Theoretical Notes
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ON THE CALCULATION OF THE EFFECTS OF HOLES AND SLOTS ON SGEMP

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20. ABSTRACT (Continued)

effects due to having an aperture in the source region and with those IEMP effects due to X rays penetrating the surface of the system. A hardware design suggestion is made to harden against what appears to be the most dangerous coupling effect.

SECTION 1

INTRODUCTION

Satellites, in general, are not completely continuous electromagnetic structures. Consequently, fields generated on the exterior of the structure can leak through the discontinuities — holes and slots — and possibly disrupt or destroy the circuitry within the satellite. In an actual SGEMP situation photoelectrons are ejected from the satellite surfaces by X-rays striking the surface. A proportion of these electrons escape to infinity leaving the satellite with a net positive charge. Thus, over the long term (times much longer than the X-ray pulse length) electrostatic fields are expected. Because of the electrical discontinuities in the satellite, interior electrostatic fields are expected also. If, in addition, the time duration for setting up the exterior fields is comparable to or smaller than the fundamental periods for the interior cavities then oscillatory fields may also be expected over the long term.

Competing with the effects due to slots and holes are similar effects (electrostatic fields and oscillatory electromagnetic fields) due to X-rays penetrating the skin of the satellite and causing the ejection of photoelectrons into the interior of the satellite. The discussion in this report will be mainly confined to the effects due to slots and holes. Where it is possible, comparisons will be made between the effects of internal photoemission and coupling through surface discontinuities.

Two specific types of discontinuities are discussed. One of these is a slot extending completely around the satellite, separating it

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ISOLATED SECTIONS

In SGEMP problems, charge is removed from the part of the satellite surface struck by X-rays. If the irradiated portion is conducting and electrically isolated, charge can leak onto the inside surface causing electrostatic fields and possibly stimulating the internal cavity modes. Computer evidence already exists[1] which shows that this effect can be considerable. We wish to address the question of the possible effects of differential charging. Three static solutions will be discussed first.

Static solutions represent the state of a system after all the charge has been removed to infinity and all the oscillations have died away. If the charge is removed from the system in a time interval which is smaller than or comparable to, the period of the lowest mode of the system (internal or external), then the mode will be stimulated. If the modes are highly damped, as they are for the external modes of many systems, then the quasi-static response of the system (a static solution which varies continuously in time as the sources vary in time) will predominate. Apart from being the correct solution in many cases,[2,3] static and quasi-static solutions enable one to make order of magnitude estimates of oscillating fields if modes are actually stimulated.

Figure 1 represents a conducting system with a cavity. Although the area of the gap connecting the exterior with the interior is into two parts which are electrically isolated from each other. Another discontinuity is that of a circular hole in the satellite skin. The radius of the hole is taken to be small relative to the dimensions of the satellite.

The method used in estimating the importance of these surface features is to solve a simple geometry which is tractable mathematically. The results and methods suggest general rules which can be extrapolated to more complex configurations. It is hoped that these rules will serve as tools or frames of reference for estimating fields in practical situations. The geometry chosen for the analysis is, for the most part, spherical.

A number of interesting results arise from the analysis. It appears that an irradiated, electrically isolated panel is probably the most dangerous from the point of view of internal fields. Depending upon the configuration, as much as 50 percent of the charge left on a panel could leak to the inside creating internal fields of the order of the exterior fields. The adverse effect of an isolated panel can be avoided in an actual system by connecting panels with conducting wires in non-spinning assemblies, or brushes, in the case of spinning assemblies. The isolating slot is effectively converted into a long narrow "hole" by means of these conducting brushes or wires. Internal electric fields due to a circular hole are of the order of magnitude: $E_o(\alpha/R)^3$, where E_o is the external field value, α is the radius of the hole, and R is the radius of the satellite. If α/R is .1, for example, then the internal fields are only about one thousandth of the external fields. A hole in the irradiated region may be a greater threat, however, especially when the X-ray fluence is large.

In Section 2 of this report, we present the results of the investigations together with some arguments to obtain order of magnitude estimates of internal fields. The mathematical derivation of the results is left to the appendices so that the discussion will not be obscured. Section 3 considers the utilization of already existing computer codes which calculate exterior fields, for exterior to interior coupling calculations.

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It is clear from Equation (1) that if the sphere is charged symmetrically, q_1 equals q_2, and no charge leaks inside; the internal fields are then zero. If only one-half the system is charged, that is, q_2 is zero, then

$$.193 < \frac{q_{1I}}{q_1} < .250 \quad . \tag{3}$$

(It is evident from Equation (1) that the ratio of q_{1I} to q_1, with q_2 equal to zero, cannot be greater than .25 no matter how large β is.) That is, for extreme differential charging ($q_2 = 0$) about 20 percent of the charge gets inside the cavity. The electrostatic field on the inside of the cavity is roughly of the order of magnitude of the fields on the surface. (This is because the right-hand inside hemisphere charges to $-q_{1I}$—see Appendix II for proof.) In a dynamic problem, the inside and outside charges distribute themselves over roughly the same time interval. In order to maintain charge neutrality (when $q_2 = 0$), the charge on the outside of the right hemisphere is equal to q_{1I}. Inside and outside surface currents are then roughly comparable, at least on certain major portions of the structure.

The effect of the right hemisphere actually enhances the internal field in the case that $q_2 = 0$. Appendix II, Equation (II.12), shows that if the right hemisphere were missing q_{1I} exactly equals $.1945 \times q_1$, almost the same amount that exists if the right hemisphere is juxtaposed.

