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Theory of Propagation of Electromagnetic Waves in Space-Time Varying Media

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Abstract

Using WKB (Wentzel-Kramers-Brillouin) methods we have considered the propagation of electromagnetic waves in isotropic lossless media which vary slowly with both position and time. It is found that in such media the meaning of various quantities, such as the group velocity, must be reinterpreted. The theory is applied to study propagation in space-time varying dielectrics and plasmas.

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1. INTRODUCTION

A difficult problem to consider, conceptually, is wave propagation in media which vary with both position and time. In such media our standard concepts of frequency, wavenumber, and group velocity no longer apply. That is, we find that frequency can only be defined as the time derivative of the phase function, and that the quantity so defined is a function of both the position and the time at which the observation is made. We also find that the group velocity no longer retains its conventional meaning. In spatially-homogeneous, time-invariant media the group velocity is interpreted as the velocity at which wave packets centered around some wavevector $\underline{k}_0$ propagate (Jeffreys and Jeffreys, 1962). In space-time varying media this is no longer true. In fact, as we shall see, values of ω and $\underline{k}$ do not propagate with the group velocity $\nabla_k \omega$; rather different quantities (which are functions of ω and $\underline{k}$) are propagated at this velocity. The same conclusions hold true for energy flow (that is, the energy flux does not propagate with the group velocity).

In this paper we will study the propagation of electromagnetic waves in lossless media which vary slowly with position and time. We will therefore employ the four-dimensional WKB (Wentzel-Kramers-Brillouin) method. The WKB method was first applied in three dimensions by Sommerfeld and Runge (1911). That is,

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in media in which the properties depended on position $\underline{x}$ but not on time, Sommerfeld and Runge considered solutions of the form $\exp [i\omega t - \int \underline{k} \cdot d\underline{x}]$. Because the phase function must be uniquely defined, Sommerfeld and Runge then concluded that $\nabla \times \underline{k} = 0$ was required. This is the original version of the Sommerfeld-Runge Law. This result can be extended to the four-dimensional case, as has been done by Whitham (1960) and Poeverlein (1962). That is, in space-time varying media the existence of a uniquely defined wave function, $\exp [i\int(\omega dt - \underline{k} \cdot d\underline{x})]$ requires that $\nabla\omega + \partial \underline{k}/\partial t = 0$. This is the four-dimensional Sommerfeld-Runge Law.

In the present paper we will first give an elementary derivation of the four-dimensional Sommerfeld-Runge Law. We will then examine its implications, and finally indicate its use in studying electromagnetic wave propagation in isotropic, lossless media which vary slowly with position and time.

2. GENERAL THEORY

2.1 Derivation of the Generalized Sommerfeld-Runge Law

To introduce the Sommerfeld-Runge Law we begin with the pair of equations for the electric field $\underline{E}$

$$\nabla \times \nabla \times \underline{E} = - \mu_o \frac{\partial^2(\epsilon \underline{E})}{\partial t^2}, \tag{1a}$$

$$\nabla \cdot (\epsilon \underline{E}) = 0, \tag{1b}$$

where the permittivity $\epsilon(\underline{x}, t)$ is a function of both position and time but the magnetic permeability is equal to that of vacuum. Equations (1a) and (1b) are appropriate for propagation in a lossless space-time varying dielectric. We will later generalize our results to include dispersive media, such as a plasma. Let us assume a solution for $\underline{E}$ of the form

$$\underline{E}(\underline{x}, t) = \underline{e}_o(\underline{x}, t)\, e^{\phi(\underline{x}, t)}, \tag{2}$$

where $\underline{e}_o$ varies slowly with position and time compared with $\phi(\underline{x}, t)$. Using (2) in (1b) we find that, if $\frac{1}{\epsilon}|\nabla\epsilon| << |\nabla\phi|$ we get

$$\nabla\phi \cdot \underline{e}_o = 0. \tag{3}$$

If we use (2) and (3) in (1a) we then find that if the dielectric varies slowly with position and time, the function ϕ satisfies

$$|\nabla\phi|^2 = \mu_o \epsilon(\underline{x}, t)\left(\frac{\partial\phi}{\partial t}\right)^2 , \tag{4}$$

where the conditions that the medium be considered slowly varying are

$$|\nabla\phi|^2 >> \nabla^2\phi , \tag{5a}$$

$$\left(\frac{\partial\phi}{\partial t}\right)^2 >> \frac{\partial^2\phi}{\partial t^2}, \tag{5b}$$

$$\frac{\partial\phi}{\partial t} >> \frac{1}{\epsilon}\frac{\partial\epsilon}{\partial t}, \tag{5c}$$

$$|\nabla\phi| >> \frac{1}{\epsilon}|\nabla\epsilon| . \tag{5d}$$

The solution of Eq. (4) can be written quite generally as

$$\phi = i\int_L (\underline{k}\cdot d\underline{x} - \omega\, dt) , \tag{6}$$

where L is a line integral in four space between some initial point $(\underline{x}_o, t_o)$ and the observation point $(\underline{x}, t)$. We can identify ω and $\underline{k}$ in (6) by realizing that since the phase function ϕ must be a single valued function, the line integral in (6) must be independent of the path joining $(\underline{x}_o, t_o)$ to $(\underline{x}, t)$. Therefore, the integrand in (6) must be a perfect differential and we have

$$i\underline{k} = (\nabla\phi)_t , \tag{7a}$$

$$i\omega = -\left(\frac{\partial\phi}{\partial t}\right)_{\underline{x}} . \tag{7b}$$

Using these expressions in Eq. (4) gives the dispersion relation between ω and $\underline{k}$ as

$$c^2k^2 = \epsilon_r\omega^2 , \tag{8}$$

where $k^2 = \underline{k}\cdot\underline{k}$ and ϵ_r is the relative permittivity of the dielectric. Equations (7a) and (7b) imply a relation between ω and k, in addition to that given by Eq. (8). Differentiating (7a) with respect to t, taking the gradient of (7b), and then adding the resulting equations gives:

$$\frac{\partial \underline{k}}{\partial t} + \nabla\omega = 0 . \tag{9}$$

Equation (9) is the four-dimensional version of the Sommerfeld-Runge Law. This result can alternately be derived by employing arguments on wave conservation, as has been done by Lighthill and Whitham (1955). Equation (9) has also been discussed by Whitham (1960), Poeverlein (1962), and Landau and Lifschitz (1959).

2.2 Discussion of the Properties of Eq. (9)

2.2.1 GENERALIZED GROUP VELOCITY

To discuss the properties of Eq. (9) in a medium which varies with both position and time, let us first write Eq. (8) in the general form

$$\omega = W(k, \underline{x}, t) \tag{10a}$$

or

$$\underline{k} = K(\omega, \underline{x}, t) . \tag{10b}$$

Equations (10a) and (10b) are valid for an arbitrary linear medium, and not just a dielectric. For example in a lossless plasma

$$k = c^{-1} \cdot \sqrt{\omega^2 - \omega_p^2(\underline{x}, t)}$$

where ω_p is the electron plasma frequency. If (10a) is substituted into (9) we obtain

$$\frac{\partial \underline{k}}{\partial t} + (\underline{V} \cdot \nabla)\, \underline{k} = -(\nabla W)_{k,t} \tag{11}$$

where

$$\underline{V} = (\nabla_k W) = \left(\frac{\partial W}{\partial \underline{k}}\right)_{\underline{x}, t} . \tag{12}$$

The quantity $\underline{V}$ in (12) can be interpreted as a generalized group velocity as we shall see in the following discussion. Upon defining $d/dt = \partial/\partial t + \underline{V} \cdot \nabla$ Eq. (11) can be rewritten as

$$\frac{d\underline{k}}{dt} = -(\nabla W)_{\underline{k},\, t} \,. \tag{13}$$

From (13) we see that if W does not depend explicitly on position, then $d\underline{k}/dt = 0$. This means that if one moves along the ray with the velocity $\underline{V}$, he will observe constant values of $\underline{k}$ (that is, the observer moving with $\underline{V}$, measures constant values of wavelength). Therefore, if W is a function of k and t, but does not depend explicitly on $\underline{x}$, then $\underline{k}$ is a constant of the motion, so that wave packets* sharply centered around some wavenumber $\underline{k}_o$ will be propagated with $\underline{V} = \partial W/\partial \underline{k}$ evaluated at $\underline{k} = \underline{k}_o$.

