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L. G. HANSCOM FIELD, BEDFORD, MASSACHUSETTS

Transient Signal Propagation in Lossless, Isotropic Plasmas Volume I

R. E. HASKELL
C. T. CASE

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OFFICE OF AEROSPACE RESEARCH
United States Air Force

Abstract

The concept of the wave packet is used to obtain some general results on the propagation of signals in dispersive media. The dispersion of a Gaussian carrier pulse and a square wave carrier pulse in an isotropic plasma is carried out using the wave packet concept. A general analysis of transient wave propagation in isotropic plasmas is given using Laplace transform methods. Solutions are given in terms of series solutions which may be expressed in terms of Lommel functions. Integral solutions which can be easily evaluated numerically are derived using the convolution theorem and using contour integration techniques. The solutions are useful for short dispersion lengths.

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Transient Signal Propagation in Lossless, Isotropic Plasmas Volume I

I. INTRODUCTION

The study of propagation of transient signals in dispersive media dates back to the early years of this century. Following Einstein's publication of his special theory of relativity, concern arose over the fact that in regions of anomalous dispersion the group velocity is greater than the free-space velocity of light, c. Since it was believed that the group velocity was the velocity at which energy is transported by the wave, this condition of anomalous dispersion appeared to violate Einstein's theory.

This paradox was correctly explained by Sommerfeld (1914) who showed by making a high frequency expansion that the very first part of a signal, called the signal wavefront, arrives at a given point with the velocity of light, c. Sommerfeld's solution is valid for only a short time after the arrival of the signal wavefront. By using a saddlepoint method of integration Brillouin (1914) found solutions which are valid in a certain time interval following the Sommerfeld region. The signal in this region is called a precursor since it precedes the arrival of the main signal. This work of Sommerfeld and Brillouin has been summarized in a book by Brillouin (1960) in which some of the important early papers have been reprinted.

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The arrival of the main signal follows the precursors. However, the standard saddlepoint method of integration cannot be used in this region since the saddlepoint is approaching a pole in the complex plane. Methods for appropriately modifying the saddlepoint method under these conditions have been discussed by Cerrillo (1950), van der Wearden (1950), and Clemmow (1950). These techniques have been used by Pearson (1953) to describe transient propagation in acoustic waveguides, which have the same dispersion equation as a lossless isotropic plasma. Electromagnetic waveguides also have the same dispersion equation and the saddlepoint method of integration has been applied to these problems by Cerrillo (1948), Namiki and Horiuchi (1952), and Karbowiak (1957).

The solutions discussed so far are approximate solutions that are valid only after the signal has propagated a long distance through the dispersive media. The detailed consideration of these types of solution will form the subject matter of a second and subsequent report on this topic.

The present report consists of two main parts. In Section 2 the well known concepts of a wave packet and group velocity are applied to the propagation of pulses in a dispersive medium. This method has been used to describe the propagation of a rectangular pulse in a waveguide by Cohn (1952), Elliott (1957), Knop and Cohn (1963), and Wanselow (1962). The propagation of a Gaussian pulse in a waveguide using this method has been described by Forrer (1958). These solutions only apply to quasi-monochromatic signals. In Section 3 more general solutions which are valid for arbitrary signals which propagate only a short distance through the plasma will be considered.

Jaeger and Westfold (1949) have considered both the propagation of an initial disturbance in a plasma when the spatial dependence is prescribed as well as the disturbance that is radiated when the time variation of the electric field is prescribed at some position. They have written solutions in terms of convolution integrals and have studied the Fourier spectra of several initial disturbances. They have applied their results to the solar corona in an attempt to explain some of the phenomena associated with bursts of solar noise. Some numerical results based on convolution integrals will be discussed in Section 3.1. Some of these results have been given by Case (1965).

Exact integral solutions may be obtained by contour integration. Cerrillo (1948) has discussed such solutions, and in Section 3.3 two possibilities will be described in detail and numerical results will be presented.

For the case of a turn-on sine wave the exact solution may be written as a series of Bessel functions. These solutions have been discussed by Cerrillo (1948), Rubinowicz (1950), Gajewski (1955), Kovtun (1958), and Knop (1964). Cerrillo (1948) and Kovtun (1958) have expressed these solutions in terms of Lommel functions. These series solutions will be discussed in Section 3.2.

Finally, the experimental work in which observations of transient dispersive effects have been observed should be mentioned. As has already been pointed out, the dispersion equation is the same for acoustic waves in fluid-filled tubes, electromagnetic waves in waveguides, and electromagnetic waves in lossless, isotropic plasmas. Each of these situations has been investigated experimentally.

The transient response of sound pulses propagating through fluid-filled tubes has been observed by Anderson and Barnes (1953) and by Proud, Tamarkin, and Kornhauser (1956). Similar experiments have also been carried out by Walther (1961), who in addition observes the pulse compression of a frequency modulated acoustic wave.

Transient signals in waveguides have been observed experimentally by Saxton and Schmitt (1963), and Ito (1964, 1965). The best measurements are those by Ito (1965) who measures the transient response of a short Gaussian envelope carrier pulse. The results show that the theoretical solutions of Forrer (1958) do not adequately predict the response. However, good agreement is obtained with a solution based on the method of stationary phase.

The dispersion of pulsed dc and rf signals in plasmas have been observed by Schmitt (1964, 1965). The response of short unidirectional pulses is oscillatory in nature with a "ringing" frequency that is characteristic of the plasma frequency. Such electromagnetic pulses can therefore be used as a diagnostic tool.

2. WAVE PACKETS AND GROUP VELOCITY IN ISOTROPIC PLASMAS

The simplest ideas concerning the propagation of signals in a dispersive medium involve the concept of a wave packet. A wave packet is a signal which contains a narrow band of frequencies centered about a carrier frequency. The frequency spectrum $E(\omega)$ is then some type of a peaked function and is related to the time function $\mathcal{E}(t)$ by the Fourier transform pair

$$\mathcal{E}(t) = \int_{-\infty}^{+\infty} E(\omega) \; e^{i\omega t} \; d\omega \tag{1}$$

$$E(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathcal{E}(t) \; e^{-i\omega t} \; dt. \tag{2}$$

For example, for a Gaussian envelope carrier pulse

$$\mathcal{E}(t) = E_o \, e^{-\frac{t^2}{2T^2}} \, e^{i\omega_o t} \tag{3}$$

the Fourier transform is

$$E(\omega) = \frac{E_o}{2\pi} \int_{-\infty}^{+\infty} e^{-\frac{t^2}{2T^2}} \, e^{i(\omega_o - \omega)t} \, dt$$

$$E(\omega) = \frac{E_o}{2\pi} \int_{-\infty}^{+\infty} e^{at - bt^2} \, dt. \tag{4}$$

where

$$a = -i\,(\omega - \omega_o)$$
$$b = \frac{1}{2T^2} \quad . \tag{5}$$

Evaluating the integral in Eq. (4) (see Appendix A) and substituting from Eq. (5) one obtains

$$E(\omega) = \frac{E_o}{\sqrt{2\pi}} \; T \; \exp\left\{ - \frac{T^2 (\omega - \omega_o)^2}{2} \right\} \; . \tag{6}$$

This frequency spectrum together with the corresponding time response is shown in Figure 1a.

Similarly, for the rectangular envelope carrier pulse shown in Figure 1b the frequency spectrum is given by

$$E(\omega) = E_o \, \frac{\sin\,(\omega - \omega_o)T}{\pi\,(\omega - \omega_o)} \, . \tag{7}$$

Note that in each of these cases the frequency spectrum becomes narrower and more peaked as the width of the time response increases. Thus the concept of a wave packet can be expected to break down when the carrier pulse becomes very short.

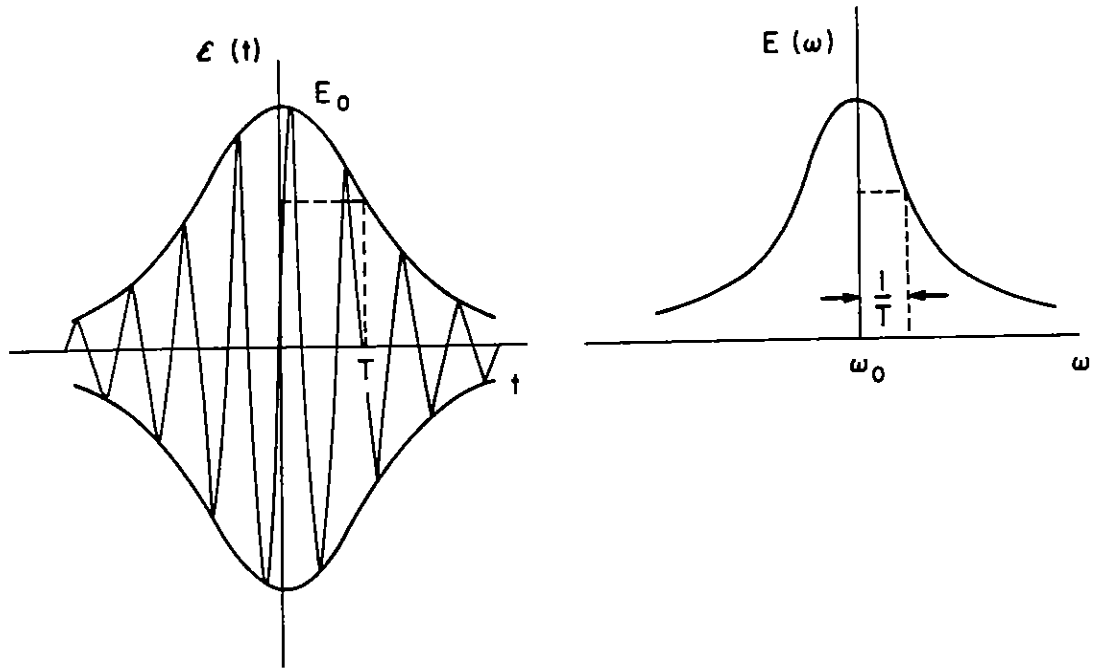

FIG. 1a GAUSSIAN ENVELOPE CARRIER PULSE

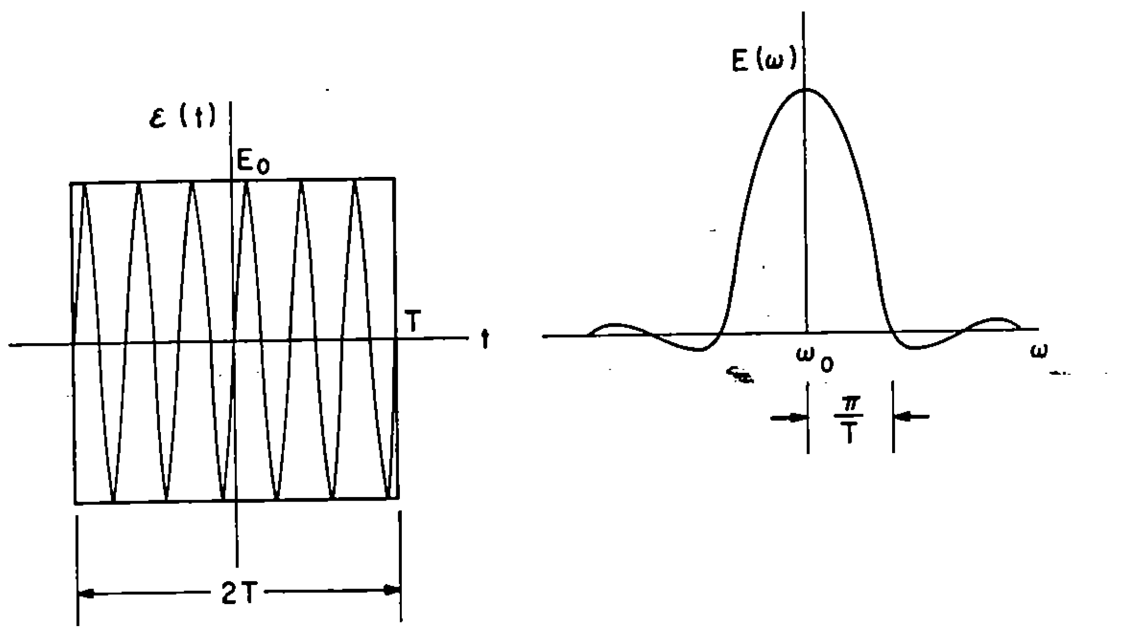

FIG. 1b RECTANGULAR ENVELOPE CARRIER PULSE

Figure 1. Frequency Spectrums for Two Types of Wave Packets

Now consider the propagation of a wave packet in a dispersive medium. Suppose that at z = 0 the time response is given by $\mathcal{E}(t, 0)$ with the corresponding frequency spectrum $E(\omega, 0)$. Then, if the medium is lossless, at some point, z, each frequency component will have undergone an appropriate phase change and the resultant time response will be given by

$$\mathcal{E}(t, z) = \int_{-\infty}^{+\infty} E(\omega, 0)\, e^{i\omega t - i\phi(\omega, z)}\, d\omega. \tag{8}$$

For an isotropic, lossless, homogeneous plasma

$$\phi(\omega, z) = k(\omega)\, z \tag{9}$$

where

$$k(\omega) = \frac{1}{c}\sqrt{\omega^2 - \Pi^2} \tag{10}$$

is the wave number and Π is the electron plasma frequency.

