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A UNIVERSAL IMPEDANCE FOR SOILS

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1. INTRODUCTION

Several years ago Scott[1] reported the results of measurements of the dielectric constant ε and the conductivity σ of many samples of soils, over the frequency range 10^2 to 10^6 Hz. He noted that the results for the many samples could be correlated quite well in terms of just one parameter, the water content. By averaging his data, he produced a set of curves $\varepsilon(f)$ and $\sigma(f)$ as functions of frequency f for various values of water content. These curves are reproduced in Figures 1 and 2. Thus if one knows the water content of a soil, one can predict what its ε and σ will be with generally useful accuracy. Or, if one measures ε or σ at some frequency, one can use Scott's curves to predict ε and σ over the frequency range 10^2 to 10^6 Hz.

In 1971, the present author proposed a time-domain method for solving Maxwell's equations in dispersive soils, based on the assumption that each volume element of the soil could be represented by an RC network. He and Longley[2] worked out the parameters for the network by fitting Scott's averaged curves. A consequence of the RC network model is that the variation of ε and σ with frequency are not independent. In fact if the variation of ε is given, the variation of σ is determined, or vice versa; only a constant remains to be chosen freely in either case. Scott's curves obey this relation rather well.

The author also noticed the fact that all of Scott's curves for $\varepsilon(f)$ will very nearly coincide with each other if displaced to the right or left, i.e., that there is just one curve for $\varepsilon(f/f_0)$, where f_0 scales with water content. In terms of the RC network, this means that as the water content is varied, only the R values change, while the C values remain fixed, as we shall see below. However, this fact was not used in Reference 2.

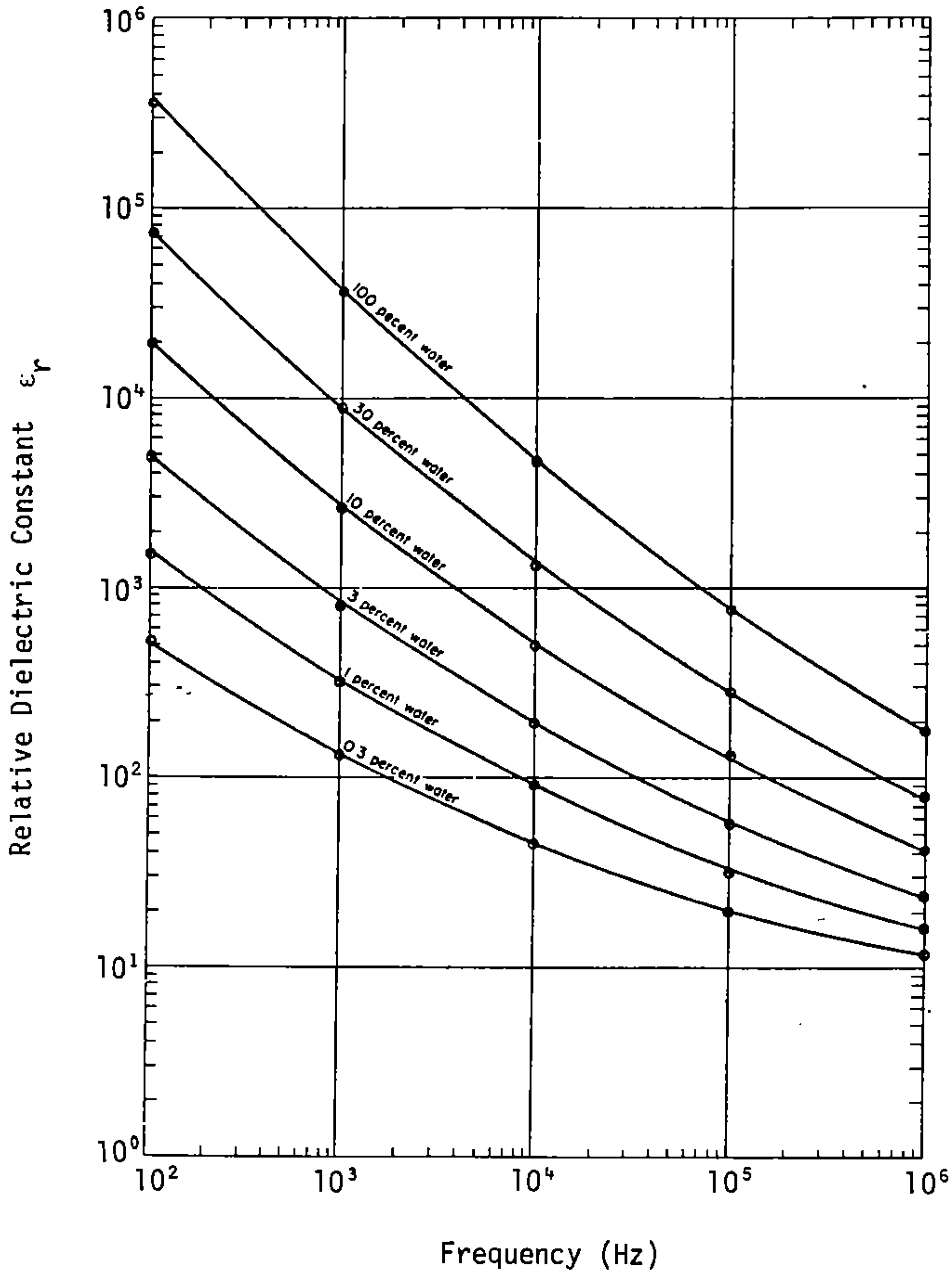

Figure 1. Relative dielectric constant of soils. Solid curves are Scott's fits to results of measurements of many samples. Circles are from our universal formulae.

