Theoretical Notes

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Interaction of a Hollow Metallic Sphere Having a Circular Aperture with an Electron Moving in an Arbitrary Trajectory Exterior to the Sphere

by

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Abstract

The effect of the moving electron on the field quantities in the interior of the sphere is calculated in the quasi-static approximation. Suitable field-quantities are plotted versus the normalized distance of the electron from the center of the sphere with the aperture size, angular position and velocity components of the electron as parameters.

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I. Introduction

When X rays or γ rays, originating from a nuclear explosion, strike the material surface of a space system in exoatmospheric regions electrons are ejected from the surface. These electrons originate in several kinds of processes (see for example reference 1) and represent a threat to the space system because of the electromagnetic pulse they generate [2]. Depending on whether the electrons are ejected into cavities or backwards into free space the electromagnetic pulse is characterized as internal or external respectively. In reference 3 the effect of the external electromagnetic pulse on a space system has been considered. The space system is modeled as a perfectly conducting sphere and the emitted electrons have prescribed trajectories. The problem is solved exactly within the realm of classical electrodynamics and validity criteria for the quasi-static solution are established in terms of the particle's speed and distance from the sphere. In the quasi-static approximation one can find the total induced charge density and total induced current density on the surface of the space system by solving an electrostatic problem and a magnetostatic problem separately. The electrostatic solution gives the total induced charge density σ_s and the "electrostatic" current density $\underline{K}'_s$ (derivable from the continuity equation), whereas the magnetostatic solution gives the divergence-free "magnetostatic" current density $\underline{K}''_s$. The total current density is equal to $\underline{K}'_s + \underline{K}''_s$.

In this note we consider the quasi-static interaction of an electron of known trajectory with a satellite modeled as a perfectly conducting hollow sphere having a circular aperture. The electron is assumed to have been ejected from the outer surface of the sphere and away from the sphere. The ejection of the electron leaves the sphere charged positively. (We ignore the effect of the ambient plasma [1], except for the fact that its presence can make the quasi-static approximation more realistic). To evaluate the effect of the aperture on the field quantities in the interior of the sphere we calculate the following quantities. The charge Q_i on the interior surface of the sphere as a function of the polar angle θ (Figure 2), the total current I_i on the interior surface crossing the circumference of a circle corresponding to a polar angle θ (Figure 3) and the total electric field E_t at the center of the sphere. The knowledge of the above quantities requires the solution of the electrostatic problem only. The reason is that the "magnetostatic" current density $\underline{K}''_s$ does not contribute to the current I_i because $\underline{K}''_s$ is divergence-free and I_i is defined over a closed contour. I_i is simply given by - (dQ_i/dt).

In Section II the electrostatic problem is formulated and solved as dual series equations for the coefficients of the potential function series expansion. Then we proceed to calculate $Q_i(\theta)$, $I_i(\theta)$ and $\underline{E}_t(\underline{r}=0)$ in terms of an appropriate subset of the expansion coefficients. The expressions we obtain can be cast in relatively simple forms for numerical calculations. Plots are presented, in Section III, of $Q_i(\theta)$, $\underline{E}_t(\underline{r}=0)$ versus r_o/a with the aperture size and angular position of the electron as parameters, and also $I_i(\theta)$ versus r_o/a with the aperture size, angular position and velocity components of the electron as parameters. The radius of the sphere is "a" and the radial distance of the electron from the center of the sphere is r_o.

II. Formulation and Solution of the Electrostatic Problem

The geometry of the problem is depicted in Fig. 1. The position of the electron is defined by the three spherical coordinates Z_o, θ_o, φ_o, a is the radius of the metallic sphere, and the circular aperture is defined by the polar angle α.

The electron induces charges on both the inside and outside surfaces of the hollow sphere and the potential due to these charges (and also the initial charge on the sphere) can be written in the form

$$\Phi_i = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} D_{\ell m} (r/a)^{\ell} Y_{\ell m}(\theta, \varphi), \qquad r \leq a \tag{1}$$

$$\Phi_o = \sum_{\ell=0} \sum_{m=-\ell} C_{\ell m} (a/r)^{\ell+1} Y_{\ell m}(\theta, \varphi) \qquad r \geq a \tag{2}$$

where $Y_{\ell m}(\theta, \varphi)$ is the usual spherical harmonic. The potential is continuous across the surface of the metallic sphere and also across the aperture and, therefore, we can immediately deduce that

$$C_{\ell m} = D_{\ell m}$$

To determine $C_{\ell m}$ we use the boundary conditions for the constancy of the potential on the metallic surface of the sphere and the continuity, across the spherical cap subtended by the aperture, of the normal component of the <u>total</u> electric field which can be written as

$$\hat{e}_r \cdot \underline{E}_{it} = -\frac{\partial}{\partial r}(\Phi_i + \Phi_{inc})$$

$$= -\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\infty} (\ell C_{\ell m}/a)(r/a)^{\ell-1} Y_{\ell m}(\theta, \varphi) - (\partial \Phi_{inc}/\partial r) \tag{3}$$

$$\hat{e}_r \cdot \underline{E}_{ot} = -\frac{\partial}{\partial r}(\Phi_o + \Phi_{inc})$$

$$= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (\ell + 1)(C_{\ell m}/r)(a/r)^{\ell+1} Y_{\ell m}(\theta,\varphi) - (\partial\Phi_{inc}/\partial r) \tag{4}$$

where Φ_{inc} is the potential due to the electron in free space.

If the potential on the surface of the sphere is denoted by V_o we can write

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} C_{\ell m} Y_{\ell m}(\theta,\varphi) + \Phi_{inc}(r = a) = V_o, \quad 0 \le \theta \le \alpha, \; 0 \le \varphi \le 2\pi \tag{5}$$

and with the aid of (3) and (4), the continuity of the normal component of the electric field gives,

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (2\ell + 1) C_{\ell m} Y_{\ell m}(\theta,\varphi) = 0, \qquad \alpha < \theta \le \pi, \; 0 \le \varphi \le 2\pi \tag{6}$$

since $\partial\Phi_{inc}/\partial r$ is continuous across the aperture.

The potential on the sphere V_o can be determined by fixing the total charge on the sphere. This calculation will be performed later.

If we multiply (5) and (6) by $e^{-in\varphi}$ on both sides and integrate from 0 to 2π we obtain

$$\sum_{\ell=0}^{\infty} C_{\ell n} A_{\ell n} P_{\ell}^{n}(\cos\theta) = G_n(\theta), \qquad 0 \le \theta \le \alpha \tag{7}$$

$$\sum_{\ell=0}^{\infty} (2\ell + 1) C_{\ell n} A_{\ell n} P_{\ell}^{n}(\cos\theta) = 0, \qquad \alpha \le \theta \le \pi \tag{8}$$

where we have used the explicit forms

$$Y_{\ell m}(\theta,\varphi) = A_{\ell m} P_{\ell}^{m}(\cos\theta)\, e^{im\varphi} \tag{9}$$

$$A_{\ell m} = \left[\frac{2\ell + 1}{4\pi}\right]\left[\frac{(\ell - m)!}{(\ell + m)!}\right]^{\frac{1}{2}} \tag{10}$$

and

$$Y_{\ell,-m}(\theta,\varphi) = (-1)^m \, Y_{\ell m}(\theta,\varphi) \tag{11}$$

$$= A_{\ell,-m} P_\ell^{-m}(\cos\theta)\, e^{-im\varphi} \tag{12}$$

$$A_{\ell,-m} = \frac{(\ell+m)!}{(\ell-m)!} A_{\ell m} \tag{13}$$

$$P_\ell^{-m}(\cos\theta) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(\cos\theta) \tag{14}$$

$$\left.\begin{aligned} G_n(\theta) &= \delta_{no} V_o - (1/2\pi)\int_o^{2\pi} \Phi_{inc}\, e^{-in\varphi}\, d\varphi \\ \Phi_{inc} &= -\frac{e}{4\pi\epsilon_o |\underline{r} - \underline{r}_o|} \end{aligned}\right\} \tag{15}$$

$$\underline{r} = a\hat{e}_r, \quad \underline{r}_o = r_o\hat{e}_r .$$

Before we proceed to solve for $C_{\ell n}$, we obtain a simple relationship between $C_{\ell n}$ and $C_{\ell,-n}$. We rewrite (7) and (8) by replacing n by -n.

$$\sum_{\ell=0}^{\infty} C_{\ell,-n} A_{\ell,-n} P_\ell^{-n}(\cos\theta) = G_{-n}(\theta)$$

$$\sum_{\ell=0}^{\infty} (2\ell+1)\, C_{\ell,-n} A_{\ell,-n} P_\ell^{-n}(\cos\theta) = 0$$

Next we use (13) and (14) to obtain

$$\sum_{\ell=0} C_{\ell,-n} (-1)^n P_\ell^n(\cos\theta) = G_{-n}(\theta)$$

$$\sum_{\ell=0}^{\infty} (2\ell+1)\, C_{\ell,-n} (-1)^n P_\ell^n(\cos\theta) = 0$$

If we take the complex conjugate of both sides of the previous equation we have

$$\sum_{\ell=0}^{\infty} C^{*}_{\ell,-n}(-1)^n P^n_\ell(\cos\theta) = G_n(\theta)$$

$$\sum_{\ell=0}^{\infty} (2\ell+1)\, C^{*}_{\ell,-n}(-1)^n P^n_\ell(\cos\theta) = 0$$

since $G^{*}_{-n}(\theta) = G_n(\theta)$ from (15).

We thus deduce that

$$C_{\ell n} = (-1)^n C^{*}_{\ell,-n} \tag{16}$$

We now return to (7) and (8), and introduce ℓ defined by

$$\ell = k + n \tag{17}$$

$$\sum_{-n}^{\infty} B_{kn} P^n_{n+k}(\cos\theta) = G_n(\theta), \qquad 0 \le \theta \le \alpha \tag{18}$$

$$\sum_{-n}^{\infty} (2k+2n+1)\, B_{kn} P^n_{n+k}(\cos\theta) = 0, \qquad \alpha < \theta \le \pi \tag{19}$$

where $$B_{kn} = C_{\ell n} A_{\ell n}. \tag{20}$$

Recalling that $P^n_{k+n}(\cos\theta) = 0$ for $k+n < n$ i.e. $k \ge 0$ $(\ell \ge n)$ we can change the summation to $k = 0, 1, 2, \ldots$. Next we use (14) to re-write (18) and (19) as

$$\sum_{k=0}^{\infty} M_{kn} P^{-n}_{n+k}(\cos\theta) = G_n(\theta), \qquad 0 \le \theta \le \alpha \tag{21}$$

$$\sum_{k=0}^{\infty} (2k+2n+1)\, M_{kn} P^{-n}_{n+k}(\cos\theta) = 0, \qquad \alpha < \theta \le \pi \tag{22}$$

where $$M_{kn} = (-1)^n \frac{(k+2n)!}{k!} B_{kn} \tag{23}$$

We have cast our original pair of equations (7) and (8) into (21) and (22) because they represent dual series relations for which an explicit solution exists. Thus from Ref. 4 (eq. 5.6.1 and 5.6.2) we obtain

$$-(e\ell/2\ell+1)(a^{\ell-1}/r_o^{\ell+1})|A_{\ell m}|^2 \; P_\ell^m(\cos\theta_o)\,P_\ell^m(\cos\theta)\,e^{-im\varphi_o+im\varphi}\Big\} \tag{33}$$

The total charge on the spherical cap corresponding to a polar angle θ (see Fig. 2) is given by

$$Q_i(\theta) = \int_0^\theta\int_0^{2\pi} \sigma_i(\theta,\varphi)\,a^2 \sin\theta\,d\theta\,d\varphi \tag{34}$$

Using (33), (34) gives

$$Q_i(\theta) = 2\pi\epsilon_o a \sum_{\ell=1}^{\infty} \frac{\ell}{2\ell+1}\left[b_{\ell o} - \frac{e}{4\pi\epsilon_o a}\left(\frac{a}{r_o}\right)^{\ell+1} P_\ell(\cos\theta_o)\right]$$
$$\left[P_{\ell-1}(\cos\theta) - P_{\ell+1}(\cos\theta)\right] \tag{35}$$

where we have used the result

$$\int_0^\theta P_\ell(\cos\theta)\sin\theta\,d\theta = \frac{1}{2\ell+1}\left[P_{\ell-1}(\cos\theta) - P_{\ell+1}(\cos\theta)\right] \tag{36}$$

and

$$\left.\begin{aligned} b_{\ell m} &= C_{\ell m}\,A_{\ell m} \\ b_{\ell o} &= C_{\ell o}\,A_{\ell o} = \frac{2\ell+1}{4\pi}\;C_{\ell o} \end{aligned}\right\} \tag{37}$$

When $n = o$, then from (17) $\ell = k$, and

$$B_{ko} = C_{\ell o}\,A_{\ell o}$$

and from (23), (37) we obtain

$$b_{\ell o} = B_{ko} = M_{ko}\,. \tag{38}$$

Equation (24) gives

$$M_{\ell o} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{\Gamma(\ell+1)}{\Gamma(\ell+\frac{1}{2})} \int_0^\alpha F_o^*(u)\,\cos\tfrac{1}{2}u\; P_\ell^{(-\frac{1}{2},\frac{1}{2})}(\cos u)\,du \tag{39}$$

and from (31) we obtain

$$I_{os}(u) = \sqrt{\pi}\,\frac{\Gamma(s+1)}{\Gamma(s+\frac{1}{2})}\,(1-\cos u)^{-\frac{1}{2}}\sin u\;P_s^{(-\frac{1}{2},\frac{1}{2})}(\cos u) \tag{40}$$

From (2.6.17) in ref. 4, we see that

$$P_n^{(-\frac{1}{2},\frac{1}{2})}(\cos u) = \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)\Gamma(\frac{1}{2})}\;\frac{\cos(n+\frac{1}{2})u}{\cos\frac{1}{2}u} \tag{41}$$

and (39) gives (the intermediate steps involve an elementary integration)

$$M_{\ell o} = b_{\ell o} = \frac{V_o}{\pi}\left[\frac{\sin \ell\alpha}{\ell} + \frac{\sin(\ell+1)\alpha}{\ell+1}\right]$$

$$+\sum_{s=0}^{\infty}\frac{1}{\pi}N_s\left[\frac{\sin(s-\ell)\alpha}{s-\ell} + \frac{\sin(s+\ell+1)\alpha}{s+\ell+1}\right] \tag{42}$$

where

$$N_s = \frac{e}{4\pi\epsilon_o r_o}\left(\frac{a}{r_o}\right)^s P_s(\cos\theta_o) \tag{43}$$

and $[\sin(s-\ell)\alpha]/(s-\ell) = \alpha$ when $s = \ell$.

We are now in a position to calculate $Q_i(\theta)$; (notice that it only depends on r_o, θ_o but not φ_o). Before we proceed to simplify the expression for $Q_i(\theta)$ we determine V_o. We will assume that the electron is ejected from the surface of the sphere; therefore, the total charge of the sphere is $+e$. However, if we consider the ejection of N electrons, the net charge of the sphere should be corrected to Ne in order to apply superposition of the results for the interaction of one electron and a sphere of net charge Ne. For this reason we will assume that the net charge on the surface (inside plus outside) of the sphere is equal to Q.

The total charge on the sphere is given by

$$Q = \epsilon_o \int_0^{2\pi}\int_0^{\alpha} \hat{e}_r \cdot (\underline{E}_{ot} - \underline{E}_{it})\,a^2 \sin\theta\,d\theta\,d\varphi \tag{44a}$$

or

$$Q = \epsilon_o \int_0^{2\pi}\int_0^{\pi} e_r \cdot \underline{E}_{ot}\, a^2 \sin\theta \, d\theta \, d\varphi \tag{44b}$$

Equation (44b) is derived by applying Gauss's Law over a spherical surface that completely surrounds and touches our sphere (including the spherical cap around the aperture).

We start with (44a) which, with the aid of (3) and (4), becomes

$$Q = 2\pi a \epsilon_o \sum_{\ell=0}^{\infty} T_\ell b_{\ell o} \tag{45}$$

where

$$\left.\begin{aligned} T_o &= 1 - \cos\alpha \\ T_\ell &= P_{\ell-1}(\cos\alpha) - P_{\ell+1}(\cos\alpha) \end{aligned}\right\} \tag{46}$$

If we combine (42) with (45) we can arrive at the following expression for V_o.

$$V_o = \frac{Q/2a\epsilon_o - \sum_{\ell=0}^{\infty}\sum_{s=0}^{\infty} T_\ell N_s \left[\frac{\sin(s-\ell)\alpha}{s-\ell} + \frac{\sin(s+\ell+1)\alpha}{s+\ell+1}\right]}{\sum_{\ell=0}^{\infty} T_\ell \left[\frac{\sin \ell\alpha}{\ell} + \frac{\sin(\ell+1)\alpha}{\ell+1}\right]} \tag{47}$$

where T_ℓ is given by (46), N_s by (43), and $(\sin k\alpha)/k = \alpha$ when $k = 0$.

We can simplify (47) as follows.

Consider the infinite series

$$S(s) = \sum_{\ell=0}^{\infty} \left[P_{\ell-1}(\cos\alpha) - P_{\ell+1}(\cos\alpha)\right]\left[\frac{\sin(s-\ell)\alpha}{s-\ell} + \frac{\sin(s+\ell+1)\alpha}{s+\ell+1}\right] \tag{48}$$

First, we rewrite the second bracketted term as

$$\frac{\sin (s - \ell)\alpha}{s - \ell} + \frac{\sin (s + \ell + 1)}{s + \ell + 1} = \int_0^\alpha [\cos (s - \ell)x + \cos (s + \ell + 1)]\, dx$$

$$= \int_0^\alpha 2 \cos (\ell + \tfrac{1}{2})x \cos (s + \tfrac{1}{2})\, x\, dx \qquad (49)$$

Thus

$$S(s) = 2\sum_{\ell=0}^{\infty} [P_{\ell - 1} (\cos \alpha) - P_{\ell + 1}(\cos \alpha)] \int_0^\alpha \cos (\ell + \tfrac{1}{2})x \cos (s + \tfrac{1}{2}) \qquad (50)$$

Again ref. 4 can be useful. This time we use (2.6.33) which has the form

$$(1 + \cos \alpha) \cos \tfrac{1}{2}x - \sum_{\ell=1}^{\infty} [P_{\ell - 1} (\cos \alpha) - P_{\ell + 1} (\cos \alpha)] \cos (\ell + \tfrac{1}{2})x$$

$$= \frac{\sqrt{2} \sin x\, H (x - \alpha)}{(\cos \alpha - \cos x)^{\frac{1}{2}}} \qquad 0 < x, \alpha < \pi \qquad (51)$$

where $H(u)$ is the Heaviside function: $H(u) = 1$ for $u > 0$, $H(u) = 0$ for $u < 0$. Combining (50) with (51) we obtain the simple result

$$\left.\begin{aligned} S(s) &= 2 \left[\frac{\sin (s\alpha)}{s} + \frac{\sin (s + 1)\alpha}{s + 1}\right] \\ S(0) &= 2 [\alpha + \sin \alpha] \end{aligned}\right\} \qquad (52)$$

Using (52) in (47) we obtain

$$V_o = \frac{Q/2a\varepsilon_o - 2\sum_{s=0}^{\infty} N_s \left[\frac{\sin s\alpha}{s} + \frac{\sin (s + 1)\alpha}{s + 1}\right]}{2(\alpha + \sin \alpha)} \qquad (53)$$

If we now start with (44b) we can use (4) to arrive at

$$Q = 4\pi \varepsilon_o a b_{oo}$$

Using (42) in the previous equation, we obtain

$$Q/4\pi \varepsilon_o a = \frac{V_o}{\pi} (\alpha + \sin \alpha) + \sum_{s=0}^{\infty} \frac{1}{\pi} N_s \left[\frac{\sin (s\alpha)}{s} + \frac{\sin (s + 1)\alpha}{s + 1}\right]$$

which is identical to (53).

We can interpret (53) as follows. Assume that we remove the electron to infinity and call the potential on the sphere V_∞. Then it is easy to show (see also ref. 4, section 8.7) that the potential everywhere is given by

$$\varphi_\infty(r, \theta) = V_\infty \sum_{s=0}^{\infty} \frac{1}{\pi} (a/r)^{s+1} \left[\frac{\sin s\alpha}{s} + \frac{\sin (s+1)\alpha}{s+1}\right] P_s(\cos \theta) \tag{54}$$

and (53) becomes

$$\begin{aligned} V_o &= \frac{Q/2a\epsilon_o - (e/2a\epsilon_o)\varphi_\infty(r_o, \theta_o)/V_\infty}{2(\alpha + \sin \alpha)} \\ &= \frac{Q - e\varphi_\infty(r_o, \theta_o)/V_\infty}{4a\epsilon_o(\alpha + \sin \alpha)} \end{aligned} \tag{55}$$

If we now recall that the capacitance of a charged spherical bowl (with an aperture defined by a polar angle α) is given by

$$C = 4a\epsilon_o(\alpha + \sin \alpha) \tag{56}$$

we can rewrite (55) as

$$V_o = V_\infty - \frac{e}{Q}\varphi_\infty(r_o, \theta_o) \tag{57}$$

since $V_\infty = Q/C$.

Equation (57) can be derived using Green's reciprocation theorem (see Appendix A).

We can now return to (35) in order to cast it into a simpler form with the aid of ref. 4 once again. In section 7.6 the problem of a spherical bowl immersed in an axisymmetric field is considered. The induced potential function $V(r, \theta)$ is given by

$$V(r, \theta) = \frac{1}{2}\int_{-\alpha}^{\alpha} \frac{g(x)}{R} \sec \tfrac{1}{2} x\, dx \tag{58}$$

where g(x) will be determined by solving the integral equation

$$V(a, \theta) = \int_0^{\theta} \frac{g(x) \sec \frac{1}{2} x \, dx}{(2 \cos x - 2 \cos \theta)^{\frac{1}{2}}} \qquad 0 \le \theta \le \alpha \tag{59}$$

and

$$R = [(r/a)^2 e^{ix} - 2(r/a) \cos \theta + e^{-ix}]^{\frac{1}{2}} \tag{60}$$

where

$$r/a \ge 1, \quad R = \rho e^{\frac{1}{2} i \tau}, \quad \rho \ge 0, \quad 0 \le \tau \le \pi \text{ for } 0 \le x < \alpha, \quad -\pi < \tau \le 0 \text{ for } -\alpha \le x < 0 \tag{61a}$$

$$\left.\begin{aligned}
\rho &= \left\{[[1 + (r/a)^2] \cos x - 2(r/a) \cos \theta]^2 + [[(r/a)^2 - 1] \sin x]^2\right\}^{\frac{1}{4}} \\
\cos \tau &= \frac{1}{\rho^2} \left\{[1 + (r/a)^2] \cos x - 2(r/a) \cos \theta\right\} \\
\cos(\tau/2) &= \left(\frac{1 + \cos \tau}{2}\right)^{\frac{1}{2}} \\
\sin(\tau/2) &= \left(\frac{1 - \cos \tau}{2}\right)^{\frac{1}{2}} \qquad 0 \le x < \alpha \\
\sin(\tau/2) &= -\left(\frac{1 - \cos \tau}{2}\right)^{\frac{1}{2}} \qquad -\alpha \le x < 0
\end{aligned}\right\} \tag{61}$$

If we recall (7) and (8) for n = 0 we understand that the forcing function is $G_o(\theta)$ given by (28)

$$G_o(\theta) = V_o + \frac{e}{4\pi \epsilon_o} \sum_{s=0}^{\infty} \frac{a^s}{r_o^{s+1}} P_s(\cos \theta_o) P_s(\cos \theta) \tag{62}$$

Thus

$$V(a, \theta) = G_o(\theta) \tag{63}$$

(To understand (63) recall the boundary condition $\Phi_o(r = a) = -\Phi_{inc}(r = a) + V_o$. If we integrate from 0 to 2π with respect to φ and divide by 2π we obtain the axisymmetric part of $\Phi_o(r = a)$ which is equal to $V(a, \theta)$.)