If $E(\omega, 0)$ is sufficiently peaked about ω_o, then $\phi(\omega)$ can be expanded in a Taylor series about ω_o and gives

$$\phi(\omega) = \phi(\omega_o) + (\omega - \omega_o)\, \phi'(\omega_o) + \frac{1}{2}(\omega - \omega_o)^2\, \phi''(\omega_o) + \ldots, \tag{11}$$

or, keeping only terms up to order $(\omega - \omega_o)^2$

$$\phi = \phi_o + V\phi' + \frac{1}{2} V^2 \phi'' \tag{12}$$

where

$$V = \omega - \omega_o. \tag{13}$$

Substitution of Eq. (12) into Eq. (8) gives

$$\mathcal{E}(t, z) = e^{i(\omega_o t - \phi_o)} \int_{-\infty}^{+\infty} E(\omega, 0)\, e^{i(t - \phi')V}\, e^{-\frac{i\phi''}{2}V^2}\, dV. \tag{14}$$

The case of a Gaussian envelope and a rectangular envelope carrier pulse will be considered separately.

2.1 Propagation of a Gaussian Envelope Carrier Pulse

Substitution of the frequency spectrum for a Gaussian envelope carrier pulse given by Eq. (6) into Eq. (14) gives

$$\mathcal{E}(t, z) = E_o \frac{T}{\sqrt{2\pi}} e^{i(\omega_o t - \phi_o)} \int_{-\infty}^{+\infty} \exp\left\{ i(t - \phi') V - \left(\frac{T^2}{2} + \frac{i\phi''}{2}\right) V^2 \right\} dV$$

$$\mathcal{E}(t, z) = E_o \frac{T}{\sqrt{2\pi}} e^{i(\omega_o t - \phi_o)} \int_{-\infty}^{+\infty} \exp\left\{ \alpha V - \beta V^2 \right\} dV$$

$$\mathcal{E}(t, z) = E_o \frac{T}{\sqrt{2\pi}} e^{i(\omega_o t - \phi_o)} \sqrt{\frac{\pi}{\beta}}\; e^{\frac{\alpha^2}{4\beta}} \tag{15}$$

(see Appendix A), where

$$\alpha = i\,(t - \phi')$$

$$\beta = \frac{1}{2}\,(T^2 + i\,\phi''). \tag{16}$$

From Eq. (16) it is found that

$$\frac{\alpha^2}{4\beta} = -\gamma^2 + i\,\zeta$$

where

$$\gamma^2 = \frac{(t - \phi')^2}{2T^2\left[1 + \left(\frac{\phi''}{T^2}\right)^2\right]}, \tag{17}$$

$$\zeta = \frac{\phi''\,(t - \phi')^2}{2T^2\left[1 + \left(\frac{\phi''}{T^2}\right)^2\right]}. \tag{18}$$

Thus, the solution Eq. (15) can be written in the form

$$\mathcal{E}(t, z) = \frac{E_o}{\sqrt[4]{1 + \left(\frac{\phi''}{T^2}\right)^2}} e^{-\gamma^2} \quad \exp\left\{i\left(\omega_o t - \phi_o - \frac{\theta_o}{2} + \zeta\right)\right\} \tag{19}$$

where

$$\theta_o = \tan^{-1}\left(\frac{\phi''}{T^2}\right) . \tag{20}$$

From Eq. (19) it is seen that the wave packet has become frequency modulated while the envelope has remained Gaussian but has decreased in amplitude and spread out. The maximum of the pulse envelope occurs at the delayed time $\phi' = z \left.\frac{\partial k}{\partial \omega}\right|_{\omega_o} = \frac{z}{v_g}$ where

$$v_g = \left.\frac{\partial \omega}{\partial k}\right|_{\omega_o} \tag{21}$$

is the group velocity of the wave packet. From Eq. (10) and Eq. (21) the group velocity for an isotropic plasma can be written as

$$v_g = c\sqrt{1 - \frac{\Pi^2}{\omega_o^2}} . \tag{22}$$

A plot of v_g/c vs Π/ω is shown in Figure 2. For this same plasma we can write $\phi''(\omega_o)$ as

$$\phi''(\omega_o) = -\frac{z}{\omega_o c}\left[\frac{\frac{\Pi^2}{\omega_o^2}}{\left(1 - \frac{\Pi^2}{\omega_o^2}\right)^{3/2}}\right] \tag{23}$$

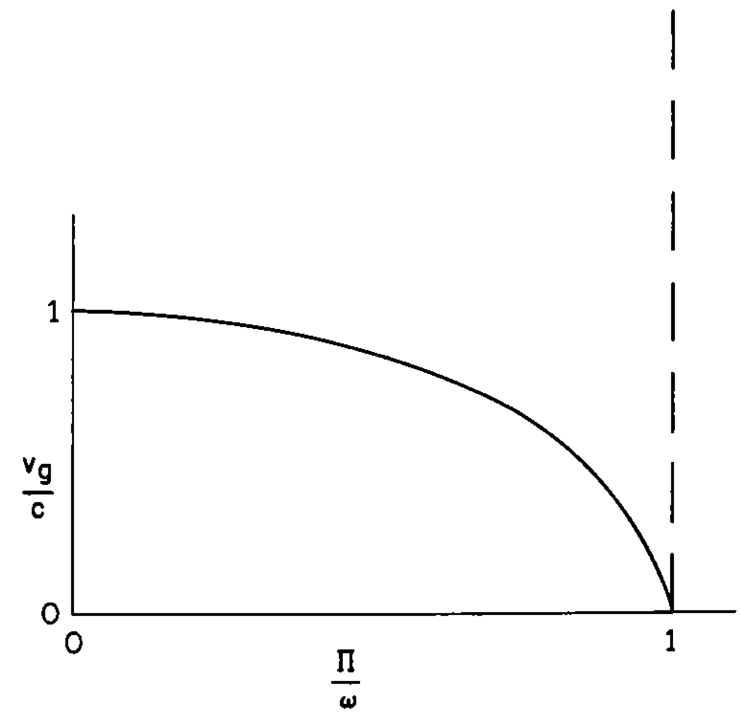

Figure 2. The Group Velocity as a Function of the Plasma Frequency

Figure 3 shows the relation between the normalized plasma frequency

$$P = \frac{\Pi}{\omega_o} \tag{24}$$

and the damping factor $\sqrt{|\phi''(\omega_o)|}$ plotted from Eq. (23).

2.1.1 APPLICATION TO PULSE DEFINITION AND SEPARATION PROBLEM

The results of the previous section will now be applied to the problem of resolving two Gaussian pulses which are close together. Consider a double Gaussian pulse as shown in Figure 4a. After traveling through a plasma a certain distance the two pulses may become dispersed to the degree shown in Figure 4b. This condition, at which the amplitude of each pulse is equal to $e^{-0.5}$ at $(t - \phi') = \frac{\tau_o}{2}$, is defined as the condition of no pulse definition. From Eq. (19) and Eq. (17) this condition occurs when

$$(t - \phi')^2 = \frac{\tau_o^2}{4} = T^2\left[1 + \left(\frac{\phi''}{T^2}\right)^2\right]$$

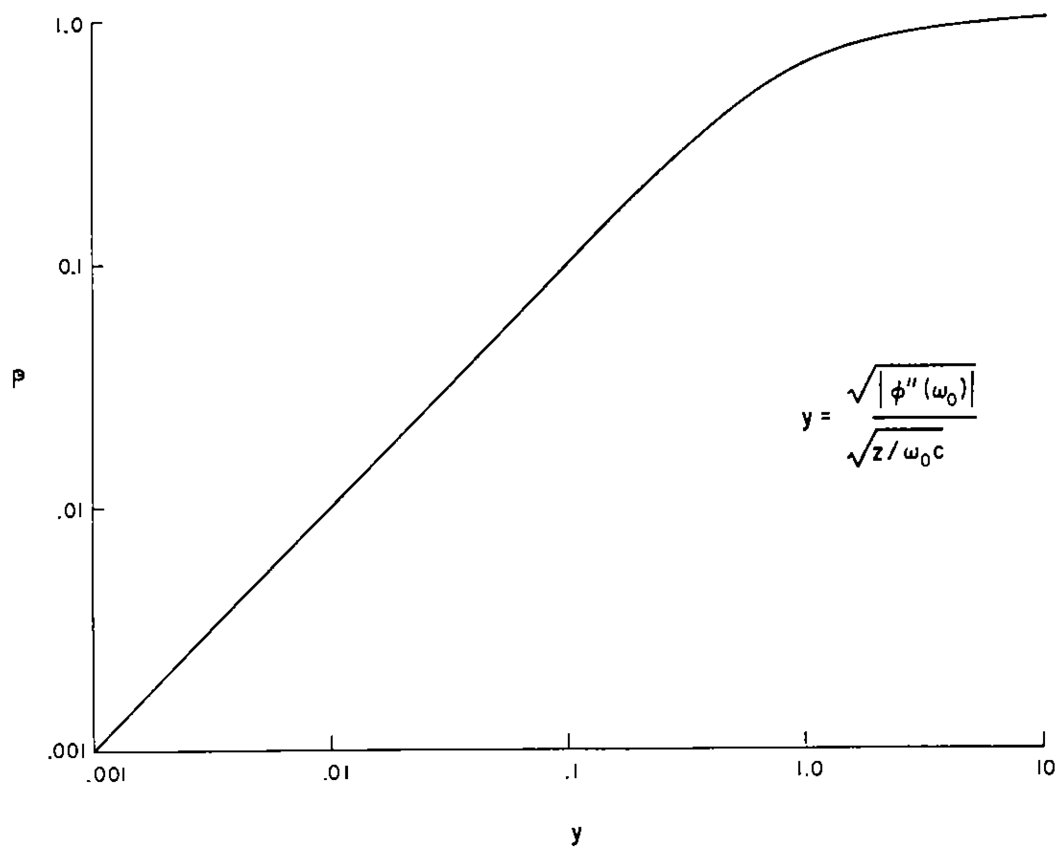

Figure 3. Relation between Normalized Plasma Frequency and the Damping Factor

or when

$$\frac{\tau_o}{2T} = \sqrt{1 + \left(\frac{\sqrt{|\phi''|}}{T}\right)^4} \quad . \tag{25}$$

Equation (25) is plotted in Figure 5 and separates the region of good pulse definition from the region of no pulse definition. This figure indicates the critical ratio of pulse separation to pulse width for different values of $\frac{T}{\sqrt{|\phi''|}}$. Values of $\sqrt{|\phi''|}$ may be conveniently determined from Figure 3 for different plasma frequencies and propagation distances. Due to the quasi-monochromatic nature of the wave packet, the pulse definition given by Eq. (25) is only a rough approximation. A more accurate treatment can be carried out using the methods of saddlepoint integration.

