Theoretical Notes

Note 202

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SCALING OF SGEMP PHENOMENA

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ABSTRACT

This note discusses exact and approximate scaling laws for the problem of electrically conducting structures excited by photoelectrons generated by an incident pulse of X rays.

1. INTRODUCTION

This report discusses scaling laws for SGEMP phenomena produced by X rays on structures in vacuum. In Sections 2 through 5, exact scaling is derived for the full set of Maxwell equations and electron equations of motion. This scaling allows changes of dimensions, but keeps the normal X-ray spectrum. In Section 5, the scaling of the electrostatic approximation is derived, which allows scaling of the X-ray spectrum.

2. THE SELF-CONSISTENT FIELD-PARTICLE EQUATIONS

We consider first a structure composed of perfect conductors in vacuum. The electron-field problem then can be described in terms of the Boltzmann phase space density $f(\vec{r}, \vec{v}, t)$ of the electrons and the electric and magnetic fields $\vec{E}(\vec{r}, \vec{t})$ and $\vec{B}(\vec{r}, t)$. Here t is the time, $\vec{r}$ is the spatial coordinate and $\vec{v}$ is the velocity coordinate. The equation for f is the "collisionless" Boltzmann equation

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f - \frac{e}{m} (\vec{E} + \frac{\vec{v}}{c} \times \vec{B}) \cdot \nabla_v f = S(\vec{r}, \vec{v}, t) \ . \qquad (1)$$

Here -e is the electron charge, m is its mass, c is the velocity of light, and S is the source density of electrons, which we shall relate to the photon flux later. The current density $\vec{J}$ resulting from the electron motions is

$$J(\vec{r}, t) = -\frac{e}{c} \int \vec{v} f(\vec{r}, \vec{v}, t) d^3v \ . \qquad (2)$$

The equations for the fields are

$$\frac{1}{c} \frac{\partial \vec{B}}{\partial t} = - \nabla \times \vec{E} \ , \qquad (3)$$

$$\frac{1}{c} \frac{\partial \vec{E}}{\partial t} = \nabla \times \vec{B} - 4\pi \vec{J} \ . \qquad (4)$$

The units employed here are cgs Gaussian units, with electric fields and charge in esu, and magnetic field and current in emu.

The equations above, when augmented by the boundary condition that the tangential component of $\vec{E}$ vanishes at the surface of conductors, are complete, i.e., they are sufficient to determine the solution. They are also exact. Let us assume that they have been solved for a given structure and source function.

We now wish to scale the size of the structure, so that the new dimensions are a fraction α of the original dimensions. We may introduce new units of length so that the new structure has the same dimensions, in the new units, as the old structure did in the original units. This is accomplished by using a new spatial variable

$$\vec{r}\,' = \frac{\vec{r}}{\alpha} \quad \text{or} \quad \vec{r} = \alpha \vec{r}\,' . \tag{5}$$

On making this change of variable in Equations 1 through 4, the new equations will of course be different, in having a factor α at each place where the spatial coordinates enter. However, it may be possible to scale other variables in such a way that all of the scale factors cancel out. Then the new equations in the new variables would have exactly the same form as the original equations in the old variables, and the solution would have to be the same.

It follows from Maxwell's Equations 3 and 4 that time must be scaled in the same way as distance, and that $\vec{E}$ must be scaled in the same way as $\vec{B}$. For if these equations are to remain invariant, the scale factors which we write temporarily as L, T, E and B must satisfy

$$\frac{B}{T} = \frac{E}{L} \quad , \quad \text{from 3} \ ,$$

$$\frac{E}{T} = \frac{B}{L} \quad , \quad \text{from 4} \ .$$

Therefore,

$$\frac{B}{E} = \frac{T}{L} = \frac{L}{T} ,$$

so that

$$L = T \quad , \quad B = E , \tag{6}$$

as stated.

Since length and time must scale in the same way, the first two terms of Equation 1 indicate that the velocity has to be unaltered, i.e., its scale factor is unity.

We therefore introduce scale factors for the rest of the variables as follows:

$$t = \alpha t' , \tag{7}$$

$$E = \beta E' , \quad B = \beta B' , \tag{8}$$

$$f = \gamma f' , \tag{9}$$

$$S = \eta S' . \tag{10}$$

Equation 2 then shows that the scaling of J is

$$J = \gamma J' . \tag{11}$$

The Maxwell Equation 4 will be invariant if

$$\frac{\beta}{\alpha} = \gamma , \tag{12}$$

and the Boltzmann Equation 1 will be invariant if

$$\frac{\gamma}{\alpha} = \beta\gamma = \eta . \tag{13}$$

The first of Equations 13 requires that

$$\beta = \frac{1}{\alpha} . \tag{14}$$

Equation 12 then requires that

$$\gamma = \frac{1}{\alpha^2} , \tag{15}$$

and the second of Equations 13 requires that

$$\eta = \frac{1}{\alpha^3} . \tag{16}$$

Since there are no other equations to satisfy, we have found a perfectly scaled problem.

If the scaled problem has lengths and times reduced by a factor α, fields are increased by a factor $1/\alpha$. Potentials, which are products of field and distance, are invariant.

The source density S is proportional to $1/\alpha^3$. When S is integrated over volume and time, the total number N of electrons produced is

$$N = \int S d^3r dt \sim \alpha . \tag{17}$$

The X-ray fluence F is proportional to the total number of electrons produced per unit area of surface. Therefore

$$F \sim \frac{N}{\alpha^2} \sim \frac{1}{\alpha} . \tag{18}$$

Thus the required fluence increases by the same factor by which the dimensions decrease. The X-ray spectrum is invariant, like electron velocities.

