is given by

$$f(1) = 1 \tag{7}$$

Solving Equation 6 subject to Equation 7 one obtains, with ξ replaced by x and f by $F_1/\sqrt{x}$,

$$F_1(x) = \sqrt{x}\, e^{\beta^2 x/2} \left\{ e^{-\beta^2/2} - \int_1^x g(u) e^{-\beta^2 u/2} du \right\} \tag{8}$$

where g is given by

$$g(u) = \frac{1}{2u\sqrt{u}} + \frac{\beta}{\sqrt{2\pi}} \frac{1}{u\sqrt{u-1}} \tag{9}$$

Let $R = x - 1$. Then, Equation 8 is simplified to

$$F_1(R,\beta) = \sqrt{R+1}\, e^{\beta^2 R/2} \left\{ 1 - \int_0^{\sqrt{R}} \left[\frac{\zeta}{(\zeta^2+1)^{3/2}} + \frac{\beta\sqrt{2/\pi}}{\zeta^2+1} \right] e^{-\beta^2\zeta^2/2} d\zeta \right\} \tag{10}$$

after changes of variables have been made. Note that

$$\int_0^\infty \left[\frac{\zeta}{(\zeta^2+1)^{3/2}} + \frac{\beta\sqrt{2/\pi}}{\zeta^2+1} \right] e^{-\beta^2\zeta^2/2} d\zeta = 1 \tag{11}$$

Equation 10 can be written in a more convenient form

$$F_1(R,\beta) = \sqrt{R+1}\, e^{\beta^2 R/2} \int_{\sqrt{R}}^\infty \left[\frac{\zeta}{(\zeta^2+1)^{3/2}} + \frac{\beta\sqrt{2/\pi}}{\zeta^2+1} \right] e^{-\beta^2\zeta^2/2} d\zeta \tag{12}$$

where $R = x - 1 \geq 0$. Another convenient form can be obtained from Equation 12 by a simple change of variables $u = \zeta^2 - R$ and is given by

$$F_1(R,\beta) = \frac{\sqrt{R+1}}{2} \int_0^{\infty} \left[\frac{1}{(u+R+1)^{3/2}} + \frac{\beta\sqrt{2/\pi}}{(u+R+1)\sqrt{u+R}} \right] e^{-\beta^2 u/2} du \qquad (13)$$

When R >> 1, one may replace R by R+1, and vice versa, in Equation 13. In doing so the resulting integral can be evaluated explicitly and the result is (Ref. 3)

$$F_1(R,\beta) \sim (1 + \beta\sqrt{2/\pi}) \left[1 - \sqrt{\pi p_1} e^{p_1} \operatorname{erfc}(\sqrt{p_1}) \right] \qquad (14)$$

where the "numerical distance" p_1 is defined as

$$p_1 = \beta^2 R/2 \qquad (15)$$

The expression in the square bracket of Equation 14 is exactly the Norton formula for the Sommerfeld problem of a dipole radiating over a finitely conducting ground (Ref. 1).

In Figure 2 the exact solution given by Equation 12 is shown together with the numerical solution given in Reference 1.