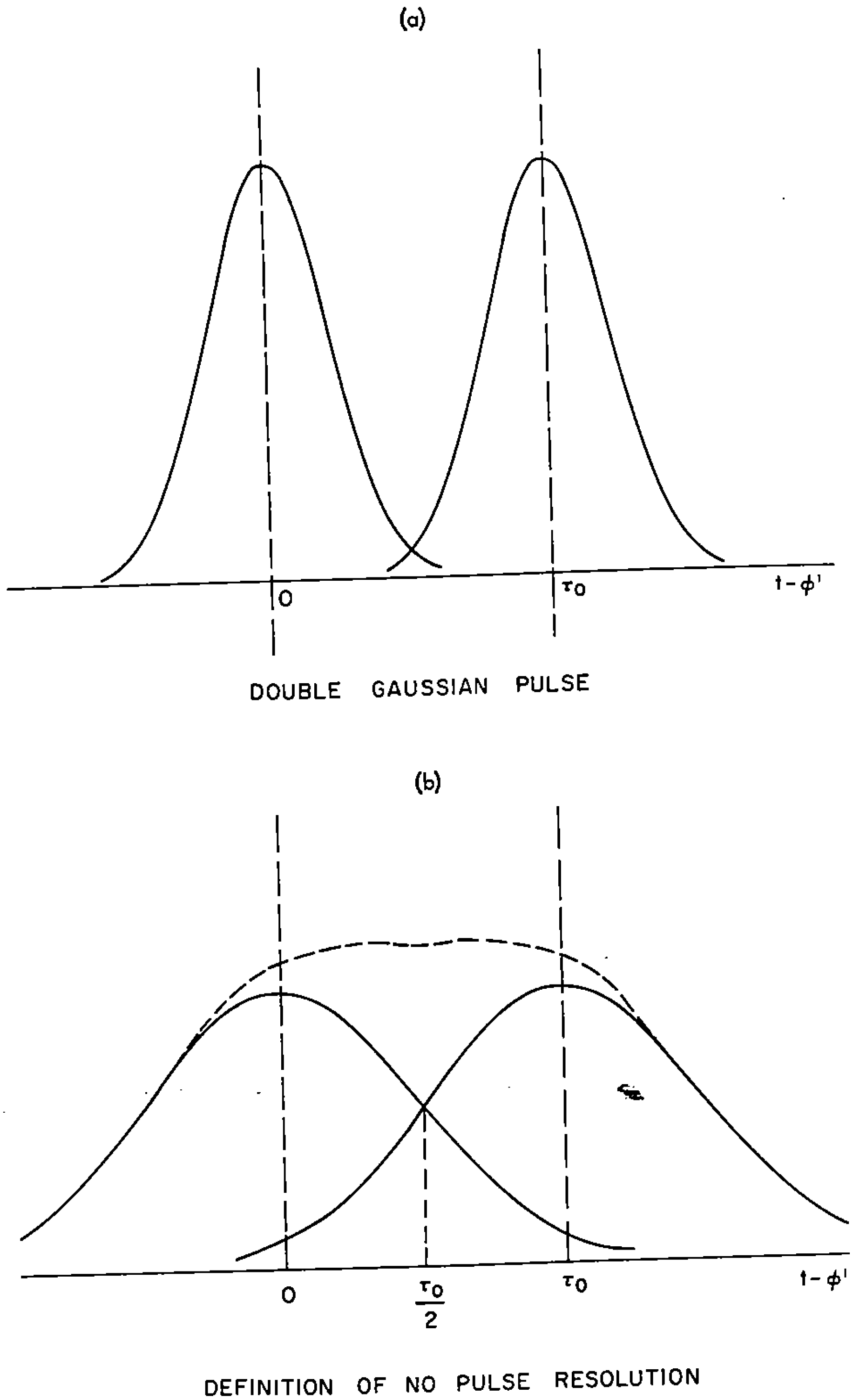

Figure 4. Pulse Definition of Two Gaussian Pulses

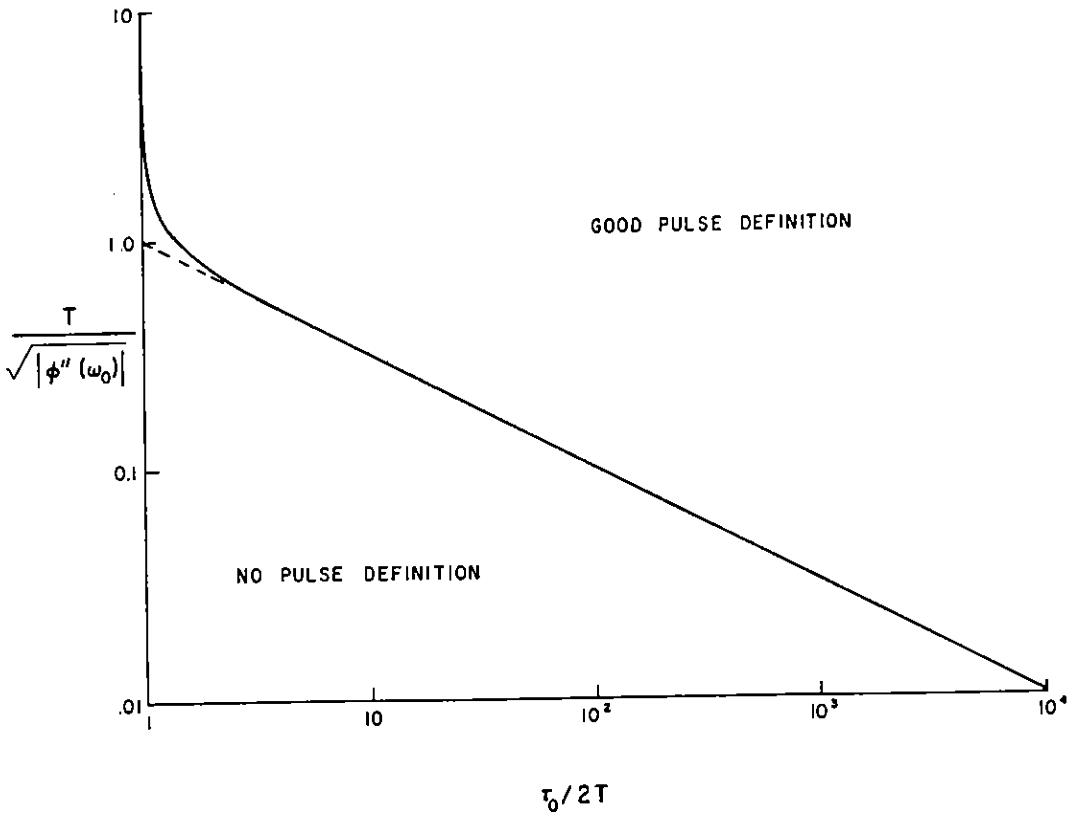

Figure 5. Regions of Good Pulse Definition and No Pulse Definition

2.2 Propagation of a Rectangular Envelope Carrier Pulse

Consider now a rectangular envelope carrier pulse which has the frequency spectrum given by Eq. (7). Using this expression in Eq. (14) one obtains

$$\mathcal{E}(t,z) = \frac{E_o}{\pi} e^{i(\omega_o t-\phi_o)} \int_{-\infty}^{+\infty} \frac{\sin VT}{V} e^{i(t-\phi')V} e^{-i\frac{\phi''}{2}V^2} \, dV$$

$$\mathcal{E}(t,z) = \frac{E_o e^{i(\omega_o t-\phi_o)}}{2\pi i}\left[\int_{-\infty}^{+\infty} \frac{\exp\left\{i(t-\phi'+T)V - i\frac{\phi''}{2}V^2\right\} dV}{V} - \int_{-\infty}^{+\infty} \frac{\exp\left\{i(t-\phi'-T)V - \frac{i\phi''}{2}V^2\right\} dV}{V}\right] \tag{26}$$

$$\mathcal{E}(t,z) = \frac{E_o\, e^{i(\omega_o t - \phi_o)}}{2\pi i}\left[\int_{-\infty}^{\infty} \frac{\exp\left\{i a_1 V - b V^2\right\} dV}{V} - \int_{-\infty}^{\infty} \frac{\exp\left\{i a_2 V - b V^2\right\} dV}{V}\right] \quad (26) \text{ (Cont)}$$

where

$$a_1 = t' + T$$
$$a_2 = t' - T$$
$$b = \frac{i\phi''}{2}$$
$$t' = t - \phi' \quad (27)$$

The integrals in Eq. (26) can be evaluated from the results of Appendix B. One then obtains

$$\mathcal{E}(t,z) = \frac{E_o}{2} e^{i(\omega_o t - \phi_o)} \left[\operatorname{erf}\left(\frac{a_1}{2\sqrt{b}}\right) - \operatorname{erf}\left(\frac{a_2}{2\sqrt{b}}\right)\right] . \quad (28)$$

From Eq. (27) it is found that

$$\frac{a_1}{2\sqrt{b}} = \frac{(1-i)}{2} \quad \frac{(t' + T)}{\sqrt{|\phi''|}} ,$$
$$\frac{a_2}{2\sqrt{b}} = \frac{(1-i)}{2} \quad \frac{(t' - T)}{\sqrt{|\phi''|}} . \quad (29)$$

Substituting Eq. (29) into Eq. (28) and using the identity

$$\operatorname{erf}\left[\frac{\sqrt{\pi}}{2}(1-i)\, z\right] = (1-i)\, F(z) \quad (30)$$

where

$$F(z) = \int_0^z \exp\left(\frac{i\pi u^2}{2}\right) du = C(z) + i\, S(z) \quad (31)$$

is the Fresnel integral function, one obtains

$$\mathcal{E}(t,z) = \frac{E_o}{2}(1 - i)\, e^{i(\omega_o t - \phi_o)} \left[F(A_1) - F(A_2)\right] \tag{32}$$

where

$$A_1 = \frac{t' + T}{\sqrt{\pi\,|\phi''|}} ,$$

$$A_2 = \frac{t' - T}{\sqrt{\pi\,|\phi''|}} . \tag{33}$$

The envelope of the time response Eq. (32) is then given by

$$\left|\mathcal{E}(t,z)\right| = \frac{E_o}{\sqrt{2}} \sqrt{X^2 + Y^2} \tag{34}$$

where

$$X = C(A_1) - C(A_2) ,$$

$$Y = S(A_1) - S(A_2) . \tag{35}$$

Knop and Cohn (1963) have plotted this envelope as a function of $\frac{t'}{2T}$ for different values of $\frac{2\sqrt{|\phi''|}}{T}$. Ginzburg (1962) has plotted the envelope as a function of $\frac{t'}{\sqrt{\pi|\phi''|}}$ for different values of $\frac{2T}{\sqrt{\pi|\phi''|}}$.

The discussion thus far has dealt with the ideas of a wave packet. This notion requires that the frequency spectrum be narrow and peaked. This is clearly not the case during the time interval in which the pulse is building up. For during this time the frequency spectrum is, in general, very wide. Thus, the results of the previous sections are limited and would not be expected to hold during the initial arrival and main buildup of the pulsed signal. In order to study this facet of the problem, a more general treatment of signal propagation in plasmas will now be discussed.

3. GENERAL ANALYSIS OF TRANSIENT SIGNAL PROPAGATION

The general method of obtaining the time response of a signal propagating in an isotropic, lossless plasma will be considered. The analysis will be based on Laplace transform techniques. The general integral to be evaluated may be obtained from Eqs. (8), (9), and (10) by letting $s = i\omega$ and picking up a factor of 2π, and is written as

$$\mathcal{E}(t,z) = \frac{1}{2\pi i} \int_{\gamma_s} E(s, 0) \exp\left\{st - \frac{z}{c}\sqrt{s^2 + \Pi^2}\right\} ds. \tag{36}$$

This expression, which is derived independently in Appendix C, is just the inverse Laplace transform of $E(s, 0) \exp\left\{-\frac{z}{c}\sqrt{s^2+\Pi^2}\right\}$ where $E(s, 0)$ is the Laplace transform of the time response at $z = 0$. The contour γ_s is a straight line between $\sigma_o - i\infty$ and $\sigma_o + i\infty$ where σ_o is to the right of all singularities in the complex s-plane ($s = \sigma + i\omega$).

There are very few initial time responses for which the inverse transform given by Eq. (36) may be found from tables. One such time response whose solution may be written down is a turn-on Bessel function whose frequency is the plasma frequency. That is,

$$\mathcal{E}(t, 0) = J_o(\Pi t)\ U(t), \tag{37}$$

where

$$U(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases} .$$

The Laplace transform of Eq. (37) is

$$E(s, 0) = \frac{1}{\sqrt{s^2 + \Pi^2}} . \tag{38}$$

Equation (36) may then be written

$$\mathcal{E}(t,z) = \mathcal{L}^{-1}\left[\frac{\exp\left\{-\frac{z}{c}\sqrt{s^2+\Pi^2}\right\}}{\sqrt{s^2+\Pi^2}}\right] , \tag{39}$$

which, from tables, is

$$\mathcal{E}(t,z) = \begin{cases} 0 & 0 < t < z/c \\ J_o\left(\Pi\sqrt{t^2-\frac{z^2}{c^2}}\right) & t > \frac{z}{c} \end{cases} . \tag{40}$$

A similar solution is obtained when the initial time response is a step H field. To see this, note from the transformed Maxwell equation

$$\frac{\partial E(s,z)}{\partial z} = -\mu_o s H(s,z) ,$$

that

$$-\frac{1}{c}\sqrt{s^2+\Pi^2}\ E(s,z) = -\mu_o s H(s,z) ,$$

from which

$$E(s,z) = \sqrt{\frac{\mu_o}{\epsilon_o}}\ \frac{s}{\sqrt{s^2+\Pi^2}}\ H(s,z) . \tag{41}$$

Thus for an initial step H field

$$\mathcal{H}(t, 0) = U(t) ,$$

$$H(s, 0) = \frac{1}{s} ,$$

so that, from Eq. (41)

$$E\ (s,\ 0) = \sqrt{\frac{\mu_o}{\epsilon_o}}\quad \frac{1}{\sqrt{s^2 + \Pi^2}}\quad . \tag{42}$$

Since this initial transform is of the same form as Eq. (38), the resulting time response will be, from Eq. (40),

$$\mathcal{E}(t,z) = \begin{cases} 0 & 0 < t < \frac{z}{c} \\ \sqrt{\frac{\mu_o}{\epsilon_o}}\ J_o\left(\Pi\sqrt{t^2 - \frac{z^2}{c^2}}\right) & t > \frac{z}{c} \end{cases}\quad . \tag{43}$$

Other forms of initial time responses which may be of more practical importance do not result in transforms with known inverses. Therefore, in the following sections several different techniques for evaluating the integral Eq. (36) will be investigated.

3.1 Convolution Integrals

The Laplace transform of a function f(t) is

$$F(s) \equiv \mathcal{L}\ [f(t)]\ \equiv \int_0^\infty f(t)\ e^{-st}\ dt \tag{44}$$

and the inverse Laplace transform of F(s) is

$$f(t)\ \equiv\ \mathcal{L}^{-1}\ [F(s)] \equiv \frac{1}{2\pi i}\int_{\gamma_s} F(s)\ e^{st}\ ds. \tag{45}$$

The convolution theorem states that if F(s) is the Laplace transform of f(t) and G(s) is the Laplace transform of g(t), then

$$\mathcal{L}^{-1}\ [F(s)\ G(s)] = \int_0^t f(t - t')\ g(t')\ dt'. \tag{46}$$

Application of this theorem to Eq. (36) by using data from tables (for example, see Erdélyi et al, 1954), which show that

$$\mathcal{L}^{-1}\left[\exp\left\{-\frac{z}{c}\sqrt{s^2+\Pi^2}\right\}\right] = \delta\left(t-\frac{z}{c}\right) - \frac{z\Pi}{c}\,\frac{J_1\left(\Pi\sqrt{t^2-\frac{z^2}{c^2}}\right)U\left(t-\frac{z}{c}\right)}{\sqrt{t^2-\frac{z^2}{c^2}}} \tag{47}$$

gives

$$\mathcal{E}(t,z) = \int_0^t \mathcal{E}(t-t',0)\left[\delta\left(t'-\frac{z}{c}\right) - \frac{z\Pi}{c}\,\frac{J_1\left(\Pi\sqrt{t'^2-\frac{z^2}{c^2}}\right)U\left(t'-\frac{z}{c}\right)}{\sqrt{t'^2-\frac{z^2}{c^2}}}\right]dt'$$

or

$$\mathcal{E}(t,z) = \left[\mathcal{E}\left(t-\frac{z}{c},0\right) - \frac{z\Pi}{c}\int_{z/c}^{t}\mathcal{E}(t-t',0)\,\frac{J_1\left(\Pi\sqrt{t'^2-\frac{z^2}{c^2}}\right)}{\sqrt{t'^2-\frac{z^2}{c^2}}}\,dt'\right]U\left(t-\frac{z}{c}\right). \tag{48}$$

As an example of the use of Eq. (48) let the time response at z = 0 be a step-carrier sine wave given by

$$\mathcal{E}(t,0) = E_o \sin \omega_o t \; U(t),$$

$$\mathcal{E}(t,0) = \mathrm{Im}\left[E_o\, e^{i\omega_o t}\, U(t)\right], \tag{49}$$

where Im means "the imaginary part of". Using Eq. (49) in Eq. (48) one obtains

$$\mathcal{E}(t,z) = \mathrm{Im}\, E_o\left[e^{i\omega_o\left(t-\frac{z}{c}\right)} - \frac{z\Pi}{c}e^{i\omega_o t}\int_{z/c}^{t}\frac{e^{-i\omega_o t'}\,J_1\left(\Pi\sqrt{t'^2-\frac{z^2}{c^2}}\right)dt'}{\sqrt{t'^2-\frac{z^2}{c^2}}}\right]U\left(t-\frac{z}{c}\right). \tag{50}$$

Normalizing all quantities to the signal frequency ω_o, one may write Eq. (50) as

$$\mathcal{E}(\tau,\eta) = \text{Im}\, E_o \left[e^{i(\tau-\eta)} - P\eta\, e^{i\tau} \int_\eta^\tau \frac{e^{-iu} J_1\left(P\sqrt{u^2-\eta^2}\right) du}{\sqrt{u^2-\eta^2}} \right] U(\tau-\eta) \qquad (51)$$

where

$$\tau = \omega_o t,$$

$$\eta = \frac{\omega_o z}{c},$$

$$P = \frac{\Pi}{\omega_o}. \qquad (52)$$

The form of the solution Eq. (51) is interesting. First, propagation of the signal front or wave front proceeds at the speed of light in free space. The solution consists of two terms. The first term represents the propagation of a plane wave in free space. The second term represents the dispersive properties of the plasma. The two terms combine to give the total or dispersed wave that is propagating in the plasma.