To examine the other limit when the properties of the medium vary with position, but not with time, it is convenient to use (10b) in (9). For this case we obtain (assuming the medium is isotropic)

$$\frac{\partial \omega}{\partial t} + \underline{V} \cdot \nabla \omega = -\left(\frac{\partial \omega}{\partial k}\right)_{\underline{x},\, t} \left(\frac{\partial K}{\partial t}\right)_{\omega,\, \underline{x}} . \tag{14}$$

From Eq. (14) it is clear that if K depends on ω and $\underline{x}$, but does not explicitly depend on time, then

$$\frac{d\omega}{dt} = 0 \,. \tag{15}$$

This means that the observer moving along the ray with the velocity $\underline{V}$ will observe constant values of ω (that is, constant wave period). Snell's Law follows immediately from Eq. (15). Therefore wave packets sharply centered about some frequency ω_o will propagate with the group velocity $\underline{V}$ (evaluated at $\omega = \omega_o$).

For the general case, when the properties of the medium depend on both position and time neither ω nor $\underline{k}$ will be invariant as one moves along the ray with the group velocity $\underline{V}$. (In Section 2.3, however, we do show that there are velocities $\underline{V}^{(k)} \neq \underline{V}$ and $\underline{V}^{(\omega)} \neq \underline{V}$ with which values of $\underline{k}$ and ω are propagated in space-time varying media.) In this case other quantities will be invariants of the motion. Consider each scalar component of Eq. (11). We have

$$\frac{\partial k_\ell}{\partial t} + (\underline{V} \cdot \nabla)\, k_\ell = -\frac{\partial W}{\partial x_\ell} \,. \tag{16}$$

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*In a spatially homogeneous, time-varying medium, pulses can be represented as $\int d\underline{k}\, A(\underline{k}) \exp\left[i\left(\underline{k} \cdot \underline{x} - \int \omega\, (\underline{k}, t)\, dt\right)\right]$ provided A(k) has significant amplitude only at values of $\underline{k}$ for which Eq. (5) is satisfied.

To solve Eq. (16) we consider the subsidiary set

$$\frac{dt}{1} = \frac{dx_1}{V_1} = \frac{dx_2}{V_2} = \frac{dx_3}{V_3} = -\frac{dk_\ell}{\left(\frac{\partial W}{\partial x_\ell}\right)} . \tag{17}$$

Let us denote the particular integrals of (17) by $f(k_\ell, \underline{x}, t) = C_1$; $g(k_\ell, \underline{x}, t) = C_2$; $h(k_\ell, \underline{x}, t) = C_3$; $\psi(k_\ell, \underline{x}, t) = C_4$. Then it can be shown that (Sneddon, 1957)

(a) The general solution of (16) is given by

$$C_1 = \Phi(C_2, C_3, C_4) , \tag{18}$$

where Φ is an arbitrary function, determined by the boundary conditions imposed.

(b) The invariants of motion, for an observer moving with the velocity $\underline{V}$ given in Eq. (12), are C_1, C_2, C_3, C_4. That is, each C_s satisfies

$$\frac{dC_s}{dt} = \left(\frac{\partial}{\partial t} + \underline{V} \cdot \nabla\right) C_s = 0 . \tag{19}$$

Therefore, for media which vary with both position and time the generalized group velocity defined in Eq. (12) is the velocity with which the quantities C_s are propagated. It is only in the limit of spatially homogeneous, time-invariant media that some of the constants C_s can be identified with ω and $\underline{k}$.

In Section 3.1 we will calculate the constants of motion for several examples of media which vary with position and time.

2.2.2 THE ANGLE BETWEEN $\underline{V}$ AND $\underline{k}$

In this section we will demonstrate that in isotropic lossless media the normal, $\hat{k} = (\underline{k}/k)$, to the phase surface $\phi(\underline{x}, t)$ lies in the same direction as the group velocity $\underline{V}$. We will also calculate the angle between $\hat{k}$ and $\underline{V}$ for the case of a simple anisotropy.

Let us consider (10b) for the case when the dispersion relation can be written as

$$k = K(\omega, \theta, \underline{x}, t) , \tag{20}$$

where θ is the angle that the vector $\underline{k}$ makes with the z axis. (This corresponds to the case when the dispersion relation is given by the Appleton-Hartree formula.)

Using Eq. (20) in Eq. (12), we have upon differentiating implicitly that the group velocity is

$$\underline{V} = \left(\frac{\partial\omega}{\partial k}\right)\left[\hat{k} - \frac{\hat{\theta}}{k}\left(\frac{\partial K}{\partial\theta}\right)\right]. \tag{21}$$

where $\hat{k}$ is a unit vector along $\underline{k}$, and $\hat{\theta}$ is the unit θ vector in a spherical coordinate system. Taking the cross product of $\underline{k}$ with (21) gives

$$|\underline{k}\times\underline{V}| = \left(\frac{\partial\omega}{\partial k}\right)\left(\frac{\partial K}{\partial\theta}\right). \tag{22}$$

From (22) we see that if the medium is isotropic and lossless so that K is independent of θ in Eq. (20), then $\underline{k}\times\underline{V} = 0$ so that k and $\underline{V}$ are in the same direction. The dot product of $\underline{k}$ with (21) gives $\underline{k}\cdot\underline{V} = k(\partial\omega/\partial k)$. From this result and (22) we can compute the angle γ between $\underline{k}$ and $\underline{V}$. We have

$$|\tan\gamma| = \frac{|\underline{k}\times\underline{V}|}{\underline{k}\cdot\underline{V}} = \frac{1}{k}\frac{\partial K}{\partial\theta} = \frac{1}{k}\frac{\partial k}{\partial\theta}. \tag{23}$$

Equation (23) is a well-known result from the study of whistlers (Holt and Haskell, 1965; Kelso, 1964). We have shown here that the result also applies to media which vary slowly with both position and time, provided we understand that $\underline{V}$ is not the velocity with which values of $\underline{k}$ and ω are propagated. One final result of interest is to use (23) and (21) to calculate the magnitude of the group velocity. We get

$$|\underline{V}| = \frac{1}{\left(\frac{\partial k}{\partial\omega}\right)\cos\gamma}. \tag{24}$$

2.2.3 THE EQUATION OF MOTION OF $\hat{k}$

In this section we shall demonstrate that the classical equation presented by Landau and Lifschitz (1959) for the motion of the normal to $\phi(\underline{x}, t)$ can be generalized to include space-time varying media. For isotropic media we write, using $\underline{k} = k\hat{k}$ in Eq. (11),

$$k\frac{d\hat{k}}{dt} + \hat{k}\frac{dk}{dt} = -(\nabla W)_{k,t}. \tag{25}$$

We next use the dispersion relation of Eq. (10b) in the second term on the left hand side of (25). We obtain, after regrouping terms

$$k\frac{d\hat{k}}{dt} + \hat{k}\left\{\frac{1}{V}\left(\frac{d\omega}{dt}\right) + \left(\frac{\partial K}{\partial t}\right)_{\omega, \underline{x}} + \underline{V} \cdot (\nabla K)_{\omega, t}\right\} = -(\nabla W)_{k, t} , \tag{26}$$

where $V \equiv |\underline{V}| = (\partial K/\partial \omega)^{-1}$. Now substitute Eq. (14) for $d\omega/dt$ into Eq. (26). The result is

$$k\frac{d\hat{k}}{dt} + V\hat{k}\hat{k} \cdot (\nabla K)_{\omega, t} = -(\nabla W)_{k, t} . \tag{27}$$

Equation (27) is valid for an arbitrary lossless isotropic medium. To obtain the analog of the ray-normal equation of Landau and Lifschitz (1959), we next specialize (27) to dielectrics. For this case $K = \omega/v$ where $v = V$ = phase velocity. In this limit Eq. (27) becomes

$$\frac{d\hat{k}}{dt} - \hat{k}(\hat{k} \cdot \nabla v) = -\nabla v(\underline{x}, t) . \tag{28}$$

Equation (28) determines the motion of the ray normal in dielectrics which vary slowly with both position and time. As demonstrated by Landau and Lifschitz, who obtained the same equation for the case when v varies with position only, Eq. (28) predicts a bending of the rays toward the region where v is smaller.

