Theoretical Notes

Note 201

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NOTE ON EQUIVALENT CIRCUITS FOR CONDUCTING SPHERE

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ABSTRACT

This note discusses exact and approximate scaling laws for the problem of electrically conducting structures excited by photoelectrons generated by an incident pulse of X rays.

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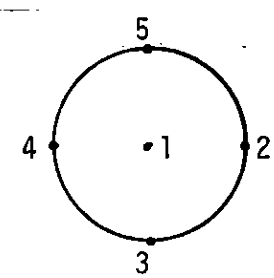
1. INTRODUCTION

This note addresses the problem of making equivalent electrical circuits for simulating the electromagnetic response of conducting structures to charges moving in space near them. We examine the most simple case of a conducting sphere, representing its surface by six nodes. In Section 2 we treat static problems, and find values for the elastance matrix which duplicate some results of exact electrostatics. In Section 3 we consider time varying problems, and find values for the inductances connecting the nodes by requiring that the frequency of the lowest mode have the correct value.

2. ELECTROSTATICS

We consider a perfectly conducting sphere of radius R centered at the origin of a Cartesian coordinate system x, y, z. Let a point charge q_0 be located in space outside the sphere at vector position $\vec{a}$.

We shall replace the sphere by six nodes, located at the points where the coordinate axes puncture the sphere. We number the nodes as in Figure 1. We imagine, for the present, that the nodes are disconnected from each other.



Node 6 is opposite node 1.

Figure 1. Sphere and node points.

If charges q_i are placed on the nodes i = 1, ··6, then the potential V_i of the i^{th} node with respect to infinity can be expressed by the linear relation

$$V_i = \sum_{j=1}^{6} \eta_{ij} q_j + \eta_{i0} q_0 \ . \tag{1}$$

Here the matrix η_{ij}, called the elastance matrix, is independent of the charges q_j, and depends only on geometry of the nodes. The terms η_{i0} are the elastance coefficients from the position of the external charge to the various nodes.

For simplicity we shall use electrostatic units in this section, in which the unit of elastance is cm^{-1}; in fact, the elastance between two points is just the reciprocal of the distance between them. Since our nodes really represent elements of area of the sphere, our situation is not quite that simple. We shall give the connection with MKS units below.

The elastance matrix is always symmetric in its indices, a result which stems from the fact that the electrostatic energy

$$W = \frac{1}{2} \sum_i q_i V_i = \frac{1}{2} \sum_{i,j} q_i \eta_{ij} q_j \ , \tag{2}$$

is independent of the order in which the charges on the various nodes are brought up to their final values q_i. Further, since the elastance matrix is real and symmetric, it can be diagonalized by a linear orthogonal transformation of the q_i. Since the energy, being also equal to the volume integral of the square of the electrostatic field, is positive definite, all of the eigenvalues η_{ii} in the diagonalized matrix must be positive. Therefore the inverse of η_{ij} exists.

In the present case, the elastance matrix has further simple properties because of the symmetry of our model. All of the nodes are obviously equivalent, if we ignore the external charge for the moment. This means that there will be only three different numbers in the 36 number array η_{ij}. One of these will be any self elastance, e.g., η_{11}. Another will be the elastance between neighboring nodes, e.g., $\eta_{12} = \eta_{13} = \eta_{14} = \eta_{15}$, etc. The third will be the elastance between a node and its antipode, e.g., η_{16}. By considering a special example, we shall now make a good guess for these three values.