Equation (51) has been evaluated numerically and several typical results for the overdense case are presented in Figure 6.

In order to investigate the effects of a finite rise-time on the dispersion of a carrier pulse, let

$$\mathcal{E}(t,0) = E_o \left(1 - e^{-\alpha\omega_o t}\right) \sin\omega_o t\; U(t),$$

$$\mathcal{E}(t,0) = \text{Im}\, E_o \left[\left(1 - e^{-\alpha\omega_o t}\right) e^{i\omega_o t}\right] U(t). \qquad (53)$$

Substituting Eq. (53) into Eq. (48) and using Eq. (52) one obtains

$$\mathcal{E}(\tau,\eta) = \text{Im}\, E_o \left[e^{i(\tau-\eta)} - e^{-(\alpha-i)(\tau-\eta)} - P\eta\, e^{i\tau} \int_\eta^\tau \frac{e^{-iu} J_1\left(P\sqrt{u^2-\eta^2}\right)}{\sqrt{u^2-\eta^2}}\, du \right.$$

$$\left. + P\eta\, e^{-(\alpha-i)\tau} \int_\eta^\tau \frac{e^{(\alpha-i)\tau} J_1\left(P\sqrt{u^2-\eta^2}\right)}{\sqrt{u^2-\eta^2}}\, du \right] U(\tau-\eta). \qquad (54)$$

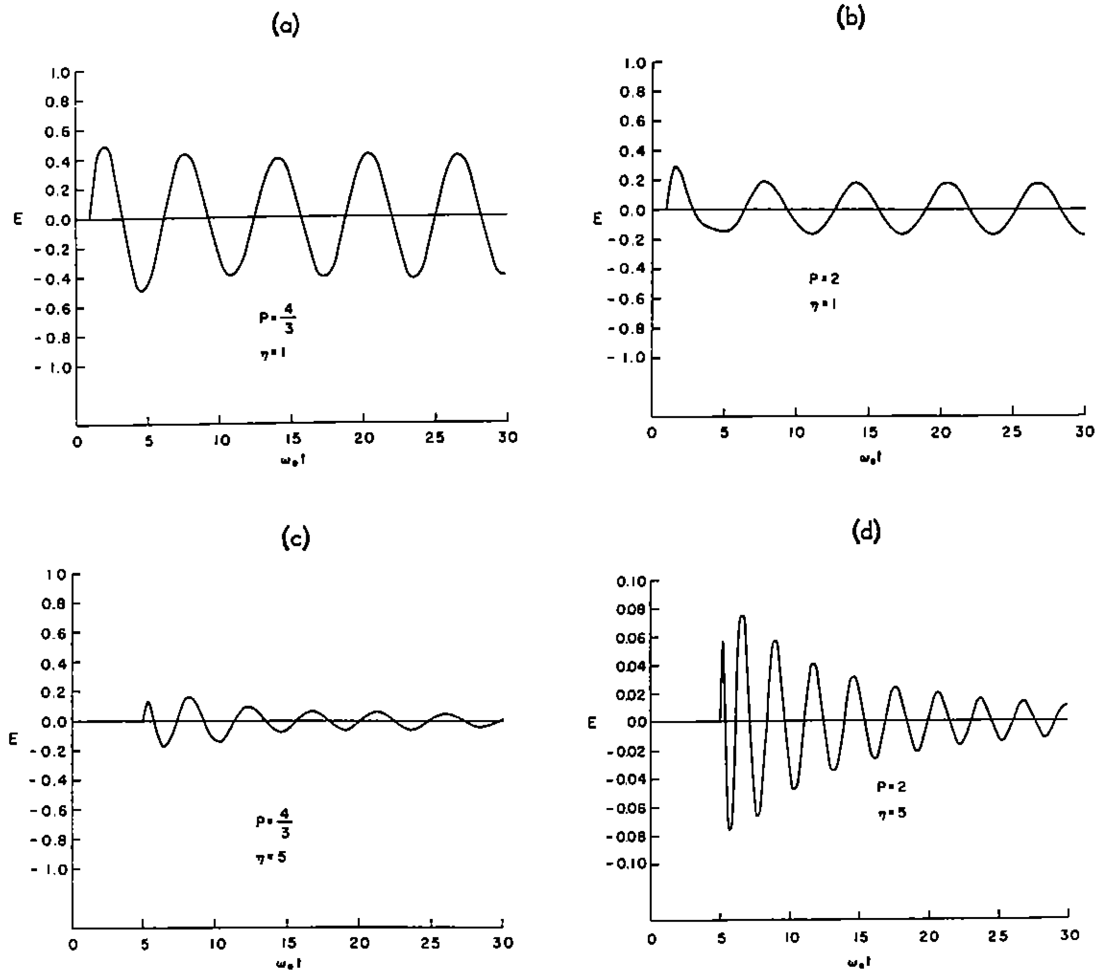

Figure 6. Propagation of a Sine Wave Electric Field in an Overdense Plasma

Equation (54) has been evaluated numerically and the results are shown in Figure 7.

As a further example consider a step E field turned on at z = 0, that is,

$$\mathcal{E}(t,0) = E_o\ U(t). \tag{55}$$

Then, from Eq. (48), the time response at z will be given by

$$\mathcal{E}(t,z) = E_o \left[1 - \frac{z\Pi}{c}\int_{z/c}^{t} \frac{J_1\left(\Pi\sqrt{t'^2 - \frac{z^2}{c^2}}\right)}{\sqrt{t'^2 - \frac{z^2}{c^2}}}\, dt'\right] U(t - \tfrac{z}{c})\ . \tag{56}$$

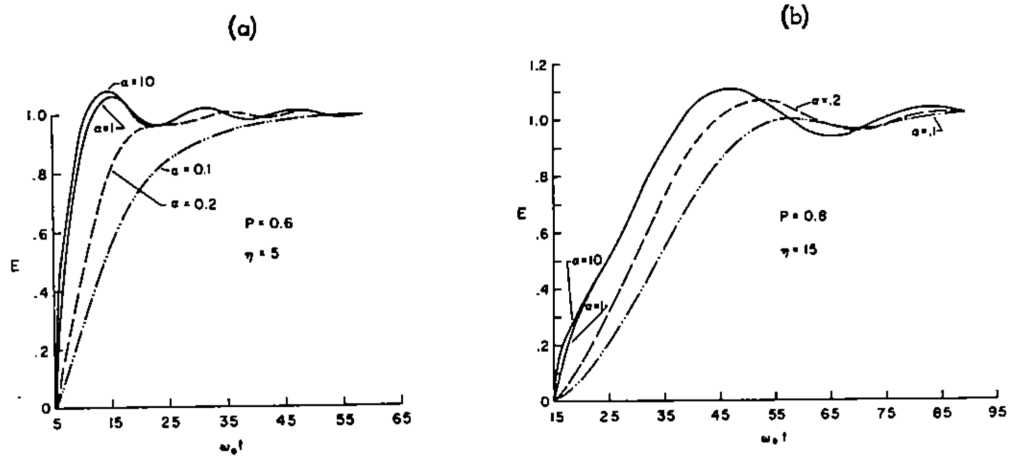

Figure 7. Envelope Response for the Propagation of a Sine Wave Electric Field with a Finite Rise Time

This expression has been evaluated numerically and typical cases are shown in Figure 8.

3.2 Series Solutions and Lommel Functions

It is possible in the case of a lossless, isotropic plasma to obtain an exact solution for a step carrier sine wave input electric field. The procedure hinges on several known inverse Laplace transforms and several identities involving Bessel functions. Let the time response of the electric field at z = 0 be given by

$$\mathcal{E}(t,0) = E_o \sin \omega_o t \; U(t) \; . \tag{57}$$

The transform of Eq. (57) is

$$E(s,0) = \mathcal{L}[\mathcal{E}(t,0)] = E_o \frac{\omega_o}{s^2 + \omega_o^2} \tag{58}$$

and the time response at z is then given from Eq. (36) by

$$\mathcal{E}(t,z) = \mathcal{L}^{-1} \left[E(s,0) \exp \left\{ - \frac{z}{c} \sqrt{s^2 + \Pi^2} \right\} \right] \; . \tag{59}$$

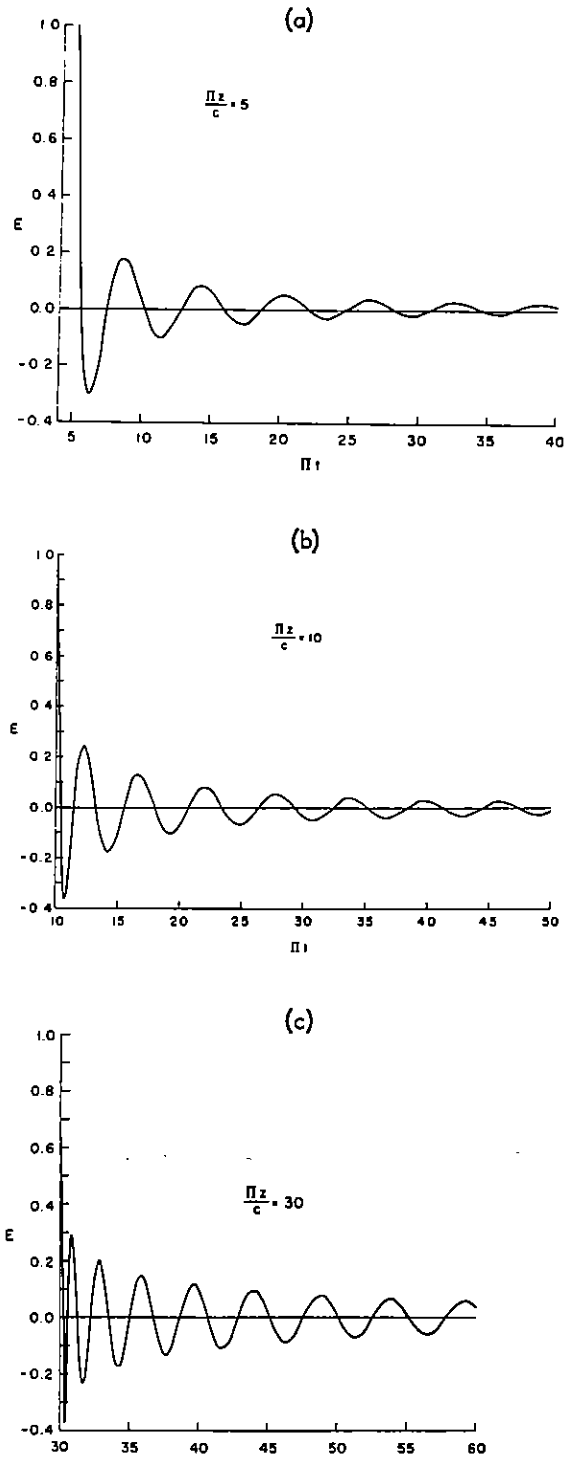

Figure 8. Propagation of a Step Function Electric Field in a Plasma

If the expression for E(s, 0) given by Eq. (58) is used in Eq. (59), the inverse transform of the resulting expression is not known. However, it has been found previously that the inverse transform of an input step Bessel function with argument Πt is known (see Equation (40)). This suggests the possibility of synthesizing $\sin \omega_o t$ from a sum of Bessel functions with argument Πt. To this end the well known identity is used (see Abramowitz and Stegun, 1964)

$$\sin (z \cos \theta) = 2 \sum_{n=o}^{\infty} (-1)^n J_{2n+1} (z) \cos \{(2n+1) \theta\} \quad . \tag{60}$$

By letting

$$\begin{aligned} z &= \Pi t \\ \cos \theta &= \frac{\omega_o}{\Pi} \qquad \omega_o \leq \Pi \\ \cosh a &= \frac{\omega_o}{\Pi} \qquad \omega_o \geq \Pi \end{aligned} \tag{61}$$

where $\theta = ia$ for $\omega_o \geq \Pi$, one can write Eq. (60) as

$$\sin \omega_o t = 2 \sum_{n=o}^{\infty} (-1)^n A_n J_{2n+1} (\Pi t) \tag{62}$$

where

$$A_n = \begin{cases} \cos (2n+1) \theta & \omega_o \leq \Pi \\ \cosh (2n+1) a & \omega_o \geq \Pi \end{cases} \tag{63}$$