The above discussion suggests that differential charging of an isolated conducting section can, in a particular geometry, yield internal fields and currents of a magnitude which is comparable with the external fields and currents. The cause of these fields comes from charges flowing into the interior. The reason only about 20 percent of the total charge appears on the internal surface of a hemisphere, as Appendix II demonstrates, is that the curvature "shields" the internal surface. If the isolated

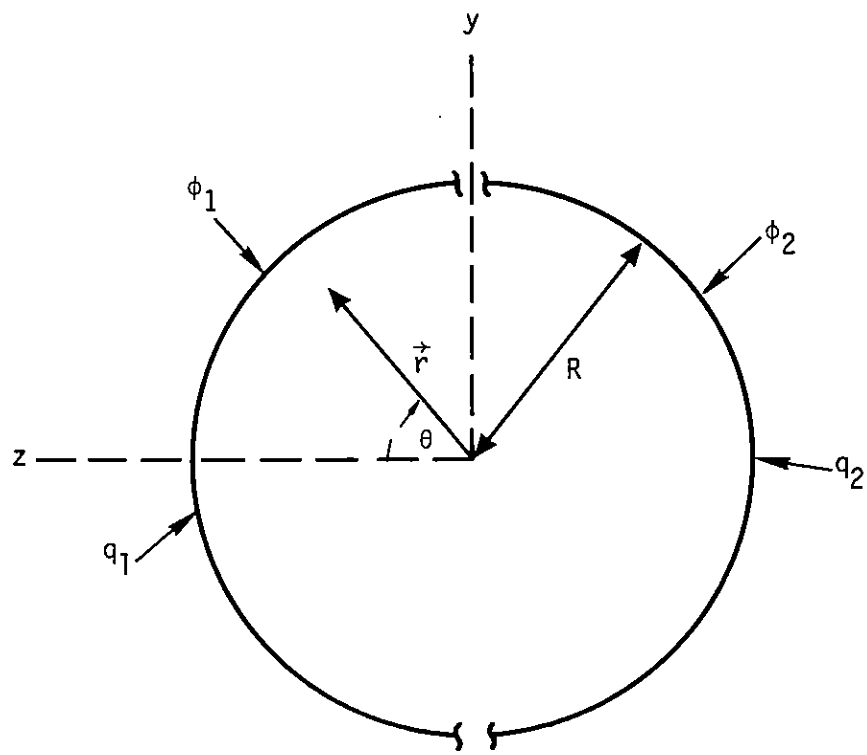

Figure 1. Cross section through x,y plane of two juxtaposed hemispheres.

infinitesimal, one-half of the system is electrically isolated from the other. We imagine that this system has been exposed to X-rays so that a charge q_1 remains on the left hemisphere and a charge q_2 on the right hemisphere. The charge on the inside of the left hemisphere q_{1I} is, from equation (I.20) of Appendix I,

$$q_{1I} = \frac{2\beta(q_1 - q_2)}{1 + 8\beta} \quad , \tag{1}$$

where β is expressed as an infinite series by Equation (I.19). Summing six terms in this series shows

$$\beta < .424 \quad . \tag{2}$$

type of solution can only be effectively done with a computer code. However, various levels of analytical approximations exist. For example, by knowing the total amount of residual charge on the conducting surfaces of Figure 1 the final electrostatic state of the system was found. By knowing the approximate time variations of the sources the skin currents can be approximated. If the time variation of a significant portion of the source is close to the lowest internal mode, then the fields associated with the oscillation could be about the order of magnitude of the electrostatic fields caused by that portion of the source.

If the internal modes of the system are known (assuming one can speak of internal modes decoupled from the exterior) and an exact or approximate quasi-static solution V exists, then the fields associated with the various modes can be approximated more accurately than the preceding paragraph by utilizing the Coulomb gauge[8]. Expressing the modes of the system by the orthonormal vector potential components $\vec{A}_{k\ell n}$ the amplitudes of these modes can be obtained by solving the equation

$$\nabla \times \nabla \times \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = - \frac{1}{c} \vec{\nabla} \frac{\partial V}{\partial t} , \tag{4}$$

where

$$\vec{A} = \sum_{k\ell n} d_{k\ell n}(t) \vec{A}_{k\ell n} , \tag{5}$$

and the coefficients $d_{k\ell n}(t)$ involve only time. The oscillatory electric and magnetic fields $\vec{E}_{k\ell n}$ and $\vec{B}_{k\ell n}$ are given by

$$\left.\begin{aligned} \vec{E}_{k\ell n} &= + \frac{i}{c} \omega_{k\ell n} \vec{A}_{k\ell n} , \\ \vec{B}_{k\ell n} &= - \vec{\nabla} \times \vec{A}_{k\ell n} , \end{aligned}\right\} \tag{6}$$

k, ℓ and n represent modal indices; $\omega_{k\ell n}$ is the angular frequency associated with the mode. For many cavities the orthonormal and complete set of modes panel has no curvature, it is likely that even more charge will flow inside. Consider Figure 2. Due to symmetry, if a charge q is placed on the exterior surface of the isolated panel, the equilibrium configuration will have a charge q/2 on both the inside and the outside of the panel. For such a special configuration, the internal fields in some portion of the interior will be of the order of magnitude of the fields on the exterior surface.

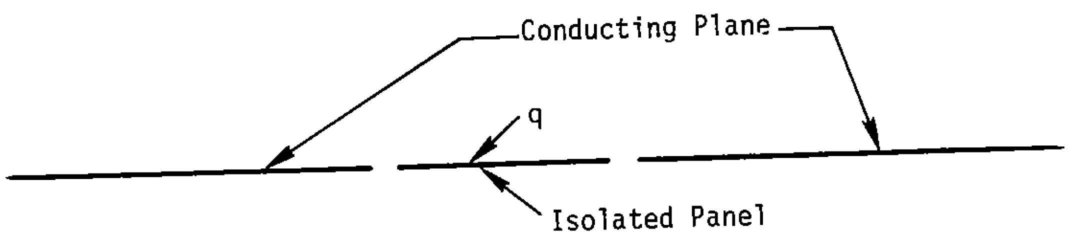

Figure 2. Cross section of an isolated circular conductor within a conducting plane.

MODAL STIMULATION FROM ISOLATED PANELS

If the sources of the fields — the current distribution of electrons exterior to the satellite in an SGEMP problem — have a time variation which is comparable to, or shorter than, the longest period of oscillation of the system, modal oscillation will be stimulated. These oscillations make a significant contribution to the internal fields if they are not severely damped. The goal of this section will be to try to estimate these oscillatory fields in a simple way.