To find g(x) in (59) we make use of Mehler's integral representation for the Legendre polynomial.

$$P_\ell(\cos\theta) = \frac{\sqrt{2}}{\pi}\int_0^\theta \frac{\cos(\ell+\frac{1}{2})x\,dx}{(\cos x - \cos\theta)^{\frac{1}{2}}} \tag{64}$$

With the aid of (64), we can obtain g(x) in the form

$$g(x)\sec\tfrac{1}{2}x = \frac{2V_o}{\pi}\cos\tfrac{1}{2}x + \frac{e}{2\pi^2\epsilon_o a}\sum_{\ell=0}^{\infty}\left(\frac{a}{r_o}\right)^{\ell+1} P_\ell(\cos\theta_o)\cos(\ell+\tfrac{1}{2})x \tag{65}$$

and from (58)

$$\frac{1}{2\pi}\int_0^{2\pi}\Phi_o(r_o,\theta_o,\varphi)\,d\varphi = V(r_o,\theta) = \frac{V_o}{\pi}\int_{-\alpha}^{\alpha}\frac{\cos\frac{1}{2}x\,dx}{R}$$

$$+\frac{e}{4\pi^2\epsilon_o a}\sum_{\ell=0}^{\infty}\left(\frac{a}{r_o}\right)^{\ell+1} P_\ell(\cos\theta_o)\int_{-\alpha}^{\alpha}\frac{\cos(\ell+\frac{1}{2})x}{R}\,dx \tag{66}$$

If we use (2) we understand that

$$\frac{1}{2\pi}\int_0^{2\pi}\Phi_o(r_o,\theta_o,\varphi)\,d\varphi = \sum_{\ell=0}^{\infty} b_{\ell o}\,(a/r_o)^{\ell+1}P_\ell(\cos\theta_o) \tag{67}$$

From section 8.7 in ref. 4 we find the relationship

$$\sum_{\ell=0}^{\infty}\frac{1}{\pi}(a/r_o)^{\ell+1}\left[\frac{\sin\ell\alpha}{\ell}+\frac{\sin(\ell+1)\alpha}{\ell}\right]P_\ell(\cos\theta) = \frac{1}{\pi}\int_{-\alpha}^{\alpha}\frac{\cos\frac{1}{2}x\,dx}{R} \tag{68}$$

If we combine (66), (67) and (68) we obtain

$$b_{\ell o} = \frac{V_o}{\pi}\left[\frac{\sin\ell\alpha}{\ell}+\frac{\sin(\ell+1)\alpha}{\ell+1}\right] + \frac{e}{4\pi^2\epsilon_o a}\int_{-\alpha}^{\alpha}\frac{\cos(\ell+\frac{1}{2})x}{R}\,dx \tag{69}$$

Comparing (42) to (69) we arrive at the relationship

$$\int_{-\alpha}^{\alpha} \frac{\cos(\ell+\frac{1}{2})x}{R}\, dx = \sum_{s=0}^{\infty}\left(\frac{a}{r_o}\right)^{s+1}\left[\frac{\sin(s-\ell)\alpha}{s-\ell}+\frac{\sin(s+\ell+1)\alpha}{s+\ell+1}\right] P_s(\cos\theta_o) \tag{70}$$

When $\alpha = \pi$, (70) assumes the simple form

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\cos(\ell+\frac{1}{2})x}{R}\, dx = \left(\frac{a}{r_o}\right)^{\ell+1} P_\ell(\cos\theta_o) \tag{71}$$

With the aid of (69) and (71), (35) can be written as

$$Q_i(\theta) = 2\pi\epsilon_o a \sum_{\ell=1}^{\infty} \frac{\ell}{2\ell+1}[P_{\ell-1}(\cos\theta) - P_{\ell+1}(\cos\theta)]\left\{\frac{e}{4\pi\epsilon_o a}\,\frac{1}{\pi}\int_{-\alpha}^{\alpha}\frac{\cos(\ell+\frac{1}{2})x}{R}\, dx\right.$$

$$\left. + \frac{V_o}{\pi}\left[\frac{\sin \ell\alpha}{\ell} + \frac{\sin(\ell+1)\alpha}{\ell+1}\right] - \frac{e}{4\pi\epsilon_o a}\,\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\cos(\ell+\frac{1}{2})x}{R}\, dx\right\} \tag{72}$$

Next we use (60) and (61) to simplify (72).

$$\frac{1}{R} = \frac{1}{\rho}\, e^{-i\tau/2} = \frac{1}{\rho}\cos\frac{\tau}{2} - i\sin\frac{\tau}{2} \tag{73}$$

and

$$\int_{-\alpha}^{\alpha}\frac{\cos(\ell+\frac{1}{2})x}{R}\, dx = \int_{-\alpha}^{0} + \int_{0}^{\alpha}$$

$$= \int_{-\alpha}^{0} f\cos\frac{\tau}{2}\, dx + \int_{0}^{\alpha} f\cos\frac{\tau}{2}\, dx - i\int_{-\alpha}^{0} f\sin\frac{\tau}{2}\, dx - i\int_{0}^{\alpha} f\sin\frac{\tau}{2}\, dx \tag{74}$$

From (61), we understand that

$$\cos\tfrac{1}{2}\tau(x) = \cos\tfrac{1}{2}\tau(-x)$$
$$\sin\tfrac{1}{2}\tau(x) = -\sin\tfrac{1}{2}\tau(-x)$$
$$\rho(x) = \rho(-x)$$

Therefore (74) reduces to

$$\int_{-\alpha}^{\alpha}\frac{\cos(\ell+\frac{1}{2})x}{R}\, dx = 2\int_{0}^{\alpha}\rho\,\cos(\ell+\tfrac{1}{2})x\cos\frac{\tau}{2}\, dx.$$

We can now rewrite (72) as

$$Q_i(\theta) = 2\pi\epsilon_o a \sum_{\ell=1}^{\infty} \frac{\ell}{2\ell+1}\left[P_{\ell-1}(\cos\theta) - P_{\ell+1}(\cos\theta)\right]$$

$$\left\{-\frac{e}{2\pi^2\epsilon_o a}\int_a^{\pi}\rho\cos(\ell+\tfrac{1}{2})x\cos(\tau/2)dx + \frac{V_o}{\pi}\left[\frac{\sin\ell\alpha}{\ell} + \frac{\sin(\ell+1)\alpha}{\ell+1}\right]\right\} \quad (75)$$

In Appendix B we evaluate the infinite series in closed form and (75) assumes the form

$$Q_i(\theta) = \frac{e}{2\pi}\left\{-2\int_{\alpha}^{\pi}\rho\cos(x/2)\cos(\tau/2)dx + \sqrt{2}\int_{\alpha}^{\pi}\frac{\rho\sin x\cos(\tau/2)}{(\cos\theta - \cos x)^{\frac{1}{2}}}dx\right.$$

$$+\frac{4\pi\epsilon_o a V_o}{e}\left[\alpha + \sin\alpha - \frac{\pi}{2}(1-\cos\theta) - (1+\cos\alpha)^{\frac{1}{2}}(\cos\theta - \cos\alpha)^{\frac{1}{2}}\right.$$

$$\left.\left. - \tfrac{1}{2}(1+\cos\theta)\cos^{-1}\left(2\,\frac{1+\cos\alpha}{1+\cos\theta} - 1\right)\right]\right\} \quad (76)$$

where $\cos^{-1} u$ is the principal value, $0 \le \cos^{-1}u \le \pi$.
The second integral in the angular brackets can be rewritten as

$$\sqrt{2}\int_{\alpha}^{\pi}\frac{\rho\sin x\cos(\tau/2)}{(\cos\theta - \cos x)^{\frac{1}{2}}}dx = 2\sqrt{2}\int_{\beta_1}^{\beta_2}\rho\cos(\tau/2)d\beta \quad (77)$$

where $\beta = (\cos\theta - \cos x)^{\frac{1}{2}}$, $\beta_1 = (\cos\theta - \cos\alpha)^{\frac{1}{2}}$ and $\beta_2 = (\cos\theta + 1)^{\frac{1}{2}}$ (78)

The potential V_o is given by (53). Because of (68) it can be rewritten as

$$V_o = \frac{Q/2a\epsilon_o - (e/2a\epsilon_o\pi)\int_{-\alpha}^{\alpha}\frac{\cos(\frac{1}{2}x)}{R}dx}{2(\alpha + \sin\alpha)} \quad (79)$$

and

$$\frac{4\pi\epsilon_o a}{e} V_o = \pi \frac{Q/e - I_o}{\alpha + \sin\alpha} \tag{80}$$

where

$$I_o = \frac{1}{\pi}\int_{-\alpha}^{\alpha} \frac{\cos(\frac{1}{2}x)}{R}\,dx \tag{81}$$

In ref. 4 section 8.7, I_o has been calculated explicitly:

$$I_o = \frac{1}{\pi}(\gamma + \gamma' a/r_o) \tag{82}$$

where

$$\sin\gamma = \frac{a}{r_o} \quad \sin\gamma' = \frac{2\sin\alpha}{r_1 + r_2} \tag{83}$$

$$r_1^2 = (r_o/a)^2 - 2(r_o/a)\cos(\alpha - \theta) + 1 \tag{84}$$

$$r_2^2 = (r_o/a)^2 - 2(r_o/a)\cos(\alpha + \theta) + 1 \tag{85}$$

The angle γ is such that $0 < \gamma < \pi/2$ at all points other than those lying on the segment of the sphere bounded by the surface of the cap and the plane $r\cos\theta = a\cos\alpha$ containing its edge; for these points $\pi/2 < \gamma < \pi$. Further, if S is the sphere through the edge of the cap and the origin (i.e. the equation of S is $r\cos\alpha = a\cos\theta$), the angle γ' satisfies $\pi/2 < \gamma' < \pi$ for those points exterior to the sphere but interior to S for $\alpha < \pi/2$ and exterior to S for $\alpha > \pi/2$. For all other points $0 < \gamma' < \pi/2$.

The first integral in (76) can also be calculated explicitly, i.e.

$$-2\int_{\alpha}^{\pi} \rho\cos(x/2)\cos(\tau/2)\,dx = \int_{-\alpha}^{\alpha}\frac{\cos(x/2)}{R}\,dx - \int_{-\pi}^{\pi}\frac{\cos(x/2)}{R}\,dx$$

$$= \gamma + \gamma' a/r_o - \pi a/r_o \tag{86}$$

In view of (77), (80), (82) and (86), $Q_i(\theta)$ given by (76) can be finally rewritten as

$$q_i(\theta) = Q_i(\theta)/e = \frac{1}{2\pi}\left[\pi(I_o - a/r_o) + I_1 + \pi(Q/e - I_o)I_2\right] \tag{87}$$

where I_o is given by (82), I_1 by (77) and

$$I_2 = 1 - \frac{1}{\alpha + \sin\alpha}\left[\frac{\pi}{2}(1 - \cos\theta) + (1 + \cos\alpha)^{\frac{1}{2}}(\cos\theta - \cos\alpha)^{\frac{1}{2}} + \frac{1}{2}(1 + \cos\theta)\cos^{-1}\left(2\,\frac{1 + \cos\alpha}{1 + \cos\theta} - 1\right)\right] \tag{88}$$

From (87) we can obtain the following limiting forms, as $r_o/a \to \infty$ (electron removed to infinity),

$$Q_i(\theta) = \frac{Q}{2} I_2$$

$$Q_i(\alpha) = \frac{Q}{2}\left[1 - \frac{\pi}{2}\,\frac{1 - \cos\alpha}{\alpha + \sin\alpha}\right] < Q/2, \qquad 0 < \alpha < \pi$$

Notice that $Q_i(\pi) = 0$, $Q_i(0) = Q/2$ as expected.

Eq. 87 can be rewritten as

$$Q_i(\theta) = \frac{Q}{2} I_2 + Q_i'(\theta)$$

where $Q_i'(\theta)$ does not depend on Q; it only depends on the electronic charge since the induced charges are proportional to it.

Calculation of the Current on the Inside Surface

In the previous section we calculated the total charge $Q(\theta)$ contained on the inside surface of a spherical cap defined by the polar angle θ (Fig. 2). As the electron moves, the value of $Q(\theta)$ changes and this gives rise to a current. We will now prove that the total current crossing the circumference of the $\theta = \theta$ circle and in the $\hat{e}_\theta$ direction (Fig. 3) is given by

$$I_i(\theta) = -\frac{dQ_i(\theta)}{dt} \tag{89}$$

We start with the Maxwell equation

$$\nabla x \underline{H} = \frac{\partial \underline{D}}{\partial t} \tag{90}$$

Consider now a surface $S(\theta)$ on the inside of the spherical cap and almost touching the inside surface of the cap (Fig. 4) then from (90) we get

$$\int_{S(\theta)} \nabla x \underline{H} \cdot \underline{dS} = \frac{d}{dt} \int_{S(\theta)} \underline{D} \cdot \underline{dS} \tag{91}$$

Applying Stokes' theorem we obtain

$$\oint \underline{H} \cdot \underline{d\ell} = \frac{d}{dt} \int D_n dS \tag{92}$$

The normal $\hat{n}$ is the outward normal, equal to $\hat{e}_r$, and the circle $\theta = \theta$ is traversed in the positive sense with respect to $\hat{n}$, i. e. in the positive φ direction. Thus

$$\oint \underline{H} \cdot d\ell = \int_0^{2\pi} H_\varphi \, a \sin \theta \, d\varphi$$

The total surface current density on the inside surface (i. e. "magneto-static" + "electrostatic") is given by

$$\underline{K}_i = (-\hat{e}_r) x \, \underline{H}$$

or

$$K_{i\theta} = H_\varphi$$

and

$$\oint \underline{H} \cdot d\ell = \int_0^{2\pi} K_{i\theta} \, a \sin \theta \, d\varphi = I_i(\theta)$$

The surface integral on the right-hand side of (92) is equal to $-Q_i(\theta)$ since $D_n = -(-\hat{e}_r \cdot \underline{D})$. Thus (89) is true. If we apply (91) assuming that $S(\theta)$ is a surface on the outside of the spherical cap and almost touching the outside surface of the cap, we can derive the relationship

$$I_o(\theta) = -\frac{dQ_o(\theta)}{dt} \tag{93}$$

where $I_o(\theta) = \int_0^{2\pi} K_\theta \, a \sin\theta \, d\varphi$, $K_\theta = -H_\varphi$ and $Q_o(\theta)$ is the outside charge $= \int D_n \, dS$.

Notice the minus sign in both (89) and (93).

At $\theta = \alpha$

$$Q_i(\alpha) + Q_o(\alpha) = Q \tag{94}$$

and

$$\frac{dQ_i(\alpha)}{dt} + \frac{dQ_o(\alpha)}{dt} = 0 \tag{95}$$

Thus

$$I_i(\alpha) + I_o(\alpha) = 0 \tag{96}$$

i.e. the current at the edge of the aperture is due to the "spill over" of the charge over the edge. Consider for example, the case $Q = e$ and the electron moving along the positive z axis. When $r_o \gg a$ most of the charge on the sphere is concentrated on the outside surface near point $P(a, \theta = 0)$. As the electron moves away, charge from the outside starts spilling over the edge into the aperture, and the inside charge increases. The inside current flows in the $-\hat{e}_\theta$ direction and $I_i(\alpha) < 0$. This can also be seen from (89) because the derivative is negative. The outside current flows in the $\hat{e}_\theta$ direction and this is consistent with (93), i.e. the outside charge decreases with time. To calculate the current, in general, it is best to use (72). The time dependent quantity is $R(r_o, \theta_o)$ and we have

$$\frac{d}{dt}\frac{1}{R} = -\frac{1}{R^2}\frac{dR}{dt} = \frac{1}{aR^3}\left[\left(\frac{r_o}{a} e^{ix} - \cos\theta_o\right) V_r + \sin\theta_o V_\theta\right] \tag{97}$$

where

$$V_r = \frac{dr_o}{dt}, \quad V_\theta = r_o \frac{d\theta_o}{dt} \tag{98}$$

are the radial and meridian components of the velocity of the electron respectively. (The current does not depend on the azimuthal component V_φ.) If we consider the real part of $\frac{d}{dt}\frac{1}{R}$ we obtain

$$Re\left(\frac{d}{dt}\frac{1}{R}\right) = -\frac{1}{a\rho^2}\left[\left[\left(\frac{r_o}{a}\cos x - \cos\theta_o\right)V_r + V_\theta \sin\theta_o\right]\cos\frac{3\tau}{2} + \frac{r_o}{a}V_r \sin x \sin\frac{3\tau}{2}\right] \tag{99}$$

The final form for $I_i(\theta)$ can now be easily obtained:

$$I_i(\theta)/(eV/a) = I_r(V_r/V) + I_\theta(V_\theta/V) \tag{100}$$

where

$$(V_r/V)^2 + (V_\theta/V)^2 = 1 \tag{101}$$

$$I_r = -\frac{1}{\pi}\left[\int_\alpha^\pi f_1(x, r_o, \theta_o)\cos\frac{x}{2}\,dx - \sqrt{2}\int_{\beta_1}^{\beta_2} f_1[x(\beta), r_o, \theta_o]\,d\beta + I_2\int_0^\alpha f_1(x, r_o, \theta_o)\cos\frac{x}{2}\,dx\right] \tag{102}$$

$$I_\theta = -\frac{1}{\pi}\left[\int_\alpha^\pi f_2(x, r_o, \theta_o)\cos\frac{x}{2}\,dx - \sqrt{2}\int_{\beta_1}^{\beta_2} f_2[x(\beta), r_o, \theta_o]\,d\beta + I_2\int_0^\alpha f_2(x, r_o, \theta_o)\cos\frac{x}{2}\,dx\right] \tag{103}$$

$$f_1(x, r_o, \theta_o) = \frac{1}{\rho^3}\left[\left(\frac{r_o}{a}\cos x - \cos\theta_o\right)\cos\frac{3\tau}{2} + \frac{r_o}{a}\sin x \sin\frac{3\tau}{2}\right] \tag{104}$$

$$\cos x = \cos\theta - \beta^2, \quad \beta_1 = (\cos\theta - \cos\alpha)^{\frac{1}{2}}, \quad \beta_2 = (\cos\theta + 1)^{\frac{1}{2}} \tag{105}$$

$$f_2(x, r_o, \theta_o) = \frac{1}{\rho^3} \sin\theta_o \cos\frac{3\tau}{2} \tag{106}$$

and $\rho(r_o, \theta_o)$, $\tau(r_o, \theta_o)$ are given by (61), I_2 by (88).

From (101), V is not the total speed because V_φ may or may not be zero. Our calculation only determines the total current that flows in the $\hat{e}_\theta$ direction. In other words if V_r, V_θ are both zero but V_φ is not, we still have currents on the sphere even though $I_i(\theta)$ given by (100) is zero. The calculation of the currents in the φ direction requires the knowledge of more expansion coefficients and also surface current densities from the magnetostatic problem.

From (100) we see that if we know $I_i(\theta)$ for $V_r/V = 1$, $V_\theta/V = 0$, i.e. $I_i(\theta)/(eV/a = I_r)$ and also for $V_r/V = 0$, $V_\theta/V = 1$, i.e. $I_i(\theta)/(eV/a) = I_\theta$, we can determine $I_i(\theta)$ in general, i.e. for any V_r/V (V_θ/V is given by 101). Notice that $I_i(\theta)$ does not depend on Q, since current is associated to the motion of the induced charges that are proportional to the charge of the moving electron.

<u>Calculation of the Electric Field at the Center of the Sphere</u>

The total electric field at the center of the sphere is given by

$$\underline{E}_t = -\nabla(\varphi_i + \varphi_{inc})\Big|_{r=0} \tag{107}$$

First we calculate $\underline{E}_i = -\nabla\varphi_i\Big|_{r=0}$. From (1) and (2) we find

$$\underline{E}_i = -\left(\frac{\partial\varphi_i}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\varphi_i}{\partial\theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\varphi_i}{\partial\varphi}\hat{e}_\varphi\right)\Bigg|_{r=0}$$

$$= -\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} C_{\ell m}\left\{\frac{\ell}{a^\ell} r^{\ell-1} A_{\ell m} P_\ell^m(\cos\theta)\, e^{im\varphi}\hat{e}_r + \right.$$

$$\frac{1}{r}(r/a)^{\ell} A_{\ell m}\, e^{im\varphi} \frac{\partial}{\partial\theta} P_{\ell}^{m}(\cos\theta)\, \hat{e}_{\theta}$$

$$+ \frac{1}{r \sin\theta}(r/a)^{\ell} A_{\ell m}(im)\, e^{im\varphi} P_{\ell}^{m}(\cos\theta)\, \hat{e}_{\varphi} \Bigg\}\Bigg|_{r=0} \tag{108}$$

From (108) we see that only the $\ell = 0$ $(m = 0)$ and $\ell = 1$ $(m = 0, \pm 1)$ terms contribute. Thus

$$\underline{E}_i = -\frac{b_{10}}{a}\left[P_1(\cos\theta)\,\hat{e}_r + \frac{\partial}{\partial\theta} P_1(\cos\theta)\,\hat{e}_{\theta}\right]$$

$$- \frac{b_{11}}{a} e^{i\varphi}\left[P_1^1(\cos\theta)\,\hat{e}_r + \frac{\partial}{\partial\theta} P_1^1(\cos\theta)\,\hat{e}_{\theta} + \frac{1}{\sin\theta}\, iP_1^1(\cos\theta)\,\hat{e}_{\varphi}\right]$$

$$- \frac{b_{1,-1}}{a} e^{-i\varphi}\left[P_1^{-1}(\cos\theta)\,\hat{e}_r + \frac{\partial}{\partial\theta} P_1^{-1}(\cos\theta)\,\hat{e}_{\theta} + \frac{1}{\sin\theta}(-i)\, P_1^{-1}(\cos\theta)\,\hat{e}_{\varphi}\right] \tag{109}$$

where $b_{\ell m} = C_{\ell m} A_{\ell m}$ (eq.(37)).

From (16) we recall that $C_{11} = -C^{*}_{1,-1}$ and from (13) $A_{1,-1} = 2A_{11}$, i. e.

$$b_{11} = -\tfrac{1}{2} b^{*}_{1,-1} \tag{110}$$

It is easy to show that

$$P_1^1(\cos\theta) = -\sin\theta, \quad P_1^{-1}(\cos\theta) = \tfrac{1}{2}\sin\theta$$

$$\frac{\partial}{\partial\theta} P_1(\cos\theta) = -\sin\theta, \quad \frac{\partial}{\partial\theta} P_1^1(\cos\theta) = -\cos\theta, \quad \frac{\partial}{\partial\theta} P_1^{-1}(\cos\theta) = \tfrac{1}{2}\cos\theta \tag{111}$$

Using (110), (111) and the relationships

$$\begin{aligned} \hat{e}_r &= \hat{e}_x \sin\theta\cos\varphi + \hat{e}_y \sin\theta\sin\varphi + \hat{e}_z\cos\theta \\ \hat{e}_{\theta} &= \hat{e}_x \cos\theta\cos\varphi + \hat{e}_y \cos\theta\sin\varphi - \hat{e}_z\sin\theta \\ \hat{e}_{\varphi} &= -\hat{e}_x \sin\varphi + \hat{e}_y\cos\varphi \end{aligned} \tag{112}$$

(109) can be rewritten as

$$\underline{E}_i(\underline{r}=0) = -\frac{b_{10}}{a}\hat{e}_z + \left[\frac{1}{a}\left(b_{11}e^{i\varphi} + b_{11}^* e^{-i\varphi}\right)\cos\varphi\right.$$

$$\left. -\frac{i}{a}\left(b_{11}e^{i\varphi} - b_{11}^* e^{-i\varphi}\right)\sin\varphi\right]\hat{e}_x + \left[\frac{1}{a}\left(b_{11}e^{i\varphi} + b_{11}^* e^{-i\varphi}\right)\sin\varphi\right.$$

$$\left. +\frac{i}{a}\left(b_{11}e^{i\varphi} - b_{11}^* e^{-i\varphi}\right)\cos\varphi\right]\hat{e}_1$$

or

$$\underline{E}_i(\underline{r}=0) = -\frac{b_{10}}{a}\hat{e}_z + \frac{1}{a}\left(b_{11} + b_{11}^*\right)\hat{e}_x + \frac{i}{a}\left(b_{11} - b_{11}^*\right)\hat{e}_y \tag{113}$$

We can always choose the position of the electron in the xz plane in which case $\varphi_o = 0$ and $b_{11} = b_{11}^*$. Thus

$$\underline{E}_i(\underline{r}=0) = \frac{2}{a}b_{11}\hat{e}_x - \frac{b_{10}}{a}\hat{e}_z \qquad (\varphi_o = 0) \tag{114}$$

The total electric field is given by

$$\underline{E}_t(r=0) = \frac{2}{a}b_{11}\hat{e}_x - \frac{b_{10}}{a}\hat{e}_z - \nabla\Phi_{inc}$$

$$= \left(\frac{2}{a}b_{11} + \frac{e}{4\pi\epsilon_o r_o^2}\sin\theta_o\right)\hat{e}_x + \left(\frac{e}{4\pi\epsilon_o r_o^2}\cos\theta_o - \frac{b_{10}}{a}\right)\hat{e}_z \tag{115}$$

(We have assumed that $x_o > 0$.)