Now, from tables

$$\mathcal{L} \left[J_{2n+1} (\Pi t) U(t) \right] = \frac{1}{\sqrt{s^2 + \Pi^2}} \left(\frac{\Pi}{s + \sqrt{s^2 + \Pi^2}} \right)^{2n+1} \tag{64}$$

so that, from Eq. (62) one may write

$$\mathcal{L} [\sin \omega_o t U(t)] = \frac{2}{\sqrt{s^2 + \Pi^2}} \sum_{n=o}^{\infty} (-1)^n A_n \left(\frac{\Pi}{s + \sqrt{s^2 + \Pi^2}} \right)^{2n+1} \quad . \tag{65}$$

One therefore obtains from Eqs. (59) and (65)

$$\mathcal{E}(t,z) = \mathcal{L}^{-1}\left[2\,E_o \frac{\exp\left\{-\frac{z}{c}\sqrt{s^2+\Pi^2}\right\}}{\sqrt{s^2+\Pi^2}} \sum_{n=o}^{\infty} (-1)^n A_n \left(\frac{\Pi}{s+\sqrt{s^2+\Pi^2}}\right)^{2n+1}\right] \quad (66)$$

Since the series in Eq. (66) is uniformly convergent, one may interchange the inverse and summation operations. If this is done, and if the transform pair (see Erdélyi, et al, 1954),

$$\mathcal{L}^{-1}\left[\frac{\exp\left\{-\frac{z}{c}\sqrt{s^2+\Pi^2}\right\}}{\sqrt{s^2+\Pi^2}}\left(\frac{\Pi}{s+\sqrt{s^2+\Pi^2}}\right)^{2n+1}\right]$$

$$= \left(\frac{t-\frac{z}{c}}{t+\frac{z}{c}}\right)^{\frac{2n+1}{2}} J_{2n+1}\left(\Pi\sqrt{t^2-\frac{z^2}{c^2}}\right) U(t-\tfrac{z}{c}), \quad (67)$$

is used one may write Eq. (66) as

$$\mathcal{E}(t,z) = 2\,E_o \sum_{n=o}^{\infty} (-1)^n A_n \left(\frac{t-\frac{z}{c}}{t+\frac{z}{c}}\right)^{2n+1} J_{2n+1}\left(\Pi\sqrt{t^2-\frac{z^2}{c^2}}\right) U(t-\tfrac{z}{c}). \quad (68)$$

Normalizing Eq. (68) to ω_o by using the relations in Eq. (52) one obtains

$$\mathcal{E}(\tau,\eta) = 2\,E_o\, U(t-\eta) \sum_{n=o}^{\infty} (-1)^n A_n \left(\frac{\tau-\eta}{\tau+\eta}\right)^{\frac{2n+1}{2}} J_{2n+1}\left(P\sqrt{\tau^2-\eta^2}\right) \quad (69)$$

where

$$A_n = \begin{cases} \cos\,(2n+1)\,\theta & P \geq 1 \\ \cosh\,(2n+1)\,a & P \leq 1 \end{cases} \quad (70)$$

and

$$\cos\theta = \frac{1}{P} \qquad P \geq 1,$$

$$\cosh a = \frac{1}{P} \qquad P \leq 1. \qquad (71)$$

It is seen then that the electric field response for this case can be represented as a convergent infinite series in terms of Bessel functions. This result has been obtained by Knop (1964). Similar results have been obtained by Cerrillo (1948), Rubinowicz (1950), and Gajewski (1955).

For the underdense case of propagation it is possible to write Eq. (69) as the sum of two Lommel functions of order one. The Lommel function of the first kind U_ν (w, z) is defined by Dekanosidze (1960) as

$$U_\nu (w, z) = \sum_{n=o}^{\infty} (-1)^n \left(\frac{w}{z}\right)^{2n+\nu} J_{2n+\nu}(z). \qquad (72)$$

For the underdense case, $P \leq 1$, the A_n in Eq. (69) is equal to cosh (2n + 1)a, where

$$a = \cosh^{-1}\left(\frac{1}{P}\right) \qquad (73)$$

or, noting that

$$e^a = \cosh a + \sinh a,$$

one may write

$$a = \ln\left[\frac{1}{P} + \sqrt{\frac{1}{P^2} - 1}\right] = \ln\left[\frac{1 + \sqrt{1 - P^2}}{P}\right].$$

If one lets

$$\alpha = \frac{1 + \sqrt{1 - P^2}}{P} \qquad (74)$$

then

$$a = \ln \alpha$$

and

$$(2n + 1)\, a = \ln \alpha^{(2n+1)}. \tag{75}$$

Using Eq. (75) one can write

$$\cosh (2n + 1) a = \frac{1}{2} \left[e^{(2n+1)a} + \frac{1}{e^{(2n+1)a}} \right]$$

or

$$\cosh (2n + 1) a = \frac{1}{2} \left[\exp \left\{ \ln \alpha^{(2n+1)} \right\} + \frac{1}{\exp \left\{ \ln \alpha^{(2n+1)} \right\}} \right]$$

and finally

$$A_n = \cosh (2n + 1)a = \frac{\alpha^{(2n+1)}}{2} + \frac{\alpha^{-(2n+1)}}{2} . \tag{76}$$

The general term of the series in Eq. (69) then becomes

$$\frac{1}{2} \left\{ (-1)^n \left(\sqrt{\frac{\tau-\eta}{\tau+\eta}} \right)^{(2n+1)} J_{2n+1} \left(P\sqrt{\tau^2 - \eta^2} \right) \left[\alpha^{(2n+1)} + \alpha^{-(2n+1)} \right] \right\} .$$

Using the identity

$$\sqrt{\frac{\tau-\eta}{\tau+\eta}} = \frac{P\,(\tau - \eta)}{P\sqrt{\tau^2 - \eta^2}}$$

one can write the general term as

$$\frac{1}{2} \left\{ (-1)^n \left(\frac{\alpha\, P\,(\tau - \eta)}{P\sqrt{\tau^2 - \eta^2}} \right)^{2n+1} J_{2n+1} \left(P\sqrt{\tau^2 - \eta^2} \right) \right.$$

$$\left. + (-1)^n \left(\frac{\frac{P\,(\tau-\eta)}{\alpha}}{P\sqrt{\tau^2-\eta^2}} \right)^{2n+1} J_{2n+1} \left(P\sqrt{\tau^2-\eta^2} \right) \right\} . \tag{77}$$

One defines the following quantities

$$w_1 = \alpha P(\tau-\eta) = \left[1 + \sqrt{1 - P^2} \right] (\tau-\eta)$$

$$w_2 = \frac{P}{\alpha}(\tau-\eta) = \frac{P^2}{\left[1 + \sqrt{1 - P^2} \right]} (\tau-\eta)$$

$$z = P\sqrt{\tau^2 - \eta^2} \qquad (78)$$

where Eq. (74) has been used. Using Eqs. (69), (72), (78) and Eq. (77) for the general term of the series one may write the solution as

$$\mathcal{E}(\tau,\eta) = E_o\, U(\tau-\eta) \left[U_1 (w_1, z) + U_1 (w_2, z) \right] . \qquad (79)$$

Lommel functions of two variables have been tabulated by Dekanosidze (1960) so that Eq. (79) may be readily evaluated for a given P, η and τ.

In a similar way it may be shown that Eq. (79) is also the solution for the overdense case, $P \geq 1$, if w_1 and w_2 are defined by the new relations

$$w_1 = \left[1 + i\sqrt{P^2 - 1} \right] (\tau - \eta),$$

$$w_2 = \frac{P^2}{\left[1 + i\sqrt{P^2 - 1} \right]} (\tau - \eta). \qquad (80)$$

The arguments of the Lommel functions are complex for this case and since these are not tabulated the usefulness of this representation is somewhat limited.

3.3 Contour Integrations

The solution to the problem of transient signal propagation in an isotropic, lossless plasma is given by Eq. (36), namely

$$\mathcal{E}(t,z) = \frac{1}{2\pi i} \int_{\gamma_s} E(s,0) \exp \left\{ st - \frac{z}{c} \sqrt{s^2 + \Pi^2} \right\} ds, \qquad (81)$$

where the contour γ_s lies to the right of all singularities as shown in Figure 9a. If the contour is closed in a large semi-circle of radius R to the right (see Figure 9b)

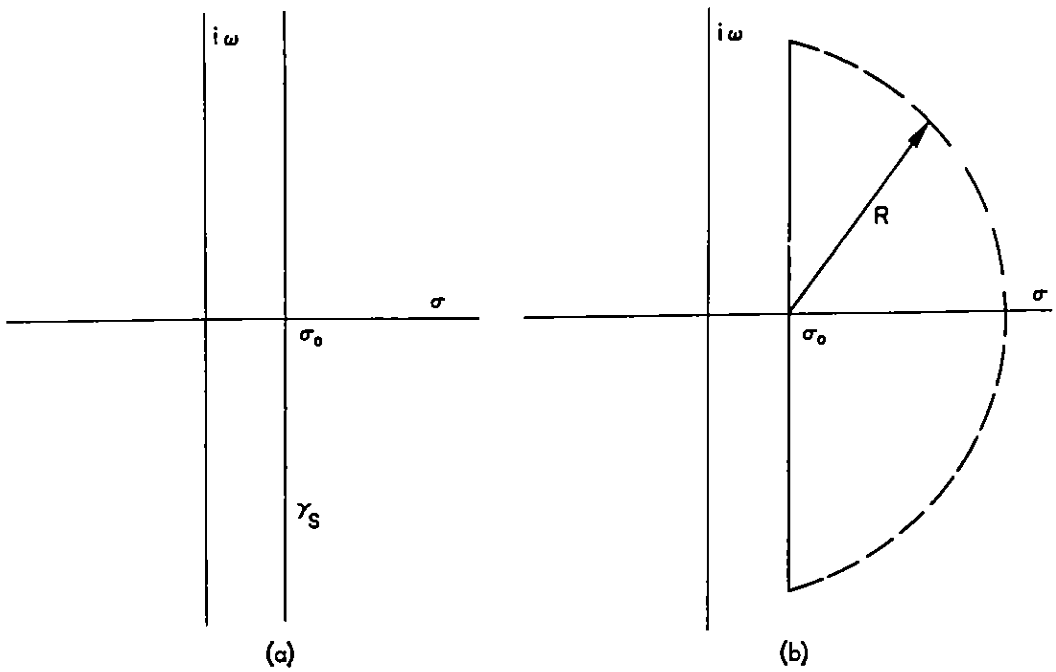

Figure 9. The Complex Frequency Plane

so that $s >> \Pi$, then the exponent in Eq. (81) becomes $s(t-z/c)$ and for this case the integral along this large semi-circle goes to zero as $R \to \infty$ when $t < z/c$. Since the integral around the closed contour in Figure 9b is zero by Cauchy's integral theorem, then the integral Eq. (81) will be zero, so that

$$\mathcal{E}(t,z) = 0 \qquad t < \frac{z}{c} \tag{82}$$

as has been found previously. Thus no signal can arrive at a point z with a speed greater than that of light in free space.

For $t > z/c$ the contour will have to be closed in a large circular arc to the left in order to make the contribution along this circle vanish. However, when this is done singularities will be enclosed in the complex plane which will contribute to the value of the integral in Eq. (81).

Two types of singularities are generally encountered. The first are poles of the function E(s, 0) which will give rise to the steady state solution. The second are branch points which arise from the double-valued nature of the function $\sqrt{s^2 + \Pi^2}$. The branch points of this function occur at $\pm i\Pi$ and in order to carry out the integral in Eq. (81) the complex plane must be cut in such a way as to make the function $\sqrt{s^2 + \Pi^2}$ single-valued. It will be found that the integration along the branch cuts gives rise to the transient solution to the propagation problem. Several ways of taking the branch cuts are possible and two of these ways will be investigated in detail.

3.3.1 BRANCH CUT ALONG IMAGINARY AXIS FROM $-i\Pi$ TO $+i\Pi$

The first step in evaluating the integral in Eq. (81) is to study the double-valued function

$$f(s) = \sqrt{s^2 + \Pi^2}\,. \tag{83}$$

One wishes to form two Riemann sheets by cutting the complex s-plane along the imaginary axis from $-i\Pi$ to $+i\Pi$. In order to properly define these two sheets one considers f(s) to be the product of the two double-valued functions

$$\sqrt{s - i\Pi} = \sqrt{\rho_1} \quad e^{i\phi_1/2}$$

and (84)

$$\sqrt{s + i\Pi} = \sqrt{\rho_2} \quad e^{i\phi_1/2}.$$

For each of these functions one takes the branch cuts shown in Figure 10. Sheet 1 of $\sqrt{s - i\Pi}$ is defined in the interval $\frac{-\pi}{2} \leq \phi_1 < \frac{3\pi}{2}$ and Sheet 2 in the interval $\frac{3\pi}{2} \leq \phi_1 < \frac{7\pi}{2}$. Sheet 1 of $\sqrt{s + i\Pi}$ is defined in the interval $\frac{-\pi}{2} \leq \phi_2 < \frac{3\pi}{2}$, while Sheet 2 is defined in the interval $\frac{3\pi}{2} \leq \phi_2 < \frac{7\pi}{2}$. Using these branch cuts, one looks at the product of the functions $\sqrt{s - i\Pi}$ and $\sqrt{s + i\Pi}$ to determine the sign distribution for the function $f(s) = \sqrt{s^2 + \Pi^2}$. The Riemann sheets for f(s) must be defined with respect to both ϕ_1 and ϕ_2. Sheet 1 of f(s) is defined by the intervals of ϕ_1 and ϕ_2 given by