The actual mathematical solution to an SGEMP problem is very complicated and involves the simultaneous and consequently nonlinear solution of Maxwell's equation with the equations of motion of the electrons. This

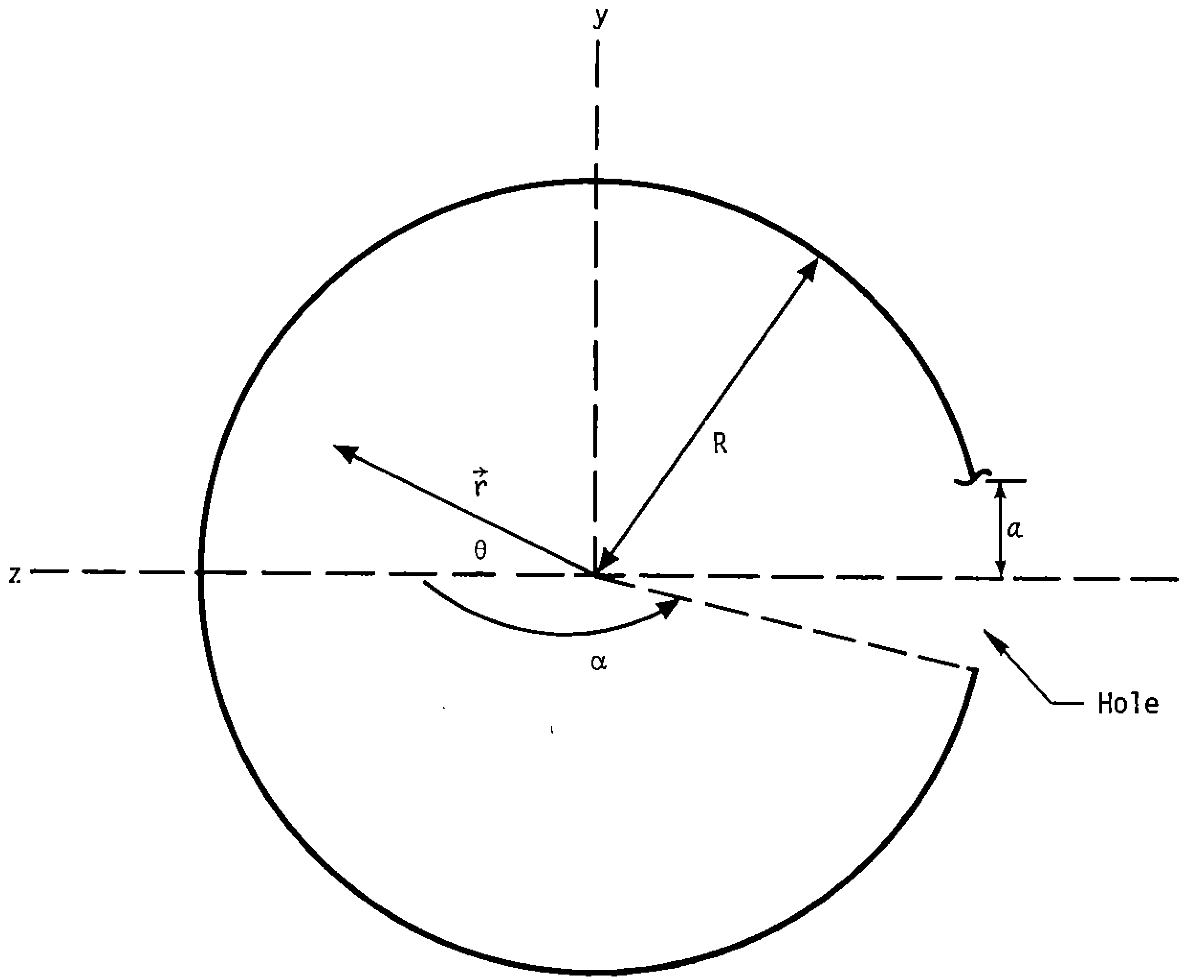

Figure 3. Cross section, through the x,y plane, of a conducting sphere with circular hole in it.

$\vec{E}_{k\ell n}$ and $\vec{B}_{k\ell n}$ are known. In practice finding the $\vec{A}_{k\ell n}$ components amounts to solving Equations (6) with the known $\vec{E}_{k\ell n}$ and $\vec{B}_{k\ell n}$. Once an orthonormal and complete set of $\vec{A}_{k\ell n}$ are found the solution of Equation (4) proceeds by utilizing the orthonormal properties in the usual way to find a second order differential equation for $d_{k\ell n}(t)$. This differential equation is then solved, again in the usual way, by Laplace transforms. The approximation expressed by Equation (4) is most accurate when the problem is approximately quasi-static. When the exterior problem is approximately quasi-static due to radiative damping, the accuracy of the approximation should be reasonably good.

The internal dynamic problem can be exactly calculated if electric fields parallel to the boundaries are known. In general, specifying the electric field parallel to the boundaries is not possible without the simultaneous solution of both the exterior and interior problem. An approximate approach for computer solutions will be discussed in Section 3. This approach decouples the interior problem from the exterior problem by specifying the interior boundary condition by modifying the exterior solution.

It should not be overlooked that many of the problems associated with isolated sections could be avoided if the panels were connected to the main body by conducting material or rotating portions of the system were conductively linked to non-rotating portions by brushes. This effectively makes the isolated panel problem a slot problem.

STATIC FIELD OF A HOLE

We wish to calculate the internal field due to a conducting satellite with a hole cut in it, after a charge q has been left on the satellite. The static solution will suggest the form the dynamic calculation will take. If we approximate the satellite by a sphere of radius R and the hole by disk of radius a (see Figure 3), then, from Appendix III Equations III.15 and III.16, the potential $V(r, \theta)$ at any point r, θ, (the problem is azimuthally symmetric) inside the sphere far from the hole, is

$$C = \frac{R}{\pi}(\pi - \beta + \sin\beta) \tag{11}$$

if β is small then we can expand $\sin\beta$ in Equation (11) obtaining

$$C \cong R\left(1 - \frac{\beta^3}{6\pi}\right) \quad . \tag{12}$$

By the definition of capacitance, the charge, q, on the sphere is

$$q \cong RV_o\left(1 - \frac{\beta^3}{6\pi}\right) \quad . \tag{13}$$

Thus, to a first approximation, the charge inside the sphere ($q_o \equiv RV_o = q$, if no hole exists) is about $q_o\beta^3/6\pi$. The field inside then is of the order of magnitude of $q_o\beta^3/6\pi R^2$. If the external field, E_o, is equal to q_o/R^2 and β is approximated by a/R, the internal field is of the order of $E_o a^3/6\pi R^3$ or the equivalent to a dipole of magnitude $E_o a^3/6\pi$.