We also note from Eqs. (27) and (28) that if v is independent of position, but does depend on time, we have

$$\frac{d\hat{k}}{dt} = 0 . \tag{29}$$

Therefore, as expected, the ray does not change its direction of propagation in media that vary only with time.

2.2.4 TEMPORAL DISCONTINUITIES

It is often desirable to know the behavior of ω and $\underline{k}$ when the properties of the medium are suddenly altered. For example, suppose we have a dielectric in which $\epsilon = \epsilon_1(\underline{x})$ for $t < t_1$, and $\epsilon = \epsilon_2(\underline{x})$ for $t > t_1$. To study the behavior of ω and $\underline{k}$ when temporal discontinuities occur let us integrate Eq. (9) from $t_1 - \delta$ to $t_1 + \delta$. (Note that Eq. (9) is not strictly valid for $\delta = 0$.) We obtain

$$\underline{k}(\underline{x}, t_1 + \delta) - k(\underline{x}, t_1 - \delta) = -\int_{t_1 - \delta}^{t_1 + \delta} (\nabla\omega)\, dt . \tag{30}$$

In the limit as $\delta \to 0$, the right hand side of (30) vanishes (unless $\nabla\omega$ has a delta function behavior). Therefore

$$\underline{k}\,(\underline{x},\ t_1 + \delta\ ) = \underline{k}\,(\underline{x},\ t_1 - \delta\ )\,. \tag{31}$$

Since Eq. (31) implies that both the magnitude and direction of $\underline{k}$ cannot change instantaneously, then from (10b) we may write

$$K_1\,[\,\omega\,(\underline{x},\ t_1 + \delta\,),\ \underline{x},\ t_1 + \delta\ ] = K_2\,[\,\omega(\underline{x},\ t_1 - \delta\ ),\ \underline{x},\ t_1 - \delta\ ]\,. \tag{32}$$

For a dielectric, in which $K = \pm\,\omega\ (\underline{x},\ t)\ \sqrt{\mu_o\,\epsilon\ (\underline{x},\ t)}$,

Eq. (32) yields

$$\omega\,(\underline{x},\ t + \delta\ ) = \pm\left[\frac{\epsilon_1\,(\underline{x})}{\epsilon_2\,(\underline{x})}\right]^{1/2} \omega(\underline{x},\ t - \delta\ )\,. \tag{33}$$

The positive sign in Eq. (33) is appropriate for the wave travelling along $\underline{k}$, whereas the negative sign is appropriate for the reflected component which travels along $-\underline{k}$. This latter component is negligible in the WKB limit. In the limit of a spatially homogeneous dielectric, Eq. (33) reduces to the previous result of Morgenthaler (1958).

2.3 Velocity of Propagation of Values of $\underline{k}$ and ω

One of the points we have made is that in a space-time varying medium neither ω nor $\underline{k}$ is an invariant as one moves with the group velocity $\underline{V} = \nabla_k \omega$. It is possible, however, to define a new velocity $\underline{V}^{(k)}$ such that $\underline{k}$ will be constant when the observer moves with this velocity. To obtain $\underline{V}^{(k)}$, let us suppose that the solution of Eq. (11) is given by $\underline{k}\,(\underline{x},\ t)$. Then we can define a velocity $\underline{V}'$ through the equation

$$(\underline{V}' \cdot \nabla\ )\,\underline{k} = (\nabla W)_{\underline{k},\ t}\,. \tag{34}$$

If this were done, Eq. (11) could be rewritten as

$$\frac{\partial\,\underline{k}}{\partial\,t} + [\,(\underline{V} + \underline{V}'\,) \cdot \nabla\ ]\,\underline{k} = 0\,, \tag{35}$$

from which we immediately identify

$$\underline{V}^{(k)} = \underline{V} + \underline{V}' \,. \tag{36}$$

Equation (34) can be solved for $\underline{V}$ by standard matrix methods. It is interesting to consider the one dimensional limit (that is, $\partial/\partial x = \partial/\partial y = 0$). Then Eq. (34) becomes

$$V'(z, t)\left(\frac{\partial k}{\partial z}\right)_t = \left(\frac{\partial W}{\partial z}\right)_{k, t} , \tag{37}$$

which is readily solved for V. Using this result, along with Eq. (12), in Eq. (36) then gives

$$V^{(k)} = \left(\frac{\partial W}{\partial k}\right)_{z, t} + \frac{\left(\frac{\partial W}{\partial z}\right)_{k, t}}{\left(\frac{\partial k}{\partial z}\right)_t} \,. \tag{38}$$

Therefore, we have demonstrated it is possible to define a velocity $\underline{V}^{(k)}$ such that the observer moving with this velocity sees constant values of $\underline{k}$. A similar argument holds for ω. That is, we can find a velocity $\underline{V}^{(\omega)}$ with which values of ω are propagated in space-time varying media. In general, $\underline{V}^{(\omega)}$ will not equal $\underline{V}^{(k)}$, except in the limit of spatially homogeneous, time-invariant media.

2.4 Comments on Energy Flow

Since in a space-time varying medium ω and $\underline{k}$ are not propagated with $\underline{V} = \nabla_k \omega$, we should not be surprised to find that the energy flux does not flow with this velocity either. To consider this problem let us limit our discussion to isotropic, lossless dielectrics. We then have, upon using Eq. (2) in (1) and employing the assumptions expressed in Eq. (5), that the equation satisfied by $\underline{e}_o$ is

$$\begin{aligned} &\underline{k}(\nabla\psi \cdot \underline{e}_o) + \underline{e}_o(\nabla \cdot \underline{k}) + 2(\underline{k} \cdot \nabla)\,\underline{e}_o \\ &= -\mu_o \epsilon \left[\underline{e}_o \left(\frac{\partial \omega}{\partial t} + 2\omega \frac{\partial \psi}{\partial t} \right) + 2\omega \frac{\partial \underline{e}_o}{\partial t} \right], \end{aligned} \tag{39}$$

where $\psi = \ln \epsilon$. (Note that Eq. (34) reduces to the results of Section 3.1.3 in Born and Wolf (1959) in the limit when the medium does not depend on time.) Now recalling that $\underline{k} \cdot \underline{e}_o = 0$, taking the scalar product of e_o^* with Eq. (39), and then adding its complex conjugate to the resulting equation gives:

$$|\underline{e}_o|^2 \frac{\partial k}{\partial \ell} + k \frac{\partial}{\partial \ell}|\underline{e}_o|^2 = -\mu_o \epsilon \left[|\underline{e}_o|^2 \left(\frac{\partial \omega}{\partial t} + 2\omega \frac{\partial \psi}{\partial t} \right) + \omega \frac{\partial |\underline{e}_o|^2}{\partial t} \right], \tag{40}$$

where $\ell = \hat{k} \cdot \underline{x}$. If we now substitute $k = \omega \sqrt{\mu_o \epsilon}$ into Eq. (40), and use Eq. (14) to write $d\omega/dt = -\frac{\omega}{2} \partial\psi/\partial t$, we obtain

$$\frac{\partial}{\partial \ell}\left(\sqrt{\frac{\epsilon}{\mu_o}} |\underline{e}_o|^2 \right) = -\frac{3}{2}\left(\frac{\partial \epsilon}{\partial t} \right) |\underline{e}_o|^2 - \epsilon \frac{\partial |\underline{e}_o|^2}{\partial t} . \tag{41}$$

Let us define an energy flux $I = \sqrt{\frac{\epsilon}{\mu_o}} |\underline{e}_o|^2$. Then in terms of I Eq. (41) becomes

$$\frac{\partial I}{\partial t} + V \frac{\partial I}{\partial \ell} = -I \frac{\partial \psi}{\partial t} . \tag{42}$$

We now consider the flux equation further in the limit when $\nabla\epsilon$ and $\underline{k}$ are in the same direction (say along the z axis). From (42) we then have that:

(a) If $\epsilon(z, t)$ is independent of time then

$$\frac{dI}{\partial t} = \left(\frac{\partial}{\partial t} + V \frac{\partial}{\partial z} \right) I = 0 , \tag{43}$$

so that the energy flux I remains constant as one moves with the group velocity $V = (\epsilon \mu_o)^{-1/2}$. That is, values of I are propagated with the group velocity.