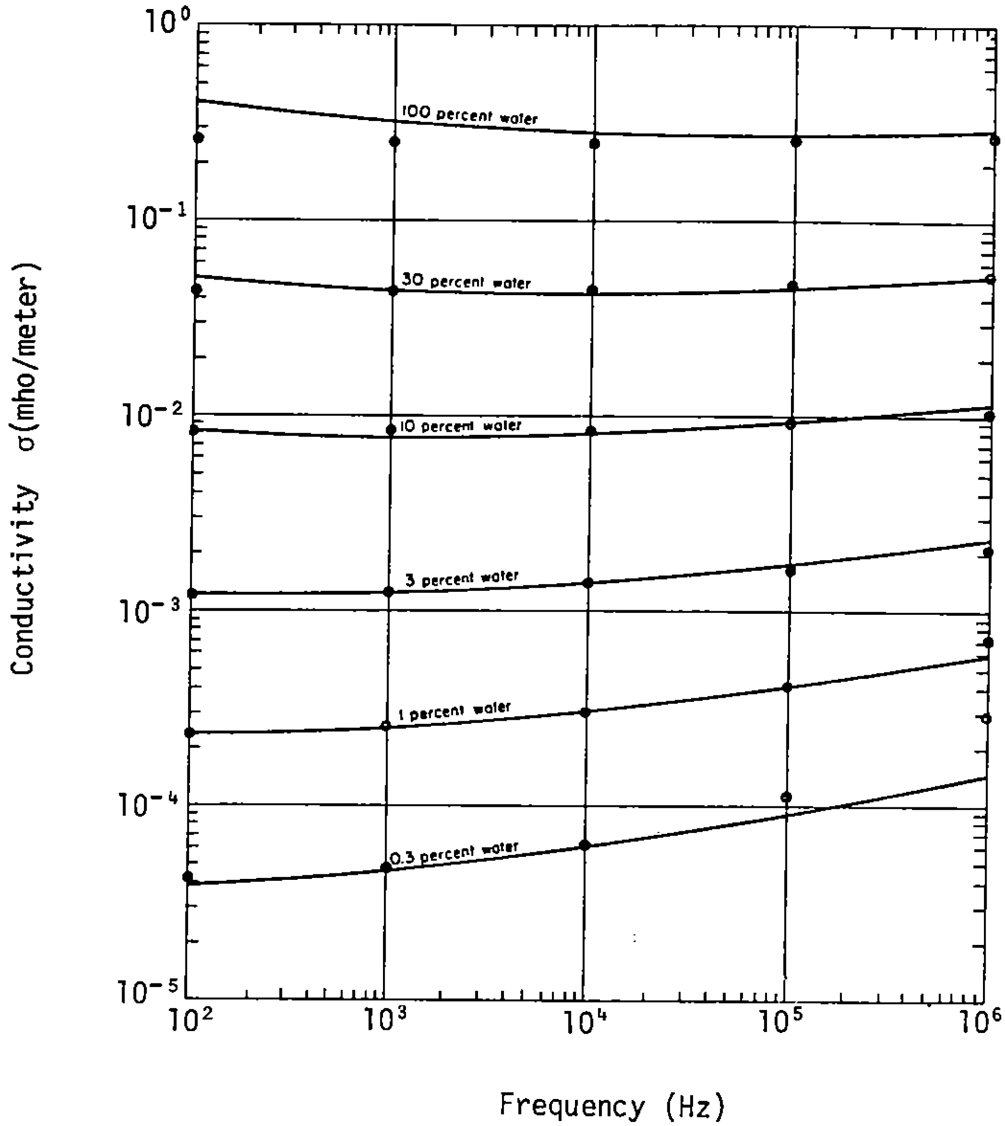

Figure 2. Electrical conductivity of soils. Solid curves are Scott's fits to results of measurements of many soils. Circles are from our universal formulae.

Recently, Wilkenfeld[3] has measured ε and σ for several samples of concrete and grout over the frequency range 10^6 to 2×10^8 Hz. In examining this data, we found that it connected rather well with Scott's curves, and also obeys the RC network relation between variations in ε and σ. We therefore decided to make a universal impedance (actually, admittance) function which would include both Scott's and Wilkenfeld's data.

2. THE RC NETWORK MODEL

Maxwell's equations in a non-magnetic medium are (MKS units)

$$\frac{\partial \vec{B}}{\partial t} = - \nabla \times \vec{E} \ , \tag{1}$$

$$\varepsilon \frac{\partial \vec{E}}{\partial t} + \sigma \vec{E} = \frac{1}{\mu_0} \nabla \times \vec{B} \ . \tag{2}$$

In the frequency domain, the properties ε and σ of the medium are contained in the relation between the (total) current density $\vec{J}$ and electric field $\vec{E}$ ($E \sim e^{i\omega t}$),

$$\vec{J} = (i\omega\varepsilon + \sigma)\vec{E} \equiv Y(\omega)\vec{E} \ . \tag{3}$$

This equation defines the admittance Y appearing between opposite faces of a cubic meter of the medium. We have assumed here that Y is isotropic, although that is not necessary. We have also made the standard assumption that the relation between $\vec{J}$ and $\vec{E}$ is linear.

The basic assumption of our model is that the cubic meter of medium is equivalent to a network of resistors and capacitors, as far as the relation between $\vec{J}$ and $\vec{E}$ is concerned. The inductance of space is included in other terms in Maxwell's equations, but our model assumes that there are no helical conduction paths in the medium—a reasonable assumption for soils.

An arbitrary two-terminal RC network can be replaced by an equivalent network of the type shown in Figure 3 with the same impedance, as a function of frequency, as the arbitrary network. R_0 is the resistance at zero frequency, C_∞ is the capacitance at infinite frequency, and the other branches provide transient responses with various time constants. The admittance of the equivalent network is

$$Y = \frac{1}{R_0} + i\omega C_\infty + \sum_{n=1}^{N} \frac{1}{R_n + \frac{1}{i\omega C_n}} \tag{4}$$

$$= \frac{1}{R_0} + i\omega C_\infty + \sum_{n=1}^{N} \frac{i\omega C_n(1 - i\omega R_n C_n)}{1 + (\omega R_n C_n)^2} . \tag{5}$$

The real and imaginary parts of Y are related to σ and ε through Equation 3. Defining, for brevity,

$$\beta_n = (R_n C_n)^{-1} , \tag{6}$$

we have

$$\varepsilon = C_\infty + \sum_{n=1}^{N} \frac{C_n}{1 + (\omega/\beta_n)^2} , \tag{7}$$

$$\sigma = \frac{1}{R_0} + \sum_{n=1}^{N} C_n\beta_n \frac{(\omega/\beta_n)^2}{1 + (\omega/\beta_n)^2} . \tag{8}$$

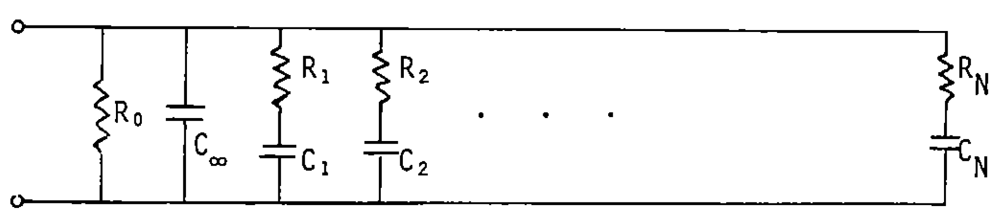

Figure 3. Equivalent network for arbitrary two-terminal RC network.

From these expressions it is clear that if $\varepsilon(\omega)$ has been specified, σ is fixed except for an additive constant, and vice versa. It is also clear that if changing the water content changes only the frequency scale of ε, then the C_n must be independent of water content and all of the β_n must scale by the same factor.