Next we calculate b_{11}. From (17) we understand that $k = 0$ and from (20) and (37)

$$b_{11} = B_{01} \tag{116}$$

Thus (eq.(23))

$$b_{11} = -\frac{1}{2}M_{01}$$

and from (24)

$$b_{11} = -\frac{1}{2}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{\Gamma(3)}{\Gamma(3/2)}\int_0^{\alpha} F_1^*(u)\left(\cos\tfrac{1}{2}u\right)^3 P_o^{(1/2,3/2)}(\cos u)\,du \tag{117}$$

$$F_1^*(u) = \frac{d}{du}\int_0^u \frac{(\tan\frac{1}{2}\theta)\,\sin\theta\; G_1\;(\theta)\,d\theta}{(\cos\theta - \cos u)^{\frac{1}{2}}} \tag{118}$$

and from (28)

$$G_1(\theta) = \frac{e}{\epsilon_o}\sum_{\ell=1}^{\infty} \frac{1}{2\ell+1}\;\frac{a^{\ell}}{r_o^{\ell+1}}\;|A_{\ell 1}|^2\;P_{\ell}^1(\cos\theta_o)\;P_{\ell}^1(\cos\theta) \tag{119}$$

$(P_{\ell}^1 = 0 \text{ for } \ell = 0)$

Combining (31), (118) and (119) we obtain

$$F_1^*(u) = \frac{e}{\epsilon_o}\sum_{\ell=1}^{\infty} \frac{1}{2\ell+1}\;\frac{a^{\ell}}{r_o^{\ell+1}}\;|A_{\ell 1}|^2\;P_{\ell}^1(\cos\theta_o)$$

$$(-\tfrac{1}{2})\sqrt{\pi}\;\frac{\Gamma(\ell+2)}{\Gamma(\ell+\frac{1}{2})}\;(1-\cos u)^{\frac{1}{2}}\;\sin u\;P_{\ell-1}^{(1/2,\,3/2)}(\cos u) \tag{120}$$

We will calculate $P_{\ell-1}^{(1/2,\,3/2)}(\cos u)$ with the aid of ref. 4 . From the equation right below (2.6.23) we see that

$$P_{\ell-1}^{(1/2,\,3/2)} = -2\;\frac{\Gamma(\ell+\frac{1}{2})}{\Gamma(\ell+2)\,\Gamma(\frac{1}{2})}\;\frac{1}{\sin u}\;\frac{d}{du}\;\frac{\cos(\ell+\frac{1}{2})u}{\cos(u/2)} \tag{121}$$

A simple calculation yields

$$\frac{d}{du}\;\frac{\cos(\ell+\frac{1}{2})u}{\cos(u/2)} = -\,\tfrac{1}{2}\;\frac{\ell\sin(\ell+1)u + (\ell+1)\sin\ell u}{\cos^2(u/2)} \tag{122}$$

and (121) gives $(\Gamma(\frac{1}{2}) = \sqrt{\pi})$

$$P_{\ell-1}^{(1/2,\,3/2)} = \frac{1}{\sqrt{\pi}}\;\frac{\Gamma(\ell+\frac{1}{2})}{\Gamma(\ell+2)}\;\frac{\ell\sin(\ell+1)u + (\ell+1)\sin\ell u}{\sin u\cos^2(u/2)} \tag{123}$$

and

$$P_o^{(1/2,\,3/2)} = \frac{1}{\sqrt{\pi}}\;\frac{(\frac{1}{2})\sqrt{\pi}}{2}\;\frac{1}{\sin u\cos^2(u/2)}\;(\sin 2u + 2\sin u) = 1 \tag{124}$$

This result is not surprising since $P_o^{(\alpha,\beta)} = 1$ for any α and β. Recalling from (10) that

$$A_{\ell 1}^2 = \frac{2\ell+1}{4\pi} \frac{(\ell-1)!}{(\ell+1)!} = \frac{2\ell+1}{4\pi\ell(\ell+1)}$$

we can rewrite (117) as

$$\frac{2b_{11}}{a} = \frac{e}{4\pi\epsilon_o a^2} \frac{2}{\pi} \sum_{\ell=1}^{\infty} \left(\frac{a}{r_o}\right)^{\ell+1} \frac{1}{\ell(\ell+1)} P_\ell^1(\cos\theta_o) \int_0^\alpha \sin u \, [\ell \sin(\ell+1)u + (\ell+1)\sin \ell u] \, du$$

$$= \frac{e}{4\pi\epsilon_o a^2} \frac{1}{\pi} \sum_{\ell=1}^{\infty} \left(\frac{a}{r_o}\right)^{\ell+1} \frac{1}{\ell(\ell+1)} P_\ell^1(\cos\theta_o) K_\ell \tag{125}$$

Where

$$K_1 = 2\alpha + \sin\alpha - \sin 2\alpha - \frac{1}{3} 3\alpha \tag{126}$$

$$K_\ell = \frac{\ell+1}{\ell-1} \sin(\ell-1)\alpha + \sin \ell\alpha - \sin(\ell+1)\alpha - \frac{\ell}{\ell+2} \sin(\ell+2)\alpha$$

Thus the x-component of the total electric field at the center of the sphere is given by

$$e_{tx} = E_{tx}(\underline{r} = o) \Big/ (e/4\pi\epsilon_o a^2) = \frac{1}{\pi} \sum_{\ell=1}^{\infty} \left(\frac{a}{r_o}\right)^{\ell+1} \frac{1}{\ell(\ell+1)} P_\ell^1(\cos\theta_o) K_\ell$$

$$+ \left(\frac{a}{r_o}\right)^2 \sin\theta_o \tag{127}$$

Notice that $E_{tx}(\underline{r} = o)$ does not depend on Q. That is easy to understand. When the electron is removed to infinity, $E_{tx}(\underline{r} = o)$ is zero and consequently

$E_{tx}(\underline{r} = o)$ in general is due to induced charges that are proportional to the charge of the electron.

When $\theta_o = 0$ or π, $P_\ell^1(\cos\theta_o) = 0$ and $\sin\theta_o = 0$ resulting in a zero x-component as it should. Another special case is of interest: $\alpha = \pi$ i.e. where there is no aperture. We can see from (126) that $K_1 = 2\pi$ and $K_\ell = 0 \quad \ell \neq 1$ i.e.

$$E_{tx} = \frac{\ell}{4\pi\epsilon_o a^2}\left[\frac{1}{\pi}\left(\frac{a}{r_o}\right)^2 \frac{1}{2} P_1^1(\cos\theta_o)\, 2\pi + \left(\frac{a}{r_o}\right)^2 \sin\theta_o\right] = 0$$

since $P_1^1(\cos\theta_o) = \sin\theta_o$.

Next we calculate E_{tz}. From (115) and (69) we see that

$$b_{10} = \frac{V_o}{\pi}\left(\sin\alpha + \frac{1}{2}\sin 2\alpha\right) + \frac{e}{4\pi\epsilon_o a}\frac{1}{\pi}\int_{-\alpha}^{\alpha}\frac{\cos(3x/2)}{R}\,dx \tag{128}$$

In section (8.7.2) of ref. 4 the integral on the right-hand side of (128) has been evaluated in closed form. It is given by (8.7.14)

$$A \equiv \frac{1}{\pi}\int_{-\alpha}^{\alpha}\frac{\cos(3x/2)}{R}\,dx = \frac{1}{\pi}\left\{\frac{r_o}{a}\gamma\cos\theta_o + \left(\cos\alpha - \frac{r_o}{a}\cos\theta_o\right)\tan\gamma\right.$$

$$\left. + \left(\frac{a}{r_o}\right)^2\gamma'\cos\theta_o + \left(\frac{a}{r_o}\right)^2\left(\frac{r_o}{a}\cos\alpha - \cos\theta_o\right)\tan\gamma'\right\} \tag{129}$$

where γ and γ' have been defined by (83), (84) and (85).

Thus (130)

$$e_{tz} = E_{tz}(\underline{r}=0)\Big/\left(e/4\pi\epsilon_o a^2\right) = \left(\frac{a}{r_o}\right)^2\cos\theta_o - \left(\frac{Q}{e} - I_o\right)\frac{\sin\alpha + (\frac{1}{2})\sin 2\alpha}{\alpha + \sin\alpha} - A$$

where I_o is given by (82) and A by (129).

When $\alpha = \pi$ then $\gamma = 0$, $\gamma' = \pi$, $A = \left(\frac{a}{r_o}\right)^2 \cos\theta_o$ and $E_{tz} = 0$ as it should.

Also when the electron is removed to infinity we can show that $A \rightarrow 0$, $I_o \rightarrow 0$ and

$$E_{tz}(\underline{r} = 0) = -\frac{Q}{4\pi\epsilon_o a^2} \frac{\sin\alpha + (\frac{1}{2})\sin 2\alpha}{\alpha + \sin\alpha} \quad (r_o \rightarrow \infty) \tag{131}$$

and from (37)

$$E_{tx}(\underline{r}=0) = 0 \tag{132}$$

Eq. (131) gives the electric field at the center of a charged spherical bowl. Notice that (130) can be written as

$$\underline{E}_{tz}(\underline{r}=0) = -\frac{Q}{4\pi\epsilon_o a^2} \frac{\sin\alpha + (\frac{1}{2})\sin 2\alpha}{\alpha + \sin\alpha} + \underline{E}'_z(\underline{r}=0) \qquad 132)$$

where $\underline{E}'_{tz}(\underline{r}=0)$ does not depend on Q. $\underline{E}'_{tz}(\underline{r}=0)$ is due to the induced charges on the sphere (proportional to the electronic charge), and the electron itself.

III. Description of the Plots

As a measure of the effect of the moving electron on the field quantities in the interior of the sphere through the aperture, we considered and plotted the following quantities:

1) The normalized charge $q_i(\theta)=Q_i(\theta)/e$ where $Q_i(\theta)$ is the charge on the interior surface at the spherical bowl defined by the polar angle θ (Fig. 2) given by eq. (88), and -e is the electronic charge. This charge $q_i(\theta)$ is plotted versus the normalized distance r_o/a where r_o is the radial distance of the electron from the center of the sphere and "a" the radius of the sphere. We have considered the cases $\theta=90^\circ, \alpha$; $\alpha=135^\circ$, 150°, 160°, 170°, 175° and $\theta_o=0^\circ$, 45°, 90°, 135°, α, 180°, where α is the polar angle defining the aperture (Fig. 1) and θ_o defines the angular position of the electron (Fig. 1).

2) The normalized currents I_r and I_θ. These currents are related to the total current $I_i(\theta)$ (Fig. 3) through eq. (100). If the velocity of the electron in a plane passing through the z-axis is $\underline{V}$ then the radial velocity component V_r and azimuthal velocity component V_θ are related to $\underline{V}$ through eq. (101). Thus I_r is the normalized current $I_i(\theta)/(eV/a)$ when $V_r/V=1$ and I_θ is the normalized current $I_i(\theta)/(eV/a)$ when $V_\theta/V=1$. For an arbitrary velocity $\underline{V}$ we can calculate the ratios V_r/V and V_θ/V and the total current is given by eq. (100). We have plotted I_r and I_θ versus r_o/a with $\theta=90^\circ$, α; $\alpha=135^\circ$, 150°, 160°, 170°, 175° and $\theta_o=0$, 45°, 90°, 135°, α, 180°. Notice that when θ_o=<u>0 or 180°</u>, the normalized current I_θ is zero even though $V_\theta=V\neq 0$.

3) The normalized electric field component $e_{tz}=E_{tz}/(e/4\pi\varepsilon_o a^2)$ given by eq. (130) and $e_{tx}=E_{tx}/(e/4\pi\varepsilon_o a^2)$ given by eq. (127). $\underline{E}_t$ is the <u>total</u> electric field at the center of the sphere, the electron lies in the xz plane and e_{tz}, e_{tx} are plotted versus r_o/a for $\alpha=135^\circ$, 150°, 160°, 165°, 170° and $\theta_o=0$, 45°, 90°, 135°, α, 180°. Notice that when

θ_o=0 or 180°, e_{tx}=0 because the induced charge on the surface of the sphere is φ -independent.

Appendix A

In this appendix we derive eq. (57) with the aid of Green's reciprocation theorem.

Consider the situation depicted in figs. 5a and 5b. In fig. 5a the electron is placed at a point P characterized by the position vector $\underline{r}_o$, whereas in fig. 5b the electron has been removed to infinity. The total charge on the sphere, in both situations, is equal to Q and V_∞ is the potential on the sphere with the electron at infinity. If we denote the potential functions by φ_a and φ_b we can write

$$\nabla^2\varphi_a = (e/\varepsilon_o)\,\delta\,(\underline{r} - \underline{r}_o) \tag{A-1}$$

$$\nabla^2\varphi_b = 0 \tag{A-2}$$

We multiply (A-1) by φ_b, (A-2) by φ_a and subtract to obtain

$$\varphi_b\nabla^2\varphi_a - \varphi_a\nabla^2\varphi_b = \varphi_b(e/\varepsilon_o)\,\delta\,(\underline{r} - \underline{r}_o) \tag{A-3}$$

Next we use the simple identity $\varphi_b\nabla^2\varphi_a - \varphi_a\nabla^2\varphi_b = \nabla\cdot(\varphi_b\nabla\varphi_a - \varphi_a\nabla\varphi_b)$ and integrate (A-3) throughout the entire space. Applying Gauss' theorem we obtain

$$\int_{S_i + S_o}\left(\varphi_b\,\frac{\partial\varphi_a}{\partial n_1} - \varphi_a\,\frac{\partial\varphi_b}{\partial n_1}\right)dS' = (e/\varepsilon_o)\,\varphi_b\,(\underline{r}_o) \tag{A-4}$$

where S_i, S_o are the inside and outside surfaces of the sphere respectively, $\hat{n}_1$ is equal to $-\hat{e}_r$ on S_o and $+\hat{e}_r$ on S_i and the surface integral at infinity can be shown to vanish.

Equation (A-4) can be rewritten as

$$V_\infty(Q/\varepsilon_o) - V_o(Q/\varepsilon_o) = (e/\varepsilon_o)\,\varphi_b\,(\underline{r}_o)$$

or

$$V_o = V_\infty - \frac{e}{Q} \varphi_b (\underline{r}_o) \tag{A-5}$$

Equation (A-5) is identical to (57) because $\varphi_\infty(r_o, \theta_o) \equiv \varphi_b(\underline{r}_o)$.

Appendix B

The purpose of this appendix is to calculate the series

$$S_1 = \sum_{\ell=1}^{\infty} \frac{\ell}{2\ell+1} \left[P_{\ell-1}(\cos\theta) - P_{\ell+1}(\cos\theta) \right] \cos(\ell + \tfrac{1}{2})x \tag{B-1}$$

and

$$S_2 = \sum_{\ell=1}^{\infty} \frac{\ell}{2\ell+1} \left[P_{\ell-1}(\cos\theta) - P_{\ell+1}(\cos\theta) \right] \left[\frac{\sin \ell\alpha}{\ell} + \frac{\sin(\ell+1)\alpha}{\ell+1} \right] \tag{B-2}$$

in closed form. From (75) we see that $\alpha \le x \le \pi$ and we also know that $\theta \le \alpha$. However, we will evaluate S_1 and S_2 in general for reasons that will become apparent later.

We start by noting that

$$\frac{\ell}{2\ell+1} = \tfrac{1}{2}\left(1 - \frac{1}{2\ell+1}\right).$$

Thus we will evaluate the following infinite sums

$$S_{11} = \sum_{\ell=1}^{\infty} R_\ell \cos(\ell + \tfrac{1}{2})x \tag{B-3}$$

$$S_{12} = \sum_{\ell=1}^{\infty} \frac{1}{2\ell+1} R_\ell \cos(\ell + \tfrac{1}{2})x \tag{B-4}$$

$$S_{21} = \sum_{\ell=1}^{\infty} R_\ell \left[\frac{\sin \ell\alpha}{\ell} + \frac{\sin(\ell+1)\alpha}{\ell+1} \right] \tag{B-5}$$

$$S_{22} = -\sum_{\ell=1}^{\infty} \frac{1}{2\ell+1} R_\ell \left[\frac{\sin \ell\alpha}{\ell} + \frac{\sin(\ell+1)\alpha}{\ell+1} \right] \tag{B-6}$$

where

$$R_\ell = \tfrac{1}{2}\left[P_{\ell-1}(\cos\theta) - P_{\ell+1}(\cos\theta)\right] \qquad \text{(B-7)}$$

From eq. (2. 6. 33) in ref. we can find immediately that

$$S_{11} = \tfrac{1}{2}(1+\cos\theta)\cos\tfrac{1}{2}x - \frac{\sin x\; H(x-\theta)}{\sqrt{2(\cos\theta - \cos x)}}, \quad 0 < x, \quad \theta < \pi \qquad \text{(B-8)}$$

where

$$H(x-\theta) = 1, \quad x > \theta$$
$$= 0, \quad x < \theta$$

To evaluate S_{12} we employ eq. (2. 6. 34) of ref.

$$\frac{1}{\sqrt{2}}(1+\cos\theta)\sin\tfrac{1}{2}x_1 - \frac{1}{\sqrt{2}}\sum_{\ell=1}^{\infty} 2R_\ell \sin(\ell+\tfrac{1}{2})x_1$$

$$= \sqrt{2}\sin\tfrac{1}{2}x_1 - \frac{\sin x_1\; H(\theta - x_1)}{\sqrt{\cos x_1 - \cos\theta}}, \qquad 0 < x_1, \theta < \pi \qquad \text{(B-9)}$$

If we integrate (B-9) from x to π we obtain

$$\int_x^\pi \frac{1}{\sqrt{2}}(1+\cos\theta)\sin\tfrac{1}{2}x_1\,dx_1 + \frac{1}{\sqrt{2}}\sum_{\ell=1}^{\infty} 2R_\ell \left.\frac{\cos(\ell+\tfrac{1}{2})x_1}{\ell+\tfrac{1}{2}}\right|_x^\pi$$

$$= \sqrt{2}\int_x^\pi \sin\tfrac{1}{2}x_1\,dx_1 - \int_x^\pi \frac{\sin x_1\; H(\theta - x_1)}{\sqrt{\cos x_1 - \cos\theta}}\,dx_1 \qquad \text{(B-10)}$$

Eq. (B-10) can be rewritten as

$$\left.\begin{aligned} 2\sqrt{2}\,S_{12} &= \sqrt{2}\cos\tfrac{1}{2}x\,(1-\cos\theta) - \int_x^\theta \frac{\sin x_1}{\sqrt{\cos x_1 - \cos\theta}}\,dx_1 && x > \theta \\ &= 2\cos\tfrac{1}{2}x\,(1-\cos\theta) && x < \theta \end{aligned}\right\} \qquad \text{(B-11)}$$

Noting that

$$d\,(\cos x_1 - \cos\theta)^{\frac{1}{2}} = -\tfrac{1}{2}\,(\cos x_1 - \cos\theta)^{-\frac{1}{2}} \sin x_1 \, dx_1$$

we finally arrive at the expression

$$S_{12} = \tfrac{1}{2}\cos\tfrac{1}{2}x\,(1-\cos\theta) - \frac{1}{\sqrt{2}}\,(\cos x - \cos\theta)^{\frac{1}{2}}\,H(\theta - x). \qquad \text{(B-12)}$$

The evaluation of S_{21} can be performed with the aid of (B-8). We multiply both sides by $\cos\frac{1}{2}x$ and integrate from 0 to α.

$$\sum_{\ell=1}^{\infty} R_\ell \int_0^\alpha \cos(\ell + \tfrac{1}{2})x \cos(x/2)\,dx$$

$$= \int_0^\alpha \tfrac{1}{2}(1+\cos\theta)\cos^2(x/2)\,dx - \int_\theta^\alpha \frac{\sin x}{\sqrt{2(\cos\theta - \cos x)}}\cos\frac{x}{2}\,dx \qquad \text{(B-13)}$$

For $\theta > \alpha$ the second integral is zero, whereas for $\theta < \alpha$ it is non-zero and can be calculated explicitly

$$\left.\begin{aligned} S_{21} &= \tfrac{1}{2}(\alpha + \sin\alpha)(1+\cos\theta) \\ &- \left\{(\cos\theta - \cos\alpha)^{\frac{1}{2}}(1+\cos\alpha)^{\frac{1}{2}} + \tfrac{1}{2}(1+\cos\theta)\cos^{-1}\left[2\,\frac{1+\cos\alpha}{1+\cos\theta} - 1\right]\right\} && \theta < \alpha \\ &= \tfrac{1}{2}(1+\cos\theta)(\alpha + \sin\alpha) && \theta > \alpha \end{aligned}\right\} \qquad \text{(B-14)}$$

where $\cos^{-1} y$ is the principal value and ranges from 0 to π. Notice that at $\theta = \alpha$ both expressions give $\frac{1}{2}(1+\cos\alpha)(\alpha + \sin\alpha)$. To evaluate the last sum we employ (B-12). We multiply both sides by $\cos\frac{1}{2}x$ and integrate from 0 to α. The result is

$$\left.\begin{aligned} S_{22} &= \tfrac{1}{2}(1-\cos\theta)(\alpha + \sin\alpha) - \tfrac{1}{2}\pi(1-\cos\theta) && \theta < \alpha \\ &= \tfrac{1}{2}(1-\cos\theta)(\alpha + \sin\alpha) \\ &- \left\{(1-\cos\alpha)^{\frac{1}{2}}(\cos\alpha - \cos\theta) + \tfrac{1}{2}(1-\cos\theta)\left[\pi - \cos^{-1}\left(2\,\frac{1-\cos\alpha}{1-\cos\theta} - 1\right)\right]\right\} && \theta > \alpha \end{aligned}\right\} \qquad \text{(B-15)}$$

Notice that at $\theta = \alpha$ both expressions yield $\frac{1}{2}(1 - \cos\theta)(\alpha + \sin\alpha) - \frac{1}{2}\pi(1 - \cos\theta)$.
Returning to our original sums we find that

$$S_1 = \cos\tfrac{1}{2}x - \frac{\sin x}{\sqrt{2(\cos\theta - \cos x)}} \qquad x \geq \theta \qquad \text{(B-16)}$$

$$S_2 = \alpha + \sin\alpha - \frac{\pi}{2}(1 - \cos\theta)$$

$$- \left\{(\cos\theta - \cos\alpha)^{\frac{1}{2}}(1 + \cos\alpha)^{\frac{1}{2}} + \tfrac{1}{2}(1 + \cos\theta)\cos^{-1}\left[2\,\frac{1 + \cos\alpha}{1 + \cos\theta} - 1\right]\right\} \qquad \theta \leq \alpha \qquad \text{(B-17)}$$

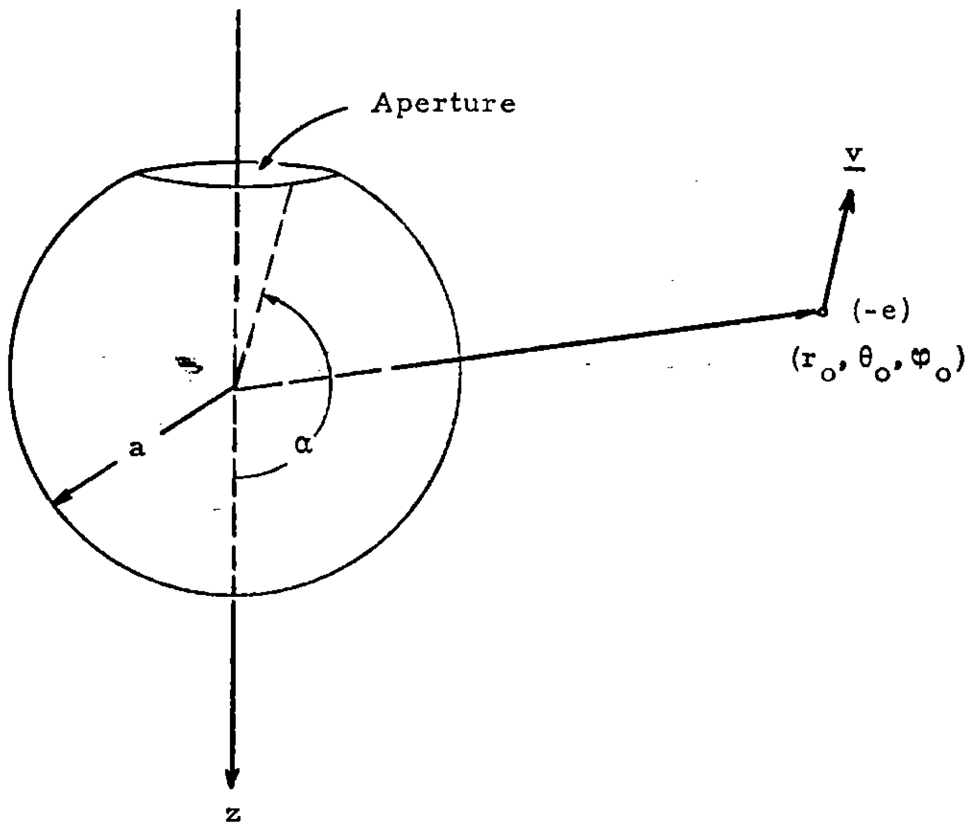


Figure 1: Geometry depicting the aperture and the relevant parameters of the problem.