$$-\frac{\pi}{2} \leq \phi_1 < \frac{3\pi}{2}$$

$$-\frac{\pi}{2} \leq \phi_2 < \frac{3\pi}{2}$$

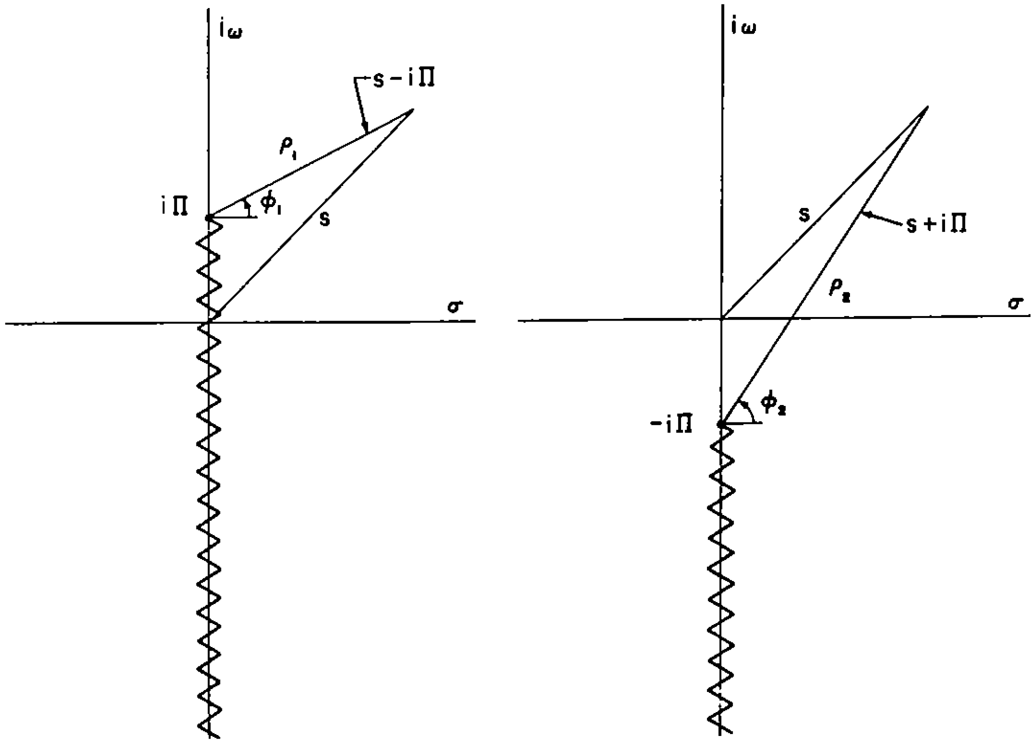

Figure 10. The Complex Frequency Plane for a Branch Cut from -iΠ to +iΠ

which are obtained from the product of Sheet 1 of $\sqrt{s - i\Pi}$ and Sheet 1 of $\sqrt{s + i\Pi}$. These conditions are identical to the intervals

$$\frac{3\pi}{2} \leq \phi_1 < \frac{7\pi}{2}$$

$$\frac{3\pi}{2} \leq \phi_2 < \frac{7\pi}{2}$$

which are obtained from the product of Sheet 2 of $\sqrt{s - i\Pi}$ and Sheet 2 of $\sqrt{s + i\Pi}$. The second sheet or Sheet 2 of f(s) is given by the intervals

$$-\frac{\pi}{2} \leq \phi_1 < \frac{3\pi}{2}$$

$$\frac{3\pi}{2} \leq \phi_2 < \frac{7\pi}{2}$$

or by the intervals

$$\frac{3\pi}{2} \leq \phi_1 < \frac{7\pi}{2}$$

$$-\frac{\pi}{2} \leq \phi_2 < \frac{3\pi}{2}$$

which are obtained by taking the product of opposite sheets of $\sqrt{s - i\Pi}$ and $\sqrt{s + i\Pi}$. With these definitions of the Riemann sheets of $\sqrt{s^2 + \Pi^2}$ one may determine the sign distribution of

$$f(s) = \sqrt{s^2 + \Pi^2} = \sqrt{\rho_1 \rho_2}\ e^{\frac{i}{2}(\phi_1 + \phi_2)} ,$$

$$f(s) = u + i\,v . \tag{85}$$

The sign distribution of u and v for each of the two sheets is shown in Figure 11.

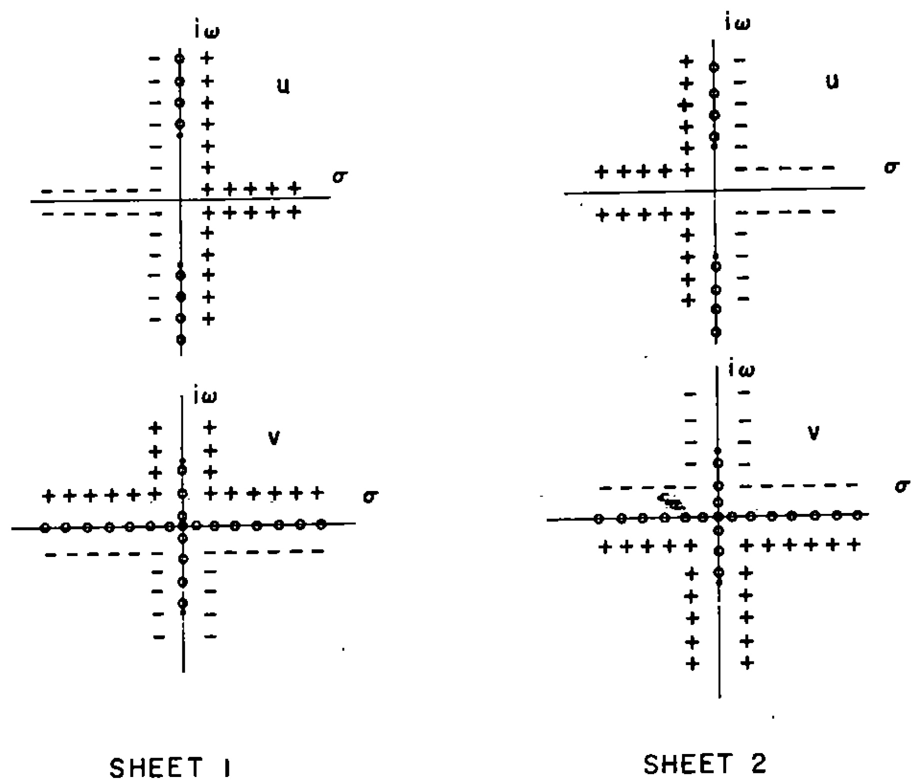

Figure 11. The Sign Distribution on Each Riemann Sheet for the Branch Cut of Figure 10

The integration on Sheet 1 will now be carried out for a step sine wave input along the contour shown in Figure 12. For this case

$$E(s, 0) = E_o \frac{\omega_o}{s^2 + \omega_o^2} \tag{86}$$

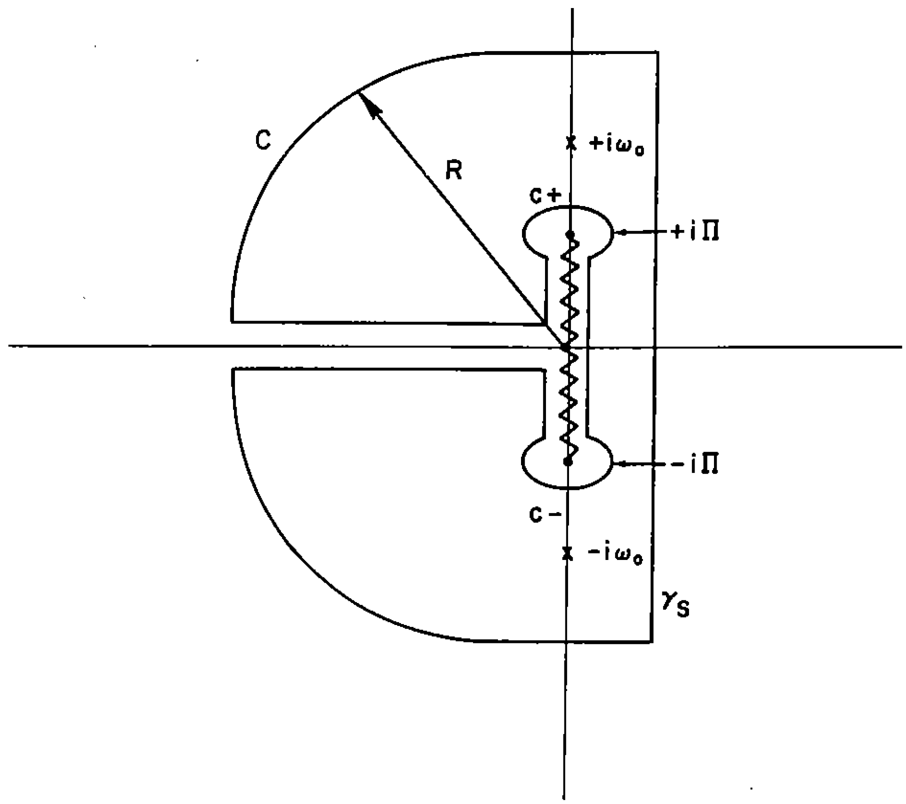

Figure 12. The Contour for the Branch Cut in Figure 10

One may readily show that as R ⟶ ∞ the integral along the large circular arc C vanishes. The two integrations along the negative real axis cancel and as the radius of the two small circles c_+ and c_- around each branch point goes to zero the contribution to the integral from these small circles also vanishes. Thus the only contributions to the integral in Eq. (81) come from the residues of the poles and from the integration along each side of the branch cut. Therefore

$$\int_{\gamma_s} + \int_{\substack{\text{Branch} \\ \text{Cuts}}} = 2\pi i \sum \text{Residues.} \tag{87}$$

Along the branch cut

$$\rho_1 = \Pi - \omega$$

$$\rho_2 = \Pi + \omega \,. \tag{88}$$

Using Eq. (88) in Eq. (85) together with the sign distribution of Sheet 1 in Figure 11 one sees that on the right side of the branch cut $f(s) = +\sqrt{\Pi^2 - \omega^2}$ while on the left side $f(s) = -\sqrt{\Pi^2 - \omega^2}$. From Eqs. (81), (86) and (87), one can then write

$$\mathcal{E}(t,z) + \frac{E_o\,\omega_o}{2\pi i} \int_{-i\Pi}^{+i\Pi} \frac{\exp\left\{st + \frac{z}{c}\sqrt{\Pi^2 - \omega^2}\right\}}{s^2 + \omega_o^2}\, ds$$

$$+ \frac{E_o\,\omega_o}{2\pi i} \int_{+i\Pi}^{-i\Pi} \frac{\exp\left\{st - \frac{z}{c}\sqrt{\Pi^2 - \omega^2}\right\}}{s^2 + \omega_o^2}\, ds = 2\pi i \sum \text{Residues},$$

or, letting $s = i\omega$

$$\mathcal{E}(t,z) + \frac{E_o\,\omega_o}{2\pi} \int_{-\Pi}^{+\Pi} \frac{e^{i\omega t}}{\omega_o^2 - \omega^2} \left(e^{\frac{z}{c}\sqrt{\Pi^2 - \omega^2}} - e^{-\frac{z}{c}\sqrt{\Pi^2 - \omega^2}} \right) d\omega$$

$$= 2\pi i \sum \text{Residues},$$

which becomes, after evaluating the residues,

$$\mathcal{E}(t,z) = E_o \sin\left(\omega_o t - \frac{\omega_o z}{c}\sqrt{1 - \frac{\Pi^2}{\omega_o^2}}\right) - \frac{E_o\,\omega_o}{\pi} \int_{-\Pi}^{+\Pi} \frac{e^{i\omega t}\sinh\frac{z}{c}\sqrt{\Pi^2 - \omega^2}}{\omega_o^2 - \omega^2}\, d\omega. \tag{89}$$

The integrand of the imaginary part of the last term in Eq. (89) is an odd function and therefore integrates to zero. One can then write Eq. (89) in normalized form, using Eq. (52), as

$$\mathcal{E}(\tau,\eta) = E_o \sin\left(\tau - \eta\sqrt{1 - P^2}\right) - \frac{E_o}{\pi} \int_{-P}^{P} \frac{\cos u\tau \sinh \eta\sqrt{P^2 - u^2}}{1 - u^2}\, du. \tag{90}$$

It is evident that the total solution is the sum of a steady state term and a transient term. The integral representing the transient term is not particularly suited for numerical integration since the integrand oscillates rapidly for large τ. In order to overcome this difficulty the solution based on a different branch cut will now be examined.

3.3.2 BRANCH CUTS IN LEFT HALF PLANE

A different branch cut which better lends itself to numerical integration will now be considered. If the s-plane is cut as shown in Figure 13 then the integrations along the branch cuts will be over negative values of the real part of s. One might expect the resulting integrands not to oscillate as much as if one integrated along the imaginary axis.