Equations (7) and (8) also show that ε decreases with increasing frequency, while σ increases. Since

$$\frac{1}{1 + (\omega/\beta_n)^2} + \frac{(\omega/\beta_n)^2}{1 + (\omega/\beta_n)^2} = 1 ,$$

the decrease in ε and the increase in σ, due to the n'th term, are related by

$$\begin{aligned} \Delta\sigma(\text{mho/m}) &= -\beta_n \Delta\varepsilon(\text{farad/m}) \\ &= \beta_n \varepsilon_0 \Delta\varepsilon_r . \end{aligned} \qquad (9)$$

Here we have introduced the relative dielectric constant ε_r and the dielectric constant ε_0 of vacuum

$$\left.\begin{aligned} \varepsilon_r &= \varepsilon/\varepsilon_0 , \\ \varepsilon_0 &= 8.85 \times 10^{-12} \text{ farad/m} . \end{aligned}\right\} \qquad (10)$$

In the neighborhood of a given central frequency f_c, the variation in ε and σ will be dominated by the term with $\beta_n \approx 2\pi f_c$, so that we may write Equation (9) as

$$\left.\begin{aligned} \Delta\sigma(\text{mho/m}) &\approx -2\pi\varepsilon_0 f_c \Delta\varepsilon_r , \\ &\approx -5.5 \times 10^{-11} f_c \Delta\varepsilon_r . \end{aligned}\right\} \qquad (11)$$

This relation provides a useful rough test of the consistency of experimental data with our model.

Since most data are given for ε_r rather than ε, and in terms of frequency f rather than ω, we rewrite Equations (7) and (8) as

$$\varepsilon_r = \varepsilon_\infty + \sum_{n=1}^{N} \frac{a_n}{1 + (f/f_n)^2} \quad \text{(relative)} , \tag{12}$$

$$\sigma = \sigma_0 + 2\pi\varepsilon_0 \sum_{n=1}^{N} a_n f_n \frac{(f/f_n)^2}{1 + (f/f_n)^2} \left(\frac{\text{mho}}{\text{m}}\right) . \tag{13}$$

The connection with Equations (7) and (8) is obviously

$$\left.\begin{array}{ll} C_\infty = \varepsilon_0\varepsilon_\infty , & C_n = \varepsilon_0 a_n , \\ R_0 = \sigma_0^{-1} , & \beta_n = 2\pi f_n . \end{array}\right\} \tag{14}$$

3. THE UNIVERSAL SOIL

In this section, we shall produce a single formula for ε_r of the type of Equation (12) in which only the f_n scale with water content, and all f_n scale by the same factor. We start with Scott's curve for 10 percent water. We extend it to higher frequencies by shifting the curves for lower water content to the right until they coincide with the 10 percent curve. It can be seen from Figure 1 that the shifted 0.3 percent curve will then extend to approximately 10^8 Hz. Similarly, we extend the 10 percent curve to lower frequencies by shifting the curves for higher water content to the left until they coincide with the 10 percent curve. It can be seen from Figure 1 that the shifted 100 percent curve will then extend down to about 5 Hz, and we extrapolated it to 1 Hz.

The curve so obtained runs from 10^0 to 10^8 Hz, and at its upper end ε_r is about 11. One of Wilkenfeld's samples was a very dry concrete for which ε_r was as low as 6.0 at 2×10^8 Hz (see Figure 7). By adjoining this data smoothly to the extended 10 percent Scott curve, we extended the latter to 3×10^{10} Hz (equivalent). For good measure, we then extrapolated ε_r to 10^{12} Hz, bringing the final ε_r down to about 5, a value appropriate for crystalline materials typically found in soils. Whether these last two decades will prove useful for some exceptionally dry soil is speculative. The entire curve is graphed in Figure 4.

It should be borne in mind that the actual data on which this curve is based was for frequencies between 10^2 and 2×10^8 Hz. Extrapolations to 5 Hz and 3×10^{10} Hz are based on the universal RC network model, and extrapolation beyond these frequencies is based simply on graphical extrapolation of the curves.

The factor F by which the frequency must be scaled to bring Scott's ε_r curves for water content P percent into coincidence with the 10 percent curve is graphed versus P in Figure 5. The function F(P) is fitted to a few percent by the formula

$$F = (P/10)^{1.28} \quad . \tag{15}$$

This factor is defined such that the frequencies f_n or β_n scale as

$$\left.\begin{aligned} f_n(P) &= F(P) f_n(10 \text{ percent}) \ , \\ \beta_n(P) &= F(P) \beta_n(10 \text{ percent}) \ . \end{aligned}\right\} \tag{16}$$

These results imply that the resistances R_n decrease with increasing water content, as $P^{-1.28}$. The sense of this variation of the R_n with P is reasonable, but why the power should be 1.28, rather than 1.00 say, is unexplained.

The very large dielectric constant at the lower frequencies, much larger than the value 80 for pure water, is puzzling if one thinks in terms of good dielectric materials. Soils typically contain a broad size spectrum of crystalline grains, with electrolytically conducting fissures between. One is reminded somewhat of the behavior of electrolytic capacitors. It has been suggested that the large dielectric constants are actually evidence