(b) If $\epsilon(z, t)$ is independent of position, Eq. (42) may be written as

$$\frac{d}{dt}(\epsilon I) = \left(\frac{\partial}{\partial t} + V \frac{\partial}{\partial z} \right)(\epsilon I) = 0 , \tag{44}$$

so that it is not the flux, but rather values of (ϵI) which are propagated with the group velocity when ϵ depends on time only.

(c) When the medium varies with both position and time neither I nor (ϵI) are propagated with V. As an example, suppose $(\mu_o \epsilon)^{1/2} = \beta_o(z) + t\ \beta_1(z)$. Then it is readily shown that the invariant of the motion is

$$I \exp\left[2 \int^z \beta_1(z')\, dz' \right] . \tag{45}$$

Therefore, we have demonstrated that it is only for media which are time invariant that the energy flux (Poynting vector) is propagated with the group velocity.

For dielectrics which vary with position and time it is not the energy flux I, but rather some new quantity f(I, $\underline{x}$, t) which is propagated with the generalized group velocity, given by Eq. (12).

2.5 Approximate Solution of Eq. (14) (Small Doppler Approximation)

In many instances it is difficult to obtain exact solutions of Eqs. (11) and (14) for space-time varying media. This is especially true when dispersion is present. However, when the frequency shift in propagating through the medium is small compared with the transmitter frequency, it is possible to solve Eq. (14) by iteration. Let us consider the limit when

$$\frac{\int \frac{\partial K}{\partial t}\, ds}{\omega} << 1 , \tag{46}$$

where, as before $k = K(\omega, \underline{x}, t)$ and $ds = \hat{k} \cdot d\underline{x}$. The integral is along the ray path in the medium between the transmitter and the observer. For the special case of a dielectric, this requires that, in addition to the condition that the medium vary slowly, the path length in the medium cannot be so large that $\int K\, ds >> 1$. When Eq. (46) is satisfied, we may neglect the right hand side of (14), and conclude that ω is approximately a constant of the motion. Furthermore, if (as is usually the case) we specify that $\omega = \omega_o$ for all time at the position of the transmitter, then $\omega \simeq \omega_o$ for all $\underline{x}$ and t (for which Eq. (46) is still satisfied). Using $\omega \simeq \omega_o$ then gives in Eq. (4)

$$|\nabla \phi| = i \frac{\omega_o}{c}\, n(\omega_o, \underline{x}, t) , \tag{47}$$

where $n(\omega, \underline{x}, t)$ is the index of refraction, defined by $|\underline{k}| = \frac{\omega}{c} n$. Equation (47) is, of course, the standard Eikonal used in ray optics. Similarly, in Eq. (6) we have (substituting $|\underline{k}| = \frac{\omega_o}{c} n(\omega_o, \underline{x}, t)$

$$\phi \simeq i \left[\frac{\omega_o}{c} \int n(\omega_o, \underline{x}, t)\, \hat{k} \cdot d\underline{x} - \omega_o t \right] , \tag{48}$$

where, as before, $\hat{k}$ is the unit vector normal to the phase surface, and is determined by solving (47). Finally, using (48) in Eq. (7b) gives for the first iteration to the instantaneous frequency

$$\omega \simeq \omega_o - \frac{\omega_o}{c} \int \frac{\partial n(\omega_o, \underline{x}, t)}{\partial t}\, ds , \tag{49}$$

where $ds = \hat{k} \cdot d\underline{x}$ = path length along the ray. Equation (49) is the result used by ionospheric researchers in studying the Doppler shift through an ionospheric region which varies slowly with path position and time (Weeks, 1958; Kelso, 1960, 1964; Ginzburg, 1964; Bennett, 1967). It is evident, from Eqs. (46) and (49) that Eq. (49) is valid only when the Doppler shift is small compared with the transmitter frequency. For most problems of propagation through the earth's ionosphere, Eq. (49) is an adequate approximation for the instantaneous frequency. However, there are laboratory plasmas and some planetary atmospheres (for example, Jupiter) where (49) may not be a good approximation. In addition, the constraint of Eq. (46) may not hold in many space-time varying dielectrics. In the next section, we will study the exact solutions of Eqs. (11) and (14) in some dielectric materials.

3. APPLICATION TO ISOTROPIC, LOSSLESS DIELECTRICS

3.1 Calculation of the Invarients ω and $\underline{k}$ in Dielectrics

We will now use the results of Section 2 to study the propagation in lossless dielectrics with permittivity varying slowly with position and time. To simplify the problem we will also assume that the propagation is in the same direction as $\nabla\epsilon$, which we choose to be along the z-axis in a rectangular coordinate system. For this case we have that

$$\underline{E} = \underline{e}_o \, e^{\,i \int_L (kdz - \omega dt)} , \tag{50a}$$

where

$$k(z, t) = \omega(z, t) \sqrt{\mu_o \, \epsilon(z, t)} . \tag{50b}$$

Also, from Eq. (14) we have that $\omega(z, t)$ satisfies

$$\frac{\partial \omega}{\partial t} + \frac{1}{\beta} \frac{\partial \omega}{\partial z} = - \omega \frac{\partial}{\partial t} \ln \beta) , \tag{51}$$

where $\beta = \sqrt{\mu_o \epsilon}$. From the theory presented in Eqs. (17) and (18) we know that the solution of (51) will have the form $C_1 = \Phi(C_2)$ where $C_1 = f(\omega, z, t)$ and $C_2 = g(\omega, z, t)$ are the particular integrals of

$$\frac{dt}{1} = \frac{dz}{\left(\frac{1}{\beta}\right)} = - \frac{d\omega}{\omega \frac{\partial (\ln \beta)}{\partial t}} . \tag{52}$$

Equation (52) is not simply solved for arbitrary variations in ϵ with z and t. However, for specific variations in ϵ solutions are readily obtained.

3.1.1 β SEPARABLE

When ϵ is separable we may write $\beta = \beta_1(z)\ \beta_2(t)$. In this case, upon combining the first and third members in (48) we get

$$\omega \beta_2(t) = C_1 \,. \tag{53}$$

Therefore, the observer moving with the velocity $V = 1/\beta$ finds that $\omega\beta_2(t)$ is an invariant. Similarly combining the first and second members of (52) we get

$$\int^{z} \beta_1(z')\, dz' - \int^{t} \frac{dt'}{\beta_2(t')} = C_2 \,, \tag{54}$$

so that the general solution of (51) when ϵ is separable is

$$\omega(z, t) = \frac{1}{\beta_2(t)}\, \Phi\left[\int^{z} \beta_1(z')\, dz' - \int^{t} \frac{dt'}{\beta_2(t')} \right], \tag{55}$$

where the arbitrary function Φ is determined by specifying boundary conditions on ω along any curve in the z - t plane. For example, if β_2 did not depend on time, and we specified $\omega = \omega_o$ at $z = 0$ for all t, then Φ = constant, and therefore ω = constant, as would be expected.

