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Transient Reflection and Transmission of a Plane Wave Normally Incident Upon a Semi-Infinite Anisotropic Plasma

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Abstract

General solutions for the transmitted and reflected waves in integral form are obtained for a plane wave of arbitrary time dependence, normally incident upon a semi-infinite, cold, collisionless, homogeneous, anisotropic ionized medium. The special case of an isotropic plasma with an incident wave of harmonic time dependence is also investigated.

Contents

Transient Reflection and Transmission of a Plane Wave Normally Incident Upon a Semi-Infinite Anisotropic Plasma

1. INTRODUCTION

The steady-state solution of wave propagation in an anisotropic plasma has been carried out by several authors.[1-3] The technique is to solve Maxwell's two curl equations along with an equation of motion for the electrons in the plasma. This set of equations is combined into a partial differential vector wave equation of both spatial and time dependence. The usual method of solution is to assume that the time dependence varies as $e^{-i\omega t}$ where ω is the angular frequency of the input wave. With this assumption it is possible to separate out the time dependence of the complete solution, and then to solve the remaining ordinary spatial differential equation. A correct solution to the partial differential wave equation is obtained with a time dependence of $e^{-i\omega t}$.

The question arises as to whether this solution is the total solution to the steady state problem. It has been suggested by Hutchital[4] that, for a collisionless plasma, an oscillation at the cyclotron frequency, $\Omega = \frac{q B_o}{m}$, should also exist. Hutchital's treatment involves solving the equation of motion for the electron orbit for a forced electric field of $e^{-i\omega t}$. This solution yields velocity components oscillating at both the signal frequency and the cyclotron frequency. Similar oscillations will exist in the current density, $\bar{\mathcal{J}} = n q \bar{v}$. The cyclotron frequency component of the current density will in turn induce similar time-dependent components of $e^{-i\Omega t}$ in the

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electric field and magnetic intensity as a consequence of Maxwell's equations

$$\nabla \times \vec{\mathcal{E}} = -\mu_o \frac{\partial \vec{\mathcal{H}}}{\partial t}$$

and

$$\nabla \times \vec{\mathcal{H}} = \vec{\mathcal{J}} + \epsilon_o \frac{\partial \vec{\mathcal{E}}}{\partial t} .$$

As yet this phenomenon of a steady-state cyclotron oscillation has not been observed experimentally.

Some closely related experiments have been conducted by Schmitt.[5,6] These experiments consist of transmitting nanosecond pulses through a plasma. In the absence of a dc magnetic field, a transient 'ringing' at the plasma frequency Π is evident. When a static magnetic field is added along the direction of propagation, a transient 'ringing' at two different frequencies is observed. The major oscillation occurs at the cyclotron frequency Ω. Superimposed is a secondary oscillation at a frequency close to $\sqrt{\Omega^2 + \Pi^2}$. It is to be noted that these experiments involve single-pulse transmission rather than a steady-state condition, but the ringing of the signal at the cyclotron frequency could be related to Hutchital's problem.

The question of a forced steady-state oscillation at the cyclotron frequency in a collisionless plasma is not fully answered. The complete rigorous solution including transient effects for a plane wave of arbitrary time dependence, which is normally incident on a cold, collisionless, homogeneous, anisotropic half-plane plasma, has not yet been reported. By solving this problem using Laplace transform methods, it is anticipated that a complete solution can be obtained. This solution should shed light on the proposed forced steady-state cyclotron oscillation.

In addition, the important situation of a plane wave of sinusoidal time dependence propagating into a homogeneous, isotropic, ionized medium has apparently not been rigorously solved. The more general solution for the anisotropic medium should apply if the static magnetic field is relaxed to zero. The solution of the anisotropic case will thus be useful in an isotropic medium.

In this paper an exact general solution to the anisotropic case is obtained in integral form.

2. THE ANISOTROPIC CASE

Consider a linearly polarized plane wave, initiated at $t = 0$, traveling in the positive z direction. At $t = z_o/c$, where c is the speed of light in free space, the

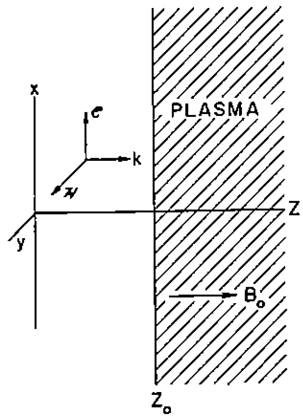

Figure 1.

wave is incident normally upon a semi-infinite, cold, collisionless plasma. A constant dc magnetic field is applied to the positive z direction (see Figure 1). It is desired to calculate the reflected and transmitted waves for all later times.

Maxwell's equations yield the manner in which the wave propagates in free space. Maxwell's equations for this case are

$$\frac{\partial \mathcal{E}_1(z,t)}{\partial z} = -\mu_o \frac{\partial \mathcal{H}_2(z,t)}{\partial t} \tag{1}$$

and

$$\frac{\partial \mathcal{H}_2(z,t)}{\partial z} = -\epsilon_o \frac{\partial \mathcal{E}_1(z,t)}{\partial t} , \tag{2}$$

where the electric field is taken along the positive x direction, the magnetic intensity along the positive y direction,

ϵ_o = free space permittivity,

and

μ_o = free space permeability.

A Laplace transformation in time yields

$$\frac{dE_1(z,p)}{dz} = -\mu_o p\, H_2(z,p) \tag{3}$$

$$\frac{d H_2(z,p)}{dz} = -\epsilon_o p\, E_1(z,p) \tag{4}$$

where the initial conditions $\mathcal{E}_1(z,o)$ and $\mathcal{H}_2(z,o)$ have been set equal to zero. The magnetic intensity can be eliminated to give

$$\frac{d^2 E_1(z,p)}{dz^2} = \frac{p^2}{c^2} E_1(z,p) \tag{5}$$

where $c = 1/\sqrt{\epsilon_o \mu_o}$.
Solving for $E_1(z,p)$ we have

$$E_1(z,p) = F(p)\, e^{-\frac{p}{c} z} \quad ,$$

where

$$F(p) = \mathcal{L}[f(t)]$$

and

$f(t)$ = the time dependence of the electric field at $z = 0$.

Then

$$\mathcal{E}_1(z,t) = \mathcal{L}^{-1}\left[F(p)\, e^{-\frac{p}{c} z}\right] = f(t - \frac{z}{c})\, U(t - \frac{z}{c}) \ , \tag{6}$$

where $U(t - \frac{z}{c}) = \begin{cases} o & \text{for } t < \frac{z}{c} \\ 1 & \text{for } t > \frac{z}{c} \end{cases}$.

For the usual sinusoidal case we have $f(t) = E_o \cos \omega t$.
Then

$$\mathcal{E}_1(z,t) = E_o \cos(\omega t - k_o z)\, U(t - \frac{z}{c}) \ , \tag{7}$$

where $k_o = \frac{\omega}{c}$.

For a plasma immersed in a dc magnetic field, Maxwell's equations must also include the current density $\vec{\mathcal{J}}$. The current density is defined by $\vec{\mathcal{J}} = nq\bar{v}$ where

n = number of particles,

q = charge of each particle,

$\bar{v}$ = average velocity of each particle.