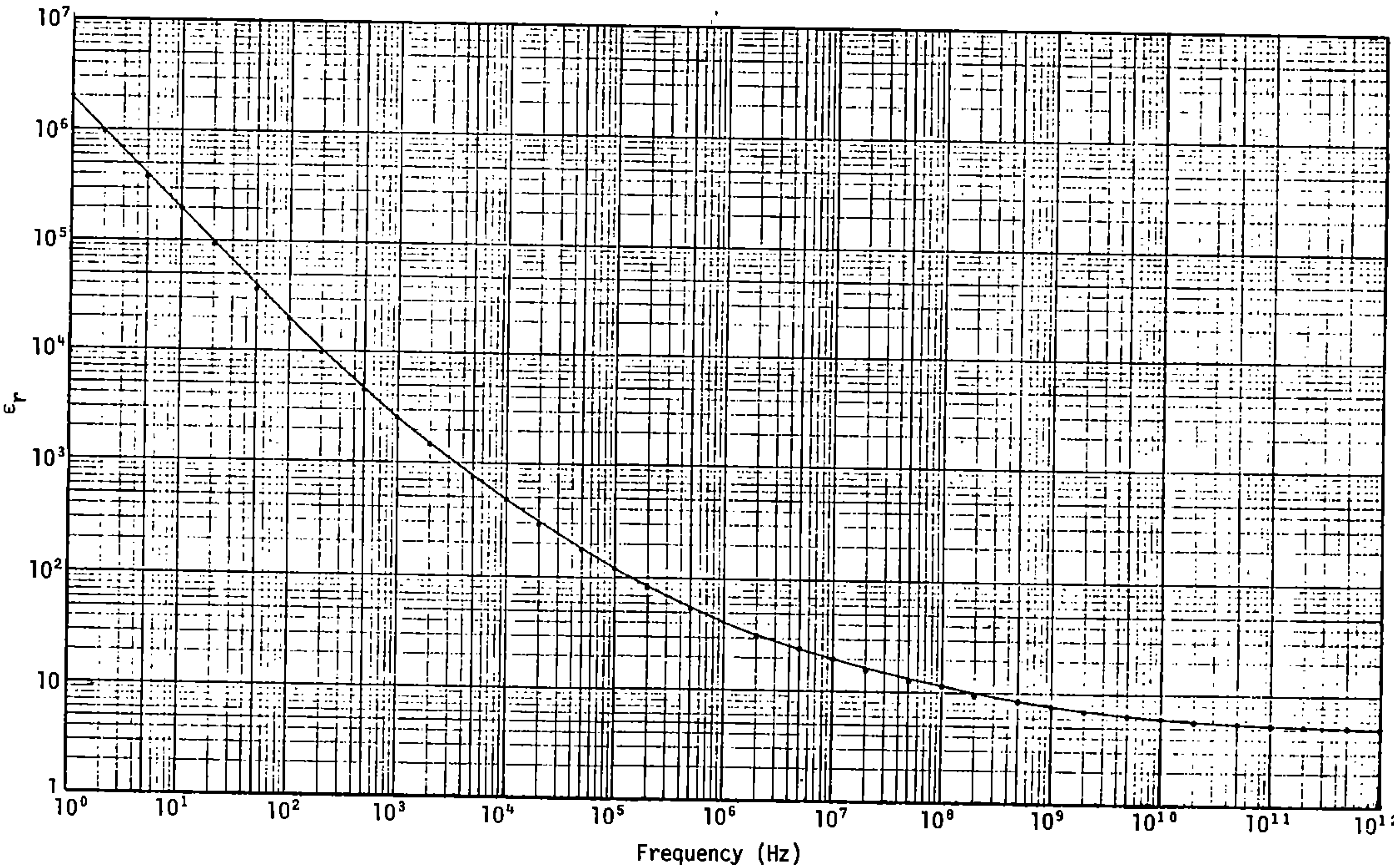

Figure 4. Universal curve for relative dielectric constant ϵ_r. Points are calculated from Equation (12), with Equations (17) and (18) and Table 1. Smooth curve is drawn through decade points.

of a phenomenon occurring at the electrodes of the sample holder, rather than in the body of the material. However, Scott went to great pains to assure good electrical contact with the sample.

Having thus obtained a curve for ε_r at P = 10 percent, we fitted it by the formula (12), choosing one f_n per decade, i.e., we let

$$f_n = 10^{n-1} \text{ Hz} \quad , \quad n = 1, 2, 3, \ldots, 13. \qquad (17)$$

The value of ε_∞ chosen was

$$\varepsilon_\infty = 5.00 \ . \qquad (18)$$

The values of the a_n are listed in Table 1. Points computed from Equation (12) using these values are indicated as dots in Figure 4. The computed points are on the curve at the decade lines where the fit values were chosen, but fall slightly below the curve in the middle of each decade. The fit could be improved by increasing the number N of f_n's, but the present fit is probably adequate, in view of the general nature of our undertaking.

Table 1. Coefficient a_n for universal soil.

n	a_n*	n	a_n	n	a_n
1	3.40(6)	6	1.33(2)	11	9.80(-1)
2	2.74(5)	7	2.72(1)	12	3.92(-1)
3	2.58(4)	8	1.25(1)	13	1.73(-1)
4	3.38(3)	9	4.80(0)		
5	5.26(2)	10	2.17(0)		

* 3.40(6) means 3.40×10^6

Having the a_n, we then computed the quantity $\sigma - \sigma_0$ from Equation (13), and the result is graphed in Figure 6. The computed points are shown as dots, and the smooth curve is drawn through the decade points. This curve is for P = 10 percent water content. For other values of P, the f_n are scaled according to Equation (16). Because of the additional factor f_n in Equation (13), the magnitude of $\sigma - \sigma_0$ also scales up and down with F as the f_n do.

For use on a computer, Equations (12), (13), (15) and (16) are sufficient. However, one can use Figures (4) and (6) for quick graphical analysis for arbitrary P without redrawing the curves, by scaling f as 1/F instead of scaling the f_n as F. Suppose, for example, that one has ε_r for a quite dry soil graphed versus frequency on the same log paper as Figure 4, preferably transparent. To make this curve of ε_r fall on top of that of Figure 4, it is likely that it will have to be moved to the right of the position relative to Figure 4 where the frequencies coincide. Let us suppose that sliding it to the right one decade makes the ε_r's coincide. We would then conclude that F = 0.1 and, from Figure 5, that P = 1.6 percent. In order to obtain a prediction for $\sigma - \sigma_0$, label another sheet of log paper as in Figure 6, i.e., with 10^{-5} mho/m at the bottom. Then slide this sheet one decade to the right (of the position where frequencies coincide) and one decade upwards, and trace the curve from Figure 6 onto the sheet. If our universal soil model applies to this sample, the traced curve should fall on or slightly below the data points for the sample σ (the difference being σ_0).

The term σ_0 is adjustable for any soil sample. However, Scott's curves of Figure 2 determine σ_0 as a function of P for his soil samples. This function is also graphed in Figure 5, and is expressed to a few percent accuracy by the relation

$$\sigma_0 = 8.0 \times 10^{-3}(P/10)^{1.54} \quad \text{(mho/m) (Scott)} . \qquad (19)$$