3.1.2 TAYLOR EXPANSION OF β

The situation when β is separable, studied above, does not usually occur in practical situations. However, there are often problems in which it is appropriate to Taylor expand $\beta(z, t)$ in either z or t. For example, we can consider the case in which we desire to study the propagation only over the time interval $t_1 \le t \le t_2$. In that interval we may expand β in Taylor series in t as

$$\beta(z, t) = \beta_o(z) + t\, \beta_1(z) + \cdots \,. \tag{56}$$

If we assume that the first two terms of the Taylor series are an accurate representation of β in the interval $t_1 \le t \le t_2$ we obtain from Eq. (52)

$$\frac{dz}{1} = \frac{dt}{\beta(z, t)} = -\frac{d\omega}{\omega \beta_1(z)} \,. \tag{57}$$

From the first and third members of Eq. (57) we obtain

$$\omega e^{S(z)} = C_1 , \tag{58}$$

where $S(z) = \int^z \beta_1(z')\, dz'$. Therefore $\omega \exp[S(z)]$ is an invariant as one moves with the group velocity $1/\beta$. A second invariant is obtained by solving the first and second members of (57). This gives

$$te^{-S(z)} - \int^z dz' \;\; \beta_o(z') e^{-S(z')} = C_2 . \tag{59}$$

Upon using Eqs. (58) and (59) in the general solution $C_1 = \Phi(C_2)$ we get

$$\omega(z, t) = e^{-S(z)} \Phi\left[te^{-S(z)} - \int^z dz' \;\; \beta_o(z') e^{-S(z')} \right]. \tag{60}$$

The wavenumber $k(z, t)$ is related to ω by $k = \omega\beta$. We can determine the arbitrary function $\Phi(\cdots)$ by specifying boundary conditions on ω. For example, suppose we specify that $\omega = \omega_o$, for all t, at $z = 0$. (This condition is appropriate for the case of a plane wave of frequency ω_o transmitted into a space-time varying half-space.) This requires that, in Eq. (56), the function Φ = constant. We therefore obtain

$$\omega = \omega_o \, e^{-\int_o^z \beta_1(z')\, dz'} . \tag{61}$$

In the limiting case when β_o and β_1 are independent of z this result can be readily shown to be identical with the previous result of Morganthaler (1958) that

$$\omega = \omega_o \frac{\beta(t_o)}{\beta(t)} = \omega_o \frac{(\beta_o + \beta_1 t_o)}{(\beta_o + \beta_1 t)} , \tag{62}$$

where t_o is the time at which the signal at (z, t) was at $z = 0$. To see this, we recall that in spatially homogeneous dielectrics

$$z = \int_{t_o}^{t} \frac{dt'}{\beta} = \int_{t_o}^{t} \frac{dt'}{\beta_o + \beta_1 t'} . \tag{63a}$$

Upon performing the integral in (63a) we obtain

$$\left(\frac{\beta_o + \beta_1 t_o}{\beta_o + \beta_1 t}\right) = e^{-\beta_1 z} \quad . \tag{63b}$$

If (63b) is substituted into (62) it is evident that Eq. (61) and (62) are identical (in the limit of spatially homogeneous dielectrics). It is also interesting to use (61) in Eq. (38) to calculate the velocity $V^{(k)}$ with which values of k are propagated when β is given by Eq. (56). Using (61) in (38) we find

$$V^{(k)} = \frac{1}{\beta}\left\{1 + \frac{\left(\frac{\partial\beta}{\partial z}\right)_t}{\left[\beta\beta_1 - \left(\frac{\partial\beta}{\partial z}\right)_t\right]}\right\}. \tag{64}$$

3.1.3 TRAVELING-WAVE-MODULATED DIELECTRICS

One other case of interest is when β depends on position and time as $z - v_o t$. This situation arises when the dielectric is modulated by a travelling wave, such as an acoustic wave. For this case, upon defining $u = z - v_o t$, we have from the first two members of Eq. (52)

$$\frac{dz}{1} = \frac{v_o dt}{v_o \beta} = \frac{dz - v_o dt}{1 - v_o \beta} = \frac{du}{1 - v_o \beta} \quad . \tag{65}$$

Combining the last member of (65) with the third member of Eq. (52) gives

$$\frac{du}{1 - v_o \beta(u)} = \frac{d\omega}{v_o \omega \frac{d\beta}{du}} \quad . \tag{66}$$

Equation (66) is readily solved to give

$$[1 - v_o \beta (z - v_o t)]\ \omega(z, t) = C_1 \quad . \tag{67}$$

Therefore $\omega(1 - v_o\beta)$ remains invariant as one travels with the group velocity $V = 1/\beta$. It is also simple to obtain the other constant of motion C_2 which is determined from

$$\frac{dt}{dz} = \beta (z - v_o t) \quad . \tag{68}$$

Letting $u = z - v_o t$, so that $v_o \frac{dt}{dz} = 1 - \frac{du}{dz}$, we find as the solution to (68)

$$\int^{(z - v_o t)} \frac{dS}{1 - v_o \beta(S)} - z = C_2 . \tag{69}$$

The general solution for ω is therefore

$$\omega(z, t) = \frac{\Phi\left[\int^{z - v_o t} \left(\frac{dS}{1 - v_o \beta(S)}\right) - z\right]}{1 - v_o \beta(z - v_o t)} . \tag{70}$$

As before, the arbitrary function Φ is determined by specification of ω along some curve in the z - t plane.

As an example, consider a sinusoidal travelling wave disturbance $\beta = \frac{1}{v_p}\left[1 - \delta \cos \alpha(z - v_o t)\right]$ where $\delta < 1$. Then Eq. (70) can be written, after performing the integration (assuming $\left|1 - \frac{v_o}{v_p}\right| \neq \delta \frac{v_o}{v_p}$) as

$$\omega(z, t) = \frac{\Phi\left[\dfrac{\sin \alpha(z - v_o t)}{\left(\frac{\delta v_o}{v_p}\right) + \left(1 - \frac{v_o}{v_p}\right)\cos \alpha(z - v_o t)} - \dfrac{\tan\left(\alpha z \sqrt{\left(1 - \frac{v_o}{v_p}\right)^2 - \left(\frac{\delta v_o}{v_p}\right)^2}\right)}{\sqrt{\left(1 - \frac{v_o}{v_p}\right)^2 - \left(\frac{\delta v_o}{v_p}\right)^2}}\right]}{\left(1 - \frac{v_o}{v_p}\right) + \left(\frac{\delta v_o}{v_p}\right)\cos \alpha(z - v_o t)} . \tag{71}$$

If we again specify that $\omega = \omega_o$ at $z = 0$ for all t, (where $\omega_o >> \alpha v_o$, $\omega_o >> \alpha v_p$) it can be shown, after considerable manipulation, that the function $\Phi(p)$ is identified as

$$\Phi(p) = \omega_o \left[\frac{\left(1 - \frac{v_o}{v_p}\right)\left(1 + \Gamma p^2\right) + \left(\frac{\delta v_o}{v_p}\right)\sqrt{1 + \Gamma p^2}}{1 + p^2\left(1 - \frac{v_o}{v_p}\right)^2}\right] \tag{72}$$

where

$$\Gamma = \left(1 - \frac{v_o}{v_p}\right)^2 - \left(\frac{\delta v_o}{v_p}\right)^2 \tag{73}$$

and p is the argument of $\Phi(\cdots)$ in Eq. (71). It appears from Eqs. (71) to (73) that ω (z, t) can exhibit some rather unusual behavior in the limit when $v_o \to v_p$. This problem has been considered by Hessel and Oliner (1961) who have shown that the difficulty arises because $v_o \to v_p$ gives rise to a singularity in the basic differential equation of Eq. (1a). That is, if (1a) is written in terms of the variable $s = z - vt$, then the coefficient of the $\partial^2 E/\partial s^2$ term becomes vanishingly small as $v_o \to v_p$. Therefore our results in (71) to (73) are limited to the regions where v_o is not too close to v_p.