The equation of motion for an average electron is simply

$$m\dot{v}_i = q\,\mathcal{E}_i + q\,\epsilon_{ijk}\,v_j\,B_k\,. \tag{8}$$

This equation gives two component equations,

$$\dot{v}_1 = \frac{q}{m}\,\mathcal{E}_1 + \Omega\,v_2 \tag{9}$$

and

$$\dot{v}_2 = \frac{q}{m}\,\mathcal{E}_2 - \Omega\,v_1\,, \tag{10}$$

where the cyclotron frequency $\Omega = \dfrac{qB_o}{m}$.

Taking the Laplace transformation in time of these equations and setting the initial electron velocities equal to zero gives,

$$p\,V_1 - \Omega\,V_2 = \frac{q}{m}\,E_1 \tag{11}$$

and

$$\Omega\,V_1 + p\,V_2 = \frac{q}{m}\,E_2\,. \tag{12}$$

Solving these equations for V_1 and V_2 and using the definition of current density, we obtain

$$J_1 = \epsilon_o\,\Pi^2 \left(\frac{1 + \dfrac{\Omega}{p}\dfrac{E_2}{E_1}}{p^2 + \Omega^2} \right) p\,E_1 \tag{13}$$

and

$$J_2 = \epsilon_o\,\Pi^2 \left(\frac{1 - \dfrac{\Omega}{p}\dfrac{E_1}{E_2}}{p^2 + \Omega^2} \right) p\,E_2\,, \tag{14}$$

where

$$\Pi = \left(\frac{nq^2}{\epsilon_o m}\right)^{1/2} = \text{the plasma frequency.}$$

Maxwell's plasma equations after a time transformation and for initial conditions of zero, become

$$\frac{dE_1}{dz} = -\mu_o p\, H_2 , \tag{15}$$

$$\frac{dH_2}{dz} = -\epsilon_o \left[\frac{\Pi^2 \left(1 + \frac{\Omega}{p} \frac{E_2}{E_1}\right)}{p^2 + \Omega^2} + 1 \right] p E_1 , \tag{16}$$

$$\frac{dE_2}{dz} = \mu_o p H_1 , \tag{17}$$

$$\frac{dH_1}{dz} = \epsilon_o \left[\frac{\Pi^2 \left(1 - \frac{\Omega}{p} \frac{E_1}{E_2}\right)}{p^2 + \Omega^2} + 1 \right] p E_2 . \tag{18}$$

If solutions of the form $E_1 = A_1 e^{-az}$, $E_2 = A_2 e^{-az}$, $H_1 = B_1 e^{-az}$, and $H_2 = B_2 e^{-az}$ are substituted into the above equations, one obtains

$$a A_1 = \mu_o p B_2, \tag{19}$$

$$a B_2 = \epsilon_o \left[\frac{\Pi^2 (pA_1 + \Omega A_2) + p(p^2 + \Omega^2) A_1}{p^2 + \Omega^2} \right] , \tag{20}$$

$$a A_2 = -\mu_o p B_1 , \tag{21}$$

$$a B_1 = -\epsilon_o \left[\frac{\Pi^2 (pA_2 - \Omega A_1) + p(p^2 + \Omega^2) A_2}{p^2 + \Omega^2} \right] . \tag{22}$$

Eliminating B_2 from Eqs. (19) and (20) and eliminating B_1 from Eqs. (21) and (22) yields

$$a^2 A_1 = \epsilon_o \mu_o p \left[\frac{\Pi^2 (pA_1 + \Omega A_2) + p(p^2 + \Omega^2) A_1}{p^2 + \Omega^2} \right] \tag{23}$$

and

$$a^2 A_2 = \epsilon_o \mu_o p \left[\frac{\Pi^2 (pA_2 - \Omega A_1) + p(p^2 + \Omega^2) A_2}{p^2 + \Omega^2} \right] . \tag{24}$$

Taking the ratio of these two equations gives

$$\frac{A_1}{A_2} = \frac{\Pi^2 (pA_1 + \Omega A_2) + p(p^2 + \Omega^2) A_1}{\Pi^2 (pA_2 - \Omega A_1) + p(p^2 + \Omega^2) A_2} . \tag{25}$$

Cross multiplication and collection of terms of Eq. (25) will result in

$$A_2 = \pm i A_1 . \tag{26}$$

Using this relation in Eq. (23) will give

$$a = \frac{p}{c} \sqrt{1 + \frac{\Pi^2}{p(p \pm i\Omega)}} . \tag{27}$$

The general solution for the fields will consist of a linear combination of the particular solutions or

$$E_1 (z,p) = A_1 \left\{ \gamma_1 \exp\left[- \frac{zp}{c} \sqrt{1 + \frac{\Pi^2}{p(p + i\Omega)}} \right] + \gamma_2 \exp\left[- \frac{zp}{c} \sqrt{1 + \frac{\Pi^2}{p(p - i\Omega)}} \right] \right\}$$

where γ_1 and γ_2 are arbitrary constants to be determined. At this point it is convenient to introduce the following transformation.

$$E^{\pm} (z,p) = 1/2 \left[E_1 (z,p) \pm i E_2 (z,p) \right]$$

$$H^{\pm} (z,p) = 1/2 \left[H_2 (z,p) \mp i H_1 (z,p) \right]$$

Use of these relations in Eqs. (15), (16), (17) and (18) will yield a transformed wave equation for both $E^{\pm}(z,p)$ and $H^{\pm}(z,p)$ which leads to

$$E^{\pm}(z,p) = A_1 \exp\left[-\frac{z}{c}\, p \sqrt{1 + \frac{\Pi^2}{p(p \pm i\Omega)}}\right] , \tag{28}$$

To obtain A_1 it is necessary to match boundary conditions at the freespace plasma interface. The following relations must hold.

$$E_I^{\pm} + E_R^{\pm} = E_T^{\pm} \tag{29}$$

and

$$H_I^{\pm} + H_R^{\pm} = H_T^{\pm} . \tag{30}$$

Also

$$\frac{E_I^{\pm}}{H_I^{\pm}} = Z_1 ; \quad \frac{E_R^{\pm}}{H_R^{\pm}} = -Z_1 ; \quad \frac{E_T^{\pm}}{H_T^{\pm}} = Z_2 .$$

If we define the transmission coefficient as

$$T^{\pm} = \frac{E_T^{\pm}}{E_I^{\pm}} , \tag{31}$$

then using Eqs. (29) and (30) and the impedance relations above, we have in the usual manner.

$$T^{\pm} = \frac{2}{1 + \dfrac{Z_1}{Z_2}} . \tag{32}$$

But

$$Z_1 = \sqrt{\frac{\mu_o}{\epsilon_o}}$$

and

$$Z_2 = \frac{A_1}{B_2} = \frac{\mu_o p}{a} \ .$$

Thus

$$T^{\pm} = \frac{2}{1 + \sqrt{1 + \frac{\Pi^2}{p(p \pm i\,\Omega)}}} \tag{33}$$

We define

$$R^{\pm} = \frac{E_R^{\pm}}{E_I^{\pm}} \ .$$

Then from Eq. (29)

$$T^{\pm} = 1 + R^{\pm} \ . \tag{34}$$

It is now convenient to make a shift of z_o/c in the time axis. At $t = 0$ we have the wave in free space incident upon the plasma at this instant. We stipulate that the time response of the wave for $t < 0$ is zero, and for $t > 0$ is $f(t)$. At the interface we have

$$T^{\pm}F(p) = A_1 \exp\left[- \frac{z_o}{c}\, p \sqrt{1 + \frac{\Pi^2}{p(p \pm i\,\Omega)}} \right]$$

where

$$F(p) = \int_o^{\infty} f(t)\, e^{-pt}\, dt.$$

The complete solution for the transmitted wave in the plasma becomes

$$E_T^{\pm}(z, p) = \frac{2\,F(p)\exp\left[-\left(\frac{z - z_o}{c}\right) p \sqrt{1 + \frac{\Pi^2}{p(p \pm i\,\Omega)}}\right]}{1 + \sqrt{1 + \frac{\Pi^2}{p(p \pm i\,\Omega)}}} . \tag{35}$$