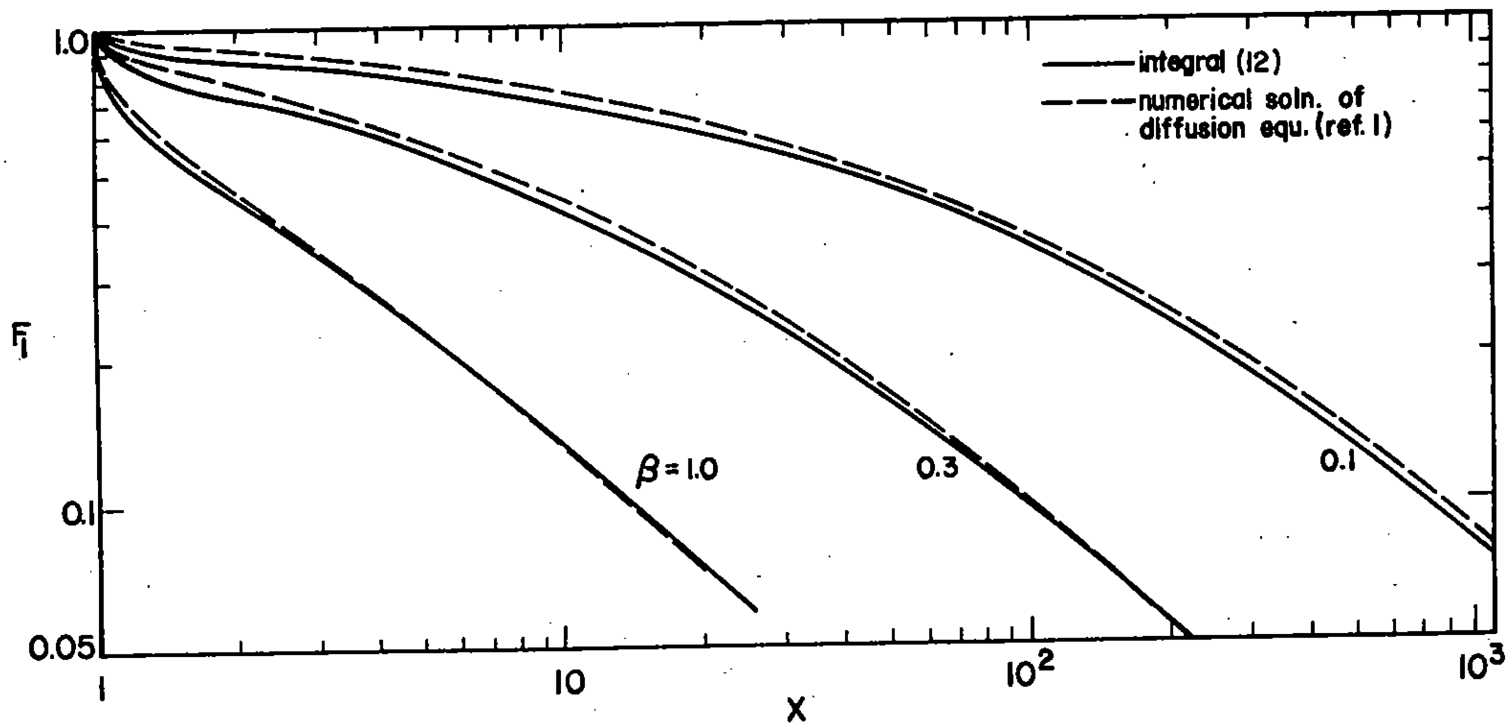

Figure 2. Comparison of the integral given by Equation 12 and the numerical solution of the diffusion equation shown in Figure 1 (Ref. 1).

III. ASYMPTOTIC SOLUTIONS

In this section various asymptotic solutions will be derived from the many different forms of the exact solution developed in the previous section.

1. R >> 1

Equation 14 gives

$$F_1 \sim (1 + \beta\sqrt{2/\pi}) \left[1 - \sqrt{\pi p_1} e^{p_1} \mathrm{erfc}\sqrt{p_1} \right], \quad \text{for any } \beta\sqrt{R} \tag{16}$$

$$\sim \frac{1}{\beta^2 R} (1 + \beta\sqrt{2/\pi}), \quad \text{for } \beta\sqrt{R} >> 1, \quad |\arg \beta| < 3\pi/4 \tag{17}$$

where $p_1 = \beta^2 R/2$.

2. R << 1

Expanding the integrand in the square bracket of Equation 10 in powers of ζ and keeping only the leading term, one gets (Ref. 3)

$$F_1 \sim e^{p_1} \mathrm{erfc}\sqrt{p_1}, \quad \text{for any } \beta\sqrt{R} \tag{18}$$

$$\sim 1 - \sqrt{2/\pi}\, \beta\sqrt{R}, \quad \text{for } \beta\sqrt{R} << 1 \tag{19}$$