3.2 Time of Transit in Space-Time Varying Dielectrics

Since the phase and group velocities in a space-time varying medium are functions of position and time, it is not immediately evident how long it would take for a disturbance to travel a distance L along a ray. To study this problem let us consider the motion of the point at which the phase $\Phi = 0$. In particular, let us suppose $\Phi = 0$ for $t = t_o$ at some point $\underline{x}_o$ on a given ray. Then the time t_L at which $\Phi = 0$ will reach another point $\underline{x}_L$ along the ray is a solution of

$$\int_{(\underline{x}_o,\, t_o)}^{(\underline{x}_L,\, t_L)} (\underline{k} \cdot d\underline{x} - \omega\, dt) = 0 \,. \tag{74}$$

To study the solution of Eq. (74) let us specialize to the case when $\underline{k}$ is in the same direction as $\nabla\epsilon$. If we assume that the initial point $\underline{x}_o$ is z = 0 and the observation point is at z = L we can rewrite Eq. (74) as

$$\int_{(0,\, t_o)}^{(L,\, t_L)} (k\, dz - \omega\, dt) = 0. \tag{75}$$

Since the line integral in Eq. (75) is independent of path it can be taken along any curve joining $(0, t_o)$ to (L, t_L) in the z-t plane. For instance, we shall find it convenient to write

$$\int_o^L k(z, o)\, dz - \int_{t_o}^{t_L} \omega(L, t)\, dt = 0 \,. \tag{76}$$

Equation (76) is an integral equation to be solved for the transit time $(t_L - t_o)$. To understand the meaning of Eq. (76) let us first suppose that $\beta(z, t)$ is independent of t, and that $\omega\,(z = 0)$ is specified to be equal to ω_o for all t. Then since $\omega\,(z = 0) = \omega\,(z = L) = \omega_o$, Eq. (76) becomes the obvious result

$$t_L - t_o = \int_o^L \frac{dz}{V(z)} \,, \tag{77}$$

where $V(z) = [\mu_o \epsilon\,(z)]^{-1/2}$. In the other limit when $\beta\,(z, t)$ is independent of z, we get [assuming $k(t = t_o) = k_o$ for all z] that t_L is a solution of

$$L = \int_{t_o}^{t_L} V(t')\, dt' \,, \tag{78}$$

which is the result obtained previously (Fante, 1971).

As an example of the application of Eq. (75) to dielectrics which vary with both position and time, let us consider the case when β is given by Eq. (56), along with the boundary condition that $\omega = \omega_o$ at $z = 0$ for all t. Then from Eq. (61) we have

$$\omega = \omega_o\, e^{-\int_o^z \beta_1(z')\, dz'} \,, \tag{79}$$

$$k = \omega_o\, \beta(z, t)\, e^{-\int_o^z \beta_1(z')\, dz'} \,. \tag{80}$$

Using these equations in (75) gives for the transit time

$$\Delta t = t_L - t_o = e^{S(L)} \int_o^L \beta_o(z')\, e^{-S(z')}\, dz' \,, \tag{81}$$

where $S(z) = \int_o^z \beta_1(g)\, dg$. From Eq. (81) one could also define an equivalent group velocity $\overline{V}(L) = L/\Delta t$, as

$$\bar{V} = \frac{L}{e^{S(L)} \int_0^L \beta_0(z')\, e^{-S(z')}\, dz'} . \tag{82}$$

In the limit of spatially homogeneous, time-invariant dielectrics ($\beta_1 = 0$, β_0 independent of z) Eq. (82) reduces to the usual result $\bar{V} = (\beta)^{-1}$.

3.3 Discussion of Transmission Through a Dielectric Slab

The results of Sections 3.1 and 3.2 can be applied to considering the transmission of plane waves through a lossless dielectric slab of thickness L in which the permittivity varies slowly with space and time. Let us denote the solutions in the slab by

$$E = e_0(z, t; z_0, t_0)\, e^{-i \int (\omega dt \pm k dz)} , \tag{83}$$

where ω and k are the appropriate solutions of Eq. (51) and $e_0(z, t)$ is the appropriate solution of Eq. (39). For example, if $\beta(z, t) = \beta_1(z)\ \beta_2(t)$ then

$$e_0(z, t; z_0, t_0) = e_0(0) \left[\frac{\beta_1(z_0)}{\beta_1(z)}\right]^{1/2} \left[\frac{\beta_2(t_0)}{\beta_2(t)}\right]^{3/2} . \tag{84}$$

Now suppose the slab occupies the region $0 \le z \le L$, and a plane wave $E_1 = \exp[i\omega_0(t - z/c)]$ is normally incident upon the slab from the region $z < 0$. Then the field transmitted through the slab at time t_s, into the region $z > L$, will consist of a number of components. First, there will be a wave which crosses (from $z < 0$) the $z = 0$ boundary at time τ_1, and is transmitted directly through the dielectric, arriving at $z = L^+$ at the time t_s. Next, there is a wave which crosses the $z = 0$ boundary at time τ_3, and arrives at $z = L^+$ at the time t_s, after being internally reflected at time τ_2 by the $z = L$ boundary and at time τ_1 by the $z = 0$ boundary. Next, there is a wave which crosses (from $z < 0$) the $z = 0$ boundary at time τ_5 and arrives at $z = L^+$ at the time t_s, after being reflected twice at the $z = 0$ boundary (at times τ_1 and τ_3) and twice by the $z = L$ boundary (at times τ_2 and τ_4), and so forth. Let us define $R(0, \tau)$ as the internal reflection coefficient at $z = 0^+$ boundary at time τ, $R(L, \tau)$ as the internal reflection coefficient at $z = L^-$ at time τ (for the case when the medium is spatially homogeneous $R(0, \tau) = R(L, \tau) = \left[\sqrt{\epsilon(\tau)/\epsilon_0} - 1\right] \left[\sqrt{\epsilon(\tau)/\epsilon_0} + 1\right]^{-1}$), $T(\tau)$ as the transmission coefficient, from $z = 0^-$ to $z = 0^+$ at time τ, and $\hat{T}(\tau)$ as the transmission coefficient from $z = L^-$ to $z = L^+$. Then, the transmitted field at $z = L^+$ can be written as

$$E(z = L^+,\ t = t_s) = \hat{T}(t_s) \left\{ T(\tau_1)\, A(\tau_1)\, e^{i\omega_o \tau_1} \right.$$

$$+ T(\tau_3)\, A(\tau_3)\, R(0, \tau_1)\, R(L, \tau_2)\, e^{i\omega_o \tau_3} \tag{85}$$

$$\left. + T(\tau_5)\, A(\tau_5)\, R(0, \tau_1)\, R(L, \tau_2)\, R(0, \tau_3)\, R(L, \tau_4)\, e^{i\omega_o \tau_5} + \cdots \right\},$$

where $A(\tau) = e_o(z = L^+,\ t = t_s;\ z_o = 0,\ t_o = \tau)$. From (85) it is clear that the nature of the transmitted field will be known once the times τ_1, τ_2, $\tau_3 \cdots$, have been determined. Extending the discussion of Section 3.2 we see that these are solutions of the equations

$$\int_0^L k(z, t_s)\, dz = \int_{\tau_1}^{t_s} \omega(0, t')\, dt' , \tag{86}$$

$$-\int_L^0 k(z, \tau_1)\, dz = \int_{\tau_2}^{\tau_1} \omega(L, t')\, dt' , \tag{87}$$

$$\int_0^L k(z, \tau_2)\, dz = \int_{\tau_3}^{\tau_2} \omega(0, t')\, dt' . \tag{88}$$

Therefore τ_1, τ_2, and so forth, could be obtained by first solving (86) for τ_1. Then using the solution of Eq. (86) in Eq. (87), the resulting equation could be solved for τ_2, and so forth.

Equation (85) can also be applied to the case of transmission into an infinite half-space. We can obtain this limit by setting $\hat{T}(t) = 1$ and $R(L, \tau) = 0$. We therefore obtain for the field at a point z_s at time t_s

$$E(z_s, t_s) = T(\tau_1)\, A(\tau_1)\, e^{i\omega_o \tau_1} , \tag{89}$$

where τ_1 is the solution of the equation

$$\int_0^{z_s} k(z', t_s)\, dz' = \int_{\tau_1}^{t_s} \omega(0, t')\, dt' . \tag{90}$$

When $\beta(z, t)$ is given by Eq. (56), we can use the results of Section 3.2 to express τ_1 as

$$\tau_1 = t_s - e^{S(z_s)} \int_0^{z_s} \beta_o(z')\, e^{-S(z')}\, dz' . \tag{91}$$

Once τ_1 is known, $T(\tau_1)$ can be obtained by applying the usual WKB methods to the profile $\epsilon(z) = \mu_o^{-1} [\, \beta_o(z) + \tau_1\ \beta_1(z)\,]^2$.