Taking the inverse Laplace transform of this expression will lead to $\mathcal{E}_T^{(1)}(z, t)$. Equation (29) can be used to obtain the reflected wave. The reflected wave in free space is

$$E_R^{\pm}(z_o, p) = \frac{2F(p)}{1 + \sqrt{1 + \frac{\Pi^2}{p(p \pm i\,\Omega)}}} - F(p) . \tag{36}$$

The inverse Laplace transform of Eq. (36) will lead to $\mathcal{E}_R^{(1)}(z_o, t)$.

Use of the convolution theorem enables us to obtain some general expressions for these inverses. The convolution theorem states that

$$\mathcal{L}^{-1}[F(p)\,G(p)] = \int_o^t f(t - \tau)\,g(\tau)\,d\tau, \tag{37}$$

where

$$F(p) = \int_o^\infty f(t)\,e^{-pt}\,dt ,$$

$$G(p) = \int_o^\infty g(t)\,e^{-pt}\,dt ,$$

and

$\mathcal{L}^{-1}[F(p)\,G(p)]$ is denoted as the inverse Laplace transform.

2.1 The Reflected Wave

The reflected wave is given by

$$\mathcal{E}_R^{\pm}(z_o, t) = \mathcal{L}^{-1}\left[\frac{2\,F(p)}{1 + \sqrt{1 + \frac{\Pi^2}{p(p \pm i\,\Omega)}}} - F(p)\right] . \tag{38}$$

Let

$$M^2 = p(p \pm i\,\Omega)$$

or

$$M^2 = (p - \alpha)^2 + \beta^2$$

where

$$\alpha = \mp \frac{i\,\Omega}{2}$$

and

$$\beta = \frac{\Omega}{2} \quad .$$

Then

$$T^{\pm} = \frac{2}{1 + \sqrt{1 + \frac{\Pi^2}{M^2}}} = \frac{2\sqrt{(p-\alpha)^2 + \beta^2}}{\sqrt{(p-\alpha)^2 + \beta^2} + \sqrt{(p-\alpha)^2 + \beta^2 + \Pi^2}} \; .$$

The shifting theorem of Laplace transforms states that

$$\mathcal{L}^{-1}\,[\,G(p - \alpha)\,] = e^{\alpha t}\;\mathcal{L}^{-1}\,[\,G(p)\,]\;.$$

Therefore

$$\mathcal{L}^{-1}\,[T^{\pm}] = e^{\alpha t}\;\mathcal{L}^{-1}\left[\frac{2\sqrt{p^2 + \beta^2}}{\sqrt{p^2 + \beta^2} + \sqrt{p^2 + \beta^2 + \Pi^2}}\right] . \tag{39}$$

We note from Campbell and Foster[7] [Eq. (576. 3)] that

$$\mathcal{L}^{-1}\left[\frac{2}{p + \sqrt{p^2 + x^2}}\right] = \frac{2\,J_1\,(xt)}{xt}$$

and also

$$\mathcal{L}^{-1}\,[\,p\,G(p) - g(o)] = \frac{d\,g(t)}{dt}\;. \tag{40}$$

Then

$$\mathcal{L}^{-1}\left[\frac{2p}{p+\sqrt{p^2+\Pi^2}}\right] = \mathcal{L}^{-1}\left[p\left(\frac{2}{p+\sqrt{p^2+\Pi^2}}\right) - g(o) + g(o)\right]$$

$$= \frac{d\left(\dfrac{2J_1(\Pi t)}{\Pi t}\right)}{dt} + g(o)\,\delta(t)$$

$$= \delta(t) - \frac{2J_2(\Pi t)}{t}\ .$$

From Erdelyi,[8] et al. [Eq. (5.1-5)] we have

$$\mathcal{L}^{-1}\left[G\left(\sqrt{p^2+\beta^2}\right)\right] = g(t) - \beta\int_o^t g\left(\sqrt{t^2-u^2}\right)J_1(\beta u)\,du\ . \qquad (41)$$

Equation (39) becomes

$$\mathcal{L}^{-1}[T^{\pm}] = e^{\alpha t}\left\{\delta(t) - \frac{2J_2(\Pi t)}{t} - \beta\int_o^t\left[\delta\left(\sqrt{t^2-u^2}\right) - \frac{2J_2\left(\Pi\sqrt{t^2-u^2}\right)}{\sqrt{t^2-u^2}}\right]\bullet\right.$$

$$\left.\bullet\left[J_1(\beta u)\right]du\right\}$$

which reduces to

$$\mathcal{L}^{-1}[T^{\pm}] = e^{\mp i\Omega\tau/2}\left\{\delta(t) - \frac{2J_2(\Pi t)}{t} + \Omega\int_o^t\frac{J_2\left(\Pi\sqrt{t^2-u^2}\right)J_1\left(\frac{\Omega u}{2}\right)du}{\sqrt{t^2-u^2}}\right\} \qquad (42)$$

Then from Eqs. (37) and (38)

$$\mathcal{E}_R^{\pm}(z_o,t) = \int_o^t f(t-\tau)\left[e^{\mp i\Omega\tau/2}\left\{\delta(\tau) - \frac{2J_2(\Pi\tau)}{\tau} + \Omega\int_o^{\tau}\frac{J_2\left(\Pi\sqrt{\tau^2-u^2}\right)J_1\left(\frac{\Omega u}{2}\right)du}{\sqrt{\tau^2-u^2}} - \delta(\tau)\right\}\right]d\tau$$

or

$$\mathcal{E}_R^{\pm}(z_o,t) = \int_o^t f(t-\tau)\left[e^{\mp i\Omega\tau/2}\left\{-\frac{2J_2(\Pi\tau)}{\tau} + \Omega\int_o^{\tau}\frac{J_2(\Pi\sqrt{\tau^2-u^2})\,J_1\left(\frac{\Omega u}{2}\right)du}{\sqrt{\tau^2-u^2}}\right\}\right]d\tau, \quad (43)$$

where f(t) is the electric field of the incident wave for all time.

Equation (43) gives the time dependence of the reflected wave at the interface. The spatial solution can be obtained using Maxwell's freespace equations. Since free space is nondispersive, the time dependence of the solution is given within a constant phase angle for each spatial point by Eq. (43).