MODAL STIMULATION INSIDE A CAVITY DUE TO A HOLE

Bethe[6] has constructed a solution for radiation penetrating a circular aperture in an infinite plane. His solution is valid in the limit that

$$\left(\frac{a}{\lambda}\right)^2 \ll 1 \quad , \tag{14}$$

where λ is the wavelength of the radiation. The fields due to the hole, at distances far from the hole, are equivalent to an electric and magnetic dipole, P_E and P_M, respectively, given by the expressions

$$P_E = \frac{1}{3\pi} E_o a^3 \quad , \tag{15}$$

$$V(r, \theta) = V_o + \frac{P}{2R} \frac{(r^2 - R^2)}{(R^2 + r^2 - 2rR \cos \theta)^{3/2}} + O[(a/R)^5] \ , \ r > a \quad (7)$$

where

$$P \equiv \frac{1}{3\pi} V_o R^2 \left(\frac{a}{R}\right)^3 \quad , \qquad (8)$$

and V_o is the potential of the sphere with the hole in it. Equation (7) expresses the fact that the internal field is that due to a dipole P. The dipole is formed by bringing a negative charge up to the surface of the sphere. The negative charge and its positive image form the dipole. To a first approximation, the potential V_o is just RE_0; where E_o is the field at the hole in the absence of the hole. Substituting for V_o in Equation (8) we obtain the expression for the dipole moment:

$$P = \frac{a^3}{3\pi} E_o \quad . \qquad (9)$$

A Green's theorem argument and intuition strongly suggest the following method of calculating the internal static field due to a hole: if the radius of curvature, R, of a system is greater than the radius a of the hole to the extent that

$$\left(\frac{a}{R}\right)^2 << 1 \qquad (10)$$

then the internal field is that due to a dipole. The dipole moment is given by Equation (9) where E_o is the field at the hole when the hole is not there. The dipole is formed by bringing a negative charge up to the surface and forming a dipole with its image. Constructing the dipole in this way ensures that the boundary conditions at the surface of the conductor are satisfied. This method of construction is equivalent to finding the internal Green's function for the body and taking its normal derivative at the surface.

We can make some qualitative arguments to find the order of magnitude of the internal fields. The capacitance[5], C, of a sphere whose hole subtends on angle β is rise to a large electric field at the surface—call this stage one. Large numbers of electrons return to the surface due to the high value of the electric field giving rise to a relatively quick drop (of the order of 10 to 50 percent) in the electric field value—call this stage two. The electric field maintains itself at a high value preventing most of the electrons from going more than the space charge distance from the surface (of the order of a centimeter); electrons which got to large distances from the surface during stage one slowly return to the surface—call this stage three.

We first assume that no X rays get through the hole. From Section 2—Static Field of a Hole—the charge (not electrons) that gets into the interior through the hole is of the order of $(\sigma A)\ a/R$, where A is the area of the hole and σ is the charge density on the surface. During stage one, when no electrons go through the hole, the dipole field of the hole dominates the interior fields (see Section 2—Static Field of a Hole and Modal Stimulation Inside a Cavity Due to a Hole). During stages two and three charge of the order of σ returns to the surface (per unit area). Therefore charge of the order of σA enters the hole. Since a/R is assumed small the electronic charge entering the hole is probably the dominant effect during these latter two stages. If the radius of the hole is of the order of magnitude of the space charge distance, probably far more charge than σA will get into the hole.

X rays directly entering a hole perpendicular to the wave front will cause a great many electrons to be ejected into the cavity when they strike the cavity walls. It is unlikely that these electrons will be as severely space charge limited as the exterior electrons if they strike a conductor. It seems possible to get, over a time equal to the pulse length of the X ray, many more electrons backscattered off the interior surfaces than coming through a hole whose radius is large compared to the space charge thickness. (If the X rays strike an interior dielectric surface, space charge limiting may be more severe than for X rays striking a conductor; positive charge and

$$P_M = \frac{2}{3\pi} B_0 a^3 \quad . \qquad (16)$$

Here E_0 and B_0 are the electric and magnetic fields in the absence of the hole. Note that Equation (15) is exactly Equation (9). E_0 and B_0 in Equation (15) and (16) are time varying but consistent with Equation (14). Given any surface S which satisfies Equation (10), where R is the radius of curvature, and where Equation (15) is satisfied the generalization is clear: the internal fields are due to an electric and magnetic dipole, whose magnitudes are given by Equations (15) and (16) and whose fields, on the inside of the cavity, satisfy the boundary conditions on S (with the hole absent):

$$\vec{E} \times n = 0 \ , \qquad (17)$$

$$\vec{B} \cdot n = 0 \ . \qquad (18)$$

An electromagnetic formulation which satisfies the boundary conditions and which should give an accurate approximation to the modal stimulation is the Coulomb gauge mentioned in Section 2—Modal Stimulation from Isolated Panels. It is necessary that the electrostatic and magnetostatic Green's functions together with the first few modes of the cavity (without the hole) be known. It is also necessary that the external fields at the hole in the absence of the hole be known. The oscillating field due to an electric dipole (at the position of the hole) can be found by substituting the electrostatic potential (constructed by utilizing the electrostatic Green's function) of the electric dipole for V in Equation (4). Constructing the formulation equivalent to the Coulomb gauge for magnetic charge would yield the modal stimulation due to the magnetic dipole.