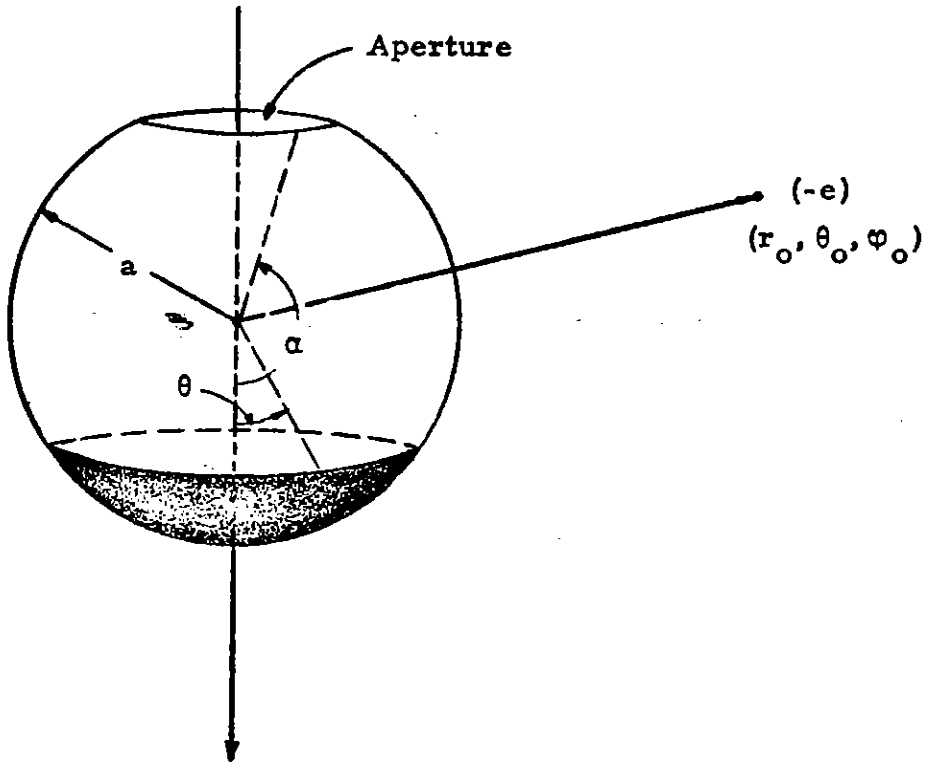


Figure 2: Geometry for the calculation of the charge $Q_i(\theta)$ on the interior surface of the (shaded) spherical bowl defined by the polar angle θ.

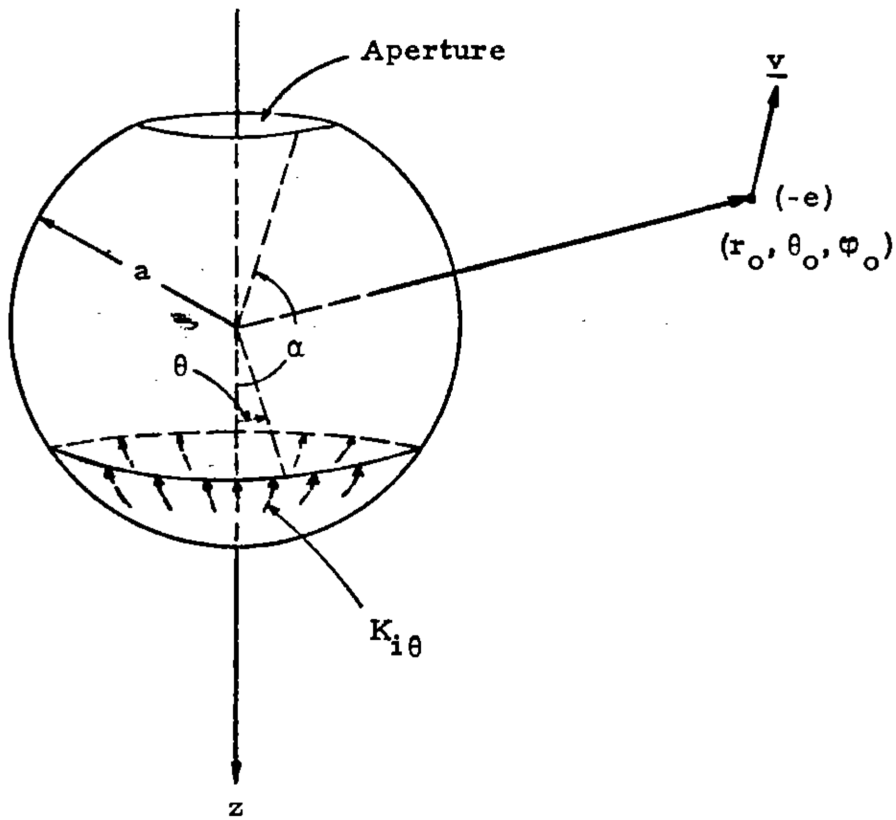


Figure 3: Geometry for the calculation of $I_i(\theta)$, the current crossing the rim of the interior surface of the spherical bowl defined by the polar angle θ. $K_{i\theta}$ is the total current density $= K'_{i\theta} + K''_{i\theta}$ = "electro-static" + "magnetostatic". $I_i(\theta) = \oint K_{i\theta}\, dl = \oint K'_{i\theta}\, dl = \int_0^{2\pi} K'_{i\theta}\, a \sin\theta\, d\varphi$. It is also given by $(-dQ_i/dt)$.

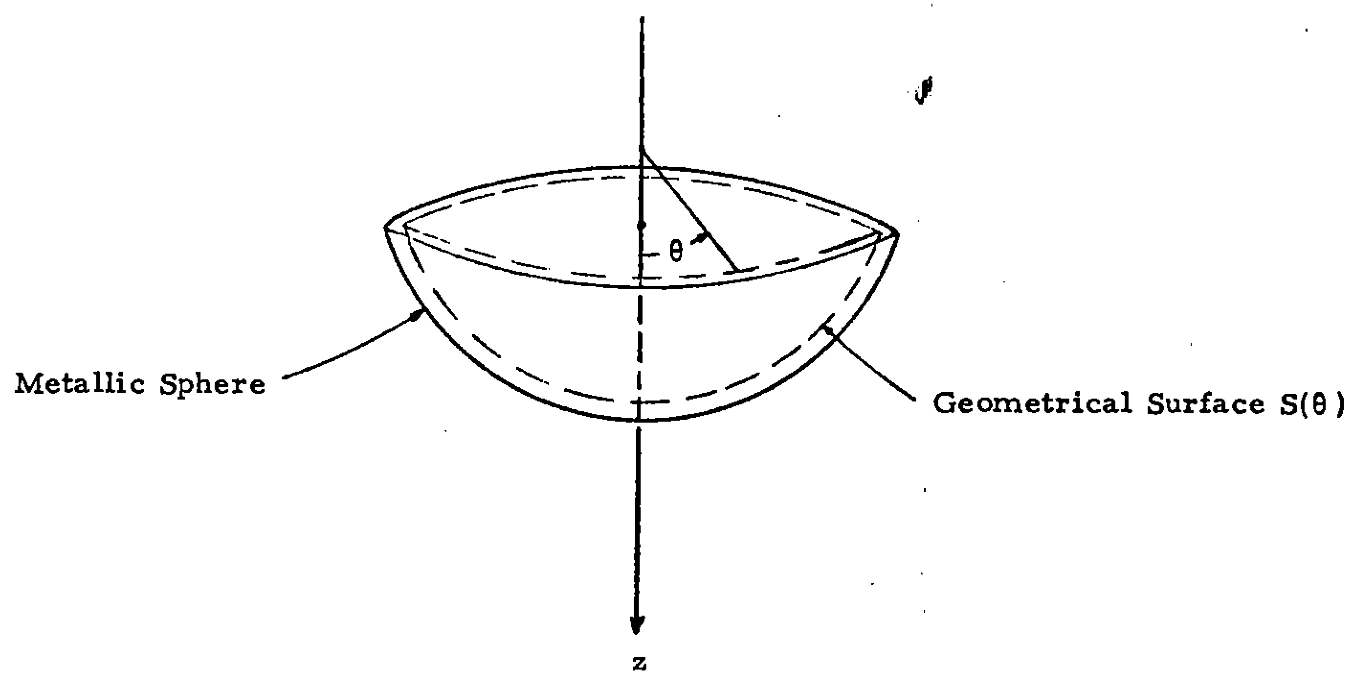


Figure 4: Geometry for the integration of eq. 90

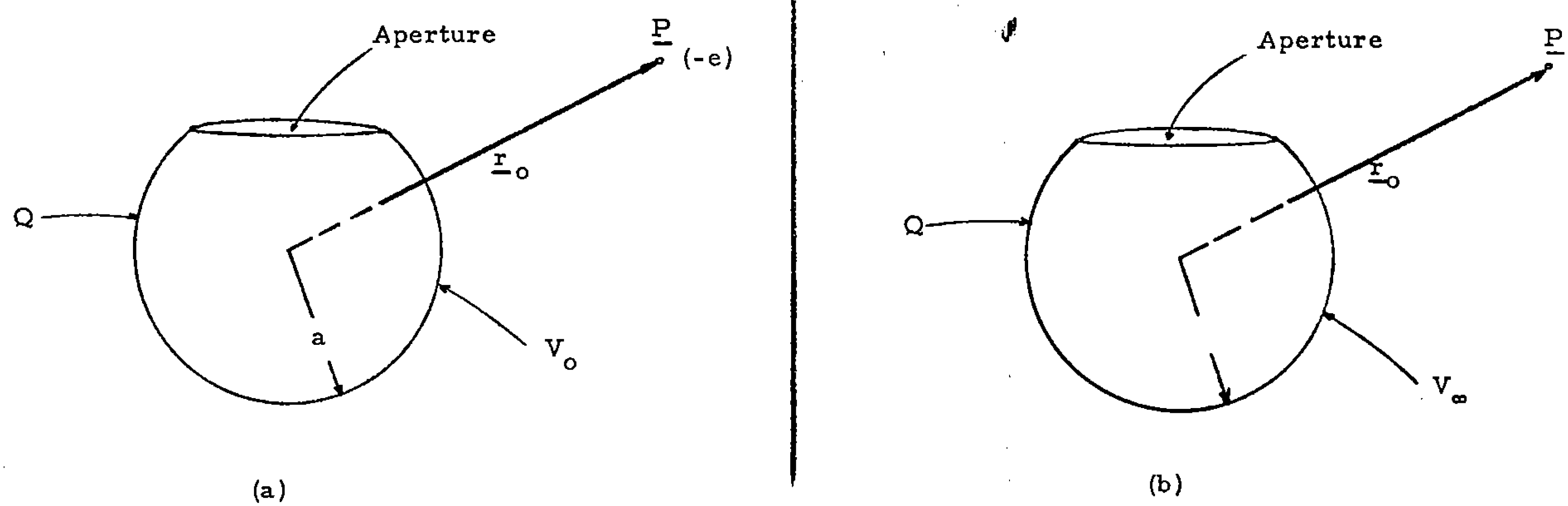


Figure 5: Geometry for the application of Green's reciprocation theorem leading to eq. A-5. In figure a the electron is placed at point $\underline{P}$, whereas in figure b it has been removed to infinity.

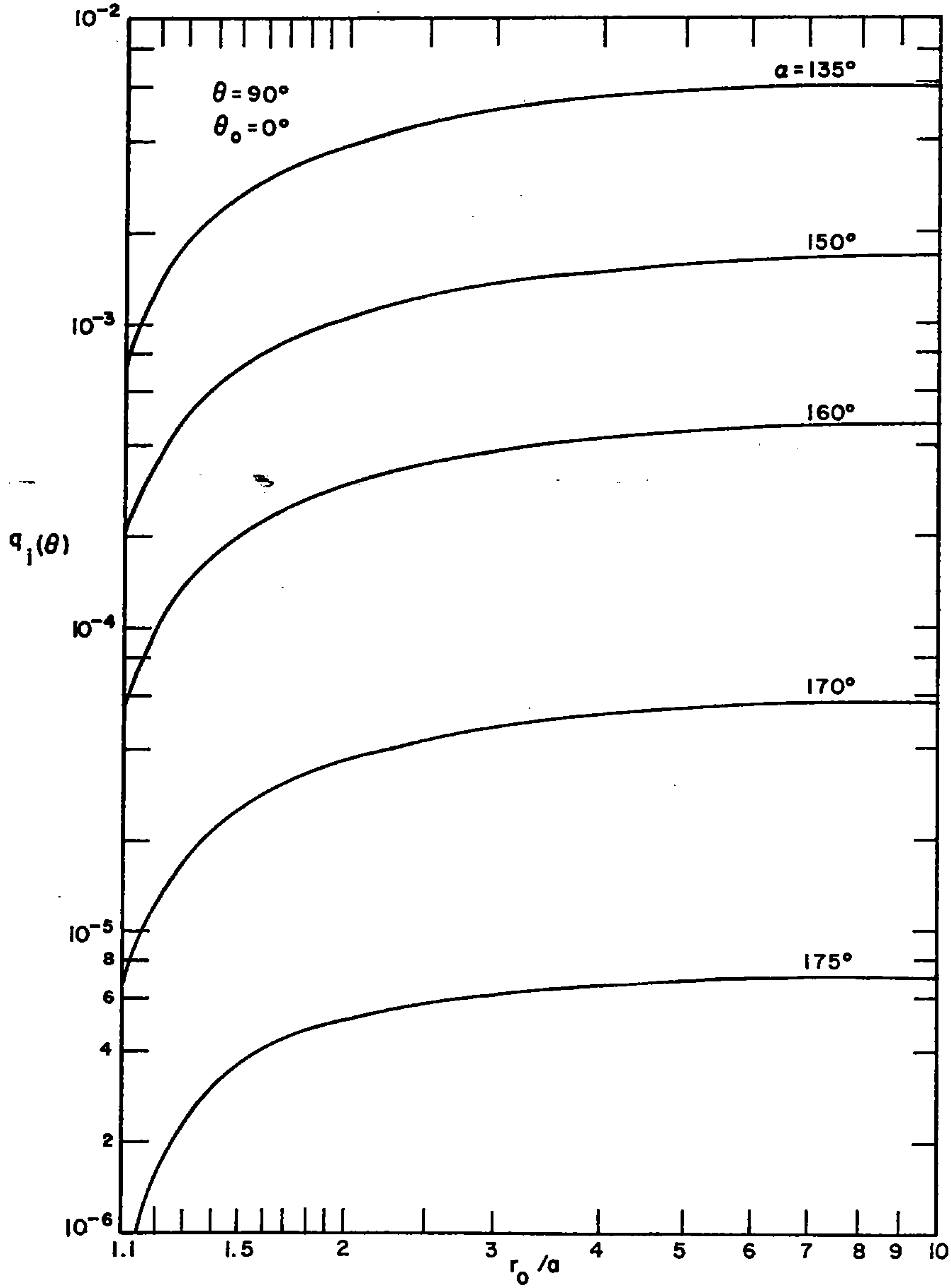

θ=90°
θ₀=0°
α=135°
150°
160°
170°
175°
q_j(θ)
10⁻²
10⁻³
10⁻⁴
10⁻⁵
8
6
4
2
10⁻⁶
1.1
1.5
2
3
4
5
6
7
8
9
10
r₀/a

Figure 6

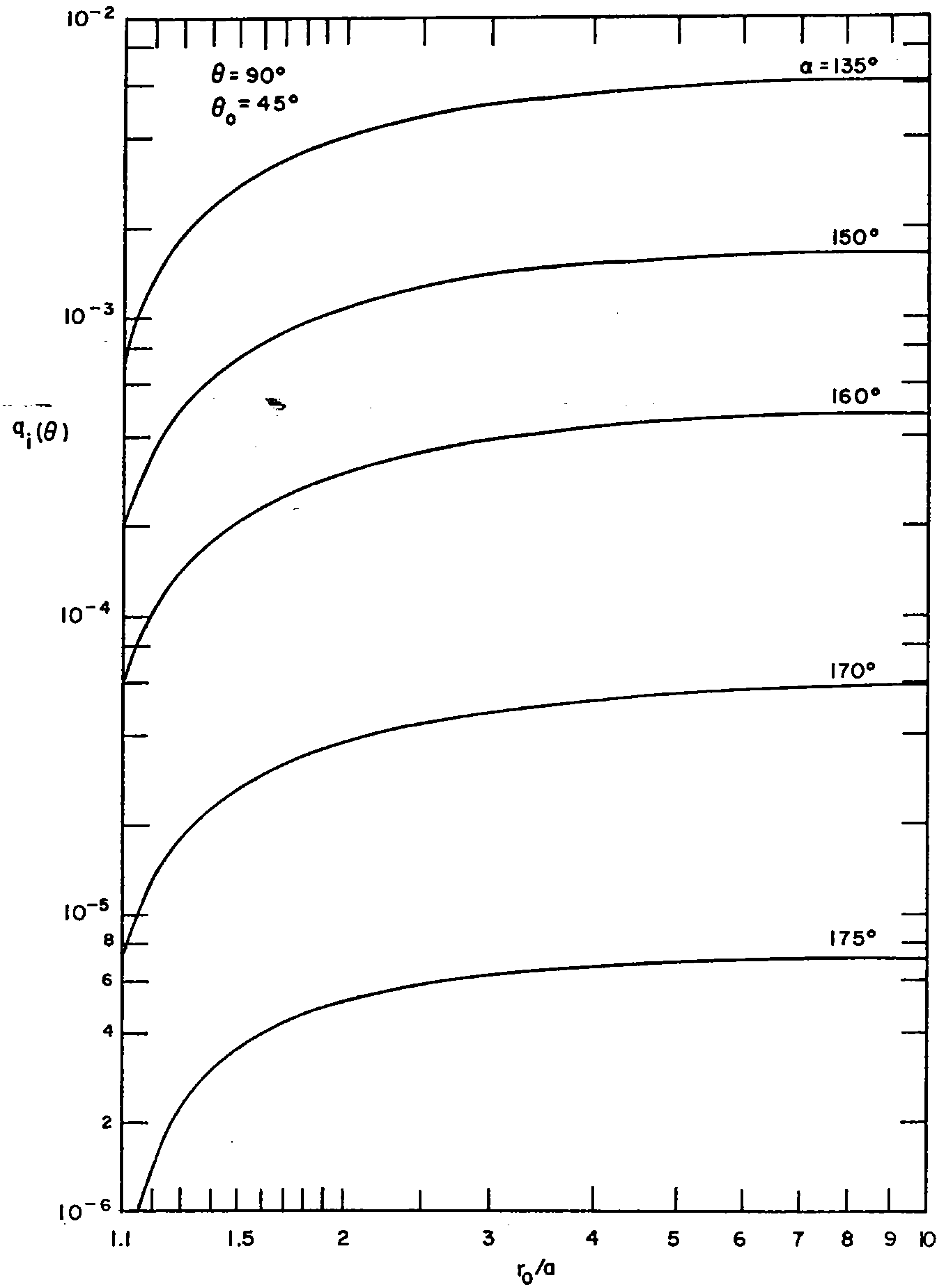

$\theta = 90°$
$\theta_0 = 45°$
$\alpha = 135°$
150°
160°
170°
175°
10^{-2}
10^{-3}
10^{-4}
10^{-5}
8
6
4
2
10^{-6}
$q_i(\theta)$
1.1
1.5
2
3
4
5
6
7
8
9
10
r_0/a