2.2 The Transmitted Wave

The transmitted wave is given by

$$\mathcal{E}_T^{\pm}(z,t) = \mathcal{L}^{-1}\left[\frac{2F(p)\exp\left\{-\left(\frac{z-z_o}{c}\right)p\sqrt{1+\frac{\Pi^2}{p(p\pm i\Omega)}}\right\}}{1+\sqrt{1+\frac{\Pi^2}{p(p\pm i\Omega)}}}\right]. \quad (44)$$

Let us examine

$$\mathcal{L}^{-1}\left[\exp\left\{-\left(\frac{z-z_o}{c}\right)p\sqrt{1+\frac{\Pi^2}{p(p\pm i\Omega)}}\right\}\right].$$

As before we call $p(p\pm i\Omega) = M^2$ and $M = \sqrt{(p-\alpha)^2+\beta^2}$.

Then

$$p\sqrt{1+\frac{\Pi^2}{p(p\pm i\Omega)}} = p\sqrt{\frac{M^2+\Pi^2}{M^2}}.$$

We now let

$$s = p - \alpha = \sqrt{M^2-\beta^2}.$$

This results in

$$p\sqrt{1+\frac{\Pi^2}{p(p\pm i\Omega)}} = \frac{s+\alpha}{M}\sqrt{M^2+\Pi^2} = \sqrt{\frac{(M^2-\beta^2)(M^2+\Pi^2)}{M^2}} + \frac{\alpha\sqrt{M^2+\Pi^2}}{M}$$

$$= \sqrt{M^2 - \frac{\Pi^2\beta^2}{M^2} + (\Pi^2-\beta^2)} + \frac{\alpha\sqrt{M^2+\Pi^2}}{M}. \quad (45)$$

Then

$$\mathcal{L}^{-1}\left[\exp\left\{-\left(\frac{z-z_o}{c}\right)p\sqrt{1+\frac{\Pi^2}{p(p\pm i\Omega)}}\right\}\right]=\mathcal{L}^{-1}\left[\exp\left\{-\left(\frac{z-z_o}{c}\right)\left(\sqrt{M^2-\frac{\Pi^2\beta^2}{M^2}+(\Pi^2-\beta^2)}+\frac{\alpha\sqrt{M^2+\Pi^2}}{M}\right)\right\}\right] \tag{46}$$

$$=e^{\alpha t}\mathcal{L}^{-1}\left[\exp\left\{-\left(\frac{z-z_o}{c}\right)\left(\sqrt{r^2-\frac{\Pi^2\beta^2}{r^2}+(\Pi^2\beta^2)}+\frac{\alpha\sqrt{r^2+\Pi^2}}{r}\right)\right\}\right]$$

where $r = \sqrt{p^2+\beta^2}$.

Considering this product of exponentials we observe that

$$\exp\left[-\left(\frac{z-z_o}{c}\right)\sqrt{p^2-\frac{\Pi^2\beta^2}{p^2}+(\Pi^2\beta^2)}\right] = \exp\left[-\left(\frac{z-z_o}{c}\right)\sqrt{\left(p+\frac{i\Pi\beta}{p}\right)^2+(\Pi-i\beta)^2}\right]. \tag{47}$$

From Goldman[9] we see that

$$\mathcal{L}^{-1}\left[\exp\left\{-\left(\frac{z-z_o}{c}\right)\sqrt{p^2+(\Pi-i\beta)^2}\right\}\right] =$$

$$\delta\left(t-\frac{z-z_o}{c}\right)-\frac{\left(\frac{z-z_o}{c}\right)(\Pi-i\beta)}{\sqrt{t^2-\left(\frac{z-z_o}{c}\right)^2}}\; J_1\left([\Pi-i\beta]\sqrt{t^2-\left(\frac{z-z_o}{c}\right)^2}\right)U\left(t-\frac{z-z_o}{c}\right) \tag{48}$$

and from Erdelyi[8] [Eq. (4. 1-36) we have

$$\mathcal{L}^{-1}\left[\frac{1}{p}G\left(p+\frac{\gamma}{p}\right)\right] = \int_o^t J_o\left(2\sqrt{\gamma ut-\gamma u^2}\right)g(u)\,du$$

so that

$$\mathcal{L}^{-1}\left[G\left(p+\frac{\gamma}{p}\right)\right] = \frac{d}{dt}\int_o^t J_o\left(2\sqrt{\gamma ut-\gamma u^2}\right)g(u)\,du$$

$$= g(t)-\gamma\int_o^t \frac{J_1\left(2\sqrt{\gamma ut-\gamma u^2}\right)g(u)\,u\,du}{\sqrt{\gamma ut-\gamma u^2}}\ . \tag{49}$$

From Eqs. (47), (48) and (49) it follows that

$$\mathcal{L}^{-1}\left[\exp\left\{-\left(\frac{z-z_o}{c}\right)\sqrt{p^2 - \frac{\Pi^2\beta^2}{p^2} + \Pi^2 - \beta^2}\right\}\right] = \delta\left(t - \frac{z-z_o}{c}\right) -$$

$$-\frac{\left(\frac{z-z_o}{c}\right)(\Pi - i\beta)}{\sqrt{t^2 - \left(\frac{z-z_o}{c}\right)^2}}\; J_1\left[(\Pi - i\beta)\sqrt{t^2 - \left(\frac{z-z_o}{c}\right)^2}\right] U\left(t - \frac{z-z_o}{c}\right) -$$

$$-i\Pi\beta_o\int_o^t\left[\delta\left(u - \frac{z-z_o}{c}\right) - \frac{\left(\frac{z-z_o}{c}\right)(\Pi - i\beta)}{\sqrt{u^2 - \left(\frac{z-z_o}{c}\right)^2}}\; J_1\left((\Pi - i\beta)\sqrt{u^2 - \left(\frac{z-z_o}{c}\right)^2}\right) U\left(u - \frac{z-z_o}{c}\right)\right] \cdot$$

$$\cdot\; \frac{J_1\left(2\sqrt{i\Pi\beta u t - i\Pi\beta u^2}\right) du}{\sqrt{i\Pi\beta u t - i\Pi\beta u^2}} \tag{50}$$

and calling the right-hand side of Eq. (50) by h(t) we have

$$\mathcal{L}^{-1}\left[\exp\left\{-\left(\frac{z-z_o}{c}\right)\sqrt{p^2 - \frac{\Pi^2\beta^2}{p^2} + \Pi^2 - \beta^2}\right\}\right] = h(t)\,. \tag{51}$$

Use of Eq. (41) will result in

$$\mathcal{L}^{-1}\left[\exp\left\{-\left(\frac{z-z_o}{c}\right)\sqrt{r^2 - \frac{\Pi^2\beta^2}{r^2} + \Pi - \beta^2}\right\}\right] = h(t) - \beta\int_o^t h\left(\sqrt{t^2 - u^2}\right) J_1(\beta u)\, du. \tag{52}$$

We shall now turn our attention back to Eq. (46) and investigate the second exponential on the right-hand side, namely

$$\exp\left[-\left(\frac{z-z_o}{c}\right)\frac{\alpha\sqrt{r^2 + \Pi^2}}{r}\right].$$