Example 1

$$\left.\begin{array}{l} q_0 = 0 \ , \\ \text{All other } q_j = \text{same} = \frac{Q}{6} \ . \end{array}\right\} \tag{3}$$

This is the case of a uniformly charged sphere, with total charge Q. Equation 1 gives

$$V_i = \frac{Q}{6} \sum_{j=1}^{6} \eta_{ij} \ . \tag{4}$$

Because of the symmetry properties of η_{ij}, the value of V_i computed from this formula will be the same for all i, a desirable result. Since from exact electrostatics we would have

$$V_i = \frac{Q}{R} \ , \tag{5}$$

it will be good to require that

$$\sum_{j=1}^{6} \eta_{ij} = \frac{6}{R} \ , \tag{6}$$

or, for example,

$$\eta_{11} + 4\eta_{12} + \eta_{16} = \frac{6}{R} \ . \tag{7}$$

Now, it is easier to guess the elastance between different nodes than the self elastance. Thus we shall guess

$$\eta_{16} \approx \frac{1}{2R} \approx \frac{0.5}{R} , \tag{8}$$

$$\eta_{12} \approx \frac{1}{\sqrt{2}R} \approx \frac{0.7}{R} . \tag{9}$$

We can now use Equation 7 to determine the self elastance and find

$$\eta_{11} \approx \frac{2.7}{R} . \tag{10}$$

Thus the equivalent spherical radius of the node 1 is

$$R_1 \approx \frac{R}{2.7} . \tag{11}$$

Note then that

$$R_1^2 \approx \frac{R^2}{7.3} , \tag{12}$$

whereas the area represented by one node is one sixth of the total area of the sphere. The difference here, 6 to 7.3, would be reduced if we took account of the fact that the distances 2R and $\sqrt{2}$R used in Equations 8 and 9 are obviously a little too large, making η_{16} and η_{12} a little too small.

Thus it is suggested that the self elastance of a node should be set equal to the inverse of the radius of a sphere having the same surface area as the area represented by the node. This rule can also be tested against the known self elastance of (both sides of) a thin circular disc of radius R, which is

$$\eta = \pi/2R \approx 1.571/R , \tag{13}$$

the rule above would give

$$\eta = \sqrt{2}/R \approx 1.414/R , \tag{14}$$

which is in error by about 10 percent. The rule does not fare as well for long, wire-like objects.

Example 2

$$\left.\begin{array}{l} q_0 \neq 0 \\ \text{All } V_i = \text{same} = V \end{array}\right\} \tag{15}$$

This is the case of a point charge and a conducting sphere, which also may be charged.

Let C_{ki} be the inverse of the elastance matrix,

$$\sum_i C_{ki}\eta_{ij} = \delta_{kj} \ , \tag{16}$$

where $\delta_{kj} = 0$ unless $k = j$, in which case $\delta_{kk} = 1$. C_{ki} is called the capacitance matrix, and is also symmetric. Multiply Equation 1 by C_{ki} and sum over i. For the case 15 we find

$$V\sum_i C_{ki} = \sum_j \delta_{kj}q_j + q_0\sum_i C_{ki}\eta_{i0} \ ,$$

or

$$q_k = -\,q_0\sum_i C_{ki}\eta_{i0} + V\sum_i C_{ki} \ . \tag{17}$$

Now sum Equation 16 over j, and find

$$\sum_i C_{ki}\sum_j \eta_{ij} = 1 \ ,$$

or, from Equation 6,

$$\sum_i C_{ki} = \frac{R}{6} \ . \tag{18}$$

Thus Equation 17 can be written

$$q_k = -q_0\sum_i C_{ki}\eta_{i0} + \frac{VR}{6} \ . \tag{19}$$

This equation gives the distribution of charge over the nodes, representing various portions of the sphere surface. The first term on the right can be viewed as the image charge induced by the external charge q_0, and the second term as a uniformly distributed charge or a charge located at the center of the sphere.

If we sum Equation 19 over k and call

$$Q = \sum_k q_k , \tag{20}$$

we find, with the help of Equation 18,

$$Q = - q_0 \frac{R}{6} \sum_i \eta_{i0} + VR . \tag{21}$$

Now, exact electrostatics for a conducting sphere and an external point charge at distance a from the center gives

$$Q = - q_0 \frac{R}{a} + VR . \tag{22}$$

Comparison of Equations 21 and 22 suggest that we should choose the external elastance coefficients so that

$$\sum_i \eta_{i0} = \frac{6}{a} , \tag{23}$$

a relation similar to Equation 6.