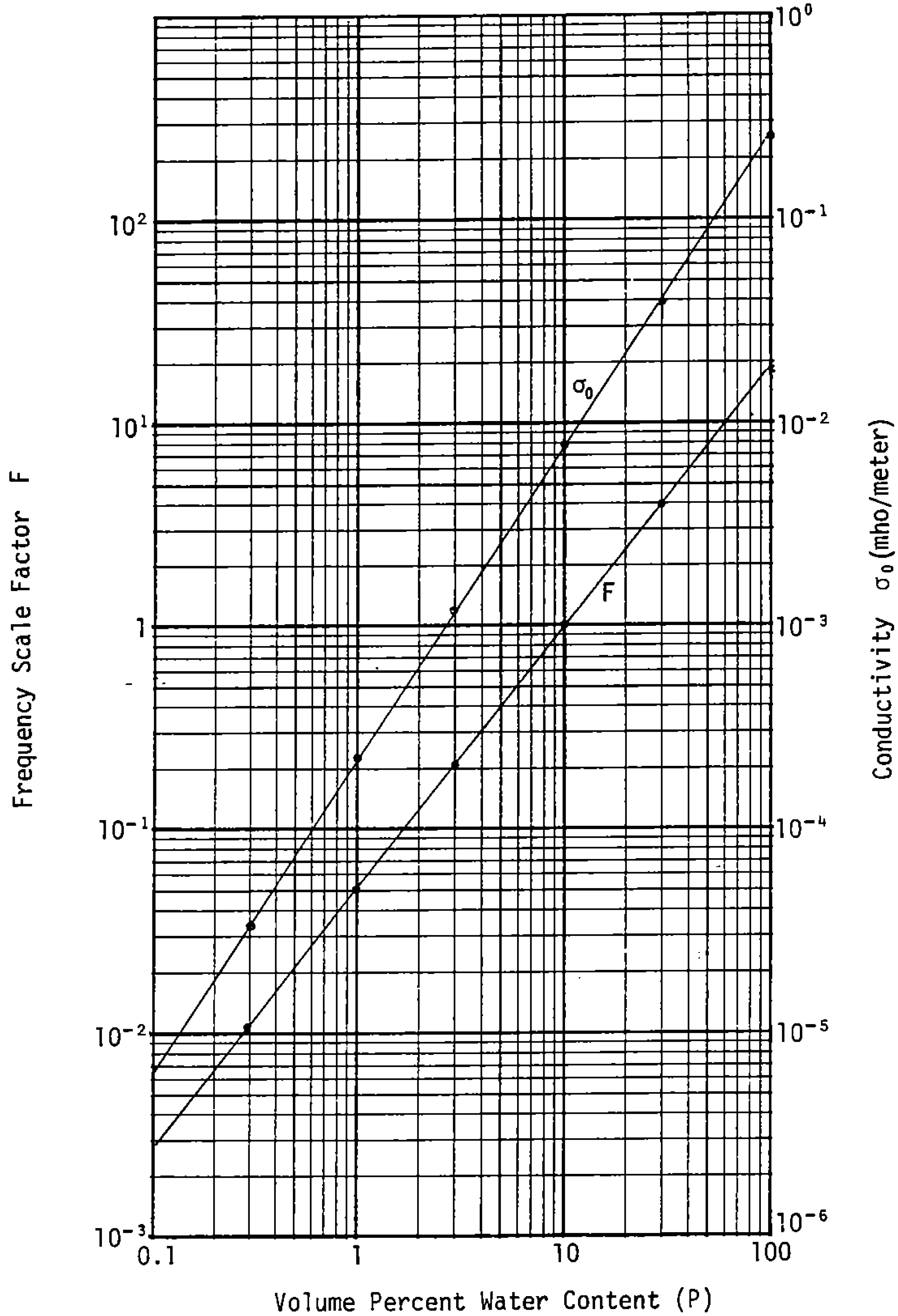

Figure 5. Frequency scale factor F and zero frequency conductivity σ_0 from Scott's results.

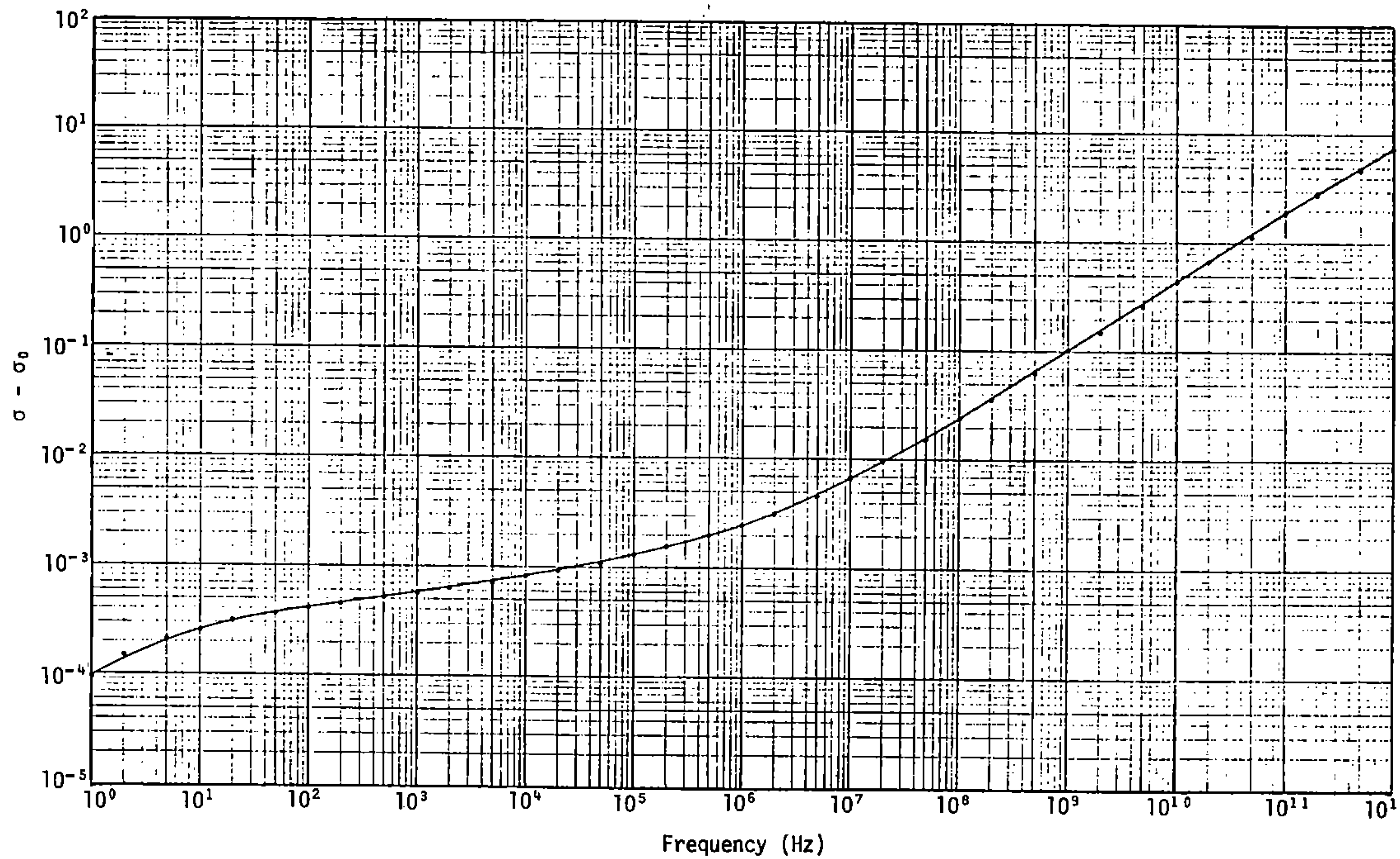

Figure 6. Universal curve for conductivity $\sigma - \sigma_0$. Points are calculated from Equation (13), with Equation (17) and Table 1. Smooth curve is drawn through decade points.

We have used Equations (12) and (13), with (15), (16), (17) and (18) and Table 1, to calculate ε_r and σ for Scott's values of P and frequencies between 10^2 and 10^6 Hz. The results are shown as the circles in Figures 1 and 2. It is seen that computed ε_r points fit Scott's curves very well. The fit to σ is quite good except for the highest and lowest water content. According to our model, σ cannot decrease with increasing frequency. Scott assumed no connection between ε_r and σ. Whether the deviations between our model and Scott's curves are statistically significant is not clear. It would be interesting to try to fit Scott's sample data directly in terms of our model.

4. WILKENFELD'S DATA

We have tested five of Wilkenfeld's concrete and grout sample results against our universal curves, using the graphical analysis. The samples are identified in Table 2.

Table 2. Identification of Wilkenfeld samples.