Observing that

$$\frac{\sqrt{p^2 + \Pi^2}}{p} = \frac{\sqrt{\left(\frac{p}{\Pi}\right)^2 + 1}}{\left(\frac{p}{\Pi}\right)} \tag{53}$$

and also from Erdelyi[8] [Eq. (5.6-22)] that

$$\mathcal{L}^{-1}\left[\frac{1}{p}\exp\left\{-\frac{1}{b}\frac{\sqrt{p^2+1}}{p}\right\}\right] = J_o(2\sqrt{bt}) - b\int_o^\infty \frac{J_o(2\sqrt{bt})\,J_1(\sqrt{u^2-b^2})\,du}{\sqrt{u^2-b^2}}$$

and, using the relation

$$\mathcal{L}^{-1}\left[G\left(\frac{p}{A}\right)\right] = A\,g\,(At),$$

we find

$$\mathcal{L}^{-1}\left[\exp\left\{-\frac{\alpha(z-z_o)}{c}\frac{\sqrt{\left(\frac{p}{\Pi}\right)^2+1}}{\left(\frac{p}{\Pi}\right)}\right\}\right] = \Pi\left\{\frac{d}{dt}\left(J_o\left(2\sqrt{\frac{\Pi ct}{\alpha(z-z_o)}}\right)\left[1-\frac{c}{\alpha(z-z_o)}\int_o^\infty\frac{J_1\left(\sqrt{u^2-b^2}\right)du}{\sqrt{u^2-b^2}}\right]\right)\right\}$$

$$\equiv \eta\,(t)\,. \tag{54}$$

It thus follows due to Eq. (41) that

$$\mathcal{L}^{-1}\left[\exp\left\{-\frac{\alpha(z-z_o)}{c}\frac{\sqrt{r^2+\Pi^2}}{r}\right\}\right] = \eta\,(t) - \beta\int_o^t \eta\left(\sqrt{t^2-u^2}\right)J_1\,(\beta u)\,du\,. \tag{55}$$

Finally from the convolution theorem and Eqs. (52) and (55),

$$\mathcal{L}^{-1}\left[\exp\left\{-\left(\frac{z-z_o}{c}\right)p\sqrt{1+\frac{\Pi^2}{p(p\pm i\Omega)}}\right\}\right] = e^{\alpha t}\int_o^t\left[h(t-\tau)-\beta_o\int_0^{t-\tau}h\left(\sqrt{(t-\tau)^2-u^2}\right)J_1(\beta u)\,du\right]$$

$$\cdot\left[\eta\,(\tau) - \beta\int_o^\tau \eta\left(\sqrt{\tau^2-u^2}\right)J_1\,(\beta u)\,du\right]d\tau. \tag{56}$$

The expression for the inverse Laplace transform for

$$\frac{2\,F(p)}{1+\sqrt{1+\frac{\Pi^2}{p\,(p\pm i\,\Omega)}}}$$

has already been found and is given from Eq. (42); that is,

$$\mathcal{L}^{-1}\left[\frac{2\,F(p)}{1+\sqrt{1+\dfrac{\Pi^2}{p(p\pm i\,\Omega)}}}\right] = \int_0^t f(t-\tau)\left[e^{\mp i\Omega\tau/2}\left\{\delta(\tau) - \frac{2\,J_2(\Pi\tau)}{\tau}\right.\right.$$

$$\left.\left. + \Omega\int_0^\tau \frac{J_2\left(\Pi\sqrt{\tau^2-u^2}\right)}{\sqrt{\tau^2-u^2}}\, J_1\left(\frac{\Omega u}{2}\right) du\right\} d\tau\right]. \tag{57}$$

Forming a convolution of Eqs. (56) and (57) will in principle yield the complete solution to the transmitted wave in the plasma. The general expression is given as

$$\mathcal{E}_T^{\pm}(z,t) = \int_0^t \left[\int_0^{t-x} f(t-x-\tau)\, e^{\alpha\tau}\left\{\delta(\tau) - \frac{2J_2(\Pi\,\tau)}{\tau} + \Omega\int_0^\tau \frac{J_2\left(\Pi\sqrt{\tau^2-u^2}\right)J_1(\beta u)du}{\sqrt{\tau^2-u^2}}\right\} d\tau\right] \bullet$$

$$\bullet \left[e^{\alpha x}\int_0^x\left\{h\,(x-\tau) - \beta\int_0^{x-\tau} h\left(\sqrt{(x-\tau)^2-u^2}\right)J_1\,(\beta u)\,du\right\} \bullet\right. \tag{58}$$

$$\left.\bullet\left\{\eta\,(\tau) - \beta\int_0^\tau \eta\left(\sqrt{\tau^2-y^2}\right)J_1\,(\beta y)\,dy\right\} d\tau\right]\, dx\,,$$

where

$$h(t) \equiv \delta\left(t - \frac{z-z_o}{c}\right) - \frac{\left(\frac{z-z_o}{c}\right)(\Pi - i\beta)}{\sqrt{t^2-\left(\frac{z-z_o}{c}\right)^2}}\; J_1\left([\Pi - i\beta]\sqrt{t^2-\left(\frac{z-z_o}{c}\right)^2}\right) U\left(t-\frac{z-z_o}{c}\right)$$

$$- i\Pi\beta\int_0^t\left[\delta\left(u-\frac{z-z_o}{c}\right) - \frac{\left(\frac{z-z_o}{c}\right)(\Pi - i\beta)}{\sqrt{u^2-\left(\frac{z-z_o}{c}\right)^2}}\; J_1\left([\Pi - i\beta]\sqrt{u^2-\left(\frac{z-z_o}{c}\right)^2}\right) U\left(u-\frac{z-z_o}{c}\right)\right] \bullet$$

$$\bullet\left[\frac{J_1\left(2\sqrt{i\Pi\beta\,(ut-u^2)}\right)}{\sqrt{i\Pi\beta\,(ut-u^2)}}\right] u\,du$$

and

$$\eta(t) \equiv \sqrt{\frac{\Pi c}{\alpha(z-z_o)t}}\; J_1\left(2\sqrt{\frac{\Pi ct}{\alpha(z-z_o)}}\right)\left[1+\sqrt{\frac{c}{2\beta(z-z_o)}}\left(\frac{c\sqrt{\pi}}{\alpha(z-z_o)}\right)J_{1/2}\left(\frac{c}{\beta(z-z_o)}\right)\right],$$

$$\alpha = \mp\frac{i\Omega}{2},$$

$$\beta = \frac{\Omega}{2}.$$

3. THE ISOTROPIC CASE

3.1 Sinusoidal Time Dependence

While in principle Eq. (58) gives the total wave solution, the expression is extremely complicated. Insight into the problem can be obtained by considering some special cases of the more general problem. First, we will consider the case with no dc magnetic field. This is equivalent to allowing $\Omega = 0$. Assume a time dependence of

$$f(t) = E_o e^{-i\omega t}\, U(t).$$

For this condition Eq. (43) for the reflected wave becomes

$$\mathcal{E}_R^{(1)}(z_o, t) = -2\,E_o e^{-i\omega t}\int_0^t \frac{e^{i\omega\tau}\, J_2(\Pi\tau)\, d\tau}{\tau}. \tag{59}$$

The transmitted wave can be obtained from Eq. (44), that is

$$\mathcal{E}_T^{(1)}(z,t) = \mathcal{L}^{-1}\left[\frac{2p\,F(p)}{p+\sqrt{p^2+\Pi^2}}\;\exp\left\{-\left(\frac{z-z_o}{c}\right)\sqrt{p^2+\Pi^2}\right\}\right]. \tag{60}$$