If the position of the external charge is moved close to the node 1, then the coefficients η_{i0} ought to approach, for any i,

$$\eta_{i0} \rightarrow \eta_{i1} . \tag{24}$$

Then, according to Equation 19, the image charges should approach

$$\begin{aligned} q_k(\text{images}) &\rightarrow - q_0 \sum_i C_{ki} \eta_{i1} \\ &\rightarrow - q_0 \delta_{k1} . \end{aligned} \tag{25}$$

Thus in the limit, all of the image charge appears on the node 1, a correct result.

In electrostatic units the capacitance of a sphere to infinity is R cm. To go to MKS units, use the relation

$$1 \text{ cm capacitance} \approx 1.11 \text{ picofarad} \;, \tag{26}$$

For elastance, the conversion is the reciprocal of Equation 26.

The elastance matrix, since its terms are approximately the reciprocals of the distances between nodes, tends to have all positive terms. For our representation of the sphere,

$$\eta \approx \frac{1}{R} \left\{ \begin{matrix} 2.7 & 0.7 & 0.7 & 0.7 & 0.7 & 0.5 \\ 0.7 & 2.7 & 0.7 & 0.5 & 0.7 & 0.7 \\ 0.7 & 0.7 & 2.7 & 0.7 & 0.5 & 0.7 \\ 0.7 & 0.5 & 0.7 & 2.7 & 0.7 & 0.7 \\ 0.7 & 0.7 & 0.5 & 0.7 & 2.7 & 0.7 \\ 0.5 & 0.7 & 0.7 & 0.7 & 0.7 & 2.7 \end{matrix} \right\} \tag{27}$$

In view of Equation 16, the capacitance matrix must have at least some negative terms. For our case

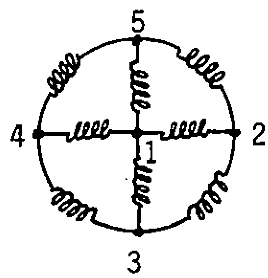
$$C \approx R \left\{ \begin{matrix} 0.440 & -0.065 & -0.065 & -0.065 & -0.065 & -0.014 \\ -0.065 & 0.440 & -0.065 & -0.014 & -0.065 & -0.065 \\ -0.065 & -0.065 & 0.440 & -0.065 & -0.014 & -0.065 \\ -0.065 & -0.014 & -0.065 & 0.440 & -0.065 & -0.065 \\ -0.065 & -0.065 & -0.014 & -0.065 & 0.440 & -0.065 \\ -0.014 & -0.065 & -0.065 & -0.065 & -0.065 & -0.440 \end{matrix} \right\} \tag{28}$$

The standard use of the capacitance matrix is in providing the inverse of Equation 1, namely

$$q_k = \sum_i C_{ki} V_i - q_0 \sum_i C_{ki} \eta_{i0} \,. \tag{29}$$

3. TIME VARYING FIELDS

To properly handle time varying problems we shall have to add inductances between the nodes. We shall use the simple model shown in Figure 2.



Node 6 is opposite node 1. Inductances shown only on front of sphere.

Figure 2. Node points and inductances.

In this section, we shall not include the external point charge. When such a charge moves at a velocity that is small compared with the velocity of light, as in the case of interest to us, it interacts with the sphere chiefly through its elastance coefficients to the nodes on the sphere. This coupling is covered by the discussion of Section 2. In the present section we shall be concerned with the natural modes of the sphere by itself. Aside from the static mode, the natural modes have fairly high frequencies, such that $2\pi R$ is close to an integral multiple of the free space wavelength. Slowly moving electrons will not excite the higher frequency modes very strongly, and our chief interest will be in the time varying modes of lowest frequency.