Sample	Material	Moisture (weight %)	F	P (volume %)	σ_0
1	Concrete	1.75%	0.007	0.2%	negligible
2	Concrete	6.4 %	1.0	10 %	1×10^{-3}
3	Concrete	7.9 %	2.50	20 %	2×10^{-3}
4	Grout	5.9 %	0.50	5.8%	negligible
5	Grout	11 %	5.0	35 %	negligible

In this table, the column labeled "Moisture" is the weight percent determined by Wilkenfeld, and should be multiplied by about 2 to give a volume percent comparable to Scott's P. Instead of using Wilkenfeld's moisture content to determine a frequency scale factor, we have chosen the scale factor F which gives the best fit with our universal ε_r curve. This value of F is given in the table, along with the value of Scott's P which yields this F (Equation (15) or Figure 5). It is seen that P is not uniformly twice Wilkenfeld's moisture content, but the correlation is not bad, except for Sample 1. For this sample, the moisture determined by Wilkenfeld is much higher than the value of P which makes the Sample 1 data lie on the universal curve.

Comparison of Wilkenfeld's data for ε_r and σ with our fitted curves are given in Figures (7) - (11). The fits are quite good, which indicates the validity of the RC network model. The value of σ_0 suggested by the σ curves is also listed in Table 2.

Sample 1 was the sample used to extend our universal curves to 3×10^{10} Hz. The fact that P disagrees with Wilkenfeld's moisture content implies that measurement of the moisture content is not always a reliable way of determining electrical properties of dry soils. In general it is wise to have measurements of ε_r and σ made at a few frequencies. Our universal curves can then be used to extend and fill in the data.

5. SUMMARY

Equations (12) and (13) give a prediction for ε_r and σ for frequencies f between 5 and 3×10^{10} Hz, for a soil containing 10 percent (by volume) water. Here ε_∞ is given by Equation (18), σ_0 by Equation (19), the f_n by Equation (17), and the a_n by Table 1. For any other water content P percent, scale the f_n by Equations (16) and (15).

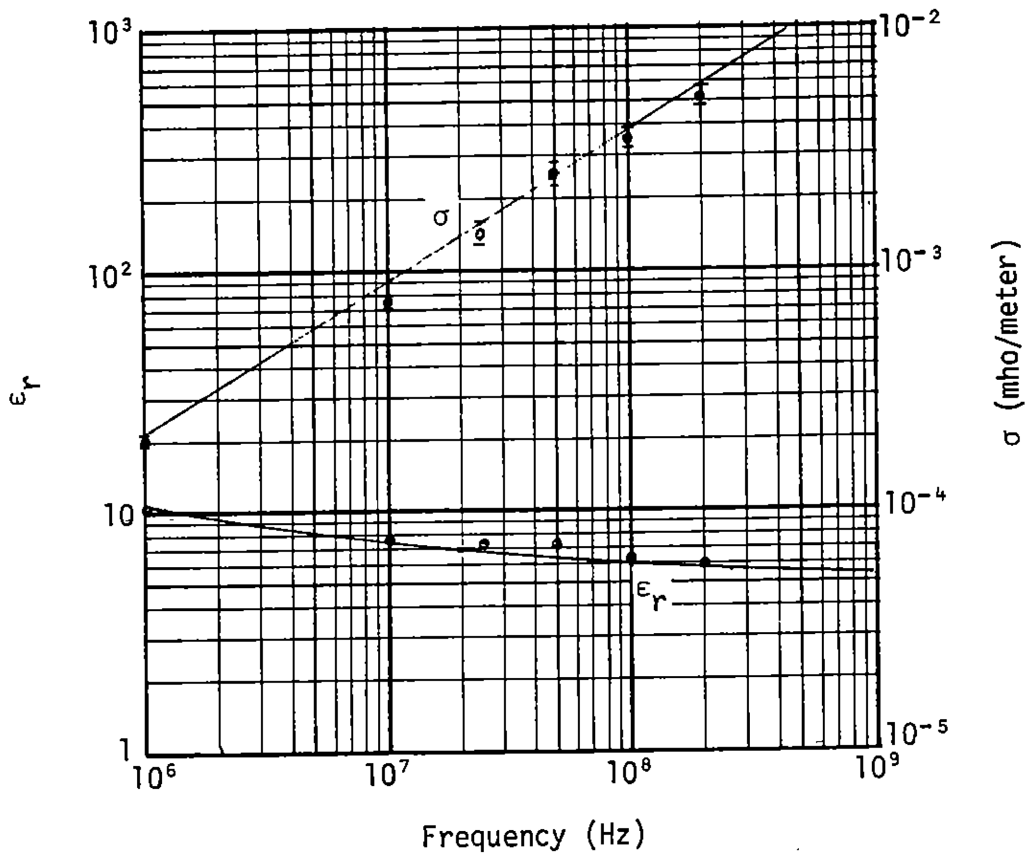

Figure 7. Relative dielectric constant and conductivity for Sample 1. Data points are Wilkenfeld's. Curves are from universal formulae with F = 0.007, and σ_0 = 0.