Utilizing Eqs. (48) and (57) we obtain

$$\mathcal{E}_T^{(1)}(z,t) = E_o\int_0^t\left[e^{-i\omega(t-\tau-x)}\left\{\delta(x) - \frac{2J_2(\Pi x)}{x}\right\}dx\right]\bullet$$

$$\bullet\left[\delta\left(\tau-\frac{z-z_o}{c}\right) - \frac{\left(\frac{z-z_o}{c}\right)\Pi\, J_1\left(\Pi\sqrt{\tau^2-\left(\frac{z-z_o}{c}\right)^2}\right)U\left(\tau-\frac{z-z_o}{c}\right)}{\sqrt{\tau^2-\left(\frac{z-z_o}{c}\right)^2}}\right]d\tau$$

or

$$\mathcal{E}_T^{(1)}(z,t) = E_o \int_o^t \left[e^{-i\omega(t-\tau)} \left\{ 1 - 2 \int_o^{t-\tau} \frac{e^{i\omega x} J_2(\Pi x)\, dx}{x} \right\} \right] \bullet$$

$$\bullet \left[\delta\left(\tau - \frac{z-z_o}{c}\right) - \frac{\left(\frac{z-z_o}{c}\right) \Pi J_1 \left(\Pi \sqrt{\tau^2 - \left(\frac{z-z_o}{c}\right)^2}\right) U\left(\tau - \frac{z-z_o}{c}\right)}{\sqrt{\tau^2 - \left(\frac{z-z_o}{c}\right)^2}} \right] d\tau \,.$$

and finally

$$\mathcal{E}_T^{(1)}(z,t) = E_o e^{-i\omega t} \left[\exp\left\{ i\omega \left(\frac{z-z_o}{c}\right) \right\} - 2 \int_o^{t - \left(\frac{z-z_o}{c}\right)} \frac{e^{i\omega x} J_2(\Pi x)\, dx}{x} \right.$$

$$- \left(\frac{z-z_o}{c}\right) \Pi \int_{\frac{z-z_o}{c}}^{t} \left\{ \frac{e^{i\omega\tau} J_1\left(\Pi \sqrt{\tau^2 - \left(\frac{z-z_o}{c}\right)^2}\right)}{\sqrt{\tau^2 - \left(\frac{z-z_o}{c}\right)^2}} - \frac{2 e^{i\omega\tau} J_1\left(\Pi \sqrt{\tau^2 - \left(\frac{z-z_o}{c}\right)^2}\right)}{\sqrt{\tau^2 - \left(\frac{z-z_o}{c}\right)^2}} \bullet \right.$$

$$\left. \left. \bullet \int_o^{t-\tau} \frac{e^{i\omega x} J_2(\Pi x)\, dx}{x} \right\} d\tau \right] \cdot U\left(t - \frac{z-z_o}{c}\right) . \qquad (61)$$

Equation (59) gives the reflected wave at the interface for all times. This wave will travel in the negative z direction undispersed in the freespace half-plane. While this integral is not listed in closed-form, for large times the Bessel function may be expanded in an asymptotic expansion yielding

$$\mathcal{E}_R^{(1)}(z_o, t) \sim 2 E_o e^{-i\omega t} \left[\int_o^{t_o} e^{i\omega\tau} \frac{J_2(\Pi\tau)\, d\tau}{\tau} \right.$$

$$\left. + \sqrt{\frac{2}{\pi}} \int_{t_o}^{t} \frac{(\cos\omega\tau + i \sin\omega\tau) \cos\left(\Pi\tau - \frac{5\pi}{4}\right) d\tau}{\tau^{3/2}} \right]$$

where t_o is large enough so that the expansion is valid. The first integral of the above expression is simply a constant. The second integral is not listed but (since $\cos\left[\Pi\tau - \frac{5\pi}{4}\right] = \cos\frac{5\pi}{4} \cos \Pi\tau + \sin\frac{5\pi}{4} \sin \Pi\tau$) yields integrals of the form

$$\int_{t_o}^{t} \frac{\cos \omega\tau \cos \Pi\tau \, d\tau}{\tau^{3/2}} ,$$

$$\int_{t_o}^{t} \frac{\cos \omega\tau \sin \Pi\tau \, d\tau}{\tau^{3/2}} ,$$

$$\int_{t_o}^{t} \frac{\sin \omega\tau \cos \Pi\tau \, d\tau}{\tau^{3/2}} ,$$

$$\int_{t_o}^{t} \frac{\sin \omega\tau \sin \Pi\tau \, d\tau}{\tau^{3/2}} .$$

Integrals of this type yield damped sinusoidal oscillations of frequencies $|\omega + \Pi|$ and $|\omega - \Pi|$. This can readily be seen by graphical integration. We therefore see that the transient reflected wave from the interface will have oscillations at the absolute magnitude of the sum and the difference of the signal frequency and the plasma frequency.

Looking at the transmitted wave we make another interesting observation. For times of the order $\left[\frac{z-z_o}{c} + \Delta\right]$, where Δ is a small increment, the integrals in Eq. (61) contribute very little to the expression. On this small time scale the transmitted wave in the plasma becomes

$$\mathcal{E}_T^{(1)}(z,t) \approx E_o \exp\left\{-i\left[\omega t - \omega \frac{(z-z_o)}{c}\right]\right\},$$

which is just the way in which the wave would have travelled in free space with no plasma present. This result is independent of the plasma frequency if the order of Δ is small compared to $1/\Pi$. This simply means that the wave must 'try out' the plasma to determine the plasma frequency and then act accordingly. This result, well known for dielectrics in general, was first discovered by Sommerfeld and Brillouin and is pointed out by Stratton.[10]

Allowing t to become very, very large in the general expressions should yield the steady-state solutions.

Considering the transmitted field at $z = z_o$, we have

$$\mathcal{E}_T^{(1)}(z_o, t) = E_o e^{-i\omega t}\left[1 - 2\int_o^t \frac{e^{i\omega x} J_2(\Pi x)\, dx}{x}\right]. \tag{62}$$

Then allowing t to become very, very large gives

$$\mathcal{E}_T^{(1)}(z_o, t)_{\text{steady state}} = E_o e^{-i\omega t}\left[1 - 2\int_o^\infty \frac{e^{i\omega x} J_2(\Pi x)\, dx}{x}\right] \tag{63}$$

or

$$\mathcal{E}_T^{(1)}(z_o, t)_{\text{steady state}} = E_o e^{-i\omega t}\left[1 - 2\left(\int_o^\infty \frac{\cos\omega x\, J_2(\Pi x)\, dx}{x} + i\int_o^\infty \frac{\sin\omega x\, J_2(\Pi x)\, dx}{x}\right)\right].$$

From Watson,* we find

$$\int_o^\infty \frac{\cos\omega x\, J_2(\Pi x)\, dx}{x} = \begin{cases} \frac{1}{2}\cos\left[2\sin^{-1}\left(\frac{\omega}{\Pi}\right)\right] & \text{for } \omega \leq \Pi \\ \dfrac{-\Pi^2}{2\left[\omega + \sqrt{\omega^2 - \Pi^2}\right]^2} & \text{for } \omega \geq \Pi \end{cases}$$

and

$$\int_o^\infty \frac{\sin\omega x\, J_2(\Pi x)\, dx}{x} = \begin{cases} \frac{1}{2}\sin\left[2\sin^{-1}\left(\frac{\omega}{\Pi}\right)\right] & \text{for } \omega \leq \Pi \\ 0 & \text{for } \omega \geq \Pi \end{cases}.$$