A HOLE NEAR THE SOURCE REGION

Since we have an idea of the magnitude of the electrostatic fields in a cavity due to a hole we can take a stab at comparing these fields with those caused by electrons or X rays entering the cavity through the hole. We consider only the circumstances of high space charge limiting. The electron scenario is proposed to be the following: just after high fluence X rays strike the surface, large numbers of electrons leave the surface giving

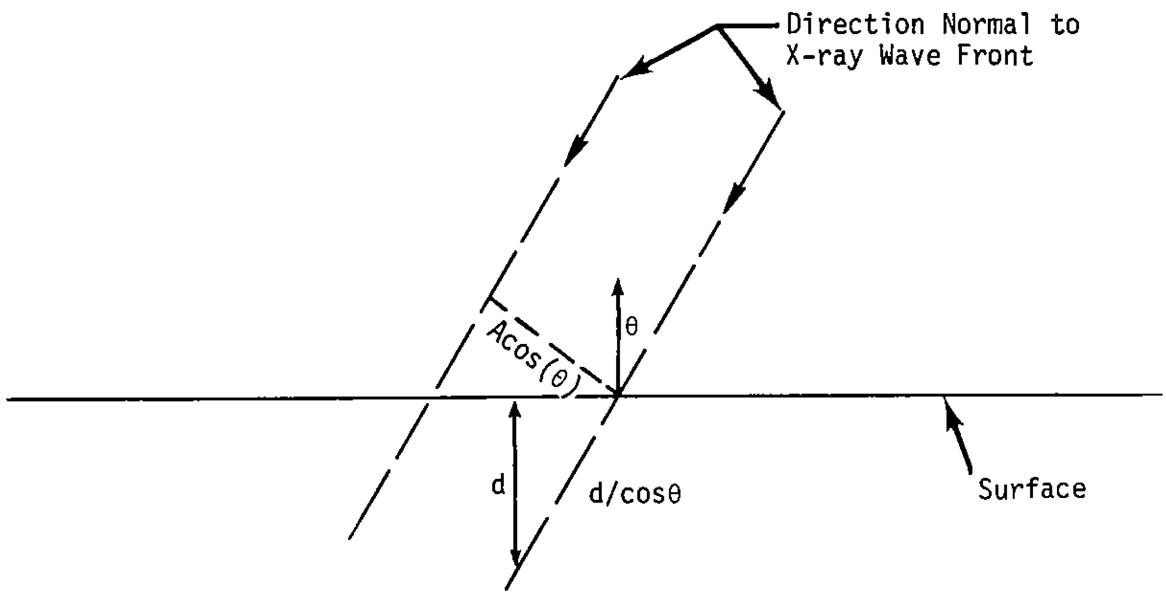

Figure 4a. X rays striking a surface.

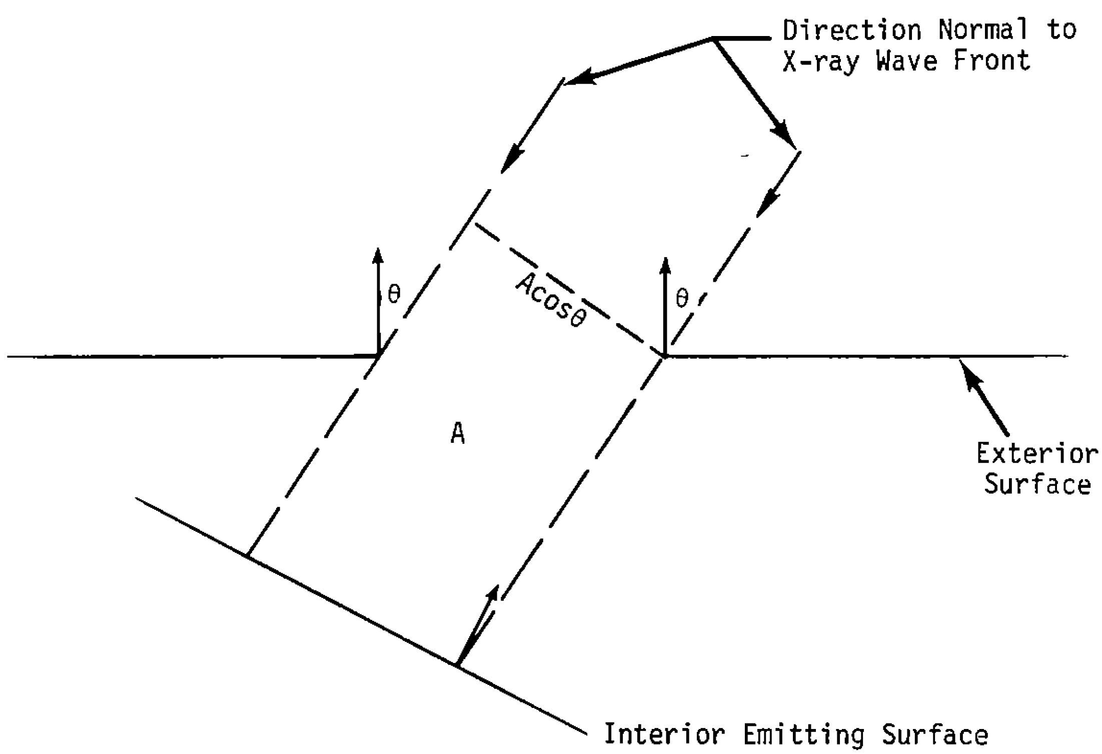

Figure 4b. X rays passing through a hole of area A.

will remain on the dielectric rather than being conducted away. Whether the positive charge is conducted away rapidly enough to reduce space charge limiting significantly depends upon the pulse length of the X rays and the magnitude of the electron flux.) How many more electrons will be emitted into the interior will depend upon the geometry, the surface involved, and the fluence.

At glancing angles of incidence with respect to the normal to the plane of the hole the X rays will be less effective in producing electrons in the cavity. If θ is the angle the normal to a surface makes with the direction of the X-ray wavefront (see Figure 4.a) and I is the energy flux, then the energy deposited per unit area per unit time is $I\cos\theta$. The number of electrons emitted from the surface is proportional to the number of atoms in the path length of the photon within an electron free path length, d, from the surface. This length is equal to $d/\cos\theta$. Even at glancing incidence the number of electrons emitted per unit area from a surface is then proportional to Id, since the number of electrons emitted per unit area is proportional both to $I\cos\theta$ and $d/\cos\theta$. It is unlikely that X rays entering a hole of area A at glancing incidence will also strike an interior surface at the same angle of glancing incidence (see Figure 4.b). The number of electrons emitted inside the cavity will then be more nearly proportional to the total energy entering the hole, $IA\cos\theta$, since the $\cos\theta$ no longer cancels out on the average. At some angle θ the electrons entering the hole due to space charge limiting will dominate those arising from emission within the cavity. This angle will probably be close to 90° (example: $\cos 85° \sim .1$).