Two remarks should be made about the model of Figure 2. First, in order to keep the circuit equations as simple as possible, we shall neglect mutual inductances in the present analysis. As a result, we may not expect the mode frequencies to bear the correct relation to each other, although by adjusting the value of the inductance we should be able to get the frequency of the lowest mode to have the correct value. Second, if we include no resistance, the modes will not be damped. The effect of radiation damping can be approximated by including some resistances, or replacing the inductances by a complex impedance. We shall investigate this possibility briefly.

We write the node equations for the circuit in the form

$$V_i = \sum_j \eta_{ij} q_j \ , \tag{30}$$

$$\frac{1}{c}\frac{dq_i}{dt} = -\sum_j I_{ij} \ , \tag{31}$$

$$\frac{1}{c}\frac{dI_{ij}}{dt} = \xi_{ij}(V_i - V_j) \ . \tag{32}$$

Here t is the time, c is the velocity of light, I_{ij} is the current flowing from node i to node j in the branch connecting those nodes. The quantity ξ_{ij} is the reciprocal of the inductance in the ij branch, and is zero if no branch connects nodes i and j. The value assigned to ξ_{ii} is immaterial, since it is always multiplied by zero, but we may take $\xi_{ii} = 0$ to avoid confusion. We shall use electromagnetic units (abamps) for the currents. The units of ξ are cm^{-1}, which are convenient for geometrical modeling. The connection with MKS units will be given below.

By taking the time derivative of Equation 31 and using the other two equations one can eliminate the V_i and I_{ij}, and obtain

$$\frac{1}{c^2}\frac{d^2 q_i}{dt^2} = -\sum_k A_{ik} q_k \ , \tag{33}$$

where

$$A_{ik} \equiv \sum_j \xi_{ij}(\eta_{ik} - \eta_{jk}) \ . \tag{34}$$

We may look for solutions that have time dependence

$$q_i(t) = q_i e^{i\omega t} \ . \tag{35}$$

We then obtain the eigenvalue equation

$$\sum_k A_{ik}q_k = \left(\frac{\omega}{c}\right)^2 q_i \ , \tag{36}$$

which will have non-trivial solutions only for certain values of ω.

In the case of our 6 node model of the sphere, it is clear that all non-zero values of ξ_{ij} should be the same, which we call ξ. Also, there are only three different values of η_{ij}, which we designate as

$$\left.\begin{aligned} \eta_0 &\equiv \eta_{11} = \eta_{22} = \text{etc.} \approx \frac{2.7}{R} \\ \eta_1 &\equiv \eta_{12} = \eta_{13} = \text{etc.} \approx \frac{0.7}{R} \\ \eta_2 &\equiv \eta_{16} = \eta_{24} = \text{etc.} \approx \frac{0.5}{R} \end{aligned}\right\} \tag{37}$$

It is easy to see also that there are only three different values of A_{ij} which we call α, β, and γ:

$$\left.\begin{aligned} \alpha \equiv A_{11} &= \xi(\eta_{11} - \eta_{21}) + \xi(\eta_{11} - \eta_{31}) \\ &\quad + \xi(\eta_{11} - \eta_{41}) + \xi(\eta_{11} - \eta_{51}) \\ &= 4\xi(\eta_0 - \eta_1) \\ \beta \equiv A_{12} &= \xi(2\eta_1 - \eta_0 - \eta_2) \\ \gamma \equiv A_{16} &= 4\xi(\eta_2 - \eta_1) \end{aligned}\right\} \tag{38}$$

In terms of α, β, and γ, the A matrix is

$$A = \begin{Bmatrix} \alpha & \beta & \beta & \beta & \beta & \gamma \\ \beta & \alpha & \beta & \gamma & \beta & \beta \\ \beta & \beta & \alpha & \beta & \gamma & \beta \\ \beta & \gamma & \beta & \alpha & \beta & \beta \\ \beta & \beta & \gamma & \beta & \alpha & \beta \\ \gamma & \beta & \beta & \beta & \beta & \alpha \end{Bmatrix} . \tag{39}$$

For this highly symmetrical case it is easy to see that there are three types of solutions to Equation 36.