3. $\beta >> 1, \; |\arg(\beta^2)| < \pi/2$

From Equation 12 one has, for large β,

$$F_1 \sim \sqrt{R+1}\; e^{\beta^2 R/2} \int_{\sqrt{R}}^{\infty} \frac{\beta\sqrt{2/\pi}}{\zeta^2 + 1} e^{-\beta^2 \zeta^2/2} d\zeta$$

Since the value of the integral comes mainly from the lower limit, one gets

$$F_1 \sim \frac{\beta\sqrt{2/\pi}}{\sqrt{R+1}} e^{\beta^2 R/2} \int_{\sqrt{R}}^{\infty} e^{-\beta^2 \zeta^2/2} d\zeta$$

$$= \frac{1}{\sqrt{R+1}} e^{p_1} \operatorname{erfc}\sqrt{p_1}, \qquad \text{for any } \beta\sqrt{R} \tag{20}$$

$$\sim \frac{1}{\beta} \sqrt{\frac{2/\pi}{R(R+1)}}\,, \qquad \text{for } \beta\sqrt{R} >> 1 \tag{21}$$

4. $\underline{\beta << 1, \ |\arg(\beta^2)| < \pi/2}$

One may neglect the second term in the integrand of Equation 13 and obtains

$$F_1 \sim \frac{\sqrt{R+1}}{2} \int_0^\infty \frac{e^{-\beta^2 u/2}}{(u+R+1)^{3/2}}\, du$$

which gives (Ref. 4)

$$F_1 \sim 1 - \sqrt{\pi p_1}\, e^{p_1} \operatorname{erfc}\sqrt{p_1}, \qquad \text{for any } \beta\sqrt{R} \tag{22}$$

$$\sim 1 - \sqrt{\pi/2}\, \beta\sqrt{R}, \qquad \text{for } \beta\sqrt{R} << 1 \tag{23}$$

IV. SIMPLIFIED REPRESENTATION OF THE EXACT SOLUTION

From the asymptotic solutions given in the previous section it is clear that for large β and any R one may take Equation 20 to approximate the exact solution. On the other hand, if one is interested in small β and any R one should take Equation 22, namely,

$$F_1 \doteq 1 - \sqrt{\pi p_1}\, e^{p_1} \operatorname{erfc}\sqrt{p_1} \tag{24}$$

with $p_1 = \beta^2 R/2$, which reduces to unity, as it should, when $R = 0$. Equation 24 is plotted in Figures 3 and 4 together with the exact solution given by the integral 12 and the approximate solution of Reference 1.

To get the time-domain counterpart of Equation 24 one recalls the definition of β, i.e.,

$$\beta = \sqrt{s r_o/c}\, \frac{Z_s}{Z_o} = \frac{s}{\sqrt{\epsilon_r}} \sqrt{\frac{r_o/c}{s + \sigma/\epsilon}} \tag{25}$$

with $\epsilon_r = \epsilon/\epsilon_o$. Assuming the radical is a constant as in Reference 1 one can rewrite Equation 25 as

$$\beta = s\tau_1 \tag{26}$$

where τ_1 can be obtained by iteration by solving

$$\tau_1 = \frac{1}{\sqrt{\epsilon_r}} \sqrt{\frac{r_o/c}{\sigma/\epsilon + 1/\tau_1}}$$

With β given by Equation 26 the inverse Laplace transform of Equation 24 is (Ref. 4)

$$F_1(R,\tau) \doteq \frac{\tau}{R\tau_1^2} \exp\left[-\tau^2/(2R\tau_1^2)\right], \quad \tau > 0 \tag{27}$$