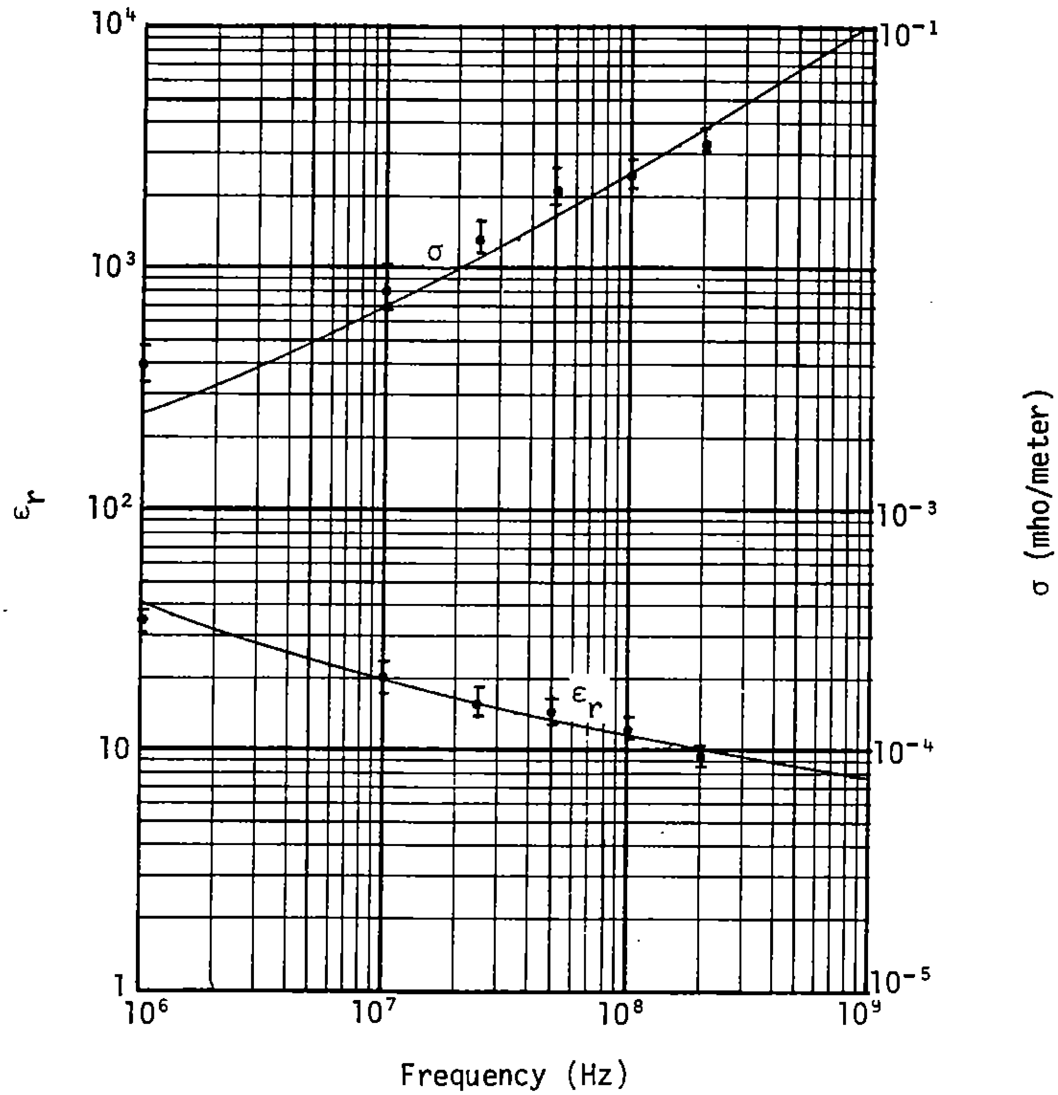

Figure 8. Relative dielectric constant and conductivity for Sample 2. Data points are Wilkenfeld's. Curves are from universal formulae with $F = 1.0$, and $\sigma_0 = 0$.

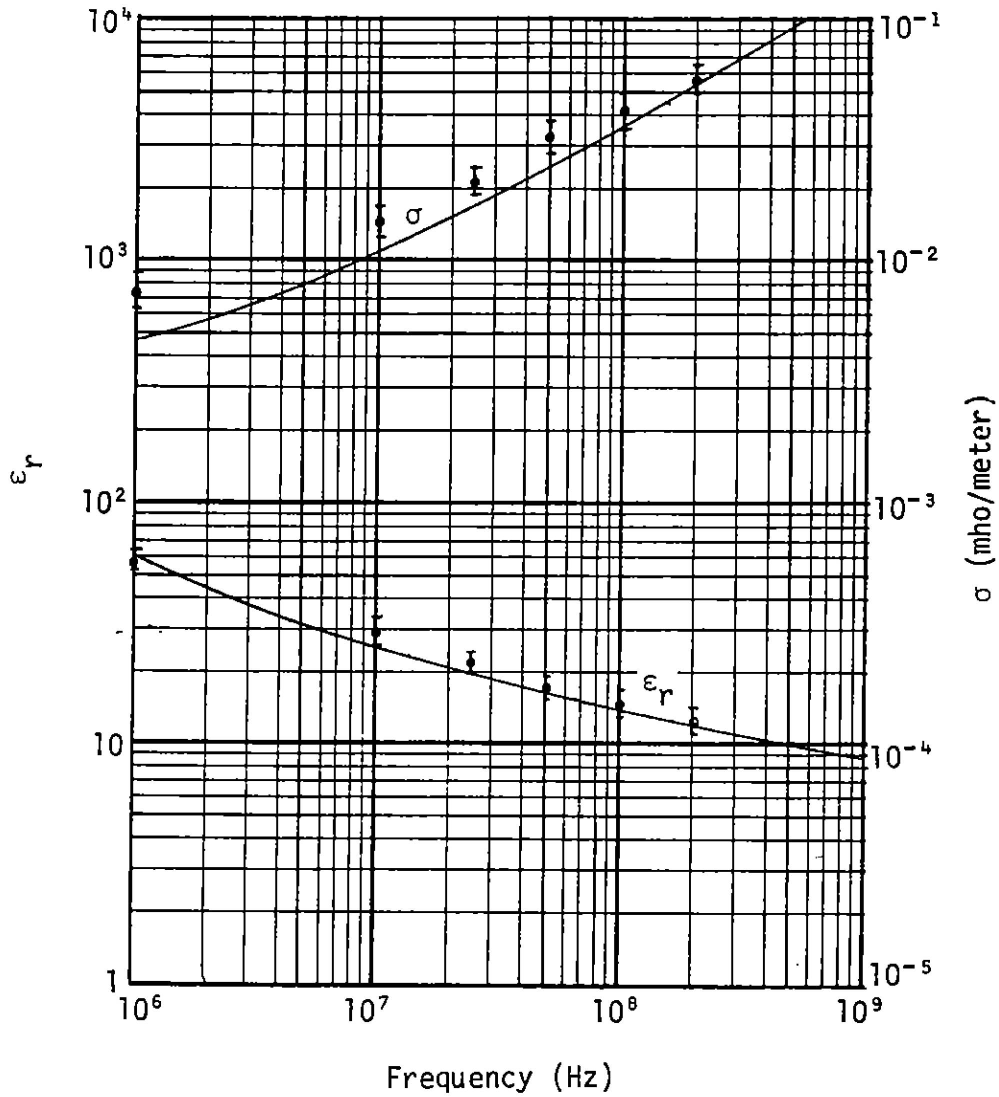

Figure 9. Relative dielectric constant and conductivity for Sample 3. Data points are Wilkenfeld's. Curves are from universal formulae with F = 2.50, and σ_0 = 0.