Figure 7

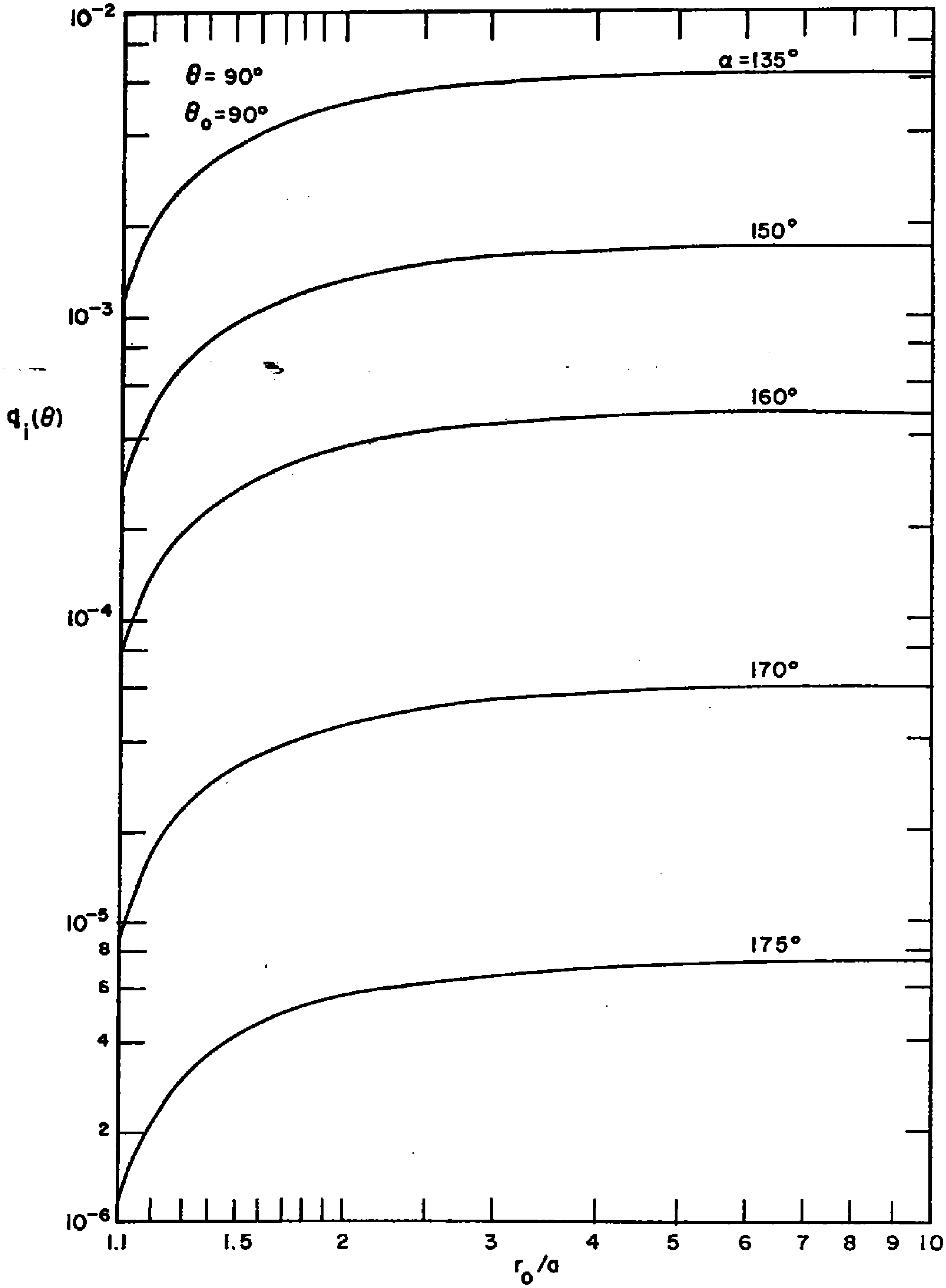

θ = 90°
θ₀ = 90°
α = 135°
150°
160°
170°
175°
$q_i(\theta)$
10^{-2}
10^{-3}
10^{-4}
10^{-5}
8
6
4
2
10^{-6}
1.1
1.5
2
3
4
5
6
7
8
9
10
r_0/a

Figure 8

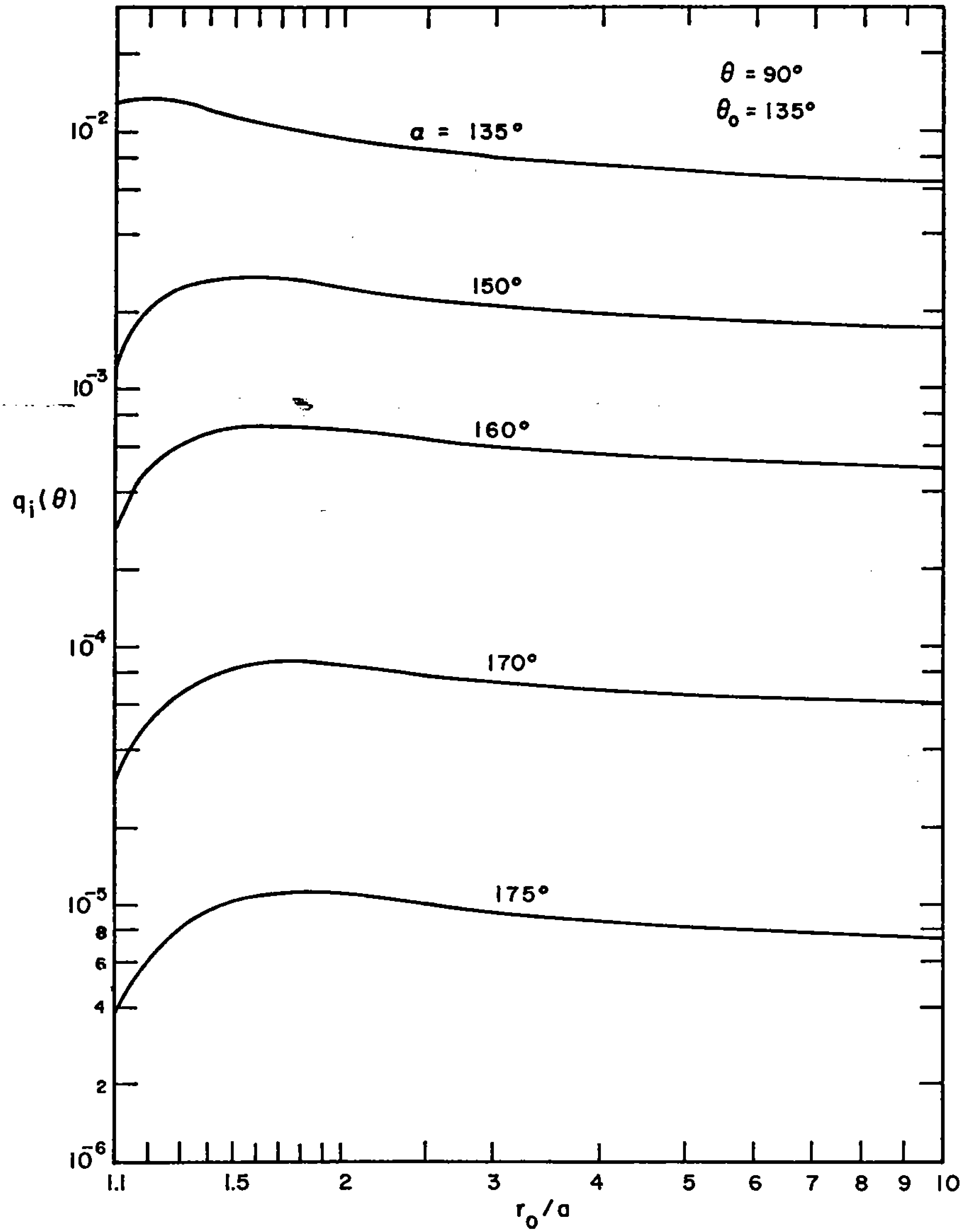
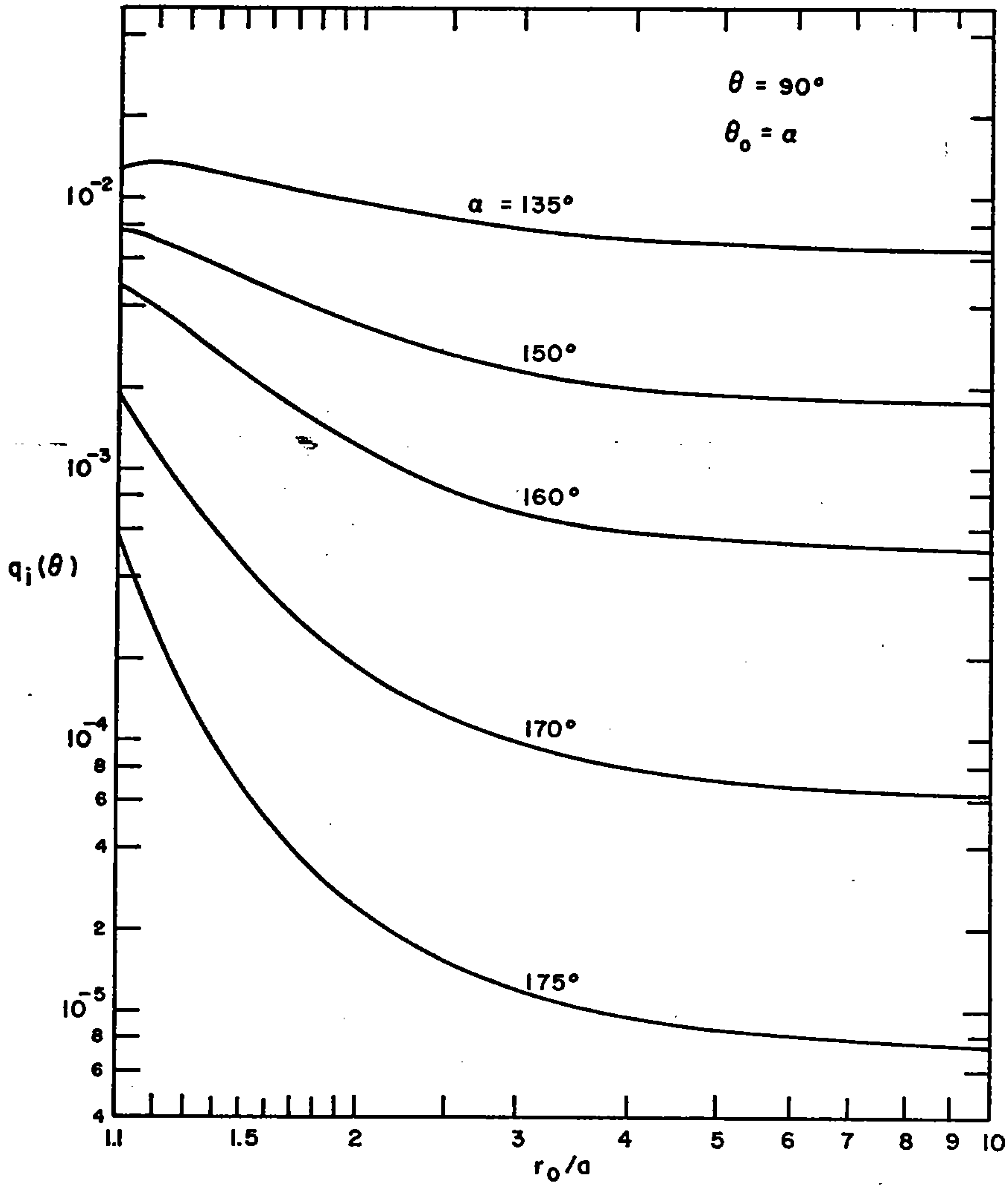

θ = 90°
θ₀ = 135°
α = 135°
150°
160°
170°
175°
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
8
6
4
2
$q_i(\theta)$
1.1
1.5
2
3
4
5
6
7
8
9
10
r_0/a

Figure 9



θ = 90°
θ₀ = α
α = 135°
150°
160°
170°
175°
10^{-2}
10^{-3}
10^{-4}
8
6
4
2
10^{-5}
8
6
4
$q_i(\theta)$
1.1
1.5
2
3
4
5
6
7
8
9
10
r_0/a

Figure 10

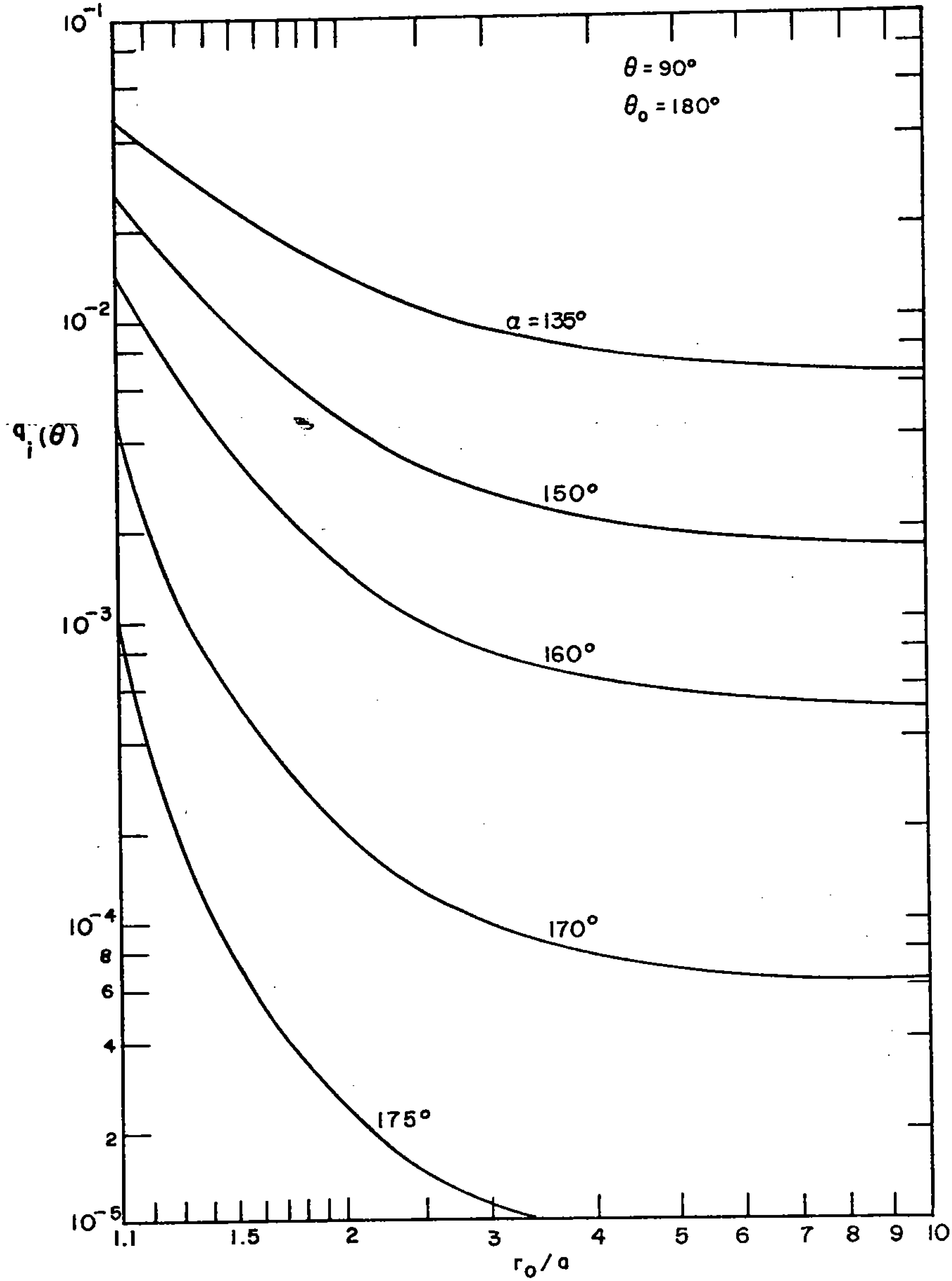
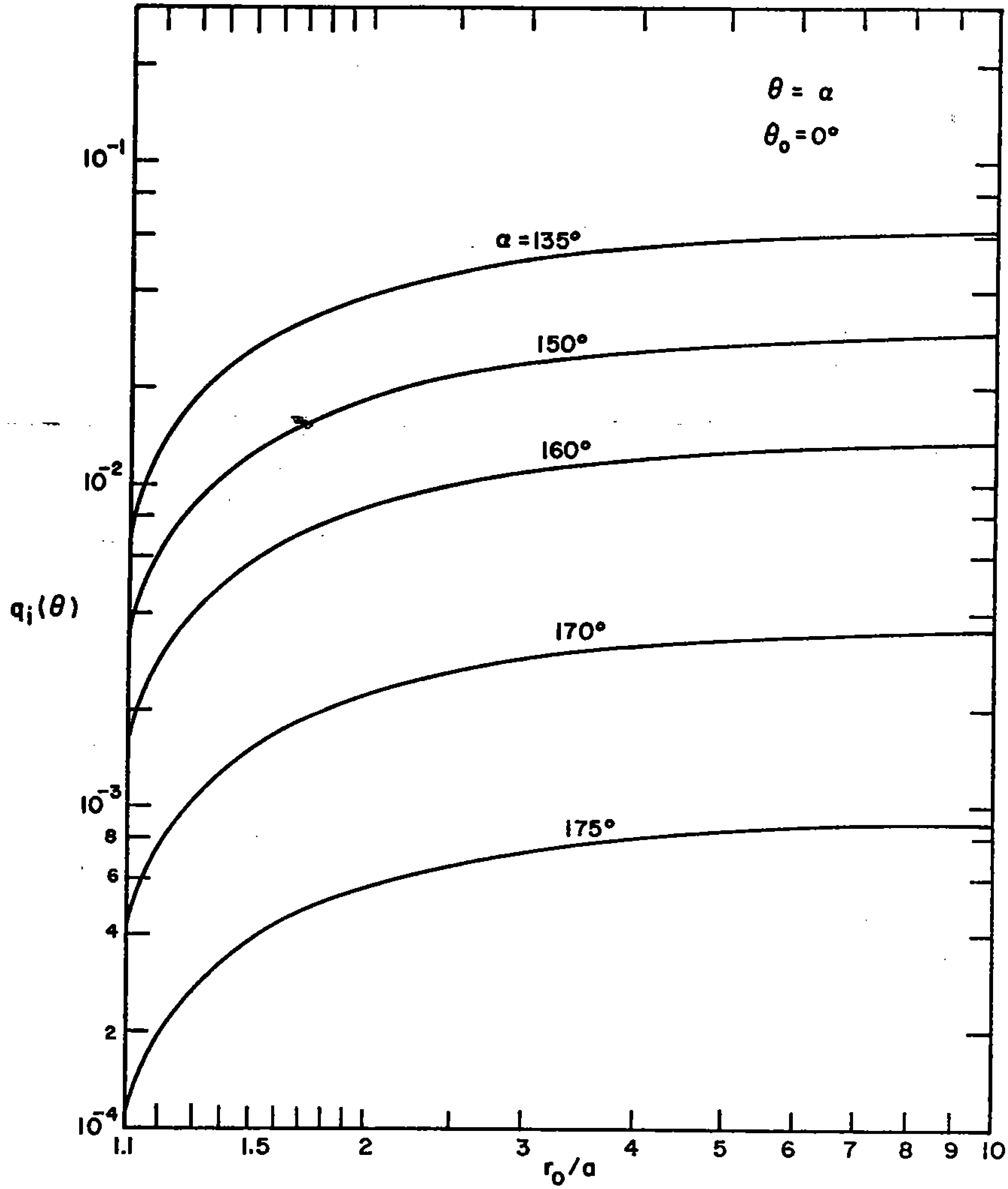

θ = 90°
θ₀ = 180°
α = 135°
150°
160°
170°
175°
q_i(θ)
10⁻¹
10⁻²
10⁻³
10⁻⁴
8
6
4
2
10⁻⁵
1.1
1.5
2
3
4
5
6
7
8
9
10
r₀/a

Figure 11



θ = α
θ₀ = 0°
α = 135°
150°
160°
170°
175°
10⁻¹
10⁻²
10⁻³
8
6
4
2
10⁻⁴
q_i(θ)
1.1
1.5
2
3
4
5
6
7
8
9
10
r₀/a

Figure 12

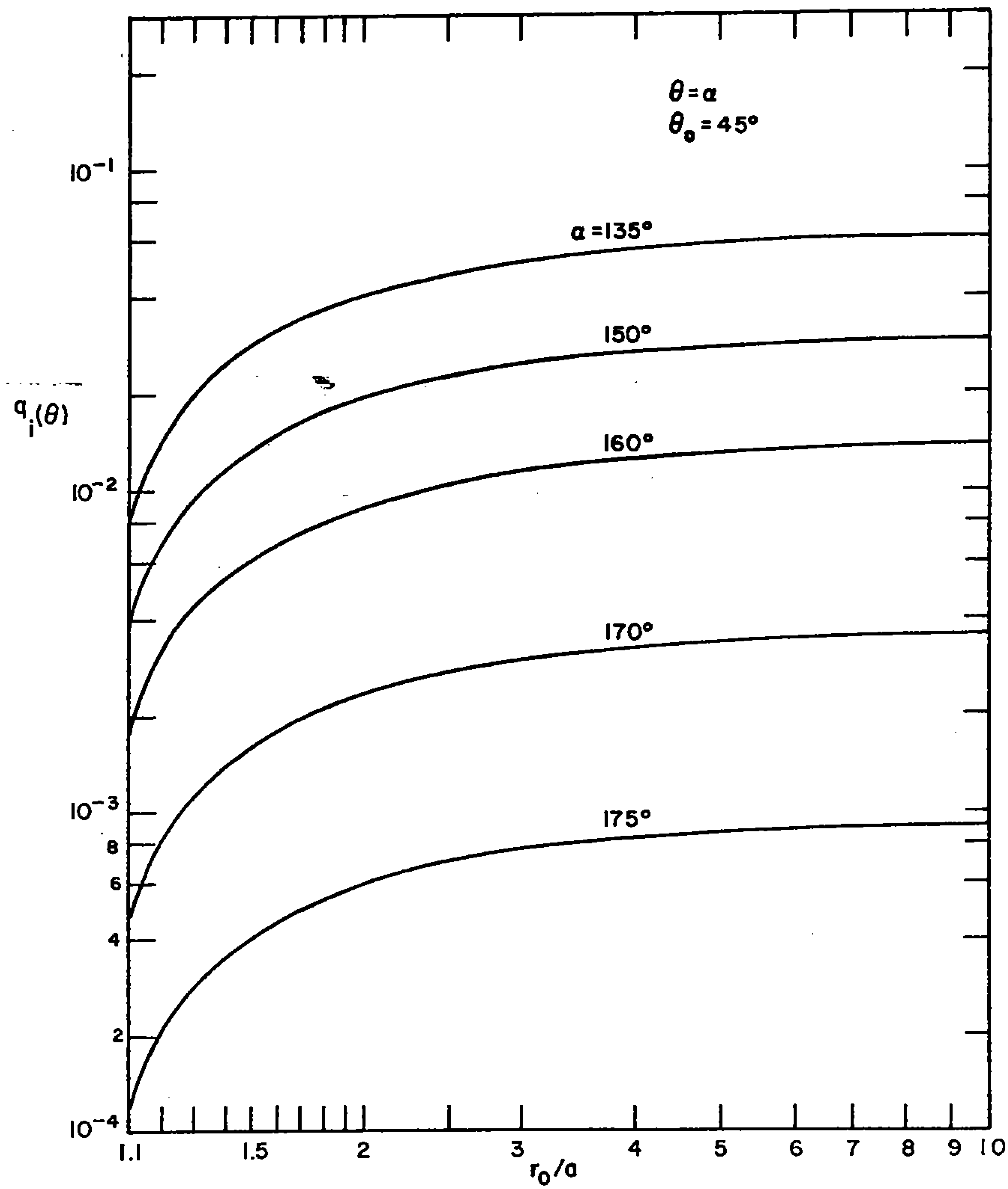

θ=α
θ₀=45°
α=135°
150°
160°
170°
175°
$q_i(\theta)$
10⁻¹
10⁻²
10⁻³
8
6
4
2
10⁻⁴
1.1
1.5
2
3
4
5
6
7
8
9
10
r_0/a