Suppose now that X rays penetrate into the cavity of interest through the walls and the electron production of the attenuated fluence is reduced by a factor γ from the exterior electron production. Since the area that the X rays fall upon in the cavity is of the order of R^2, in order for the "hole" processes to compete effectively with the electrons produced by X rays penetrating the cavity one would expect ($A \sim a^2$) that

SECTION 3

COMPUTER AIDED SOLUTIONS

In Section 2—Modal Stimulation Inside a Cavity Due to a Hole—it was suggested that the fields inside a cavity due to a hole can be found by a knowledge of the fields at the hole if the hole were absent. Fields at the surface of satellites are exactly the output of dynamic SGEMP code calculations. Internal fields due to a hole in the skin of a satellite can be calculated in two ways using the SGEMP output: either an internal finite difference Maxwell equation code, which can model a complicated interior is written, or the Green's function and a few internal modes for the interior are found.

If the satellite consists of an electrically isolated panel it may be possible, by correctly choosing the capacitance of the gap between the panel and the main body, to obtain the electric field in the gap from SGEMP codes. Finding the field in the gap would constitute a boundary condition for the internal problem. The internal problem could then be solved by means of a computer code or analytical techniques. The computer code SEMP has the capability of attributing to a given set of surface points a capacitance, inductance and resistance. This technique has been quite successful with capacitance between two solid objects but, as yet, no problems have been run with parameters representing a cavity.

$$\frac{1}{\gamma}\left(\frac{a}{R}\right)^2 > 1 \ , \tag{19}$$

at the least. Under some circumstances—particularly for low energy photons—Equation (19) may be satisfied. It is more likely, however, that internal fields caused by an exposed isolated section will compete with those due to X rays penetrating the cavity walls. An example of this phenomena can be found in Reference 1.

It should be noted that the frequency content of the electron currents due to X rays penetrating cavity walls, will be roughly that of the X-ray time history; little space charge limiting will occur. Internal fields arising from electrons entering through the hole or arising from dipole fields will contain higher frequency components. If cavity mode stimulation is important, in certain circumstances, the restrictions of Equation (19) will be relaxed.

Knowing the general material contents of a cavity, the composition of its walls and general geometrical shapes could aid the development of relevant mathematical tools.

SECTION 4

SUMMARY AND SUGGESTION FOR THE FUTURE

By solving some tractable mathematical problems order of magnitude estimates and general techniques have been suggested for analyzing the effect of circular holes and slots which electrically isolate one section of a conducting body from another. If the hole is not circular, and the slot is not long enough to separate one section from another, an estimate of its effect can probably be made by approximating the aperture by a circular hole of equivalent area. This suggestion along with some of the techniques suggested in the report may require a deeper mathematical investigation as proof.

Where a hole is in the source region, electrons pulled back into the hole and X rays passing through the hole also constitute an electromagnetic threat. Although it appears that the X-ray fluence passing through a cavity wall would have to be considerably reduced in order for the "hole" processes to compete with it electrostatically, some of the "hole" processes may be more efficient in stimulating cavity modes. Internal fields due to strong differential charging of isolated sections of a spacecraft are the greatest surface discontinuity threat.

The extent to which surface discontinuities present a problem will depend on the particular system. Nonetheless it would be useful if a systematic study could be made of general surface discontinuities and cavity shapes for a number of representative systems. Analysis of the effects of discontinuities rests, apart from computer codes, on a knowledge of Green's functions and modes. A modal analysis is relevant if the modes are not highly damped.

APPENDIX I

ELECTROSTATIC FIELD OF TWO JUXTAPOSED CONDUCTING HEMISPHERES

We consider the geometry of Figure 1 and, assuming that a charge has been placed on each hemisphere, we wish to find what percent of that charge has leaked inside. Both hemispheres are conducting although insulated from each other. We consider the problem electrostatically and so solve Laplace's equation both inside and outside the sphere. The potential inside the sphere, $V_I(r, \theta)$, is given by the expression

$$V_I(r, \theta) = \sum_{n=0}^{\infty} a_n \left(\frac{r}{R}\right)^n P_n(\cos\theta) \quad , \qquad (I.1)$$

where r and θ are the spherical coordinates defined in Figure 1 (the problem is independent of the azimuthal angle), P_n are Legendre polynomials, R is the radius of the sphere and a_n are expansion coefficients. Letting a subscript "1" designate the left hand side and a subscript "2" the right hand side, the potential on the left hemisphere, ϕ_1, and the potential on the right hemisphere, ϕ_2, are written:

$$\phi_1 = \sum_{n=0}^{\infty} a_n P_n(\cos\theta) \qquad 0 \le \theta \le \pi/2 \ , \qquad (I.2)$$

$$\phi_2 = \sum_{n=0}^{\infty} a_n P_n(\cos\theta) \qquad \frac{\pi}{2} \le \theta \le \pi \ . \qquad (I.3)$$

Using the orthogonality properties of P_n, and the fact that

$$\int_{-1}^{0} P_n(x)dx = (-1)^n \int_{0}^{1} P_n(x)dx \ , \qquad (I.4)$$

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The above references were used in the research reported. A number of other relevant reports have just recently been brought to my attention by D. F. Higgins. These appear in the later contributions to the AFWL Interaction Note series. Of particular relevance are Note 170 by C. E. Baum and the Note 45 by L. W. Chen, referred to in the former note.

$$\frac{q_{10}}{R^2} = \frac{\phi_1 + \phi_2}{4R} + \frac{(\phi_1 - \phi_2)}{4R} \sum_{\substack{n=1 \\ odd}} (2n^2 + 3n + 1)\left[\int_0^1 P_n(x)\right]^2 . \qquad (I.10)$$

It is clear from Equations (I.9) and (I.10) that

$$\frac{q_{10}}{R^2} = + \frac{\phi_1 + \phi_2}{4R} \frac{q_{1I}}{R^2} \frac{(\phi_1 - \phi_2)}{4R} \sum_{\substack{n=1 \\ odd}} (2n + 1)\left[\int_0^1 P_n(x)\right]^2 . \qquad (I.11)$$

By integrating Equation (I.1) from θ equal zero to $\pi/2$, at r equal to R, we find, using Equation (I.5) that

$$1 = \sum_{n=1}^{\infty} (2n + 1)\left[\int_0^1 P(x)dx\right]^2 . \qquad (I.12)$$