3. SCALING OF CIRCUIT PARAMETERS

The scaling of the capacitances and inductances represented by the conducting structure is covered by the analysis of the field equations of Section 2. In order to see what scaling should be applied to lumped elements, note that the inductance L of a wire is

$$L \sim \ell \log\left(\frac{b}{a}\right) , \qquad (19)$$

where ℓ is the wire length, a is its radius, and b is the return path radius. Thus the proper scaling of inductances is

$$L \sim \alpha . \qquad (20)$$

Since the formulas for capacitance C is also given by Equation 19 with lag $(\frac{b}{a})$ replaced by its reciprocal, the proper scaling of capacitance is

$$C \sim \alpha . \qquad (21)$$

The scaling of impedance Z is

$$Z = \omega L \sim \frac{1}{\alpha} \alpha = \text{invariant} , \qquad (22)$$

Resistances are therefore also invariant.

If in a scaled model the diameters of wires are not scaled (but lengths are), then the inductance of the wire will be in error through the log (b/a) term. This will not be a big error if the wire is many diameters away from any ground surface, but it could be substantial for a cable bundle lying against a ground surface.

Battery voltages are invariant, like the potentials noted in Section 2. Currents, which are the ratio of voltage to impedance, are also invariant.

4. OTHER STRUCTURAL CONSIDERATIONS

Thin membranes or sheets, which transmit a non-negligible fraction of X rays, should have the same thickness as in the real system. In most cases, the lack of thickness scaling of these thin members should have negligible effect on the electromagnetic scaling.

The thickness of solar cells presumably cannot be scaled. This means that the capacitance of the solar cells in the scaled model will be too low, by a factor α. This capacitance, if important, could be made right by adding lumped capacitors. The X-ray induced solar cell current will scale properly (i.e., be invariant), assuming that the current per unit area is linearly proportional to the X-ray power per unit area (which scales like α^2). The resistance of the solar cells should also scale properly, if it is linear with X-ray power per unit area.

Insulators scale properly if their dimensions are made proportional to α. (However, insulating membranes which are not thick to X rays should keep the same thickness.)

In order to get the proper current in a wire by direct ejection of photoelectrons, both length and diameter should be scaled proportional to α, if the wire is either very thick or very thin to X rays. For wires of intermediate thickness to X rays, perfect scaling is not possible, although it should be possible to hold errors in ejected current to about 1.5 db.

The use of these scaling laws to subject reduced scale structures to the normal X-ray spectrum appears quite promising.

5. APPROXIMATE SCALING

The scaling laws deduced in Section 2 are exact. It is interesting to see if a wider range of scalings are possible if Equations 1 through 4 are replaced by some common approximations to them.

The magnetic force on the electron is of order $(v/c)^2$ times the electric force, and is usually not very important. However, dropping the magnetic force from Equation 1 allows no additional freedom in scaling.

Another approximation is to neglect retardation, i.e., use electrostatics for the field problem. Then Equations 3 and 4 are replaced by

$$\nabla \times \vec{E} = 0 \,, \tag{3'}$$

$$\frac{1}{c}\frac{\partial}{\partial t}(\nabla \cdot \vec{E}) = -\, 4\pi\nabla \cdot \vec{J} \,. \tag{4'}$$

The magnetic force term is also dropped. In this approximation time and distance can be scaled independently, i.e., the velocity can be scaled. If we let

$$\left.\begin{aligned} r &= \alpha r' \,, \\ t &= \tau t' \,, \\ v &= \frac{\alpha}{\tau} v' \,, \quad \text{(required by Equation 1)} \\ E &= \beta E' \,, \\ f &= \gamma f' \,, \\ S &= \eta S' \,, \end{aligned}\right\} \tag{23}$$

then Equation 2 shows that J scales as

$$J = \gamma\left(\frac{\alpha}{\tau}\right)^4 J' \,. \tag{24}$$

Then Equation 4' requires

$$\beta = \gamma\frac{\alpha^4}{\tau^3} \,, \tag{25}$$

whereas the electric force term in Equation 1 requires

$$\beta = \frac{\alpha}{\tau^2} \,. \tag{26}$$

Therefore

$$\gamma = \frac{\tau}{\alpha^3} \,, \tag{27}$$

and the J scaling is

$$J = \frac{\alpha}{\tau^3} J' \quad . \tag{28}$$

Now the photoelectric yield is roughly proportional to the reciprocal of the photon energy. Thus the photo current density is approximately

$$J = K \frac{P}{X^2} \quad , \quad J' = K \frac{P'}{X'^2} \ , \tag{29}$$

where K is a constant for a given material, P is the X-ray power per unit area and X is the photon energy, which must scale as the electron energy

$$X = \left(\frac{\alpha}{\tau}\right)^2 X' \quad . \tag{30}$$

Combining Equations 28, 29 and 30, we find the scaling of P,

$$P = \frac{\alpha^3}{\tau^5} P' = \frac{1}{\alpha^2}\left(\frac{\alpha}{\tau}\right)^5 P' \quad . \tag{31}$$

Note that this scaling of the photoelectric yield neglects absorption edges and Compton effect.

An example of the use of this scaling might be in the exposure of a full scale system to harder X rays than the normal spectrum. Then $\alpha = 1$, but $\tau < 1$ in order that the scaled X-ray and electron energies be larger than normal. Equation 31 shows that the X-ray power is proportional to (quantum energy)$^{5/2}$, a serious drawback. For softer than normal X rays the power is reduced, unless the dimensions are also scaled down.

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