Solution 1: All q_i = same = q .

In this case

$$\sum_k A_{ik} q_k = q \sum_k A_{ik} = 0 , \tag{40}$$

as may be verified from Equation 38. This is the static mode, $\omega = 0$.

Solution 2: $q_1 = -q_6$, $q_2 = q_3 = q_4 = q_5 = 0$. (41)

For this case, Equation 36 becomes

$$\alpha - \gamma = \left(\frac{\omega}{c}\right)^2 .$$

Thus the eigenvalue is

$$\frac{\omega_1}{c} = \pm\sqrt{\alpha - \gamma} = \pm\sqrt{4\xi(\eta_0 - \eta_2)} . \tag{42}$$

This is the lowest time-varying mode, in which one pole of the sphere is positive when the opposite pole is negative, and the equator remains neutral. There are three such solutions, corresponding to polarizations along the three axes.

Solution 3: $q_1 = q_6 = -2q_2$, $q_2 = q_3 = q_4 = q_5$. (43)

For this case, Equation 36 becomes

$$\alpha - 2\beta + \gamma = \left(\frac{\omega}{c}\right)^2 ,$$

and the eigenvalue is

$$\frac{\omega_2}{c} = \pm\sqrt{6\xi(\eta_0 - 2\eta_1 + \eta_2)} . \quad (44)$$

In this mode, both poles oscillate in opposite phase to the equator. There appear to be three of these modes, corresponding to polarization along the three axes, but only two are linearly independent: the sum of two of them produces the third.

Thus there are six linearly independent modes, including the static mode, as we should expect.

We shall now compare these frequencies with those from exact analysis of the sphere problem. Stratton, "Electromagnetic Theory," gives for the corresponding two modes:

$$\frac{\omega_1}{c} = [\pm 0.866 + i\,0.50]/R ; \quad (45)$$

$$\frac{\omega_2}{c} = [\pm 1.81 + i\,0.70]/R . \quad (46)$$

In order to compare in the case without damping, we need to make an adjustment of these frequencies to correct for the damping effect. For either of the circuits of Figure 3, the natural frequency is

$$\omega = \pm\sqrt{\omega_0^2 - \nu^2} + i\nu , \quad (47)$$

where (here R_1 is resistance, not sphere radius)

$$\omega_0 = \frac{1}{\sqrt{LC}} ,$$

$$\nu = \frac{R}{2L} , \quad \text{Figure 3a} , \qquad (48)$$

$$= \frac{1}{2R_1C} , \quad \text{Figure 3b} .$$

We may argue that the frequency without damping would be ω_0. Thus we would have

$$\frac{\omega_1}{c} \approx 1/R , \qquad (49)$$

$$\frac{\omega_2}{c} \approx 2/R . \qquad (50)$$

Now, if we let

$$\xi = \frac{k}{R} . \qquad (51)$$

with k a constant to be determined, and use the values 37 for the η's, Equation 42 becomes

$$\frac{\omega_1}{c} = \frac{1}{R}\sqrt{8.8k} . \qquad (52)$$

We therefore need

$$k = \frac{1}{8.8} = 0.114 , \qquad (53)$$

in order to agree with Equation 49. This implies an inductance, corresponding to ξ, of

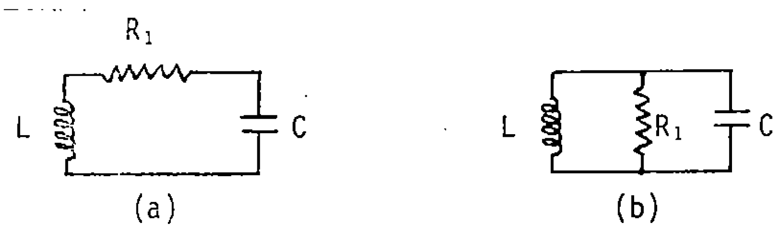


Figure 3. Simple resonant circuits.