As an example of the application of the more general result of Eq. (85) let us consider the case when the dielectric slab is spatially homogeneous. Then (85) becomes

$$\begin{aligned} E(z = L^+,\ t = t_s) = \hat{T}(t_s)\ \Big\{ & T(\tau_1)\, A(\tau_1)\, e^{i\omega_o \tau_1} \\ & + T(\tau_3)\, A(\tau_3)\, R(\tau_1)\, R(\tau_2)\, e^{i\omega_o \tau_3} \\ & + T(\tau_5)\, A(\tau_5)\, R(\tau_1)\, R(\tau_2)\, R(\tau_3)\, R(\tau_4)\, e^{i\omega_o \tau_5} + \cdots \Big\} , \end{aligned} \tag{92}$$

where

$$A(\tau) = \left[\frac{\epsilon(\tau)}{\epsilon(t_s)} \right]^{3/4} \tag{93}$$

and τ_m is the solution of

$$\int_{\tau_m}^{t_s} v(t')\, dt' = mL \qquad m = 1, 2, 3, \cdots . \tag{94}$$

To examine the various frequency components present in the transmitted wave of Eq. (92) we can Taylor expand the functions $\tau_m(t)$. That is

$$\tau_m(t) = \tau_m^o + (t - t_s)\left(\frac{\partial \tau_m}{\partial t}\right)_{t_s} + \cdots , \tag{95}$$

where

$$\left(\frac{\partial \tau_m}{\partial t}\right)_{t_s} = \frac{v(t_s)}{v(\tau_m)} = \left[\frac{\epsilon(\tau_m)}{\epsilon(t_s)}\right]^{1/2} . \tag{96}$$

Using this result we may rewrite (92) as:

$$E(z = L^+, t) = \sum_{\ell} B_\ell \, e^{i\Omega_\ell (t - t_s)} , \tag{97}$$

where

$$B_\ell = T(\tau_{2\ell-1})\,\hat{T}(t_s)\left[\frac{\epsilon(\tau_{2\ell-1})}{\epsilon(t_s)}\right]^{3/4} e^{i\omega_o \tau_{2\ell-1}^o} \prod_{j=0}^{2(\ell-1)} R(\tau_j)$$

and

$$\Omega_\ell = \omega_o\left[\frac{v(t_s)}{v(\tau_{2\ell-1})}\right] = \omega_o\left[\frac{\epsilon(\tau_{2\ell-1})}{\epsilon(t_s)}\right]^{1/2} . \tag{98}$$

From Eq. (97) we see that the transmitted signal consists of components at the instantaneous frequencies Ω_1, Ω_2, Ω_3, $\Omega_4 \cdots$. This interpretation assumes that B_ℓ is a slowly varying function of time in comparison with exp ($i\Omega_\ell t$), which is the case in the WKB approximation. The importance of the frequency components at Ω_2, Ω_3, Ω_4, and so forth, in comparison with that at Ω_1, will depend on the amplitude of the reflection coefficient R. For $R << 1$ only Ω_1 will be significant, but for R near unity this conclusion is clearly not true.

Therefore, we see that, because of the spatial boundaries, the transmitted signal has components at Ω_1, Ω_2, $\Omega_3 \cdots$, and not just at Ω_1 as found by Morganthaler (1958), who did not account for boundary effects. For relative permittivities near unity the components at Ω_2, Ω_3, and so forth, will be negligible compared with that at Ω_1. For large relative permittivities, however, the higher order frequency components will be significant.

3.4 Further Discussion – Ray Tracing Methods

In the general case when the direction of propagation is not along $\nabla\epsilon$, it is not a simple matter to solve Eq. (11) or (14) for space-time varying dielectrics. In many cases it is acceptable to use the approximation of Eq. (49), but for others this may not be possible. In such cases it appears most appropriate to approximate the temporal behavior of the dielectric in a stepwise fashion. For example, in the time interval $0 \le t \le t_N$ the function $\beta(\underline{x}, t)$ can be approximated by: $\beta = \beta_1(\underline{x})$ for $0 \le t < t_1$ $\beta = \beta_2(\underline{x})$ for $t_1 < t < t_2 \cdots$, $\beta = \beta_N(\underline{x})$ for $t_{N-1} < t \le t_N$, where we assume that $|(\beta_j - \beta_{j-1})/\beta_j| << 1$. Now suppose we have a signal of frequency ω_1 which enters the dielectric along the direction $\underline{k}_1$ at time $t = 0$. Then in the interval $0 \le t < t_1$ we can ray trace (using Snell's Law) to determine the path along which the signal will travel. The distance S_1 along the ray, which the phase $\phi = \phi_o$ will traverse in time interval t_1, is obtained by solving

$$t_1 = \int_o^{S_1} \beta_1(\underline{x})\, dS\,, \tag{99}$$

where $dS = \hat{k} \cdot d\underline{x}$, and the integral is taken along the ray path. At $t = t_1$, β is suddenly changed to $\beta_2(x)$. The new frequency ω_2 associated with the point we are following along the ray is (see Eq. 33)

$$\omega_2 = \omega_1 \frac{\beta_1(\underline{x}_1)}{\beta_2(\underline{x}_1)}, \tag{100}$$

where $\underline{x}_1$ is the coordinate of the point $S = S_1$ on the ray. Also, if $\hat{k}(t_1 - \delta)$ is the normal to the phase surface at $t_1 - \delta$, then at $t_1 + \delta$ the normal is still in the same direction, by virtue of Eq. (31). That is, if the angle between $\hat{k}(t_1 - \delta)$ and the z axis is θ_1, then the angle between $\hat{k}(t_1 + \delta)$ and the z axis is also θ_1. The new ray path (valid during $t_1 < t < t_2$) is determined by ray tracing starting at $\underline{x} = \underline{x}_1$, along the initial direction $\hat{k}(t_1 + \delta) = \hat{k}(t_1 - \delta)$. The distance $S_2 - S_1$ which the wavefront will travel along the new ray path in the interval $t_1 < t < t_2$ is obtained by solving

$$t_2 - t_1 = \int_{S_1}^{S_2} \beta_2(\underline{x})\, dS\,. \tag{101}$$

We can continue the above process until we have traced the progress of the ray point we have chosen for all values of time in the interval $0 \le t \le t_N$.

4. APPLICATION TO DISPERSIVE MEDIA

4.1 Discussion of the Invariants

The propagation in dispersive media is far more complex than in dielectrics. As the simplest example, consider the propagation in a lossless, isotropic plasma in which the propagation is in the same direction as $\nabla\omega_p$, where ω_p is the electron plasma frequency. In this case the dispersion relation is $c^2k^2 = \omega^2 - \omega_p^2(x,t)$, so that Eq. (14) becomes (assuming ω is not too close to ω_p)

$$\frac{\partial\omega}{\partial t} + c\sqrt{1 - \frac{\omega_p^2}{\omega^2}}\ \frac{\partial\omega}{\partial z} = \frac{1}{2\omega}\ \frac{\partial\omega_p^2}{\partial t}\ . \tag{102}$$

From Eqs. (17) and (18) we have that the solution of (102) will have the form $C_1 = \Phi(C_2)$, where $C_1 = f(\omega, z, t)$ and $C_2 = g(\omega, z, t)$ are the particular integrals of

$$\frac{dt}{1} = \frac{dz}{c\sqrt{1 - \frac{\omega_p^2}{\omega^2}}} = \frac{2\omega\, d\omega}{\frac{\partial}{\partial t}(\omega_p^2)}\ . \tag{103}$$

Unfortunately, Eq. (103) is not readily solved for the constants of motion, except in the limits when ω_p depends on z or t, but not on both. For example (as expected from the discussion of Section 2) we have that $\omega = C_1$ when ω_p is independent of time. This means that ω is an invariant as one moves with the group velocity $V = c(1-\omega_p^2/\omega^2)^{1/2}$. When ω_p is independent of position we find that the particular integral $C_1 = \omega^2 - \omega_p^2(t)$, which is just a statement of the fact that values of k are propagated with the group velocity on spatially homogeneous media.

We have not been able to obtain the exact invariants of the motion when ω_p is an arbitrary function of z and t. In most practical plasma problems, however, we are fortunate that the inequality of Eq. (46) is satisfied so that ω can be approximated by the result of Eq. (49). One can easily imagine problems where Eq. (46) is not satisfied, however, and in this case ray tracing methods seem most appropriate. In the next section we will discuss how ray tracing can be applied to space-time varying plasmas.