Calling

$$I = \int_o^\infty \frac{e^{i\omega x} J_2(\Pi x)\, dx}{x},$$

we see that, for $\omega \geq \Pi$,

*Reference 11, p. 405

$$[1\text{-}2I] = 1 + \frac{\Pi^2}{\left(\omega + \sqrt{\omega^2 - \Pi^2}\right)^2} = \frac{\left(1 + \sqrt{1 - \frac{\Pi^2}{\omega^2}}\right)^2 + \frac{\Pi^2}{\omega^2}}{\left(1 + \sqrt{1 - \frac{\Pi^2}{\omega^2}}\right)^2}$$

or

$$[1 - 2I] = \frac{1 + 2\sqrt{1 - \frac{\Pi^2}{\omega^2}} + 1 - \frac{\Pi^2}{\omega^2} + \frac{\Pi^2}{\omega^2}}{\left(1 + \sqrt{1 - \frac{\Pi^2}{\omega^2}}\right)^2} = \frac{2}{1 + \sqrt{1 - \frac{\Pi^2}{\omega^2}}} \,. \tag{64}$$

For the case $\omega \leq \Pi$ we find

$$[1 - 2I] = 1 - \left(\cos\left[2 \sin^{-1}\left(\frac{\omega}{\Pi}\right)\right] + i \sin\left[2 \sin^{-1}\left(\frac{\omega}{\Pi}\right)\right]\right).$$

Letting

$$\theta = 2 \sin^{-1}\left(\frac{\omega}{\Pi}\right) ,$$

then

$$\sin\frac{\theta}{2} = \sqrt{\frac{1 - \cos\theta}{2}} = \frac{\omega}{\Pi} ,$$

$$\cos\theta = 1 - 2\left(\frac{\omega}{\Pi}\right)^2,$$

and

$$\sin\theta = 2\,\frac{\omega}{\Pi}\sqrt{1 - \left(\frac{\omega}{\Pi}\right)^2} \,.$$

Equation (65) takes the form

$$[1\text{-}2I] = 1 - 1 + 2\left(\frac{\omega}{\Pi}\right)^2 - i\,2\left(\frac{\omega}{\Pi}\right)\sqrt{1 - \left(\frac{\omega}{\Pi}\right)^2} = 2\,\frac{\omega}{\Pi}\left[\frac{\omega}{\Pi} - i\sqrt{1 - \frac{\omega^2}{\Pi^2}}\right]$$

$$= 2\,\frac{\omega}{\Pi}\left[\frac{\omega}{\Pi} - i\sqrt{1 - \frac{\omega^2}{\Pi^2}}\right]\left[\frac{\frac{\omega}{\Pi} + i\sqrt{1 - \frac{\omega^2}{\Pi^2}}}{\frac{\omega}{\Pi} + i\sqrt{1 - \frac{\omega^2}{\Pi^2}}}\right] = \frac{2\left(\frac{\omega}{\Pi}\right)}{\frac{\omega}{\Pi} + \sqrt{-1 + \frac{\omega^2}{\Pi^2}}} \,.$$

and finally

$$[1 - 2I] = \frac{2}{1 + \sqrt{1 - \frac{\Pi^2}{\omega^2}}} \,. \tag{65}$$

Equations (63), (64), and (65) now give

$$\mathcal{E}_T^{(1)}(z_o, t)_{\text{steady state}} = E_o e^{-i\omega t} \left(\frac{2}{1 + \sqrt{1 - \frac{\Pi^2}{\omega^2}}} \right) . \tag{66}$$

Equation (66) is exactly the same result for the steady-state solution obtained by the usual means. Thus in the steady state the reflected wave will have simply an $e^{-i\omega t}$ dependence.

We now look at the solution, for a wave launched at $z = 0$, of

$$f(t) = E_o e^{-i\omega t}\, U(t)$$

in an infinite isotropic plasma. The solution is given by the inverse Laplace transform of Eq. (28), that is,

$$\mathcal{E}_T(z, t) = E_o e^{-i\omega t} \int_o^t e^{i\omega\tau} \left[\delta\left(\tau - \frac{z}{c}\right) - \frac{z\Pi}{c} \, \frac{J_1\left(\Pi \sqrt{\tau^2 - \frac{z^2}{c^2}}\right) U\left(t - \frac{z}{c}\right)}{\sqrt{\tau^2 - \frac{z^2}{c^2}}} \right] d\tau$$

or

$$\mathcal{E}_T(z, t) = E_o \exp\left\{ -i\left(\omega t - \frac{\omega z}{c}\right)\right\} \left[1 - \frac{z\Pi}{c} \int_{z/c}^t \frac{e^{i\omega\tau} J_1\left(\Pi \sqrt{\tau^2 - \frac{z^2}{c^2}}\right) d\tau}{\sqrt{\tau^2 - \frac{z^2}{c^2}}} \right] U\left(t - \frac{z}{c}\right).$$

We find the approximate steady-state solution for small values of z/c and for both small and large values of Π/ω by assuming that, as time becomes very large, the steady-state solution becomes

$$\mathcal{E}_T(z,t)_{\text{steady state}} \approx E_o \exp\left[-i\left(\omega t - \frac{\omega z}{c}\right)\right]\left[1 - \frac{z\Pi}{c}\int_0^\infty \frac{e^{i\omega\tau} J_1(\Pi\tau)\,d\tau}{\tau}\right]. \tag{67}$$

From Watson,* we have

$$\int_0^\infty \frac{J_1(\Pi t)\sin\omega t}{t}\,dt = \begin{cases} \dfrac{\omega}{\Pi} & \text{for } \omega \le \Pi \\ \dfrac{\Pi}{\omega + \sqrt{\omega^2 - \Pi^2}} & \text{for } \omega \ge \Pi \end{cases}$$

and

$$\int_0^\infty \frac{J_1(\Pi t)\cos\omega t}{t}\,dt = \begin{cases} \cos\left[\sin^{-1}\left(\dfrac{\omega}{\Pi}\right)\right] & \text{for } \omega \le \Pi \\ 0 & \text{for } \omega \ge \Pi \end{cases}.$$

Then Eq. (67) becomes for $\omega \ll \Pi$

$$\mathcal{E}_T(z,t)_{\text{steady state}} \approx E_o \exp\left[-i\left(\omega t - \frac{\omega z}{c}\right)\right]\left[1 - \frac{z\Pi}{c}\left(1 - i\frac{\omega}{\Pi}\right)\right]$$

$$\approx E_o e^{-i\omega t}\left(1 - \frac{z\Pi}{c}\right). \tag{68}$$

The usual steady-state solution for $\omega < \Pi$ is given as

$$\mathcal{E}_T(z,t)_{\text{steady state}} = E_o e^{-i\omega t} \exp\left\{-\frac{z}{c}\sqrt{\Pi^2 - \omega^2}\right\}$$

and for $\omega \ll \Pi$ and small z/c

$$\mathcal{E}_T(z,t)_{\text{steady state}} \approx E_o e^{-i\omega t} e^{-z\Pi/c} = E_o e^{-i\omega t}\left[1 - \frac{z\Pi}{c}\right]. \tag{69}$$

so that for this order of approximation Eqs. (68) and (69) agree.