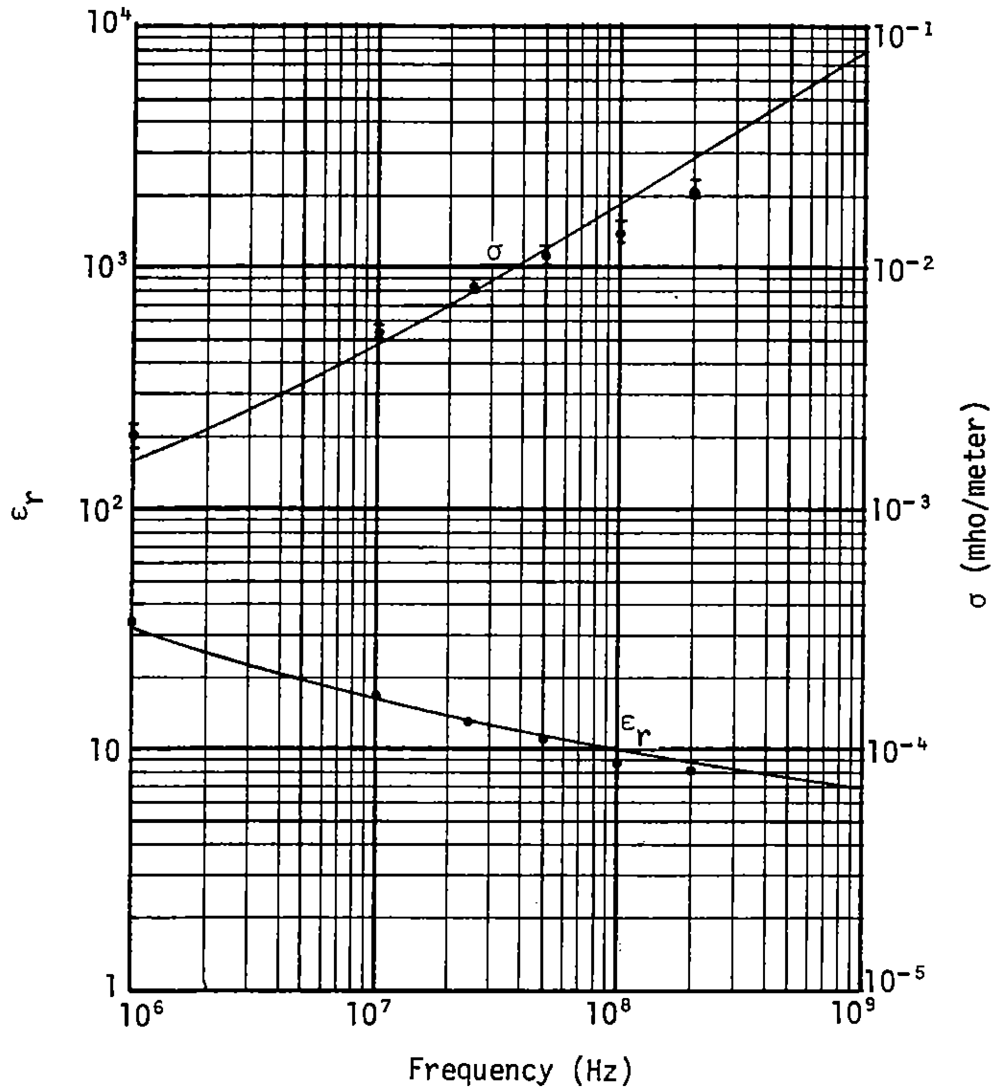

Figure 10. Relative dielectric constant and conductivity for Sample 4. Data points are Wilkenfeld's. Curves are from universal formulae with F = 0.50, and σ_0 = 0.

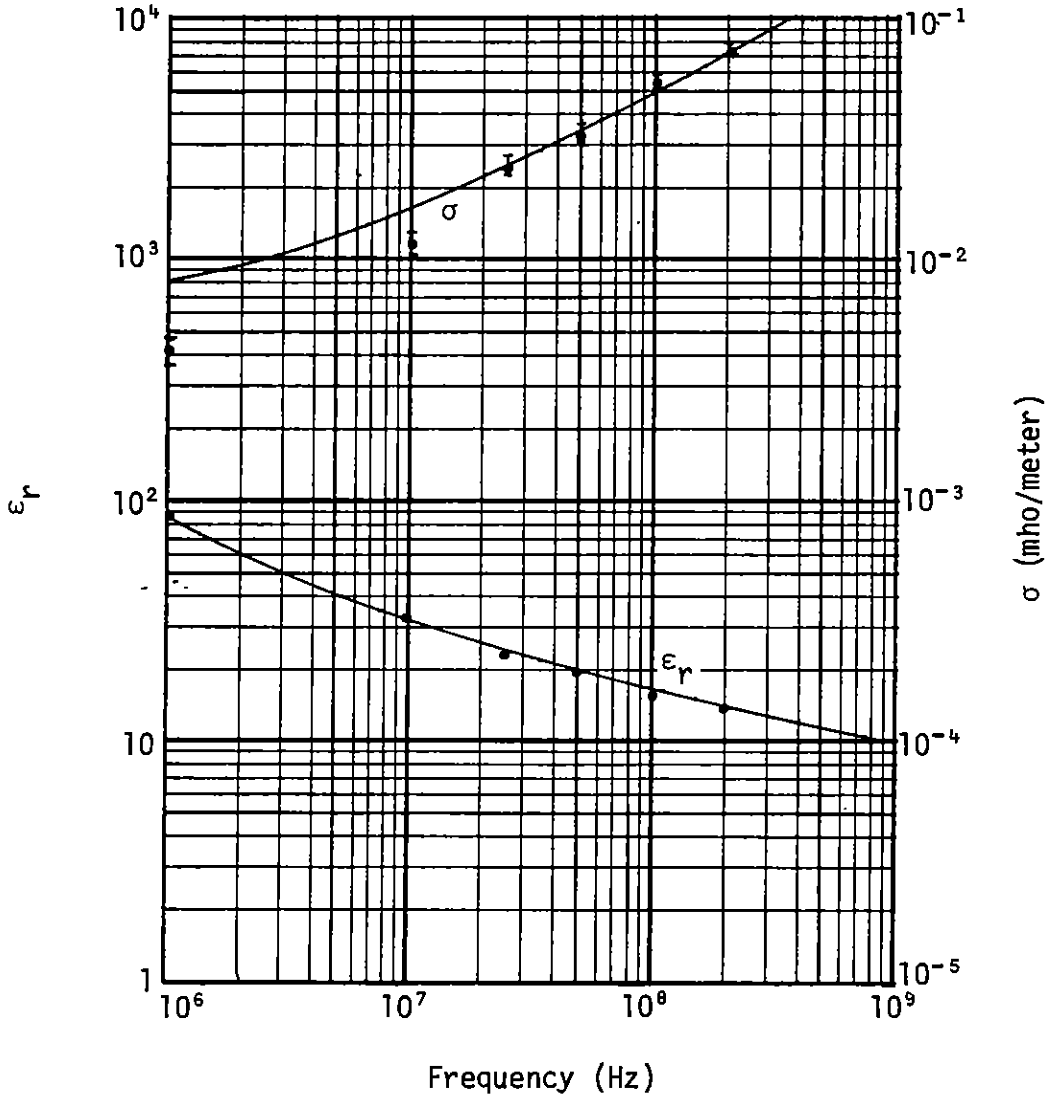

Figure 11. Relative dielectric constant and conductivity for Sample 5. Data points are Wilkenfeld's. Curves are our universal formulae with F = 5.0, and σ_0 = 0.

If one has a soil for which the water content is unknown, but one has a measurement of ϵ_r and σ at some frequency f_m, proceed as follows. Read from Figure 4 the value of f at which ϵ_r equals the measured value. Then determine the frequency scale factor F from

$$F = f_m/f \quad (20)$$

Use this value in Equation (16) to scale the f_n. Then compute from Equations (12) and (13) the values of ϵ_r and $\sigma - \sigma_0$ at any frequency of interest. Adjust σ_0 to agree with the measured value at f_m. The water content of the soil can be determined from the value of F above and Equation (15) or Figure 5.

It is more reliable to use measured ϵ_r and σ data at a few frequencies, in establishing the fit with the universal curve, than to use a measured water content, especially for dry soils.

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