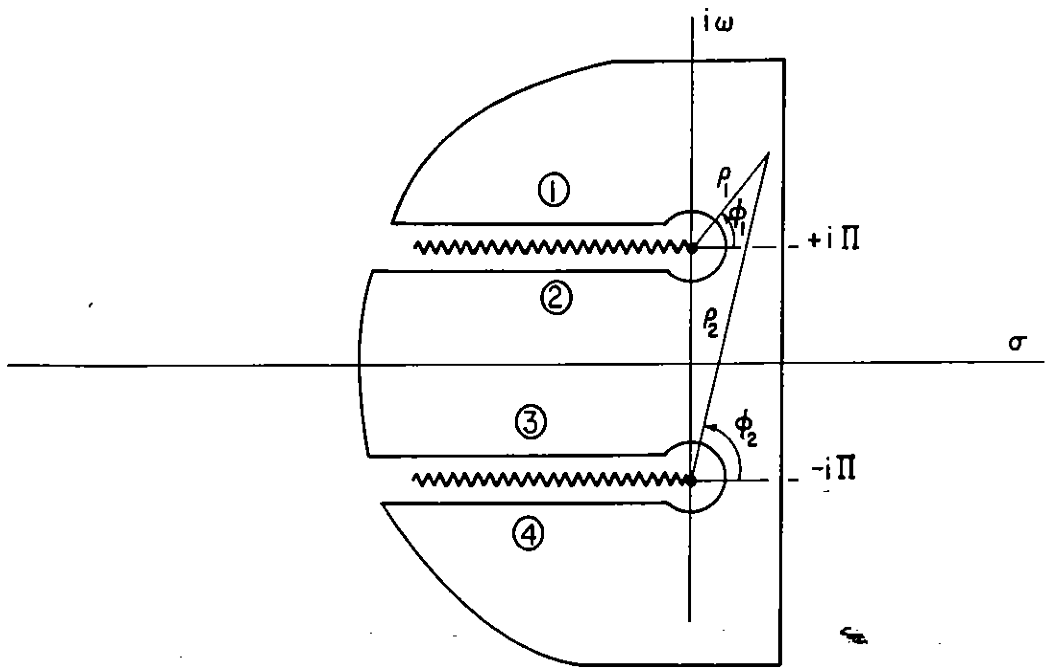

Figure 13. The Contour for Branch Cuts in the Left Half Frequency Plane

Let the initial time response be an exponential function of the form $E_o e^{i\omega_o t}$ and take the imaginary part of the result to give the response to a sine wave input. The integral to be evaluated may then be written from Eq. (81) as

$$\mathcal{E}(t,z) = \frac{E_o}{2\pi i} \int_{\gamma_s} \frac{\exp\left\{st - \frac{z}{c}\sqrt{s^2 + \Pi^2}\right\}}{s - i\omega_o} ds. \tag{91}$$

The integration will be carried out on Sheet 1 of Figure 13, defined by the relations

$$\begin{aligned} s - i\Pi &= \rho_1 e^{i\phi_1} & -\pi \leq \phi_1 < \pi \\ s + i\Pi &= \rho_2 e^{i\phi_2} & -\pi \leq \phi_2 < \pi . \end{aligned} \tag{92}$$

One must now determine the value of

$$\sqrt{s^2 + \Pi^2} = \sqrt{s - i\Pi}\ \sqrt{s + i\Pi} = \sqrt{\rho_1 \rho_2}\ e^{\frac{i}{2}(\phi_1 + \phi_2)} \tag{93}$$

for each of the four sides of the branch cuts in Figure 13.

Consider first Side ① of the upper branch cut. Along this side $\phi_1 = +\pi$ and $\phi_2 = \frac{\pi}{2} + \psi$ where $\psi = \tan^{-1}\left(\frac{|\sigma|}{2\Pi}\right)$. One then obtains from Eq. (93)

$$\begin{aligned} \sqrt{s^2 + \Pi^2} &= \sqrt{\rho_1 \rho_2}\ e^{i\frac{3\pi}{4}}\ e^{i\frac{\psi}{2}} \\ &= \sqrt{\frac{\rho_1 \rho_2}{2}} \left(-1 + i\right) \left(\cos\frac{\psi}{2} + i \sin\frac{\psi}{2}\right) \\ &= \sqrt{\frac{\rho_1 \rho_2}{2}} \left[-\left(\cos\frac{\psi}{2} + \sin\frac{\psi}{2}\right) + i\left(\cos\frac{\psi}{2} - \sin\frac{\psi}{2}\right)\right] \end{aligned}$$

or

$$\sqrt{s^2 + \Pi^2} = \sqrt{\frac{\rho_1 \rho_2}{2}}\ (-a + i b) \qquad [\text{Side } ①] \tag{94}$$

where

$$\begin{aligned} a &= \cos\frac{\psi}{2} + \sin\frac{\psi}{2} \\ b &= \cos\frac{\psi}{2} - \sin\frac{\psi}{2} . \end{aligned} \tag{95}$$

Since $0 \leq \psi < \frac{\pi}{2}$, so that $0 \leq \frac{\psi}{2} < \frac{\pi}{4}$ it is seen that $a > 0$ and $b > 0$.

In a similar manner one can readily show that for Sides ②, ③ and ④ one obtains

$$\sqrt{s^2 + \Pi^2} = \sqrt{\frac{\rho_1 \rho_2}{2}} \; (a - i\,b) \qquad [\text{Side } ②]$$

$$\sqrt{s^2 + \Pi^2} = \sqrt{\frac{\rho_1 \rho_2}{2}} \; (a + i\,b) \qquad [\text{Side } ③]$$

$$\sqrt{s^2 + \Pi^2} = \sqrt{\frac{\rho_1 \rho_2}{2}} \; (-a - i\,b) \qquad [\text{Side } ④]. \tag{96}$$

For each of the four sides it is also noted that

$$\sqrt{\frac{\rho_1 \rho_2}{2}} = \sqrt{\frac{|\sigma| \sqrt{|\sigma|^2 + 4\Pi^2}}{2}} \quad . \tag{97}$$

Consider again Side ① . Along this side $s = \sigma + i\Pi$ and the integral in Eq. (91) becomes, using Eq. (94)

$$\frac{E_o}{2\pi i} \int_{-\infty}^{o} \frac{1}{\sigma + i\,\Pi - i\,\omega_o} \exp\left\{(\sigma + i\Pi)t - \frac{z}{c}\sqrt{\frac{\rho_1 \rho_2}{2}}\;(-a + i\,b)\right\} d\sigma$$

$$= \frac{E_o e^{i\Pi t}}{2\pi i} \int_{o}^{\infty} \frac{1}{-\sigma + i\,(\Pi - \omega_o)} \exp\left\{A - i\,B - \sigma t\right\} d\,\sigma \tag{98}$$

where

$$A = a\,\frac{z}{c}\sqrt{\frac{\rho_1 \rho_2}{2}}$$

$$B = b\,\frac{z}{c}\sqrt{\frac{\rho_1 \rho_2}{2}} \quad . \tag{99}$$

Similarly, one can evaluate the integral along the other three sides of the branch cuts and obtain for the total contribution from the branch cuts

$$\int_{\substack{\text{Branch}\\\text{Cuts}}} = \int_{①} + \int_{②} + \int_{③} + \int_{④}$$

$$= \frac{E_o e^{i\Pi t}}{2\pi i} \int_o^\infty \left[\frac{\exp\{(A - iB)\} - \exp\{-(A - iB)\}}{-\sigma + i(\Pi - \omega_o)} \right] e^{-\sigma t} \, d\sigma$$

$$+ \frac{E_o e^{-i\Pi t}}{2\pi i} \int_o^\infty \left[\frac{\exp\{(A + iB)\} - \exp\{-(A + iB)\}}{\sigma + i(\Pi + \omega_o)} \right] e^{-\sigma t} \, d\sigma. \quad (100)$$

Noting that

$$\frac{1}{\mp \sigma + i(\Pi \mp \omega_o)} = \frac{\mp \sigma - i(\Pi \mp \omega_o)}{\sigma^2 + (\Pi \mp \omega_o)^2}$$

$$e^{(A \mp iB)} - e^{-(A \mp iB)} = 2 \left[\cos B \sinh A \mp i \sin B \cosh A\right] \quad (101)$$

one can write the imaginary part of Eq. (100) as

$$\text{Im} \int_{\substack{\text{Branch}\\\text{Cuts}}} = - \frac{E_o \cos \Pi t}{\pi} [I_1 - I_2] - \frac{E_o \sin \Pi t}{\pi} [I_3 - I_4] \quad (102)$$

where

$$I_1 = \int_o^\infty \frac{\sigma \alpha + \beta(\Pi + \omega_o)}{\sigma^2 + (\Pi + \omega_o)^2} e^{-\sigma t} \, d\sigma$$

$$I_2 = \int_o^\infty \frac{\sigma \alpha + \beta(\Pi - \omega_o)}{\sigma^2 + (\Pi - \omega_o)^2} e^{-\sigma t} \, d\sigma$$

$$I_3 = \int_o^\infty \frac{\sigma \beta - \alpha(\Pi + \omega_o)}{\sigma^2 + (\Pi + \omega_o)^2} e^{-\sigma t} \, d\sigma$$

$$I_4 = \int_o^\infty \frac{\sigma \beta - \alpha(\Pi - \omega_o)}{\sigma^2 + (\Pi - \omega_o)^2} e^{-\sigma t} \, d\sigma \quad (103)$$

and

$$\alpha = \cos B \sinh A$$
$$\beta = \sin B \cosh A . \tag{104}$$

As in the previous section it can readily be shown that the contributions to the integral from the large circular arc in left-half plane and from the small circles around the branch points vanish, so that the total response to a sine wave input is found from the relation

$$\mathcal{E}(t,z) + \text{Im} \int_{\substack{\text{Branch}\\\text{Cuts}}} = \text{Im} \left[2\pi i \sum \text{Residues} \right]$$

or

$$\mathcal{E}(t,z) = \text{Im} \left\{ E_o \exp \left[i \left(\omega_o t - \frac{z}{c} \sqrt{\omega_o{}^2 - \Pi^2} \right) \right] - \int_{\substack{\text{Branch}\\\text{Cuts}}} \right\} . \tag{105}$$

Normalizing all quantities to ω_o according to Eq. (52) one obtains by substituting Eq. (102) into Eq. (105)

$$\mathcal{E}(\tau,\eta) = E_o \left\{ \sin \left[\tau - \eta \sqrt{1-P^2} \right] + \sqrt{M^2 + N^2} \sin \left[P\tau + \theta(\tau) \right] \right\}, \tag{106}$$

where

$$M = \frac{1}{\pi} (I_1 - I_2)$$
$$N = \frac{1}{\pi} (I_3 - I_4)$$
$$\theta = \tan^{-1} \frac{M}{N} \tag{107}$$

and where the integrals I_1, I_2, I_3, and I_4 are given by the following relations

$$I_1 = \int_0^\infty \frac{\alpha x + \beta (P + 1)}{x^2 + (P + 1)^2} e^{-\tau x} dx$$

$$I_2 = \int_0^\infty \frac{\alpha x + \beta (P - 1)}{x^2 + (P - 1)^2} e^{-\tau x} dx$$

$$I_3 = \int_0^\infty \frac{\beta x - \alpha (P + 1)}{x^2 + (P + 1)^2} e^{-\tau x} dx$$

$$I_4 = \int_0^\infty \frac{\beta x - \alpha (P - 1)}{x^2 + (P - 1)^2} e^{-\tau x} dx \qquad (108)$$

where

$$\alpha = \cos B \sinh A$$

$$\beta = \sin B \cosh A$$

$$A = \eta \left(\cos \frac{\psi}{2} + \sin \frac{\psi}{2} \right) \sqrt{\frac{x \sqrt{x^2 + 4 P^2}}{2}}$$

$$B = \eta \left(\cos \frac{\psi}{2} - \sin \frac{\psi}{2} \right) \sqrt{\frac{x \sqrt{x^2 + 4 P^2}}{2}}$$

$$\psi = \tan^{-1} \left(\frac{x}{2 P} \right) . \qquad (109)$$

The integrands in Eq. (108) decay exponentially for large τ and for large x. They therefore lend themselves to numerical integration since the upper limit of integration can be cut off at some large value of x without appreciable error. These integrals were evaluated numerically by cutting off the upper limit at a value $x = 15/(\tau - 1)$. The results are shown in Figures 14, 15, and 16. Figure 14 is a plot of the transient envelope $\sqrt{M^2 + N^2}$ as a function of τ. The total transient solution $\sqrt{M^2 + N^2} \sin [P \tau + \theta]$ is plotted in Figure 15. Note that it is a phase modulated oscillation about the plasma frequency. The total response found by adding Figure 15 to the steady state solution is shown in Figure 16.

All of the numerical results discussed thus far in this report lose their usefulness for very large values of η. This is due to the fact that under these conditions it is necessary to take the difference of two very large numbers in order to obtain a small number. Thus the errors can become very large. However, it is just in this region of very large η that asymptotic solutions to the transient propagation problem work very well. Asymptotic solutions to the transient wave propagation problem will be considered in a second and subsequent report.