For the case of small z/c and $\omega \gg \Pi$, we have the freespace solution from Eq. (67)

$$\mathcal{E}_T(z,t)_{\text{steady state}} \approx E_o \exp\left[-i\left(\omega t - \frac{\omega z}{c}\right)\right]. \tag{70}$$

*Reference 11, p. 405.

The usual steady-state solution for the condition $\omega > \Pi$ is

$$\mathcal{E}_T(z,t)_{\text{steady state}} = E_o e^{-i\omega t} \exp\left(i \frac{z}{c} \sqrt{\omega^2 - \Pi^2}\right),$$

which for $\omega >> \Pi$ reduces to

$$\mathcal{E}_T(z,t)_{\text{steady state}} \approx E_o \exp\left[-i\left(\omega t - \frac{\omega z}{c}\right)\right]. \tag{71}$$

We see that Eqs. (70) and (71) also agree for this order of approximation. In fact Eqs. (59) and (61) are exact equations and should in general reduce to the steady state solutions for large times, although to show that this is true may be a formidable problem.

We have thus far seen, for waves of sinusoidal time dependence incident on a semi-infinite isotropic plasma, that the waves initially travel through the plasma as if it were free space. This is true even for an over-dense plasma. This condition exists on a time scale small compared with the plasma frequency period. On this time scale there is essentially no reflected wave. At larger times the reflected signal will 'ring' at beat frequencies of the sum and difference of the signal frequency and the plasma frequency. This effect will also occur in the transient transmitted wave as evidenced by the first integral in Eq. (61). For very, very large times the waves should approach the usual steady-state solutions. This has been demonstrated for the reflected wave and also for the transmitted wave at the boundary.

3.2 Step-Input Case

We now consider the step-input case. The time dependence of the wave at $z = o$ becomes $f(t) = E_o U(t)$. This is equivalent to letting the frequency approach zero in the sinusoidal case. For the reflected wave Eq. (59) becomes

$$\mathcal{E}_R^{(1)}(z_o, t) = -2 E_o \int_o^t \frac{J_2(\Pi \tau)}{\tau} d\tau = E_o \left[\frac{2 J_1(\Pi t)}{\Pi t} - 1\right]$$

and the transmitted wave at the interface becomes

$$\mathcal{E}_T^{(1)}(z_o, t) = E_o \frac{2 J_1(\Pi t)}{\Pi t} .$$

These results have been previously stated by Schmitt.[5]

4. CONCLUSIONS

General expressions for the transmitted and reflected waves in integral form are obtained for a linearly polarized plane wave of arbitrary time dependence propagating along a static magnetic field and normally incident upon a semi-infinite, cold, collisionless, homogeneous, anisotropic plasma. The solutions yield integrals involving products of Bessel functions with arguments related to $\Omega t/2$ and Πt. For a complex time dependence, $e^{-i\omega t}$, products of cos ωt and sin ωt with the Bessel functions occur. These expressions will give complicated beat frequencies involving ω, Π, and Ω. These integrals are not listed, and have not yet been reduced to a simpler form. It has not been possible to obtain the steady-state solution in closed form from the general solution.

In the general expressions for reflected and transmitted waves, if Ω is relaxed to zero the isotropic solutions are immediately obtained. Examination of this case, for an incident wave of $e^{-i\omega t}$ dependence, has been carried out. For this case the wave initially travels through the plasma as if the plasma were free space. This occurs regardless of whether the plasma is over dense or under dense. The time scale for this propagation is small compared with the plasma frequency period. At later times both the reflected and transmitted waves will 'ring' at frequencies $|\omega \pm \Pi|$. These 'ringings' are a transient effect and vanish as the steady-state condition is approached. The steady-state case is investigated at the plasma-free-space interface and also in the plasma for shallow depths. The solutions for both cases reduce to the standard solutions. It is seen that the step-input case is a special case of the sinusoidal case where the signal frequency is set equal to zero.

It is planned to try reducing the integrals obtained for the anisotropic general case by analytical methods. If this fails numerical integration will be carried out to determine the behavior of the complete solution including both transient and steady-state solutions.

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ERRATA(cont)

Page 24: Equation immediately preceding Eq.(68) should read

$$\mathcal{E}_T(z,t)_{\text{steady state}} \approx E_o\, e^{-i\omega t}\left[1-\frac{z\Pi}{c}\left(1-i\frac{\omega}{\Pi}\right)\right]$$

instead of

$$\mathcal{E}_T(z,t)_{\text{steady state}} \approx E_o\, e^{-i\left(\omega t-\frac{\omega z}{c}\right)}\left[1-\frac{z\Pi}{c}\left(1-i\frac{\omega}{\Pi}\right)\right]$$

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ERRATA

Page 18: at top of page, for $\eta(t)$ should read

$$\eta(t) \equiv \Pi\sqrt{\frac{\Pi c}{\alpha(z-z_o)t}}\; J_1\left(2\sqrt{\frac{\Pi ct}{\alpha(z-z_o)}}\right)\left[-1+\sqrt{\frac{c}{2\beta(z-z_o)}}\left(\frac{c\sqrt{\pi}}{\alpha(z-z_o)}\right)J_{1/2}\left(\frac{c}{\beta(z-z_o)}\right)\right]$$

instead of

$$\eta(t) \equiv \sqrt{\frac{\Pi c}{\alpha(z-z_o)t}}\; J_1\left(2\sqrt{\frac{\Pi ct}{\alpha(z-z_o)}}\right)\left[1+\sqrt{\frac{c}{2\beta(z-z_o)}}\left(\frac{c\sqrt{\pi}}{\alpha(z-z_o)}\right)J_{1/2}\left(\frac{c}{\beta(z-z_o)}\right)\right]$$

Page 23: Bottom of page should read:

$$\mathcal{E}_T(z,t) = E_o\, e^{-i\omega t}\left[e^{\frac{i\omega z}{c}} - \frac{z\Pi}{c}\int_{z/c}^{t}\frac{e^{i\omega\tau} J_1\left(\Pi\sqrt{\tau^2-\frac{z^2}{c^2}}\right)d\tau}{\sqrt{\tau^2-\frac{z^2}{c^2}}}\right]U\left(t-\frac{z}{c}\right)$$

instead of

$$\mathcal{E}_T(z,t) = E_o\, e^{-i\left(\omega t-\frac{\omega z}{c}\right)}\left[1-\frac{z\Pi}{c}\int_{z/c}^{t}\frac{e^{i\omega\tau}\, J_1\left(\Pi\sqrt{\tau^2-\frac{z^2}{c^2}}\right)d\tau}{\sqrt{\tau^2-\frac{z^2}{c^2}}}\right]U\left(t-\frac{z}{c}\right)$$

Page 24: Equation (67) should read:

$$\mathcal{E}_T(z,t)_{\text{steady state}} \approx E_o\, e^{-i\omega t}\left[1-\frac{z\Pi}{c}\int_o^{\infty}\frac{e^{i\omega\tau}\, J_1(\Pi\tau)\, d\tau}{\tau}\right]$$

instead of

$$\mathcal{E}_T(z,t)_{\text{steady state}} \approx E_o\, e^{-i\left(\omega t-\frac{\omega z}{c}\right)}\left[1-\frac{z\Pi}{c}\int_o^{\infty}\frac{e^{i\omega\tau}\, J_1(\Pi\tau)\, d\tau}{\tau}\right]$$