4.2 Ray Tracing in Space-Time Varying Dispersive Media

The simplest method of applying ray tracing techniques to a lossless, isotropic, space-time varying plasma is to approximate the temporal variations in a stepwise fashion as was done in Section 3.4. That is, we approximate the plasma frequency

$\omega_p(x, t)$ by: $\omega_p = \omega_{p1}(\underline{x})$ for $0 \le t \le t_1$, $\omega_p = \omega_{p2}(\underline{x})$ for $t_1 < t < t_2$, $\omega_p = \omega_{p3}(\underline{x})$ for $t_2 < t < t_3$, and so forth. We assume that

$$\left| \frac{\omega_{p1} - \omega_{p2}}{\omega_{p1}} \right| << 1 ,$$

and so forth. To illustrate the method we will further assume that the plasma occupies the half-space $z > 0$, and is spatially stratified in the z direction only. At $t = 0$, a point p on the envelope of a signal of frequency ω_1 enters the medium at an angle θ_o relative to the z axis. Then by Snell's Law we have that the ray path during the time interval $(0, t_1)$ is determined from

$$\left[1 - \frac{\omega_{p1}^2(0)}{\omega_1^2} \right]^{1/2} \sin \theta_o = \left[1 - \frac{\omega_{p1}^2(z)}{\omega_1^2} \right] \sin \theta(z) .$$

The component* at frequency ω_1 moves along this path at the group velocity

$$\left| \underline{V}_1 \right| = c \left[1 - \omega_{p1}^2 / \omega_1^2 \right]^{1/2} ,$$

so that during the time interval $(0, t_1)$ the distance S_1 travelled along the ray by the point p on the envelope of the signal is the solution of

$$t_1 = \int_o^{S_1} \frac{dS}{V_1(z)} , \tag{104}$$

where the integration is along the ray path (in isotropic media the ray and group paths are the same). We could also have chosen to follow the progress of a particular value of phase $\phi = \phi_1$. In the time interval t_1, the distance S_1' travelled along the ray by this value of phase is then the solution of

$$t_1 = \int_o^{S_1'} \frac{dS}{v_{p1}} , \tag{105}$$

*The above discussion would apply to the motion of a wave packet with frequency spectrum initially centered sharply about $\omega = \omega_1$.

where $v_{p1} = c/\sqrt{1 - (\omega_{p1}/\omega_1)^2}$.

Now suppose that at $t = t_1 - \delta$ ($\delta \to 0$) the point p on the envelope has reached the point $\underline{x} = \underline{x}_1$, and that the ray path at that point makes an angle θ_1 with the z-axis. At $t = t_1$ the plasma frequency is suddenly changed from $\omega_{p1}(z)$ to $\omega_{p2}(z)$. By virtue of Eq. (31), θ cannot change instantaneously so that $\theta(t_1 - \delta) = \theta(t_1 + \delta) = \theta_1$. Therefore, the new ray path in the plasma during the interval $t_1 < t < t_2$ is determined from

$$\left(1 - \left[\frac{\omega_{p2}(z_1)}{\omega_2}\right]^2\right)^{1/2} \sin\theta_1 = \left(1 - \left[\frac{\omega_{p2}(z)}{\omega_2}\right]^2\right)^{1/2} \sin\theta(z), \tag{106}$$

where the new frequency ω_2 associated with the point p we are following is

$$\omega_2 = \left[\omega_1^2 - \omega_{p1}^2(z_1) + \omega_{p2}^2(z_1)\right]^{1/2}. \tag{107}$$

This component now moves along the ray trajectory with the group velocity

$$\left|\underline{V}_2\right| = V_2 = c\left[1 - \left(\frac{\omega_{p2}(z)}{\omega_2}\right)^2\right]^{1/2} \tag{108}$$

and during the interval $t_1 < t < t_2$ traverses the distance $S_2 - S_1$ along the ray, where $S_2 - S_1$ is the solution of

$$t_2 - t_1 = \int_{S_1}^{S_2} \frac{dS}{V_2} \tag{109}$$

and the integral is along the ray.

The above procedure can be repeated continually until the position and frequency of the point p of interest have been determined everywhere in the time interval of interest. Of course, if in any region of space (or time) we reach the situation where ω is close to ω_p, the WKB method is no longer valid, and one must do a more careful analysis (see, for example, Kelso, 1964; Ginzburg, 1964).

5. COMMENT ON ABSORPTION

To keep our discussion relatively simple we have avoided considering the effect of absorption. When absorption is present we find that [since $\phi(x, t)$ must be unique even when losses are present] Eq. (9) still holds, except that ω and $\underline{k}$ may both be complex numbers (and in addition the real and imaginary parts of the $\underline{k}$ vector will generally lie along different directions). However, our interpretation of various quantities, such as the group velocity and the ray paths is no longer valid. For example, when absorption is present $|\underline{V}| = |\nabla_k \omega|$ may be greater than the speed of light, and consequently no longer retains any physical meaning even in the limit of spatially-homogeneous, time invariant media.

In order to understand why ω and $\underline{k}$ may both be complex in lossy space-time varying media, we first realize that since Eq. (9) is valid when Eq. (31) must also still hold, so that if the properties of a lossy medium are abruptly changed at $t = t_1$ we still have that $\underline{k}(t_1 + \delta) = \underline{k}(t_1 - \delta)$. In a lossy dielectric k and ω are related through $k^2 = \omega^2 \mu_o \epsilon + i\omega \mu_o \sigma$, where σ is the conductivity of the medium. We now suppose that $\epsilon = \epsilon_1$, $\sigma = \sigma_1$, and $\omega = \omega_1$ = real for $t < t_1$; and $\epsilon = \epsilon_2$, $\sigma = \sigma_2$, for $t > t_1$. Then since $k = |\underline{k}|$ is continuous at $t = t_1$ we can determine the new frequency ω_2 during $t > t_1$ from

$$\omega_1^2 \mu_o \epsilon_1 + i\omega_1 \mu_o \sigma_1 = \omega_2^2 \mu_o \epsilon_2 + i\omega_2 \mu_o \sigma_2 . \tag{110}$$

If ω_2 were real we would require that

$$\omega_2 = \left(\frac{\epsilon_1}{\epsilon_2}\right)^{1/2} \omega_1 , \tag{111a}$$

and

$$\omega_2 = \left(\frac{\sigma_1}{\sigma_2}\right) \omega_1 . \tag{111b}$$

Equations (111a) and (111b) cannot be simultaneously satisfied, except for the special case when $(\epsilon_1/\epsilon_2)^{1/2} = (\sigma_1/\sigma_2)$. We therefore conclude that for Eq. (110) to be satisfied in general, ω_2 must be complex. In particular denoting $\omega_2 = \omega_2' + i\omega_2''$ we have

$$\omega_1^2 \epsilon_1 = \left[(\omega_2')^2 - (\omega_2'')^2\right] \epsilon_2 - \omega_2'' \sigma_2 , \tag{112a}$$

$$\omega_1 \sigma_1 = 2\omega_2' \omega_2'' \epsilon_2 + \omega_2' \sigma_2 . \tag{112b}$$

Equations (112a) and (112b) can be solved for ω_2' and ω_2''. We will not probe too deeply into the effects of absorption. This will be deferred to a later paper. We do comment that the analysis becomes relatively straightforward in the limit when the absorption is small ($\sigma/\omega\epsilon << 1$) since then the relevant equations may be studied using perturbation methods. For example, the ray paths can be determined by neglecting absorption, and so forth.

6. SUMMARY

We have studied the properties of the WKB solutions in lossless, isotropic space-time varying media. It was found that, in principle, one can always obtain constants of the motion which lead to a complete determination of the frequency and wavenumber, once appropriate boundary conditions have been specified. Once ω and k are known one can readily study the transmission through space-time varying media, as was illustrated in Section 3.3. In general, however, the constants of motion are not always easily obtained. In such problems we have shown in Sections 3.4 and 4.2 that one can obtain solutions by modelling the medium by a series of temporal steps. That is, the index of refraction $n(\underline{x}, t)$ is approximated by $n_1(\underline{x})$ in $0 \le t \le t_1$, $n_2(\underline{x})$ in $t_1 < t < t_2$, and so forth, and ray tracing techniques are applied during each time interval.

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