Figure 13

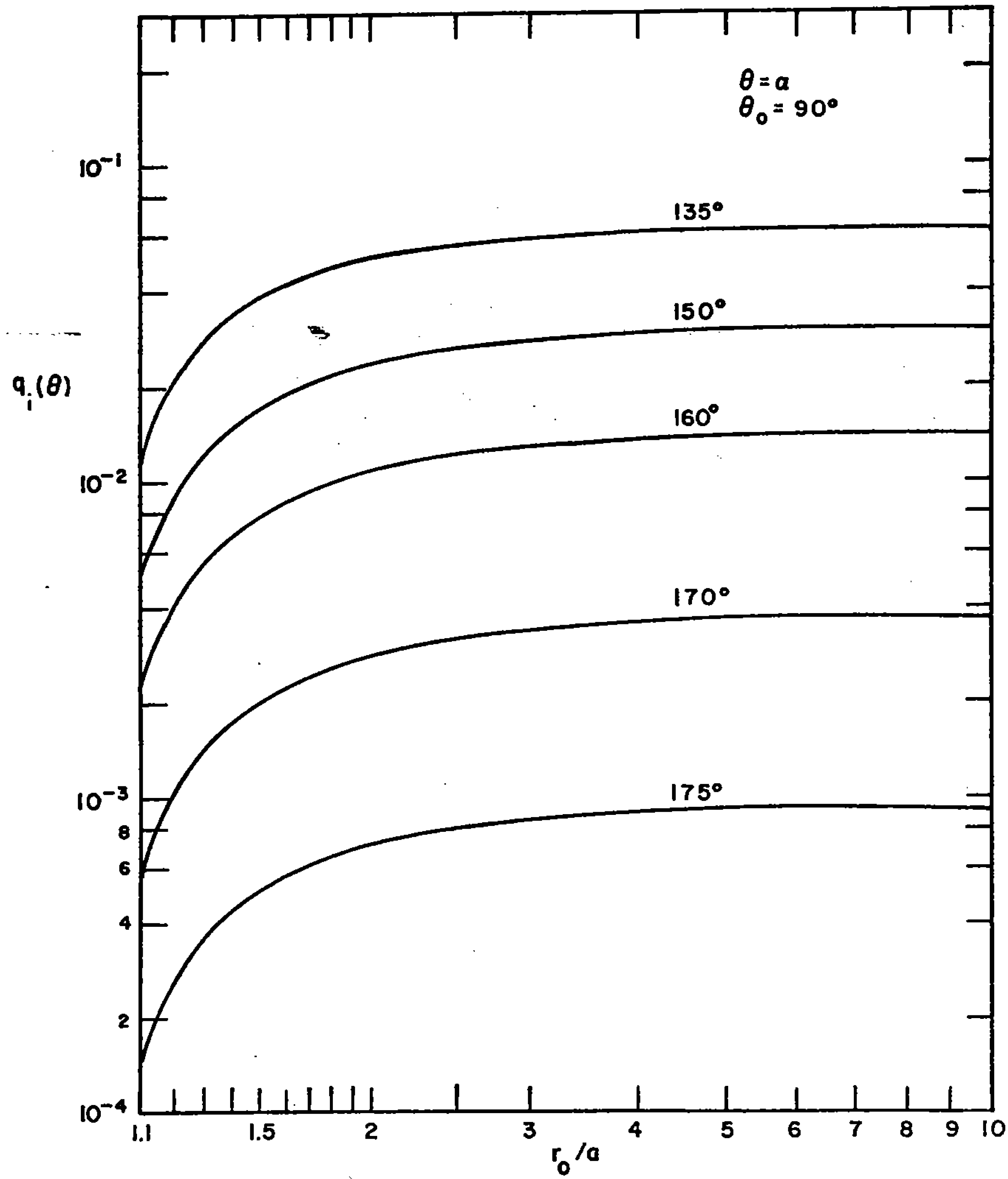

θ = a
θ₀ = 90°
135°
150°
160°
170°
175°
q_i(θ)
10⁻¹
10⁻²
10⁻³
8
6
4
2
10⁻⁴
1.1
1.5
2
3
4
5
6
7
8
9
10
r₀/a

Figure 14

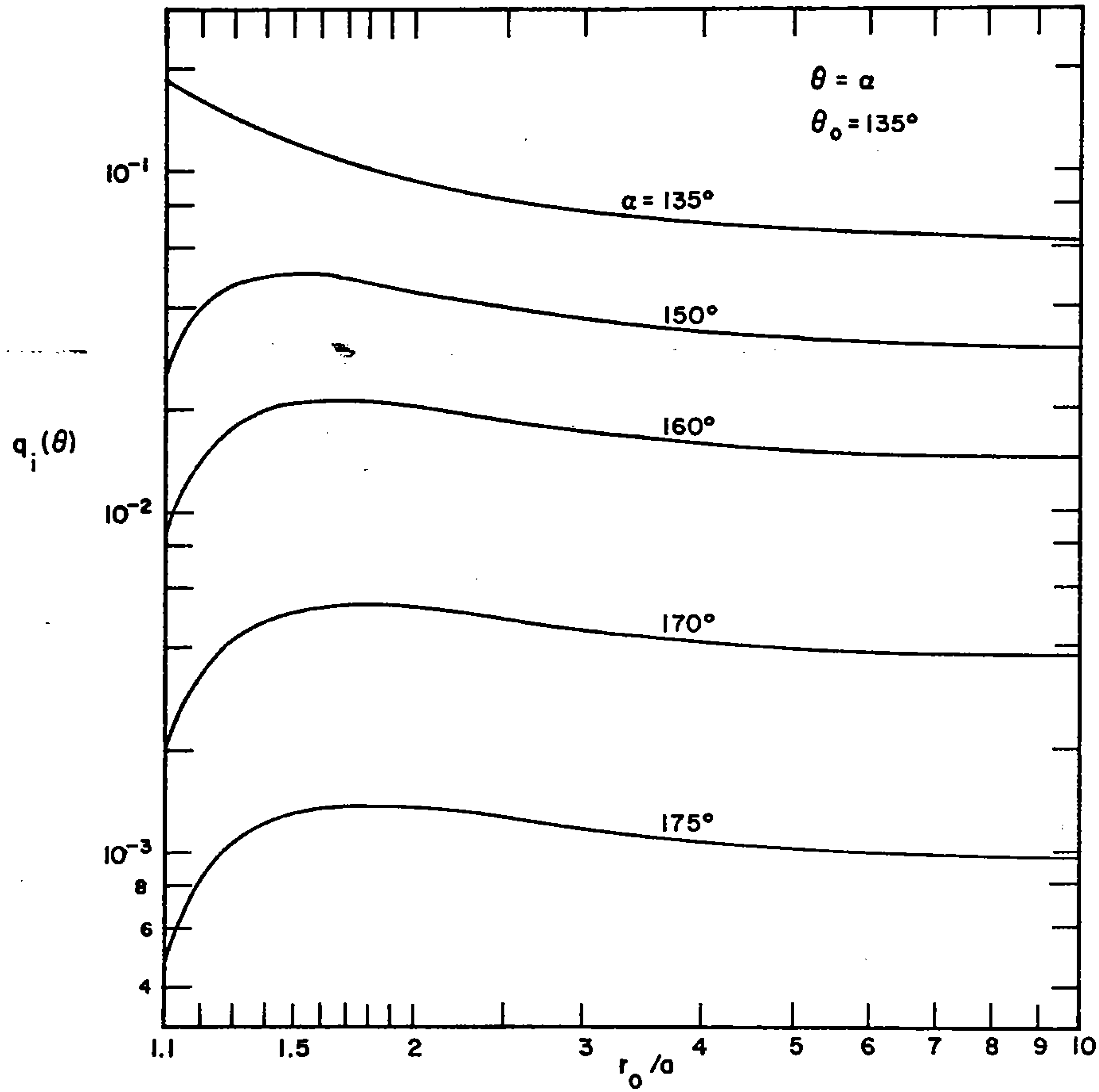

θ = α
θ₀ = 135°
α = 135°
150°
160°
170°
175°
q_i(θ)
10⁻¹
10⁻²
10⁻³
8
6
4
1.1
1.5
2
3
4
5
6
7
8
9
10
r₀/a

Figure 15

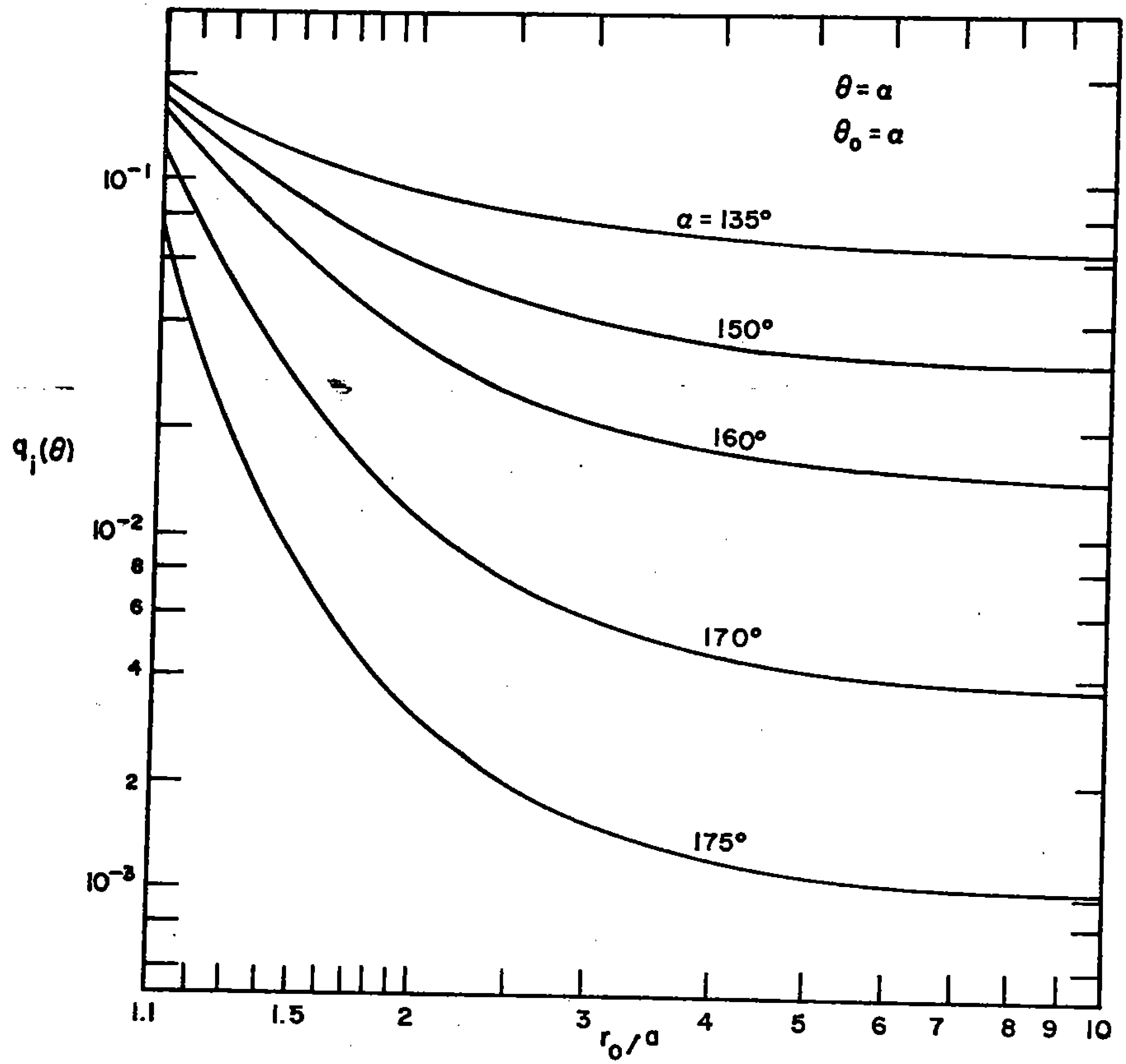

θ = α
θ₀ = α
α = 135°
150°
160°
170°
175°
$q_j(\theta)$
10⁻¹
10⁻²
8
6
4
2
10⁻³
1.1
1.5
2
3
4
5
6
7
8
9
10
r_0/a

Figure 16

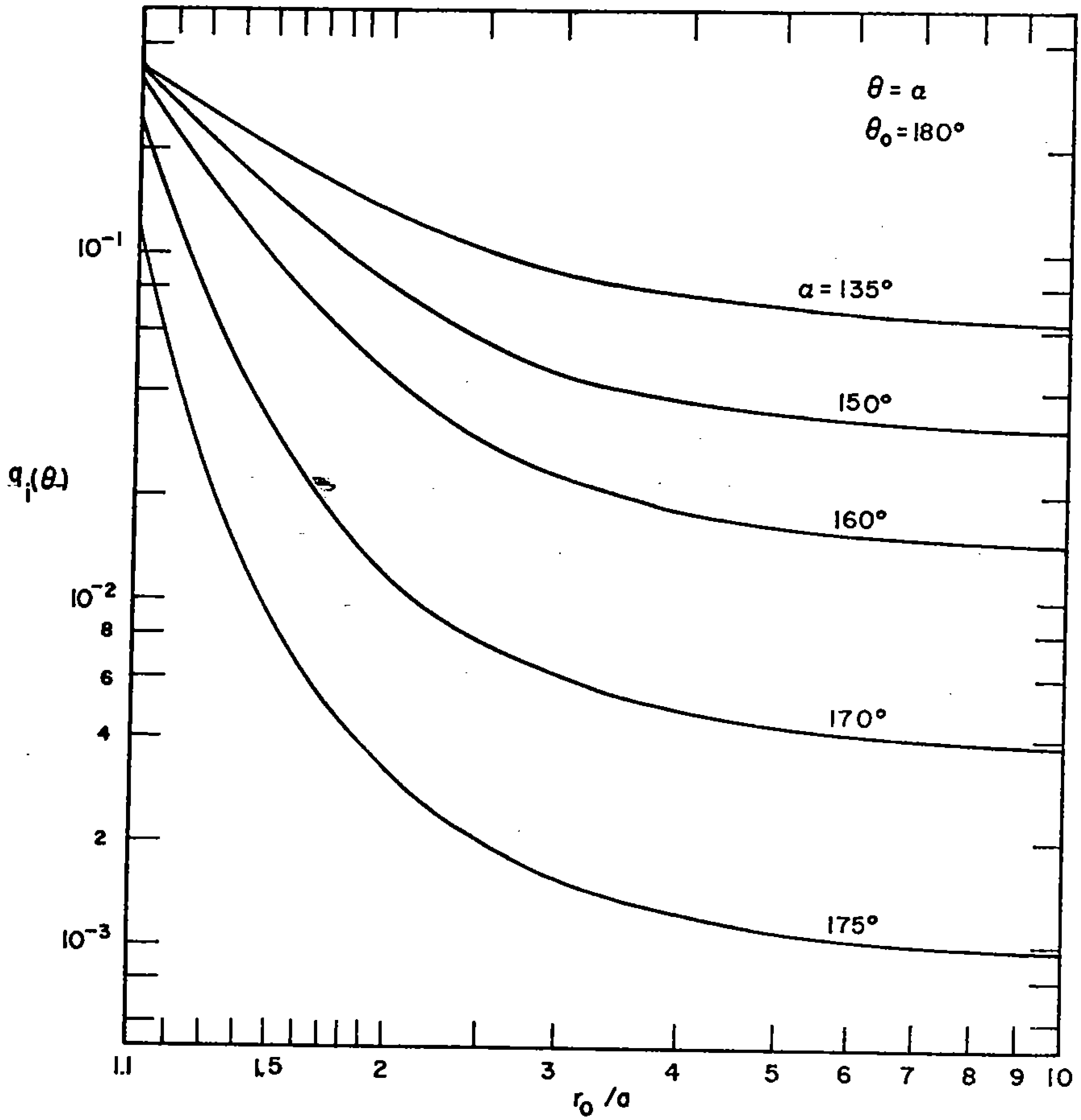

θ = α
θ₀ = 180°
α = 135°
150°
160°
170°
175°
10⁻¹
10⁻²
8
6
4
2
10⁻³
q_i(θ)
1.1
1.5
2
3
4
5
6
7
8
9
10
r₀ /a

Figure 17

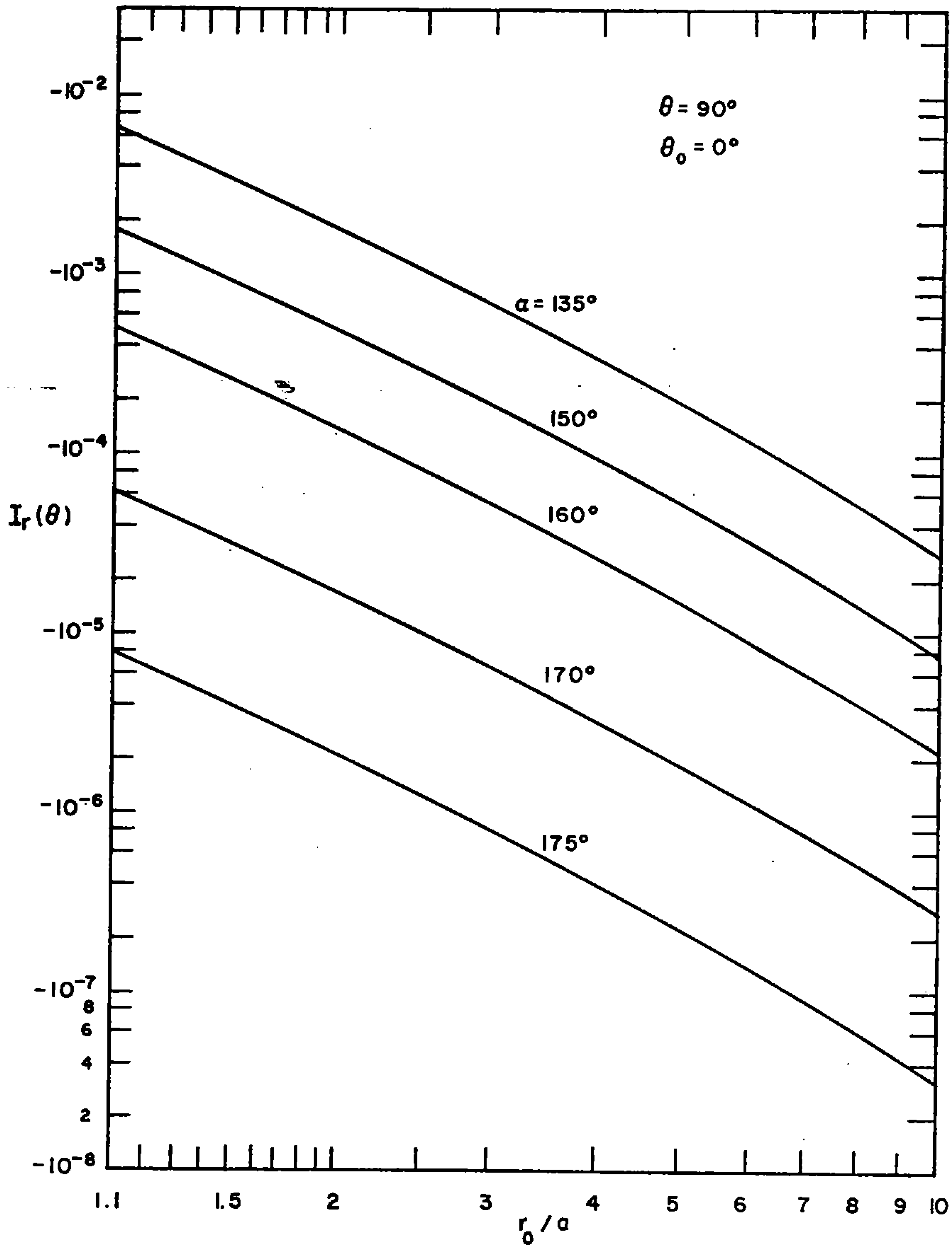
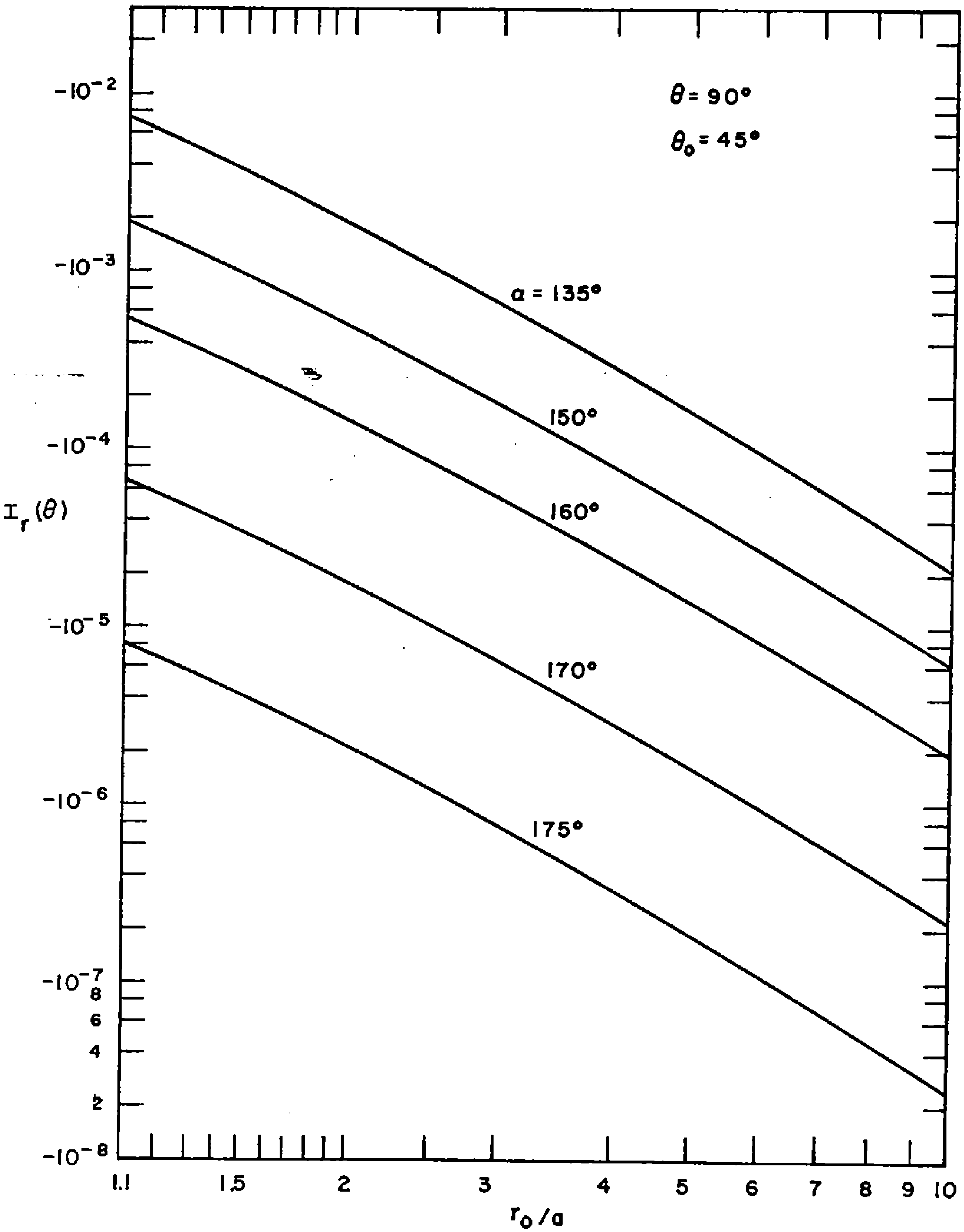

θ = 90°
θ₀ = 0°
α = 135°
150°
160°
170°
175°
-10^-2
-10^-3
-10^-4
-10^-5
-10^-6
-10^-7
8
6
4
2
-10^-8
I_r(θ)
1.1
1.5
2
3
4
5
6
7
8
9
10
r₀ / a

Figure 18



θ = 90°
θ₀ = 45°
α = 135°
150°
160°
170°
175°
$I_r(\theta)$
-10⁻²
-10⁻³
-10⁻⁴
-10⁻⁵
-10⁻⁶
-10⁻⁷
8
6
4
2
-10⁻⁸
1.1
1.5
2
3
4
5
6
7
8
9
10
r_0/a