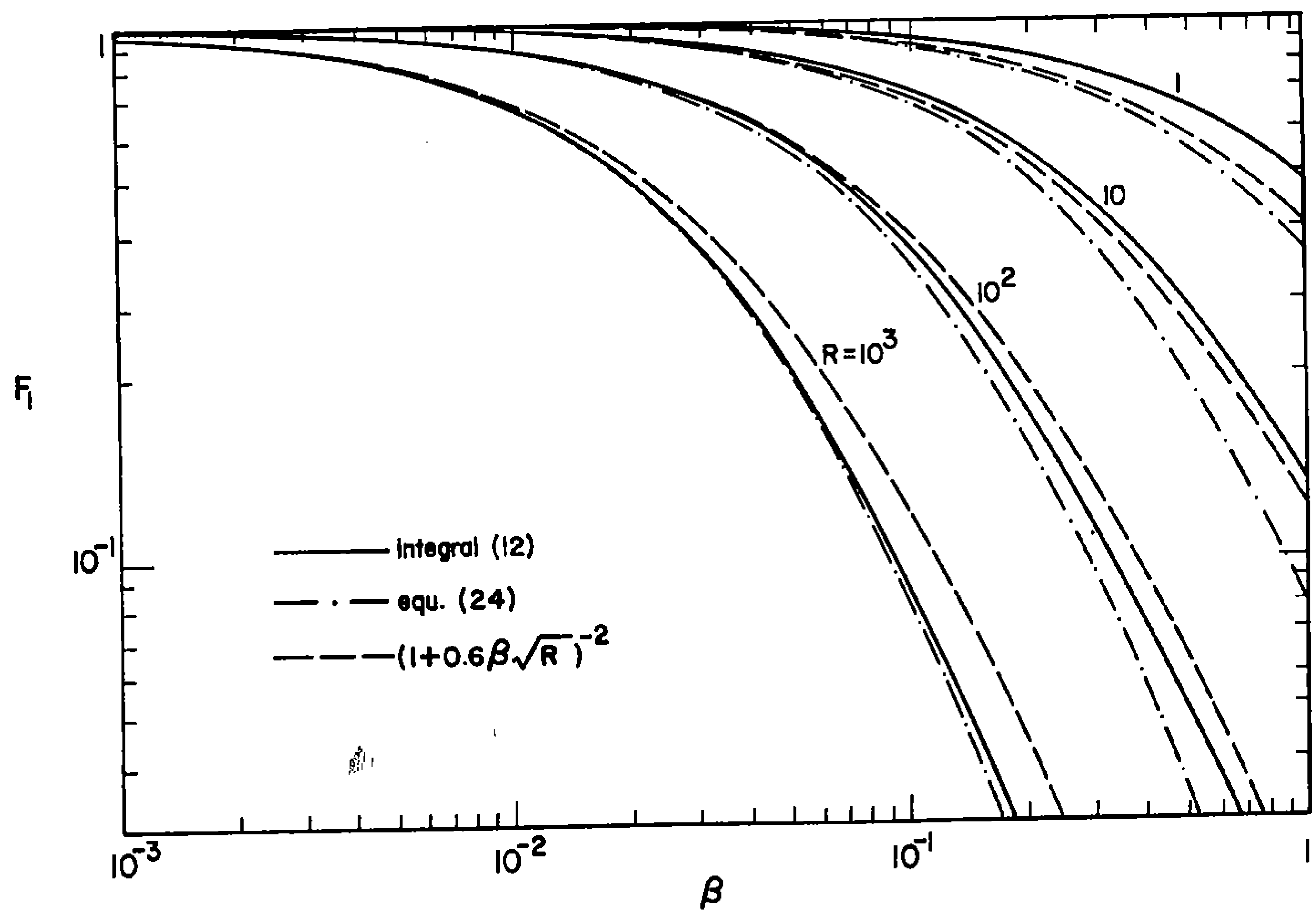

Figure 3. Comparison of two approximate solutions with respect to the exact solution given by Equation 12. The approximate solution indicated by broken lines is given in Reference 1.