Substituting Equation (I.12) into Equation (I.11) we see that

$$q_{10} - q_{1I} = \frac{\phi_1 R}{2} . \qquad (I.13)$$

If q_1 is the total charge on the left hemisphere then

$$q_{10} + q_{1I} = q_1 . \qquad (I.14)$$

Denoting the charge on the inside and outside of the right hand hemisphere by q_{2I} and q_{20} respectively we know that

$$q_{20} + q_{2I} = q_2 , \qquad (I.15)$$

where q_2 is the total charge on the right hand hemisphere. By integrating the charge density on the inside of the sphere from θ equals zero to π we find that

$$q_{1I} + q_{2I} = 0 . \qquad (I\text{-}16)$$

By integrating the charge density on the outside over the whole θ range we find that we find

$$a_n = \frac{2n+1}{2}\,(\phi_1 + (-1)^n\phi_2)\int_0^1 P_n(x)dx \ . \qquad \text{(I.5)}$$

Since the integral on the right hand side of Equation (I.5) vanishes for even n, a_n exists only for odd integers and zero. This checks with the fact that if ϕ_1 equals ϕ_2 the potential inside the sphere is a constant ϕ_1.

The potential outside the sphere V_0 can be written as

$$V_0(r,\ \theta) = \sum_{n=0}^{\infty}\left(\frac{R}{r}\right)^{n+1} b_n P_n(\cos\theta) \ . \qquad \text{(I.6)}$$

Analogously we find

$$b_n = \frac{2n+1}{2}\,(\phi_1 + (-1)^n\phi_2)\int_0^1 P_n(x)dx \ , \qquad \text{(I.7)}$$

$$b_n = 0 \quad , \quad n \text{ odd} \ .$$

The total charge q on a conducting surface is given by the expression

$$q = \frac{-1}{4\pi}\int_S (\vec{\nabla}V)\cdot\hat{n}\,dS \ , \qquad \text{(I.8)}$$

where $\hat{n}$ is the normal to the surface S and V is the electrostatic potential. Substituting expressions (I.1) and (I.6) into (I.8) while using (I.5) and (I.7) we find the charges on the inside of the left hemisphere q_{1I} and outside of the left hemisphere, q_{1O};

$$\frac{q_{1I}}{R^2} = + \frac{(\phi_1 - \phi_2)}{4R}\sum_{\substack{n=1\\ \text{odd}}} \frac{2n^2 + n}{n}\left[\int_0^1 P_n(x)\right]^2 \ , \qquad \text{(I.9)}$$

APPENDIX II

ELECTROSTATIC FIELD OF ONE CONDUCTING HEMISPHERE

We consider the effect of the right hand hemisphere on the left hand hemisphere in Figure 1. The right hand hemisphere is removed and a charge q_1 is placed on the left hand hemisphere. The solution for the problem can be obtained by a dual series approach.[5] The potential V is given by

$$V(r, \theta) = \frac{\phi_1}{\pi}(\pi/2 + 1) + \frac{\phi_1}{\pi}\sum_{\substack{n=1\\ \text{odd}}} \frac{(-1)^{\frac{n-1}{2}}}{n}\left(\frac{r}{R}\right)^n P_n$$

$$+ \frac{\phi_1}{\pi}\sum_{\substack{n=2\\ \text{even}}} \frac{(-1)^{n/2}}{(n + 1)}\left(\frac{r}{R}\right)^n P_n \qquad r < R \qquad \text{(II.1)}$$

$$V(r, \theta) = \frac{\phi_1}{\pi}(\pi/2 + 1)\frac{R}{r} + \frac{\phi_1}{\pi}\sum_{\substack{n=1\\ \text{odd}}} \frac{(-1)^{\frac{n-1}{2}}}{n}\left(\frac{R}{r}\right)^{n+1} P_n$$

$$+ \frac{\phi_1}{\pi}\sum_{\substack{n=2\\ \text{even}}} \frac{(-1)^{n/2}}{n + 1}\left(\frac{R}{r}\right)^{n+1} P_n \qquad r > R \qquad \text{(II.2)}$$

where in Equations (II.1) and (II.2) the variables have the same definition as in Appendix I. We find by a procedure similar to Appendix I that

$$q_1 + q_2 = \frac{\phi_1 + \phi_2}{2} R \ . \qquad \text{(I.17)}$$

Equations (I.13) - (I.17) together with equation (I.9) are six equations for the six unknowns q_{10}, q_{1I}, q_{20}, ϕ_1 and ϕ_2. From Equation (I.9) we will find

$$q_{1I} = R(\phi_1 - \phi_2)\beta \ , \qquad \text{(I.18)}$$

where

$$\beta \equiv \frac{1}{4} \sum_{n=1}^{\infty} 2n^2 + n\left[\int_0^1 P_n(x)dx\right]^2 \ , \qquad \text{(I.19)}$$

and β is positive. Solving the system of equations by simple algebraic manipulations we find that

$$q_{1I} = \frac{2\beta(q_1 - q_2)}{1 + 8\beta} \ , \qquad \text{(I.20)}$$

$$q_{10} = \frac{q_1(1 + 6\beta) + 2\beta q_2}{1 + 8\beta} \ , \qquad \text{(I.21)}$$

and

$$\phi_1 = \frac{2}{R}\left(\frac{q_1(1 + 4\beta) + 4\beta q_2}{1 + 8\beta}\right) , \qquad \text{(I.22)}$$

The expressions for q_{2I}, q_{20} and ϕ_2 can be found by substituting q_2 for q_1 and q_1 for q_2 in Equations (I.20) - (I.21).