Figure 19

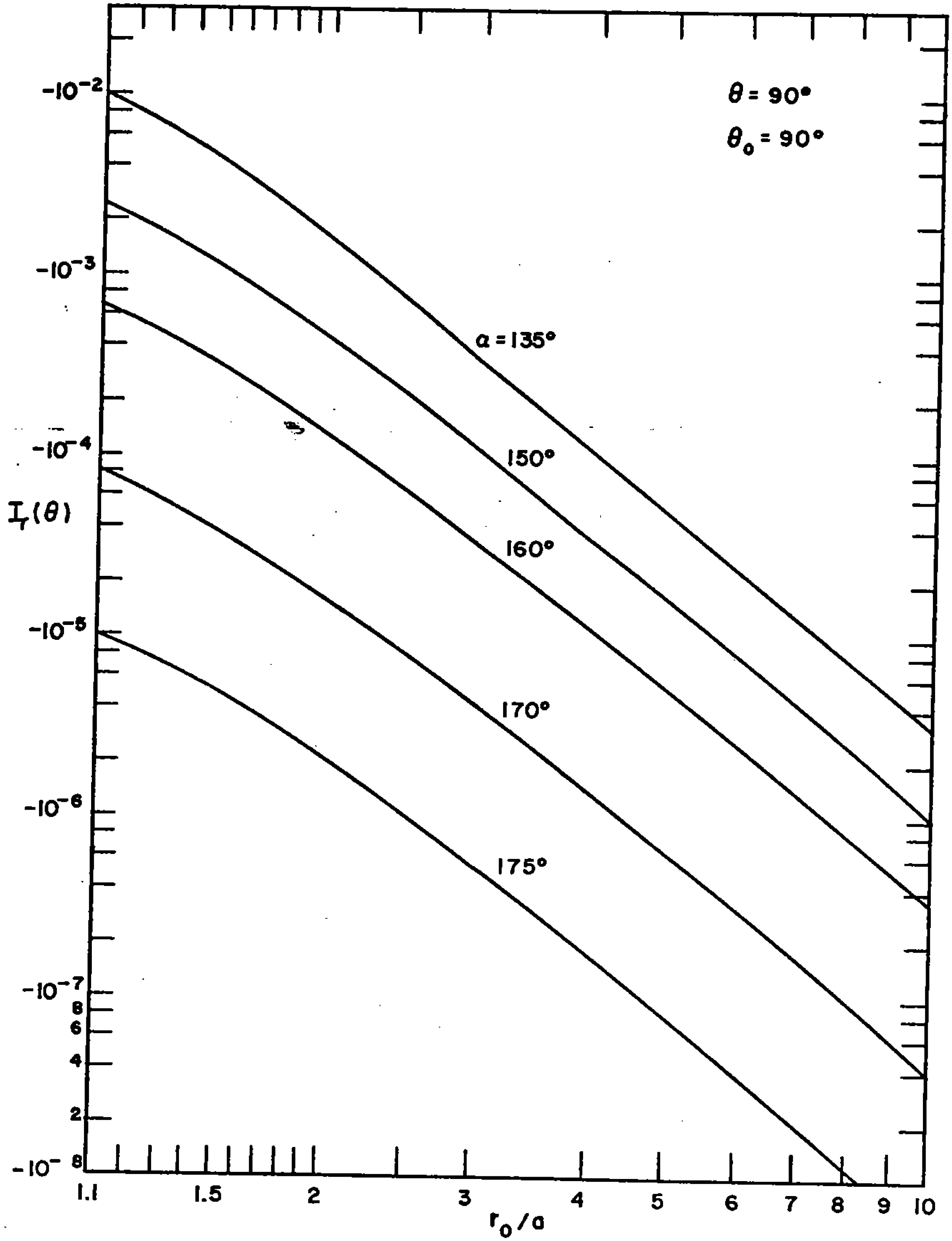
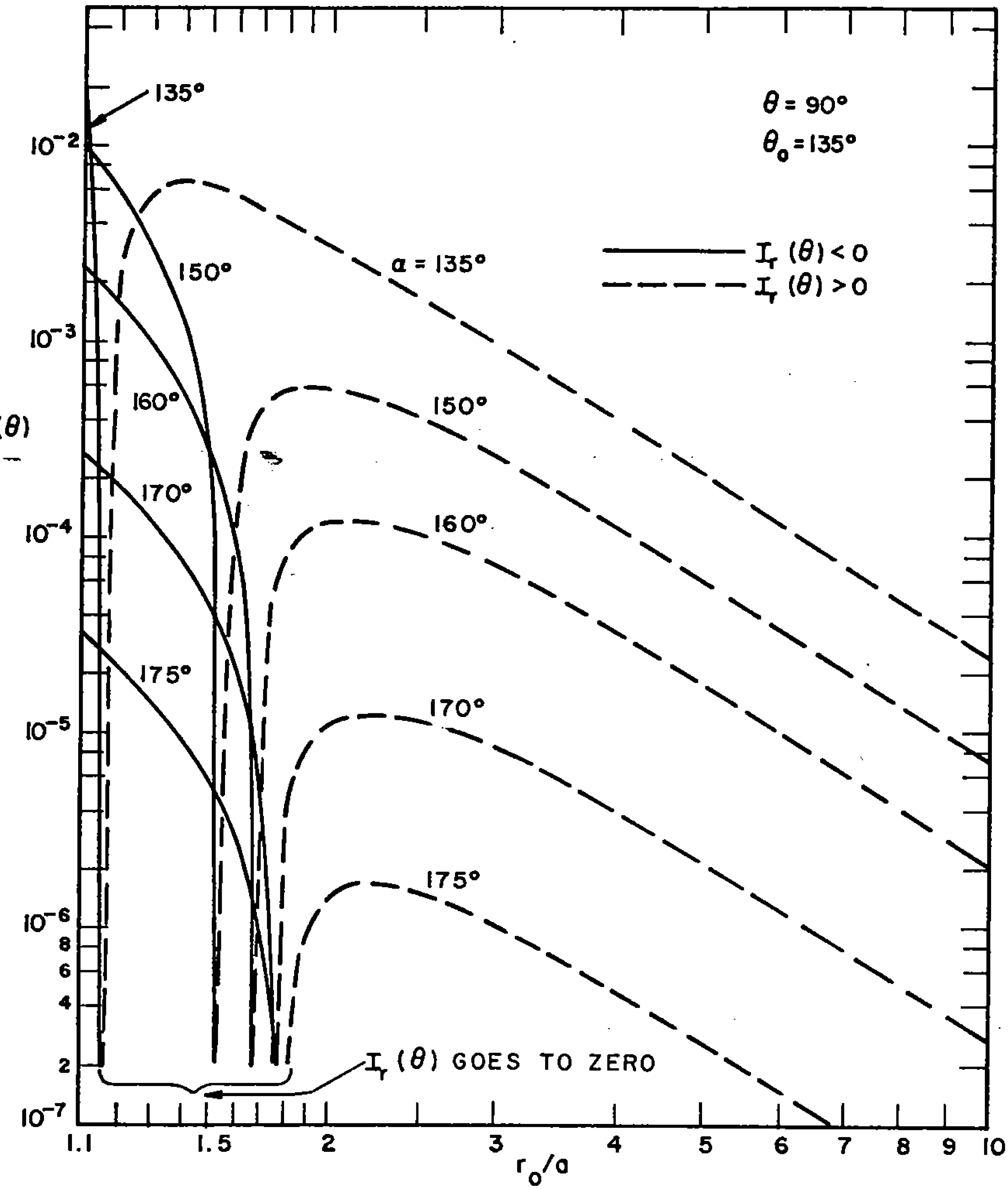
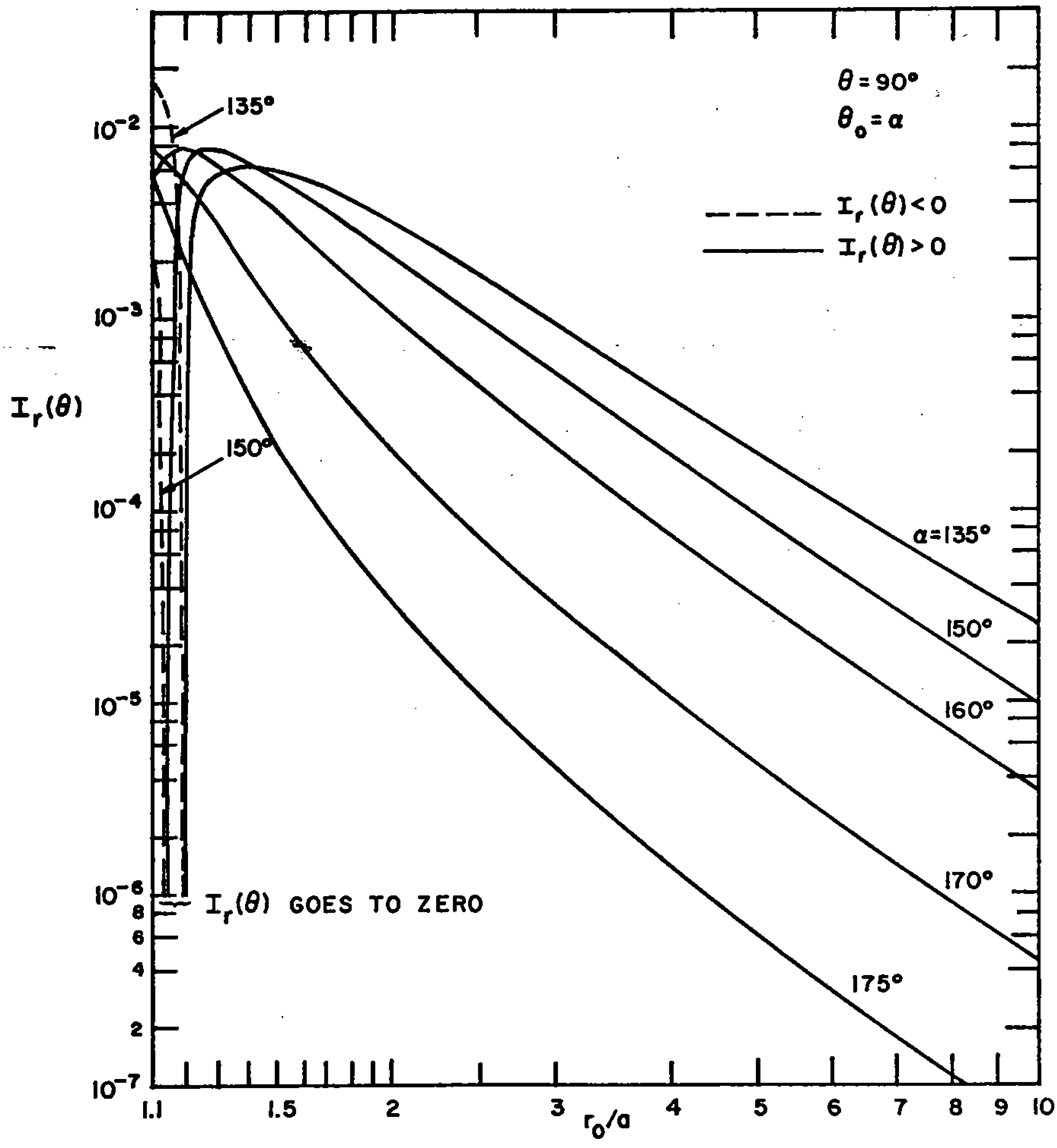

θ = 90°
θ₀ = 90°
α = 135°
150°
160°
170°
175°
-10⁻²
-10⁻³
-10⁻⁴
-10⁻⁵
-10⁻⁶
-10⁻⁷
8
6
4
2
-10⁻⁸
I_r(θ)
1.1
1.5
2
3
4
5
6
7
8
9
10
r₀/a

Figure 20



135°
θ = 90°
θ₀ = 135°
10⁻²
150°
α = 135°
I_r (θ) < 0
I_r (θ) > 0
10⁻³
160°
150°
I_r(θ)
170°
10⁻⁴
160°
175°
10⁻⁵
170°
175°
10⁻⁶
8
6
4
2
I_r (θ) GOES TO ZERO
10⁻⁷
1.1
1.5
2
3
4
5
6
7
8
9
10
r₀/a

Figure 21



θ = 90°
θ₀ = α
$I_r(\theta) < 0$
$I_r(\theta) > 0$
135°
150°
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
8
6
4
2
$I_r(\theta)$
$I_r(\theta)$ GOES TO ZERO
α=135°
150°
160°
170°
175°
1.1
1.5
2
3
4
5
6
7
8
9
10
r_0/a

Figure 22

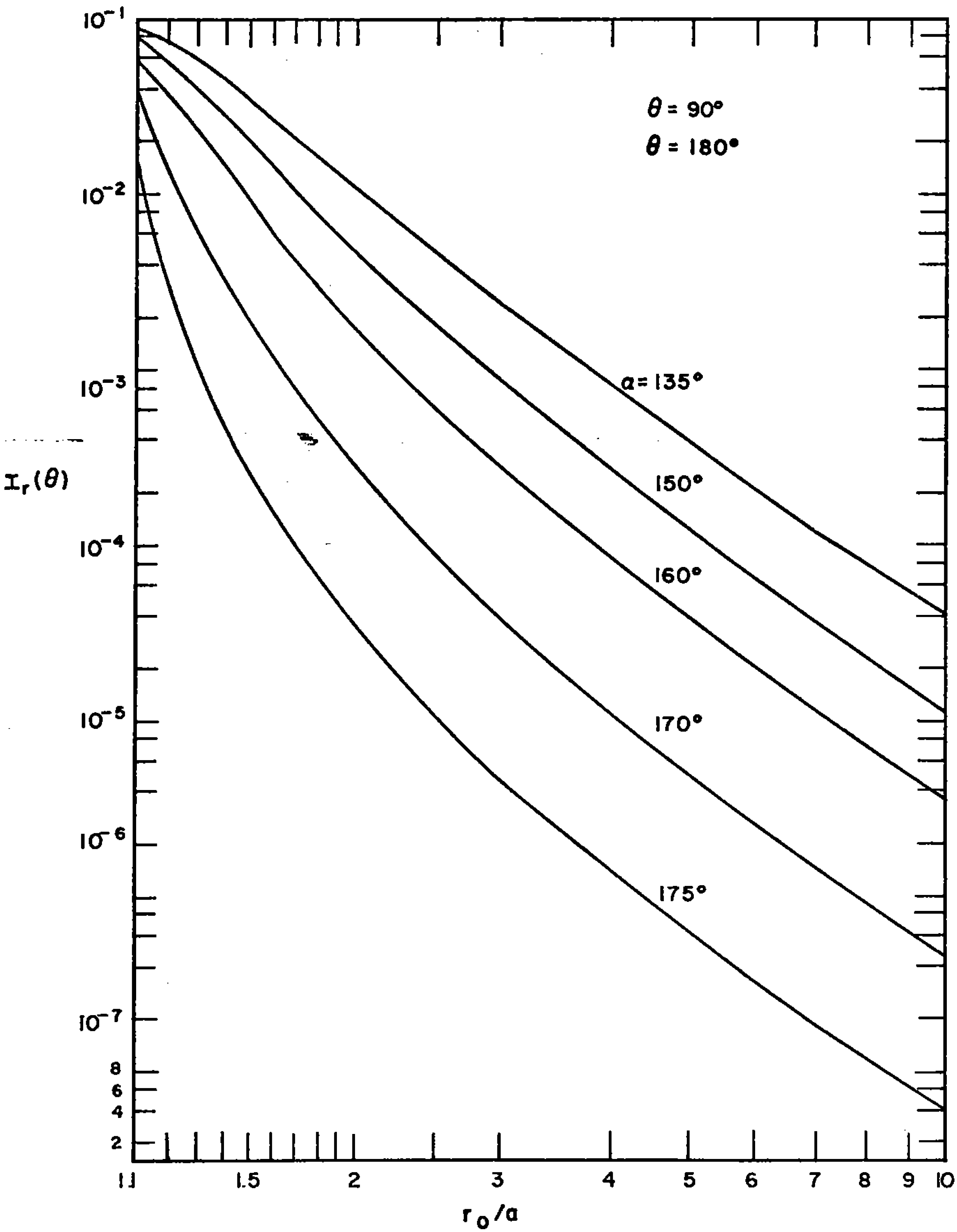

θ = 90°
θ = 180°
α = 135°
150°
160°
170°
175°
10⁻¹
10⁻²
10⁻³
10⁻⁴
10⁻⁵
10⁻⁶
10⁻⁷
8
6
4
2
$I_r(\theta)$
1.1
1.5
2
3
4
5
6
7
8
9
10
r_0/a

Figure 23

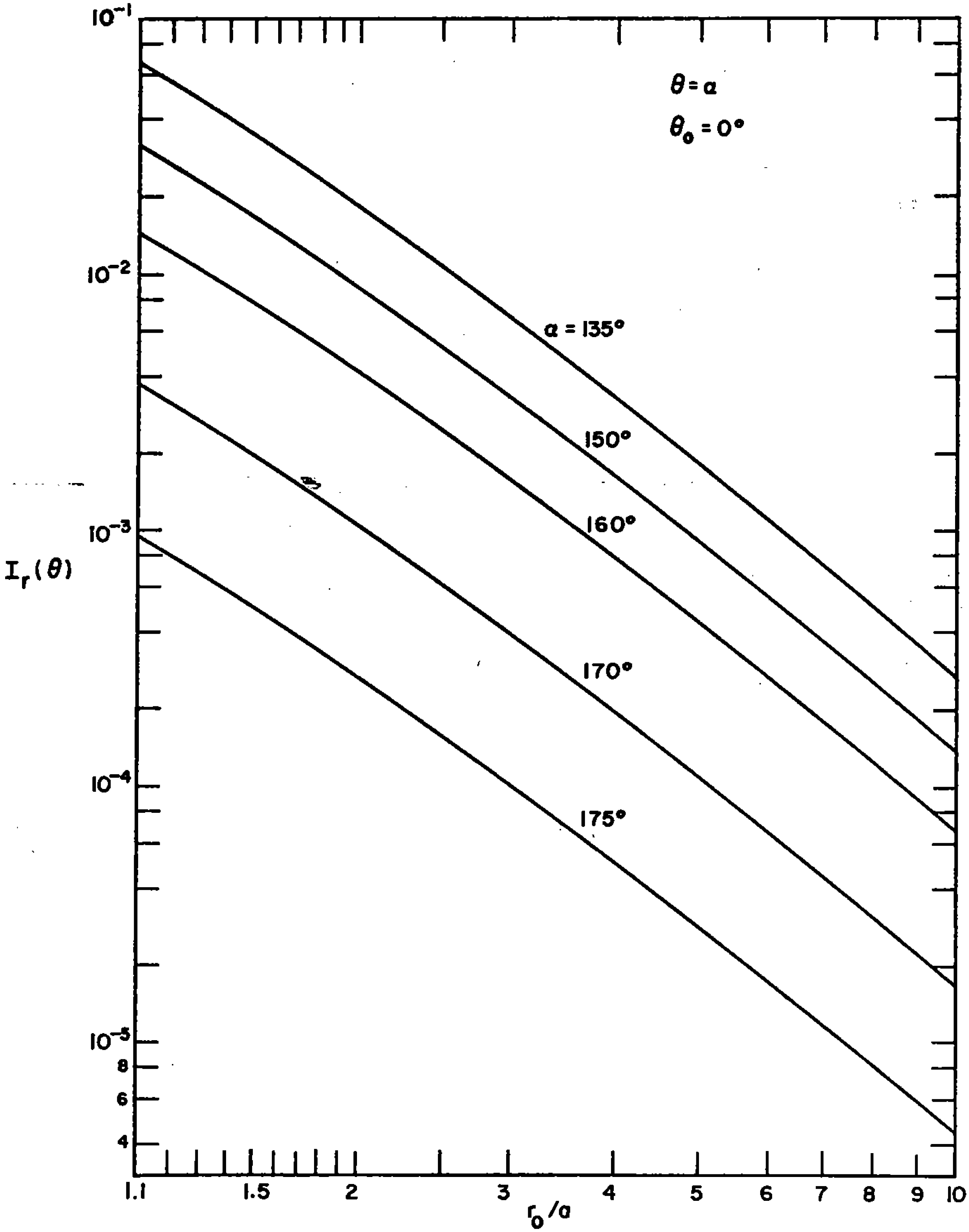
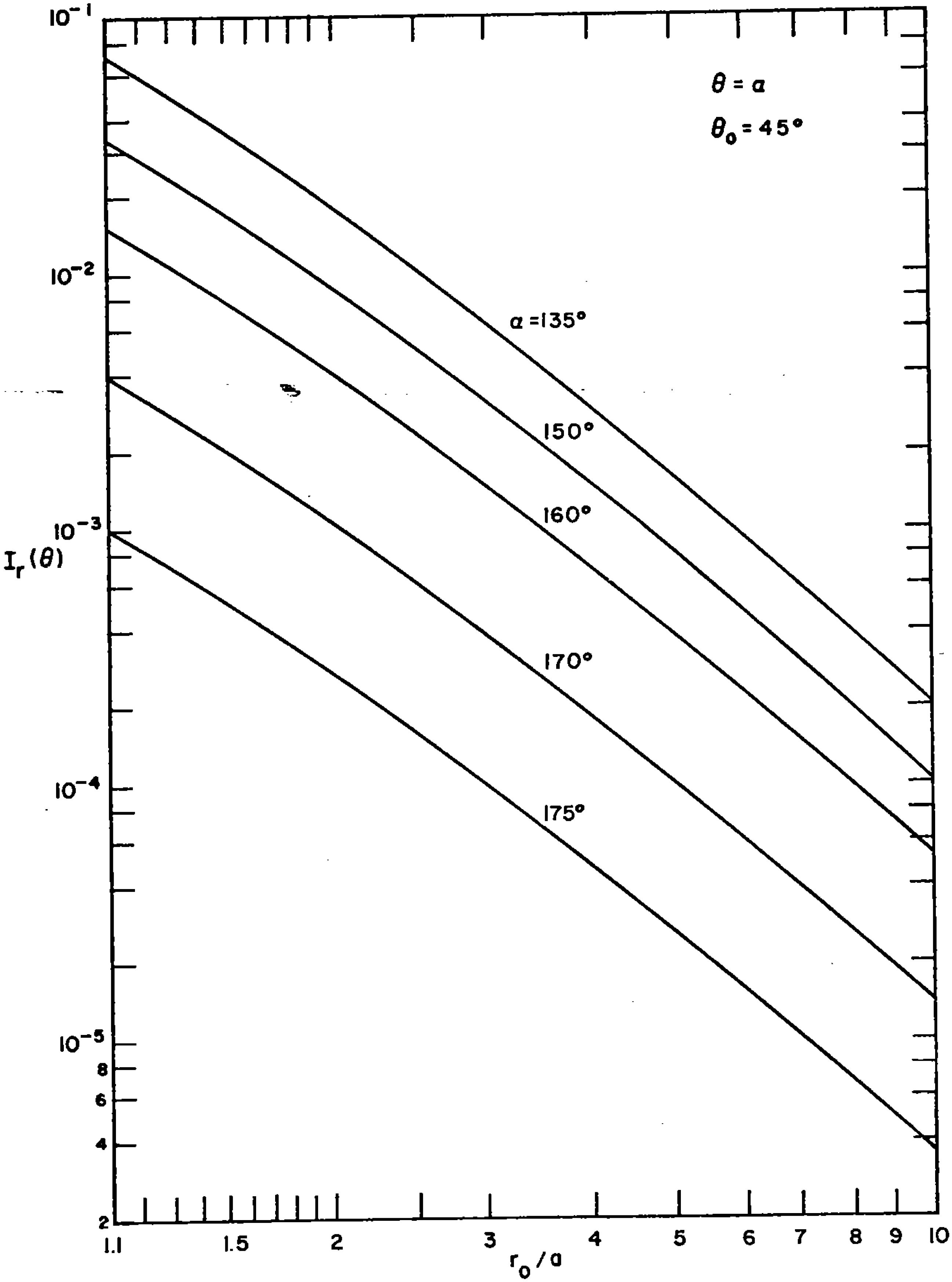
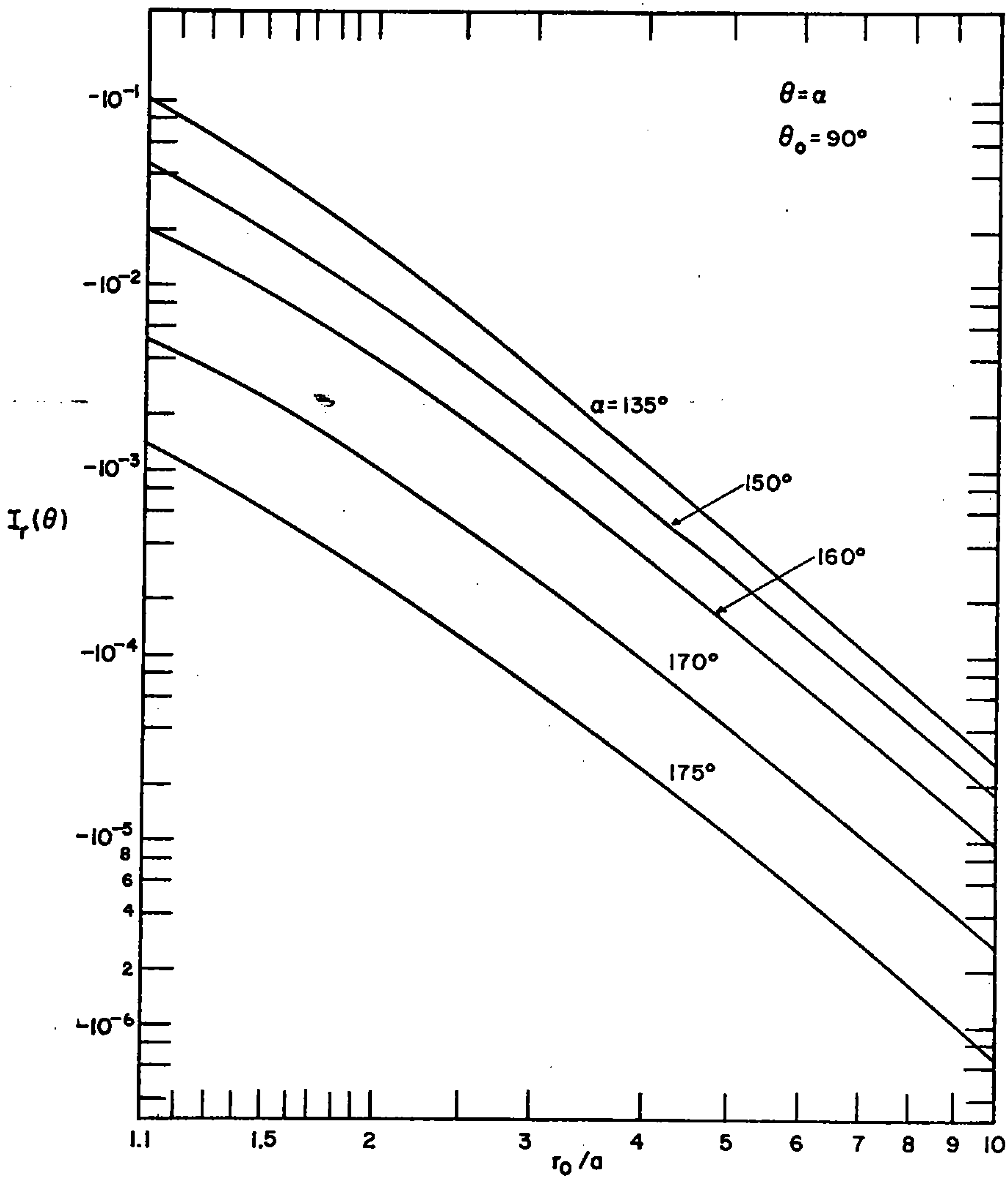