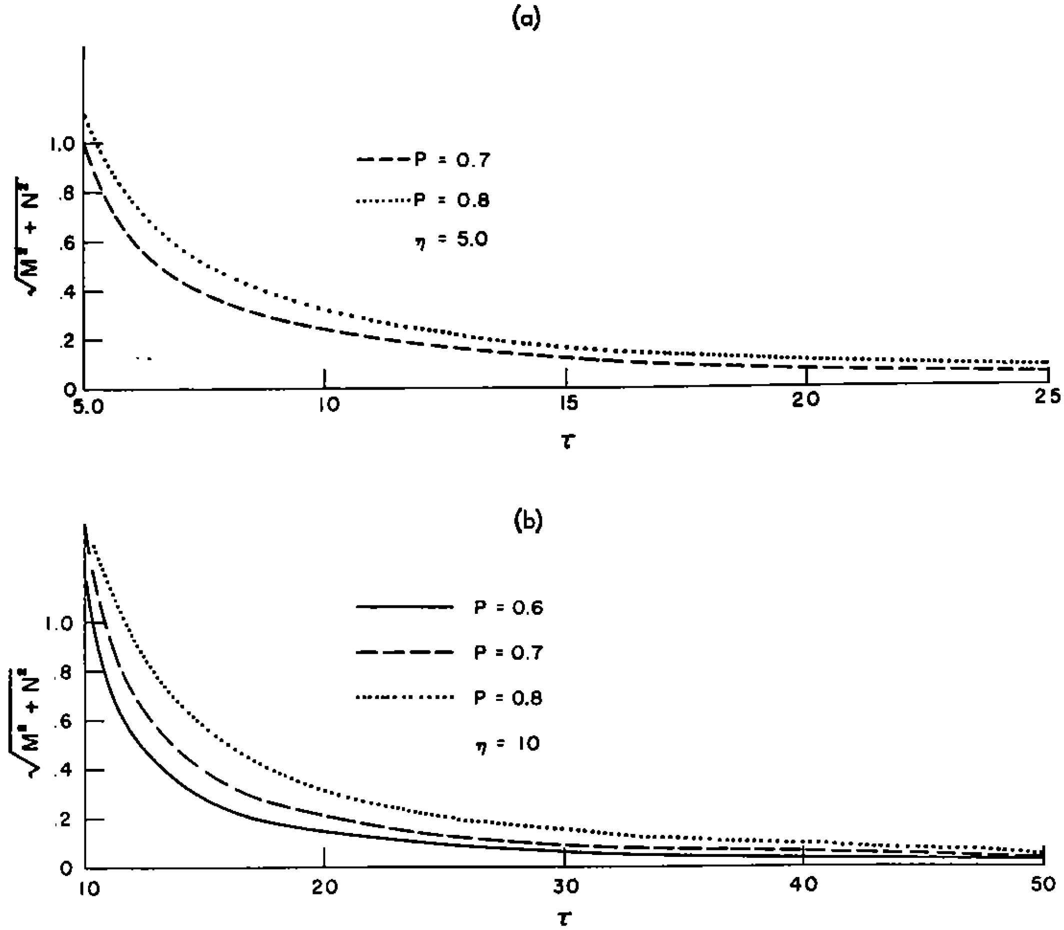

Figure 14. The Amplitude of the Transient Envelope

4. CONCLUSION

In Section 2 the propagation of wave packets in dispersive media was discussed. The wave packet is a useful concept since it enables one to define the group velocity. The group velocity definition is given by Eq. (21). Two types of wave packets, the Gaussian envelope carrier and the square pulsed carrier, were investigated. The wave packets were found to propagate in the plasma with the group velocity. The second derivative of the wave number with respect to frequency gives rise to a distortion of the wave packet. This distortion places an upper limit on the maximum rate at which repetitive pulses can be transmitted and distinguished in dispersive media. A crude method for estimating this maximum rate for Gaussian pulses in an isotropic plasma was presented. The treatment in Section 2 is limited to quasi-monochromatic signals.

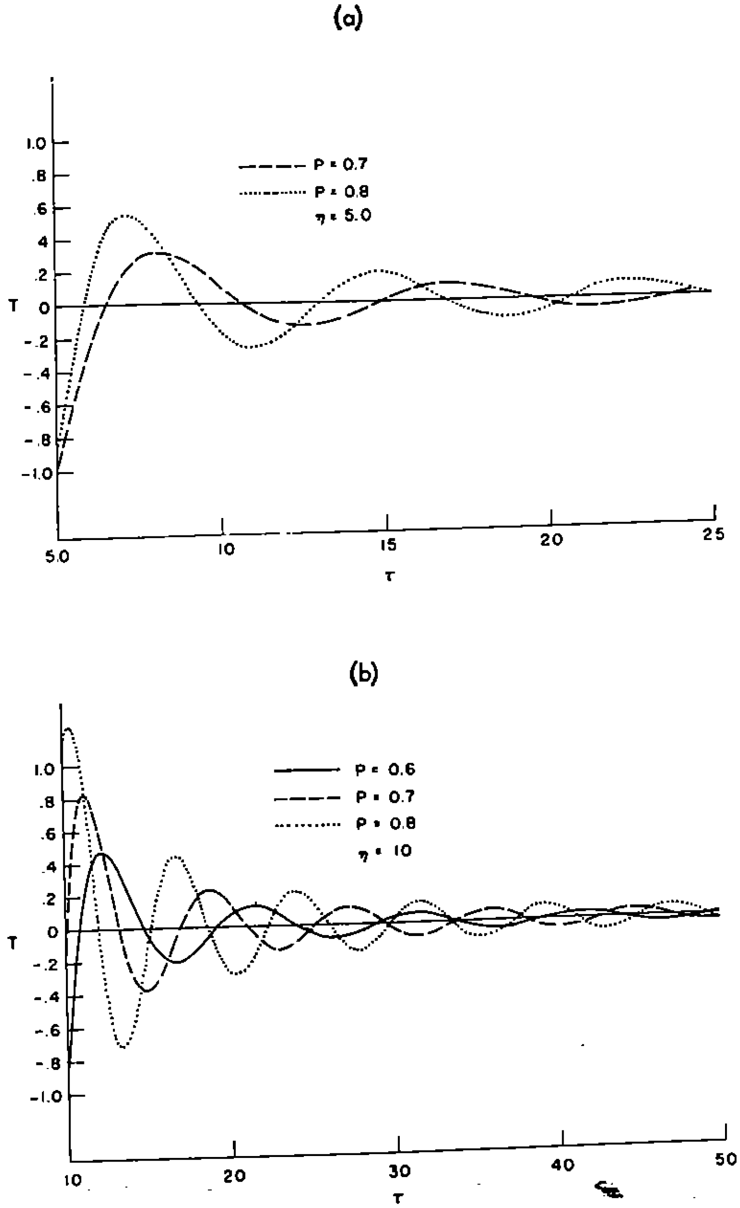

Figure 15. The Total Transient Solution Obtained by Integration Along the Branch Cuts

In Section 3 a more exact treatment of the propagation of transient waves in isotropic plasmas was given. Laplace transform methods were used, and the problem was reduced to evaluating an inverse Laplace transform. The problem of a sine wave electric field turned on at t = 0 in an isotropic plasma was investigated in detail. The solution can be expressed as a series of Bessel functions, and

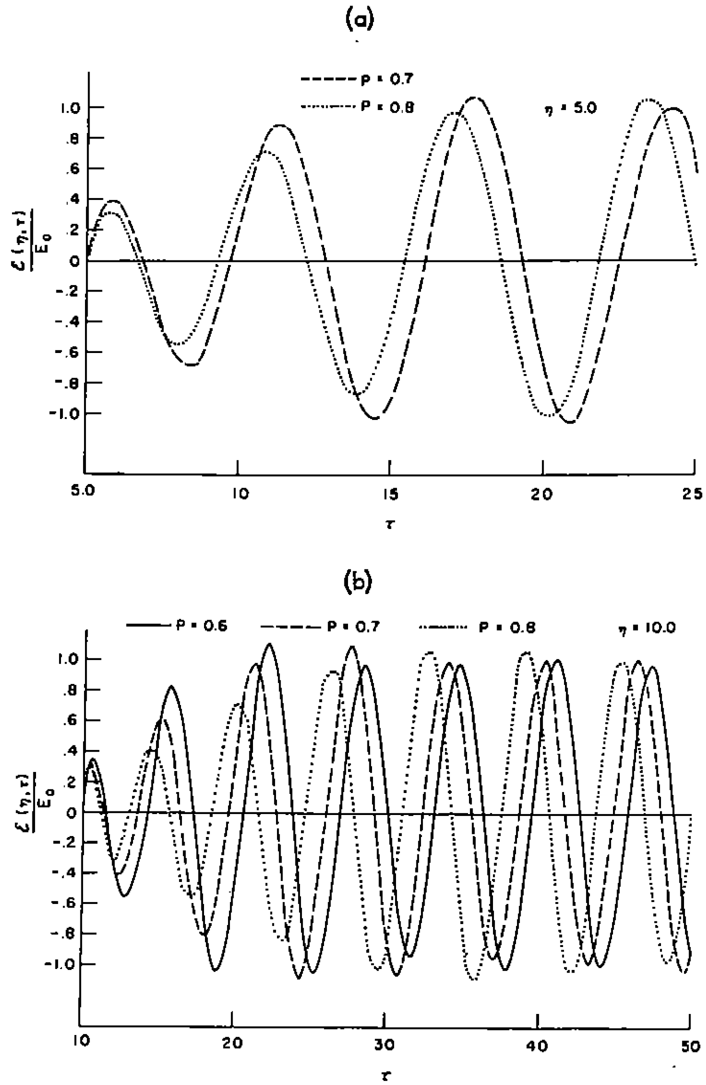

Figure 16. The Total Solution of a Propagating Sine Wave Electric Field

this solution can be expressed as the sum of two Lommel functions. The solution can also be given as a convolution integral representation. By using methods of contour integration, still other integral representations can be obtained, and two such representations were derived. Results from numerical evaluation of the integral solutions were presented. Due to limitations in the numerical methods of integration, the integral solutions are useful only for relatively small values of the parameter η. For larger values of η, asymptotic solutions can be obtained, and this will be the subject of a subsequent report.

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Appendix A

Evaluation of $I_1 = \int_{-\infty}^{\infty} e^{at-bt^2} dt$

Consider the integral

$$I_1 = \int_{-\infty}^{\infty} e^{at-bt^2} dt. \tag{A1}$$

If one completes the square in the exponent of Eq. (A1) and makes the change of variables $y = \sqrt{b}\,(t - \frac{a}{2b})$ one obtains

$$I_1 = \int_{-\infty}^{\infty} \exp \left\{ -b\,(t - \frac{a}{2b})^2 + \frac{a^2}{4b} \right\} dt = \frac{\exp\left\{\frac{a^2}{4b}\right\}}{\sqrt{b}} \int_{-\infty}^{\infty} e^{-y^2} dy,$$

from which

$$I_1 = \sqrt{\frac{\pi}{b}} \exp \left\{ \frac{a^2}{4b} \right\} . \tag{A2}$$

Appendix B

Evaluation of $I_2 = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\exp\{iaV - bV^2\}}{V} dV$

Consider the integral

$$I_2 = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\exp\left\{iaV - bV^2\right\}}{V} dV. \tag{B1}$$

Letting $V = \frac{u}{\sqrt{b}}$ and $X = \frac{a}{\sqrt{b}}$ one can rewrite Eq. (B1) as

$$I_2 = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\exp\left\{iXu - u^2\right\}}{u} du. \tag{B2}$$

Noting that

$$\int_0^X e^{iux} dx = \frac{e^{iuX}}{iu} - \frac{1}{iu}$$

so that

$$\frac{e^{iuX}}{u} = \frac{1}{u} + i \int_{0}^{X} e^{iux}\, dx$$

one can write Eq. (B2) in the form

$$I_2 = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{-u^2}}{u}\, du + \frac{1}{2\pi} \int_{0}^{X} dx \int_{-\infty}^{\infty} \exp\left\{ixu - u^2\right\}\, du, \tag{B3}$$

where the order of integration has been interchanged in the second term.

The first integral in Eq. (B3) can be evaluated by using Cauchy's integral theorem and integrating around the pole at $u = 0$. One readily finds the value of the integral to be equal to 1/2. In the second integral in Eq. (B3) the integral over du is equal to $\sqrt{\pi}\, \exp\left(-\frac{x^2}{4}\right)$ from the results of Appendix A. Therefore,

$$I_2 = \frac{1}{2} + \frac{1}{2\sqrt{\pi}} \int_{0}^{X} e^{-\frac{x^2}{4}}\, dx. \tag{B4}$$

Making the change of variable $w = x/2$ and using the definition of the error function

$$\operatorname{erf}\left(\frac{X}{2}\right) = \frac{2}{\sqrt{\pi}} \int_{0}^{X/2} e^{-w^2}\, dw \tag{B5}$$

one obtains

$$I_2 = \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{X}{2}\right),$$

from which, since $X = a/\sqrt{b}$,

$$I_2 = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{a}{2\sqrt{b}}\right)\right] \tag{B6}$$

Appendix C

Derivation of Integral Expression for $\mathcal{E}(t,z)$

Maxwell's equations which describe the propagation of electromagnetic waves in an isotropic plasma are

$$\text{curl}\, \underset{\sim}{\mathcal{E}} = - \mu_0 \frac{\partial \underset{\sim}{\mathcal{H}}}{\partial t}$$

$$\text{curl}\, \underset{\sim}{\mathcal{H}} = \underset{\sim}{J} + \varepsilon_0 \frac{\partial \underset{\sim}{\mathcal{E}}}{\partial t} \tag{C1}$$

where

$$\underset{\sim}{J} = - N e \underset{\sim}{v} . \tag{C2}$$

N is the electron number density and $\underset{\sim}{v}$ is determined from the equation of motion

$$\frac{\partial \underset{\sim}{v}}{\partial t} = - \frac{e}{m} \underset{\sim}{\mathcal{E}} . \tag{C3}$$

Consider the one-dimensional problem in which $\underset{\sim}{\mathcal{E}}$ is linearly polarized in the x-direction and is propagating in the z-direction. The x-component of $\underset{\sim}{\mathcal{E}}$ (t, z) will be written as $\mathcal{E}$ (t, z) and the Laplace transform of $\mathcal{E}$ (t, z) as E(s, z). If one then takes a Laplace transform in time of Eqs. (C1), (C2) and (C3) and solves for E(s, z) setting all initial conditions equal to zero, one readily obtains the equation

$$\frac{\partial^2 E(s, z)}{\partial z^2} - \frac{1}{c^2} (s^2 + \Pi^2) E(s, z) = 0. \tag{C4}$$

The object is to determine the time response $\mathcal{E}(t, z)$ in the semi-infinite region $z > 0$ when the time response $\mathcal{E}(t, 0)$ is prescribed at $z = 0$. The solution of Eq. (C4) may then be written as

$$E(s, z) = E(s, 0) \exp \left\{ - \frac{z}{c} \sqrt{s^2 + \Pi^2} \right\} \tag{C5}$$

where $E(s, 0)$ is the Laplace transform of $\mathcal{E}(t, 0)$. The time response $\mathcal{E}(t, z)$ is then obtained by taking the inverse of Eq. (C5). That is,

$$\mathcal{E}(t, z) = \frac{1}{2\pi i} \int_{\gamma_s} E(s, 0) \exp \left\{ st - \frac{z}{c} \sqrt{s^2 + \Pi^2} \right\} ds, \tag{C6}$$

which is Eq. (36) in Section 3 of the text.