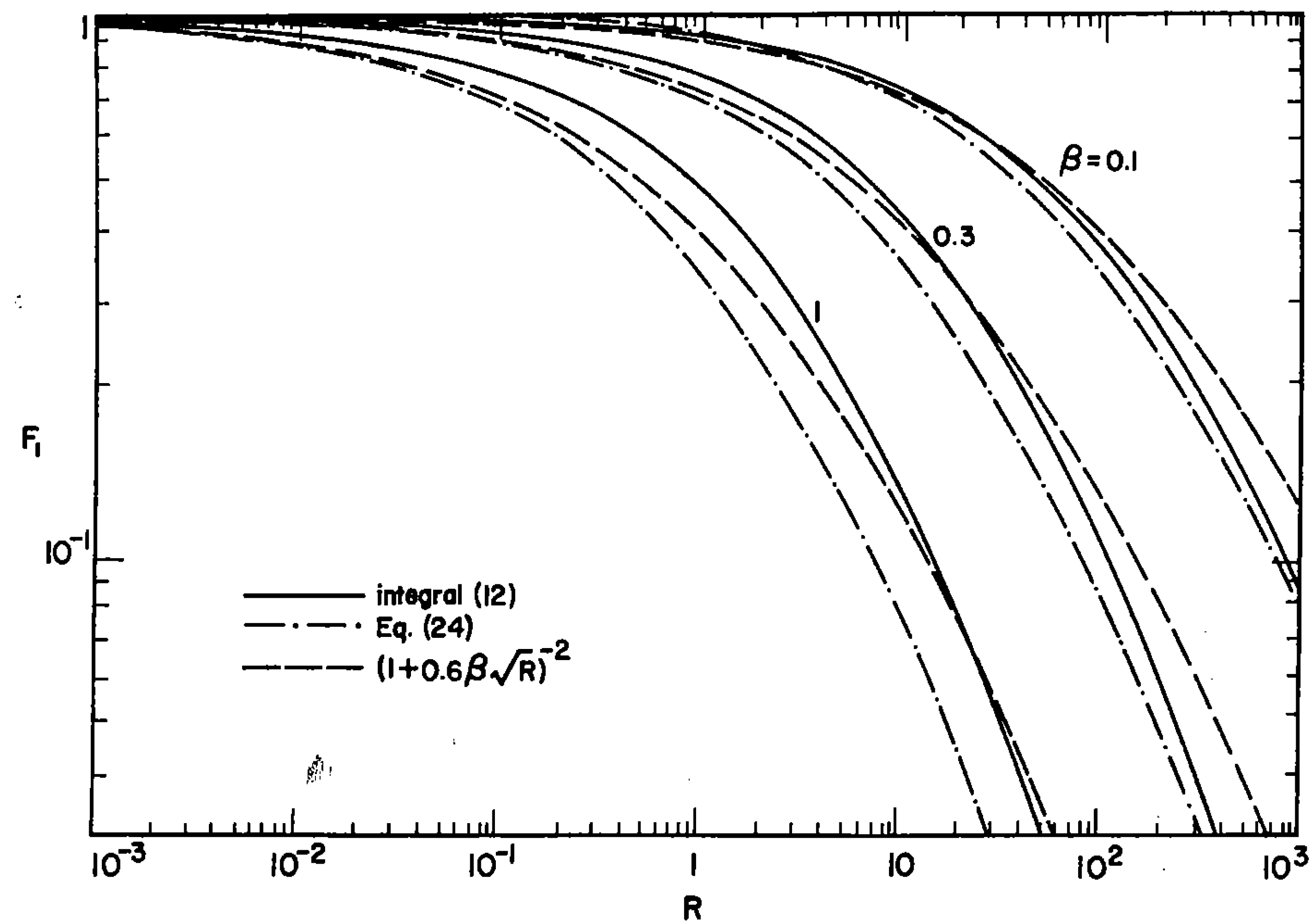

Figure 4. Comparison of two approximate solutions with respect to the exact solution given by Equation 12. The approximate solution indicated by broken lines is given in Reference 1.

REFERENCES

1. Longmire, C.L., "Attenuation of EMP in the Wave Zone due to Imperfectly Conducting Ground," DNA 4634T, Defense Nuclear Agency, Washington, D.C., June 1978.

2. Srivastav, R.P., "A Note on Certain Integral Equations of Abel-Type," Proc. Edinburgh Math. Soc. 13, 271, 1963.

3. Abramowitz, M. and I.A. Stegun, Handbook of Mathematical Functions, AMS 55, NBS, U.S. Government Printing Office, Washington, D.C., 1964.

4. Erdélyi, A., Editor, Tables of Integral Transforms, vol. I, McGraw-Hill, New York, 1954.