We now find a numerical value for β. Noting that[4]

$$\int_0^1 P_n(x)dx = \frac{(2\ell - 1)!!(-1)^\ell}{2^{\ell+1}(\ell + 1)!} \ , \quad n = 2\ell + 1 \ . \qquad \text{(I.23)}$$

We see, summing the series in Equation (I.19) from $n = 1$ to $n = 11$, that

$$\beta < .4238 \ . \qquad \text{(I.24)}$$

$$q_{01} + q_{I1} = q_1 \ . \qquad \text{(II.10)}$$

Solving (II.10) we have

$$q_{01} = q_1 \frac{(\pi + 1)}{\pi + 2} \ . \qquad \text{(II.11)}$$

$$q_{I1} = q_1 \frac{1}{\pi + 2} \cong .1945 q_1 \ . \qquad \text{(II.12)}$$

$$q_{01} = + \frac{\phi_1}{2\pi} (\pi/2 + 1)R + \frac{\phi_1}{2\pi} R \sum_{\substack{n=1 \\ odd}} (-1)^{\frac{n-1}{2}} (1 + 1/n) \int_0^1 P_n dx \qquad \text{(II.3)}$$

and

$$q_{I1} = + \frac{\phi_1}{2\pi} R \sum_{\substack{n=1 \\ odd}} (-1)^{\frac{n-1}{2}} P_n \ , \qquad \text{(II.4)}$$

combining II.3 and II.4 we find

$$q_{01} = \frac{\phi_1}{2\pi} R(\pi/2 + 1) + q_{I1} + \frac{\phi_1}{2\pi} R \sum_{\substack{n=1 \\ odd}} (-1)^{\frac{n-1}{2}} \frac{1}{n} \int_0^1 P_n dx \ . \qquad \text{(II.5)}$$

Integrating Equation (II.2) over the hemisphere we have

$$\phi_1 = \frac{\phi_1}{\pi} (\pi/2 + 1) + \frac{\phi_1}{\pi} \sum_{\substack{n=1 \\ odd}} \frac{(-1)^{\frac{n-1}{2}}}{n} \int_0^1 P_n dx \ . \qquad \text{(II.6)}$$

Substituting Equation (II.6) into (II.5) we find

$$q_{01} = q_{I1} + \frac{R}{2} \phi_1 \ . \qquad \text{(II.7)}$$

Integration of the radial derivative of Equation (II.2) over the entire sphere we find

$$R\phi_1 = \frac{\pi q_1}{\pi/2 + 1} \ . \qquad \text{(II.8)}$$

Substituting Equation (II.8) into Equation (II.7) we have

$$q_{01} - q_{I1} = \frac{q_1}{1 + 2/\pi} \ . \qquad \text{(II.9)}$$

We also know that the total change on the hemisphere q_1 is given by

APPENDIX III

ELECTROSTATIC FIELD OF A SPHERE WITH A SMALL CIRCULAR HOLE

In this appendix we will demonstrate that the electrostatic field of a small circular hole in a sphere is equivalent, in the first approximation, to a dipole field. The potential, V, of a spherical cap can be solved by a dual series approach[5]. In spherical coordinates, for points inside the sphere the solution (see Figure 3, page 11, for a definition of the variables) is:

$$V = \sum_{n=0}^{\infty} a_n r^n P_n(\cos\theta) \quad , \tag{III.1}$$

where

$$a_0 = \frac{V_0}{\pi} (\alpha + \sin\alpha) \quad , \tag{III.2}$$

$$a_n = \frac{V_0}{\pi} R^{-n}\left(\frac{\sin(n+1)\alpha}{n+1} + \frac{\sin n\alpha}{n}\right) \quad , \tag{III.3}$$

V_0 is the potential of the sphere while P_n are Legendre polynomials. Making the transformation

$$\alpha = \pi - \delta \quad , \tag{III.4}$$

we can expand a_n in a power series in n:

$$a_n = \frac{V_0}{R^n\pi}\left((-1)^{n+1} \frac{2n+1}{3!} \delta^3 + \frac{(-1)^n}{5!} (4n^3 + 6n^2 + 4n + 1) + ..\right) \tag{III.5}$$

and

If a charge $-q$ is placed a distance "d" from the inside surface of a sphere the internal potential ϕ, to order d^2, can easily be shown to be

$$\phi = -\frac{2qd}{2R}\frac{(R^2 - r^2)}{(R^2 + r^2 - 2rR\cos\theta')^{3/2}} . \qquad \text{(III.15)}$$

Let d approach zero and q approach infinity in such a way that

$$-2qd \rightarrow P , \qquad \text{(III.16)}$$

where P is a dipole moment opposite in direction to the surface normal. Then Equation (III.15) becomes the potential of a dipole. From Equation (III.14) it is clear that the hole is equivalent to a dipole P whose magnitude is given by

$$P = \frac{1}{3\pi} V_0 R^2 \delta^3 . \qquad \text{(III.17)}$$

(Since δ is small it can be approximated by a/R.)

$$a_0 = \frac{V_0}{\pi} \left(\pi - \frac{\delta^3}{3!} + \frac{\delta^5}{5!} + \ldots\right) . \qquad \text{(III.6)}$$

Letting

$$\theta' = \pi - \theta , \qquad \text{(III.7)}$$

and noting that

$$P_n(-\cos\theta') = (-1)^n P_n(\cos\theta') , \qquad \text{(III.8)}$$

we find to order δ^3 that

$$V = V_0 - \frac{\delta^3 V_0}{3!\pi} \sum_{n=0}^{\infty} (r/R)^n (2n + 1) P_n(\cos\theta') . \qquad \text{(III.9)}$$

The error in Equation (III.9) is small where the field point is not near the hole and where

$$\delta^2 < 1 . \qquad \text{(III.10)}$$

We know, from the Legendre polynomial generating function, that

$$(r^2 + R^2 - 2rR\cos\theta')^{-1/2} = \frac{1}{R} \sum_{n=0}^{\infty} \left(\frac{r}{R}\right)^n P_n(\cos\theta') , \qquad \text{(III.11)}$$

for

$$R > r . \qquad \text{(III.12)}$$

Taking the derivative of Equation III.11 with respect to r we find that

$$\frac{r(R\cos\theta' - r)}{(R^2 + r^2 - 2r\,R\cos\theta')^{3/2}} = \frac{1}{R} \sum_{n=0}^{\infty} n\left(\frac{r}{R}\right)^n P_n(\cos\theta') . \qquad \text{(III.13)}$$

Substituting Equation (III.11) and (III.13) into Equation (III.9) yields

$$V = V_0 - \frac{\delta^3 V_0 R}{6\pi} \left(\frac{R^2 - r^2}{(R^2 + r^2 - 2rR\cos\theta')^{3/2}}\right) . \qquad \text{(III.14)}$$