$$L = \frac{1}{\xi} = 8.8R \text{ cm} . \qquad (54)$$

In these units the inductance of a wire of radius a and length ℓ is

$$L \approx 2\ell \ln\left(\frac{\ell}{a}\right) \text{ cm} . \qquad (55)$$

Since it is hard to imagine that the logarithm in our case is larger than about unity, the factor 8.8 in 54 looks surprisingly large. However, this large factor is connected with our neglect of mutual inductance. In the lowest mode, the inductance that matters is that between opposite poles of the sphere. Since this inductance is that of a series connection of two groups of four parallel elementary inductances $(1/\xi)$, the effective inductance for this mode is one-half of 54, or

$$L_{eff} = 4.4R . \qquad (56)$$

This formula is easily reconciled with Equation 55, if we say that the length ℓ is about 2R. Consideration of mutual inductance would make the inductance of the four parallel paths, acting together, be not much less than that of a single path. Neglect of the mutual inductance has forced us to choose a high value for the inductance of each single path, in order to get ω_1 right.

If we choose ξ according to Equation 54, then we find for ω_2,

$$\frac{\omega_2}{c} \approx \frac{1.1}{R} , \qquad (57)$$

which is not in very good agreement with Equation 50, and is in fact not very much higher than ω_1.

Now let us include radiation damping by imagining that each inductance is replaced by an inductance and a resistance in parallel. To accomplish this we make the replacement

$$\xi \rightarrow \xi + i\,\frac{\omega}{c}\,g = \frac{k}{R} + i\,\frac{\omega}{c}\,g \ , \tag{58}$$

where g is the reciprocal of the parallel resistance; we have again replaced the inductive term by the expression 51, where k is again to be determined. When the replacement 58 is made in Equations 42 and 44 and the resulting equations are solved for ω_1 and ω_2, one finds

$$\frac{\omega_1}{c} = [\pm \sqrt{8.8k - (4.4g)^2} + i\,4.4g]/R \ , \tag{59}$$

$$\frac{\omega_2}{c} = [\pm \sqrt{10.8k - (5.4g)^2} + i\,5.4g]/R \ . \tag{60}$$

We choose k and g by fitting the exact result 45 for the lowest mode. The fit will be perfect if we put

$$4.4g = 0.5 \quad \text{or} \quad g = \frac{1}{8.8} = 0.114 \ , \tag{61}$$

$$8.8k = 1 \quad \text{or} \quad k = \frac{1}{8.8} = 0.114 \ . \tag{62}$$

Note that the value of k is the same as before. With these values, ω_2 turns out to be

$$\frac{\omega_2}{c} = [\pm\, 0.92 + i\,0.61]/R \ . \tag{63}$$

Thus the damping rate for the second mode turns out fairly well, but the real frequency is low by a factor of about two. The reason for the error is again the neglect of mutual inductance.

The connection between the units of inductance and resistance used here and MKS units is:

$$\left.\begin{array}{ll} \text{Inductance:} & 1\ \text{cm} = 1\ \text{nanohenry} \\ \text{Resistance:} & 1\ \text{unit}^* = 30\ \text{ohms} \end{array}\right\} \tag{64}$$

Thus the resistance corresponding to 8.8 units is 264 ohms.

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* Our unit of resistance is dimensionless.

4. CONCLUSIONS

The model used in this report gives a good representation of the static charges induced on a sphere. For time varying problems, the elementary inductance between nodes must be chosen artificially high to obtain the correct frequency for the lowest mode, due to neglect of mutual inductance. The frequency of the second mode then comes out too low, only slightly higher than the lowest mode instead of about twice it.

The natural modes are not excited very strongly by a passing slow (compared with the velocity of light) electron. It is possible that a single choice of the elementary inductance, not very different from the value deduced above, would give reasonably good skin currents for practical problems. Some comparisons of results from our model with those from the SEMP code would be useful.