θ = α
θ₀ = 0°
α = 135°
150°
160°
170°
175°
10⁻¹
10⁻²
10⁻³
10⁻⁴
10⁻⁵
8
6
4
Iᵣ(θ)
1.1
1.5
2
3
4
5
6
7
8
9
10
r₀/a

Figure 24



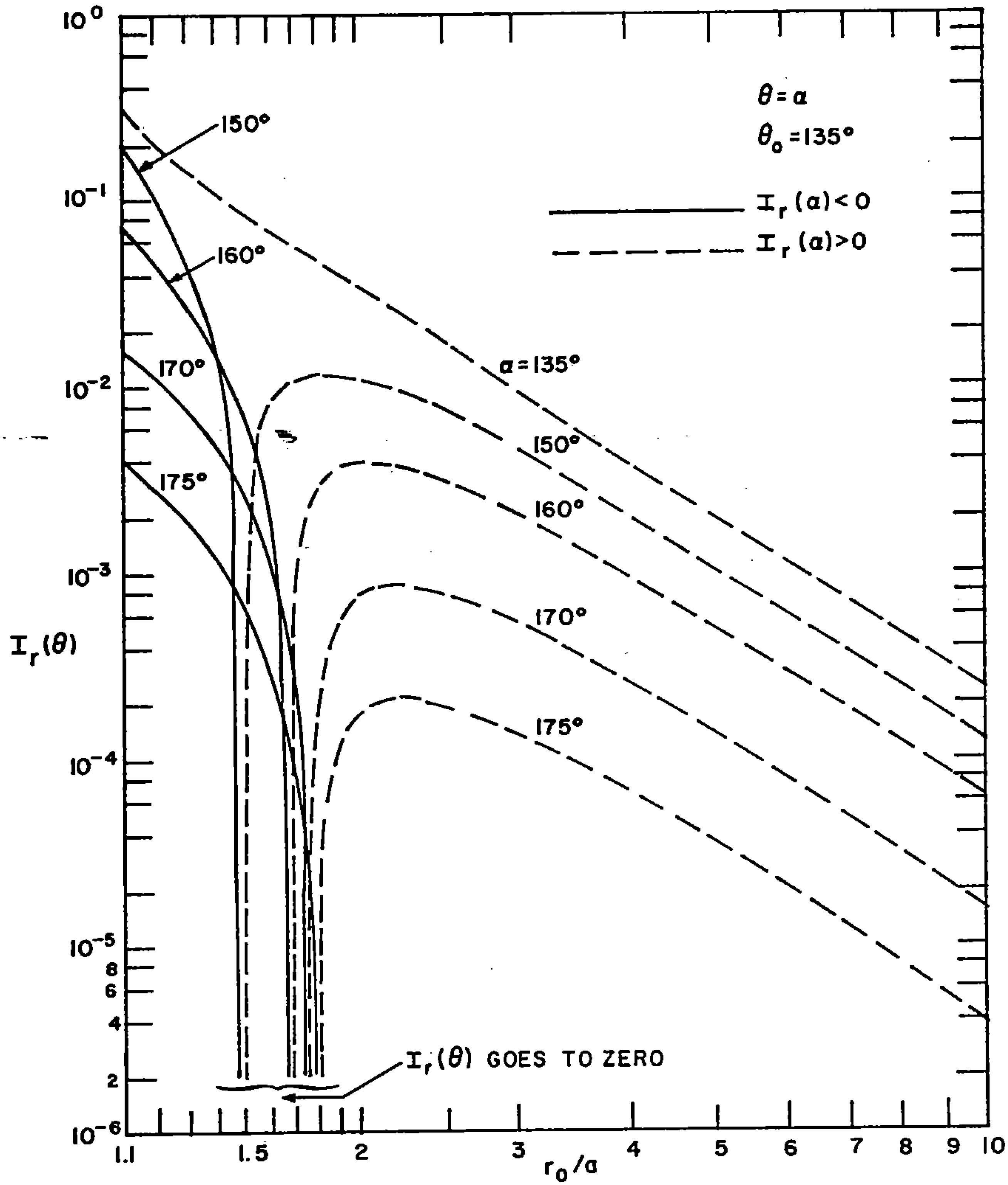
θ = α
θo = 45°
α = 135°
150°
160°
170°
175°
Ir(θ)
10⁻¹
10⁻²
10⁻³
10⁻⁴
10⁻⁵
8
6
4
2
1.1
1.5
2
3
4
5
6
7
8
9
10
ro/a

Figure 25



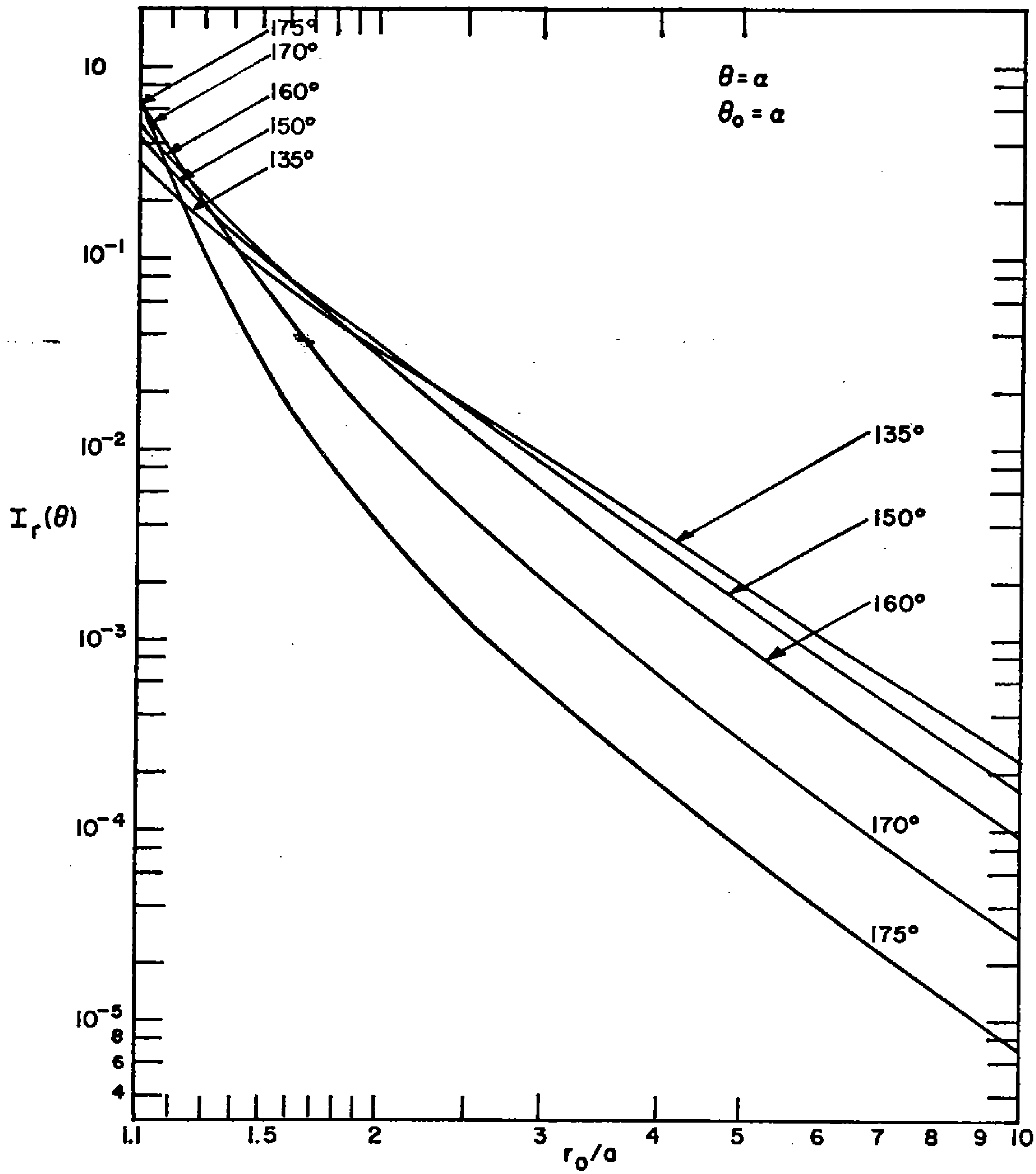
θ=a
θ₀=90°
α=135°
150°
160°
170°
175°
-10⁻¹
-10⁻²
-10⁻³
-10⁻⁴
-10⁻⁵
8
6
4
2
-10⁻⁶
I_r(θ)
1.1
1.5
2
3
4
5
6
7
8
9
10
r₀/a

Figure 26



θ = α
θ₀ = 135°
$I_r(\alpha) < 0$
$I_r(\alpha) > 0$
150°
160°
170°
175°
α = 135°
150°
160°
170°
175°
$I_r(\theta)$
$I_r(\theta)$ GOES TO ZERO
r_0/a

Figure 27



175°
170°
160°
150°
135°
θ = α
θ₀ = α
10
10⁻¹
10⁻²
10⁻³
10⁻⁴
10⁻⁵
8
6
4
Iᵣ(θ)
135°
150°
160°
170°
175°
1.1
1.5
2
3
4
5
6
7
8
9
10
r₀/a

Figure 28

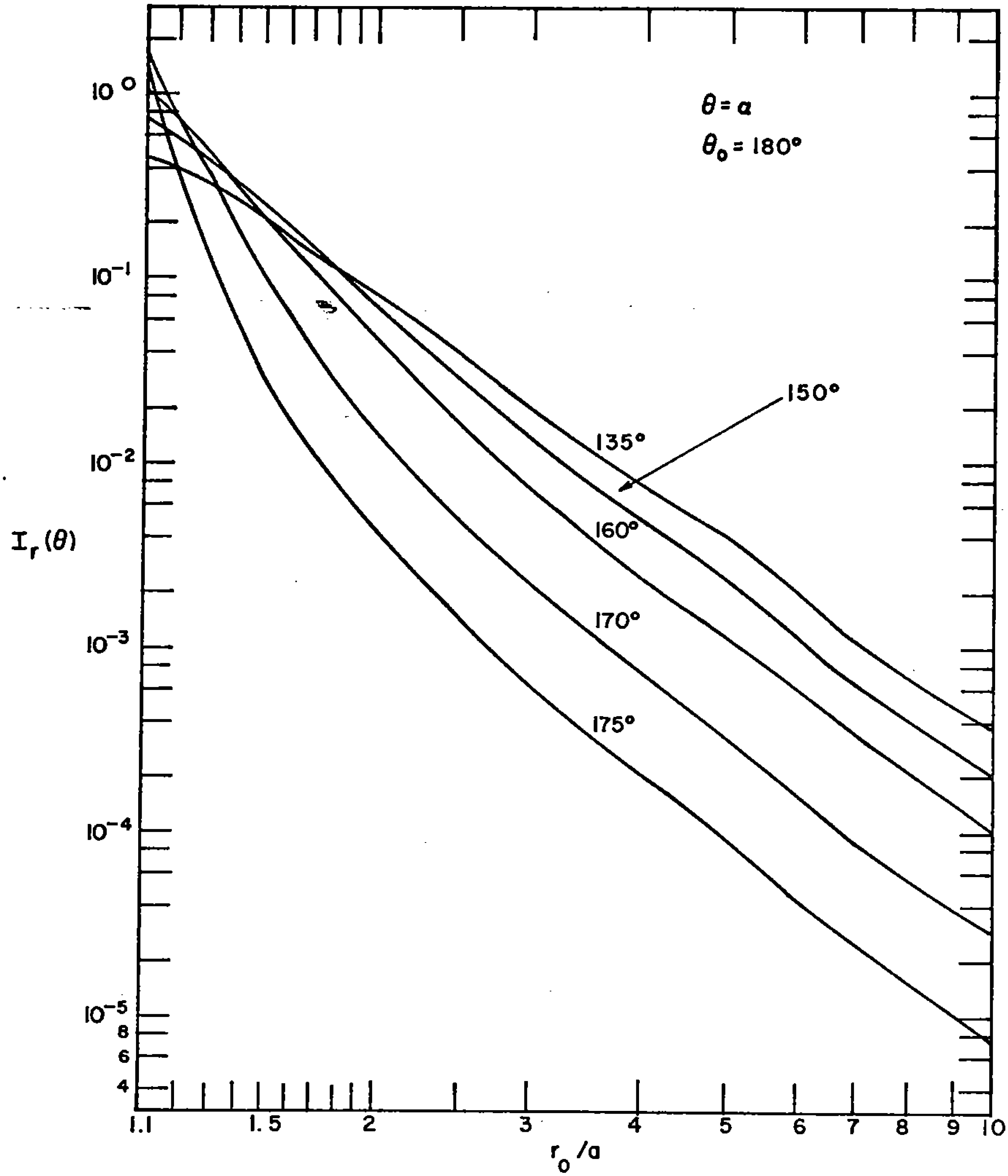
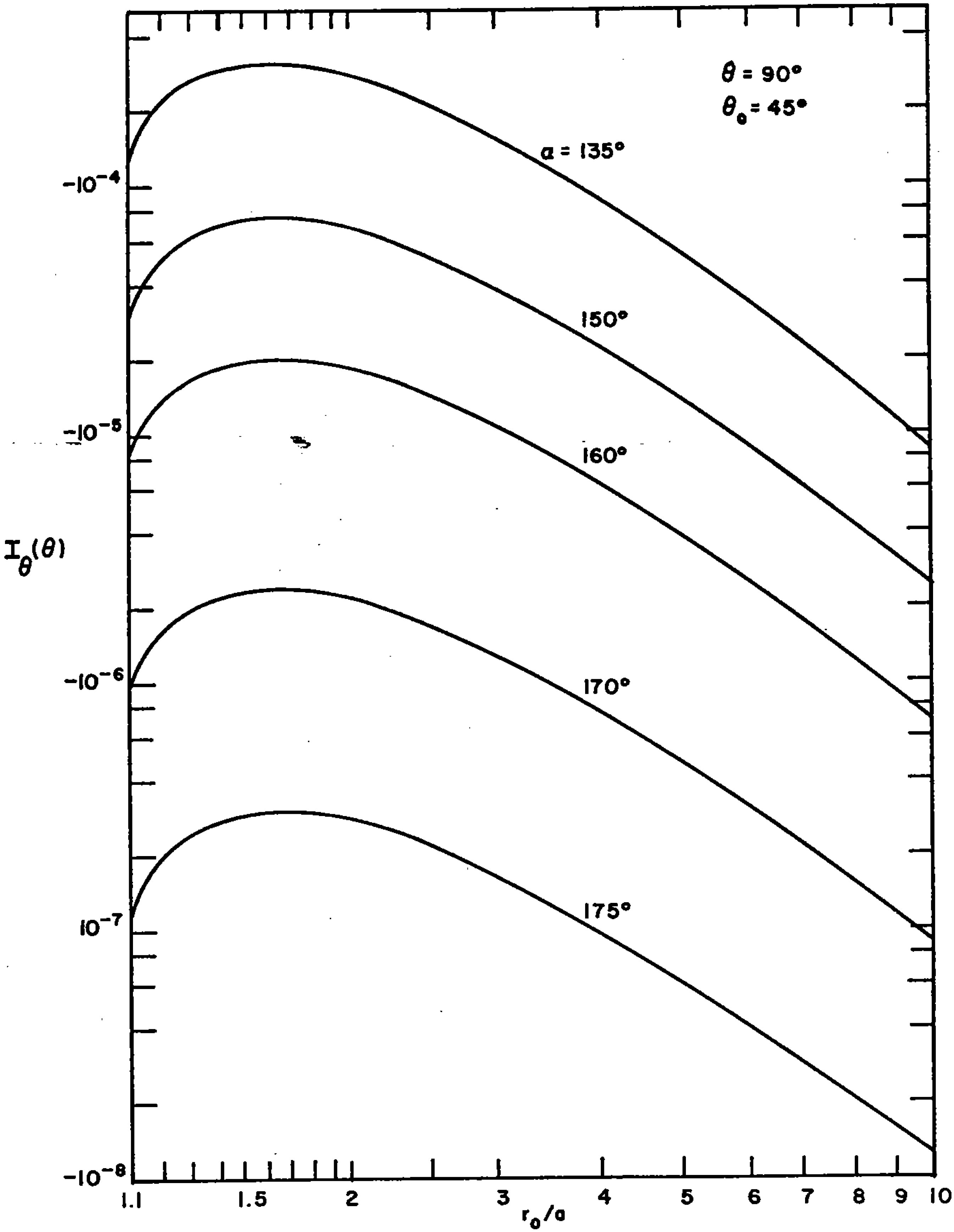

θ = a
θ₀ = 180°
150°
135°
160°
170°
175°
10 0
10⁻¹
10⁻²
10⁻³
10⁻⁴
10⁻⁵
8
6
4
I_r (θ)
1.1
1.5
2
3
4
5
6
7
8
9
10
r₀ /a

Figure 29



θ = 90°
θ₀ = 45°
a = 135°
150°
160°
170°
175°
-10^-4
-10^-5
-10^-6
10^-7
-10^-8
I_θ(θ)
1.1
1.5
2
3
4
5
6
7
8
9
10
r₀/a

Figure 30

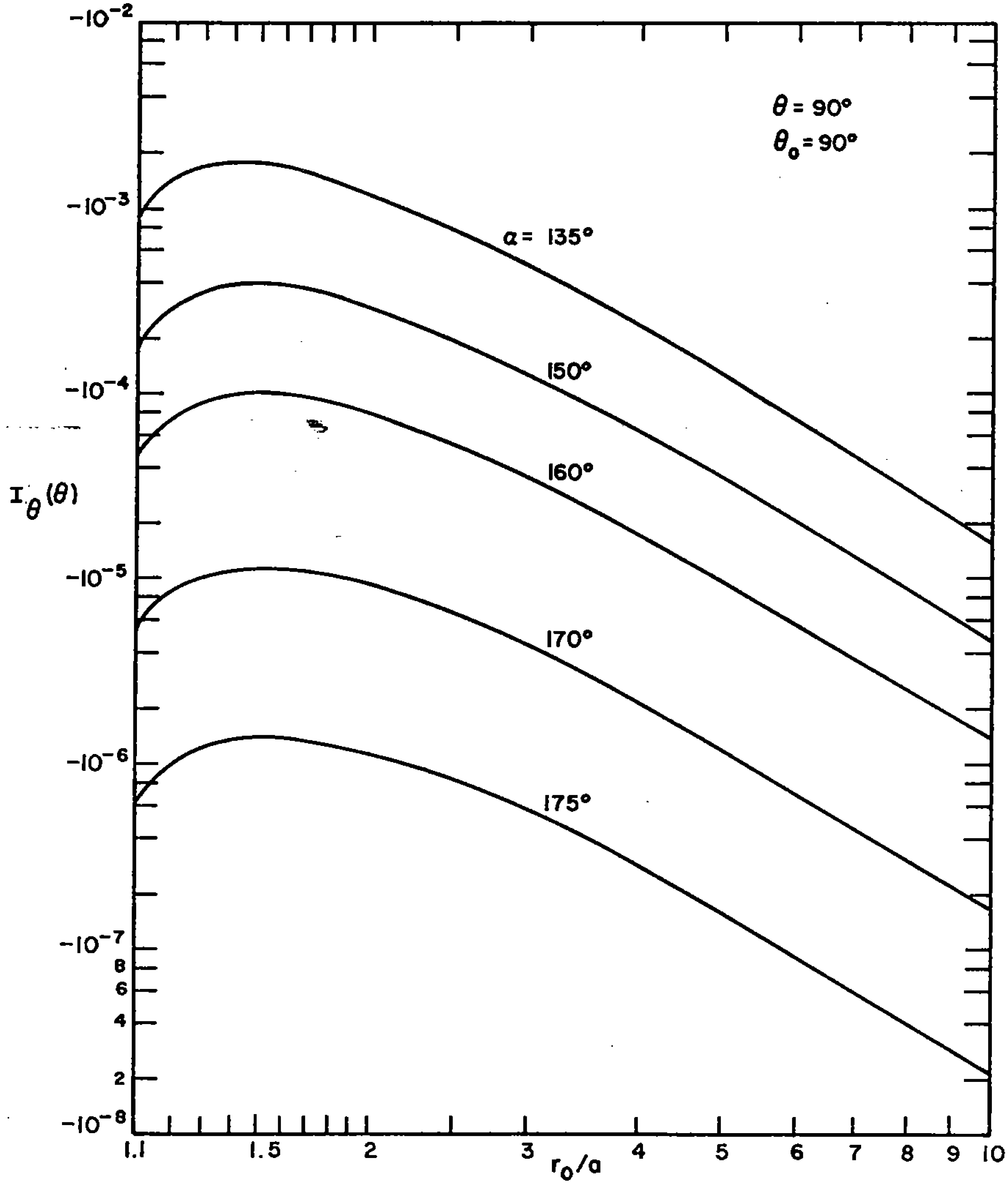
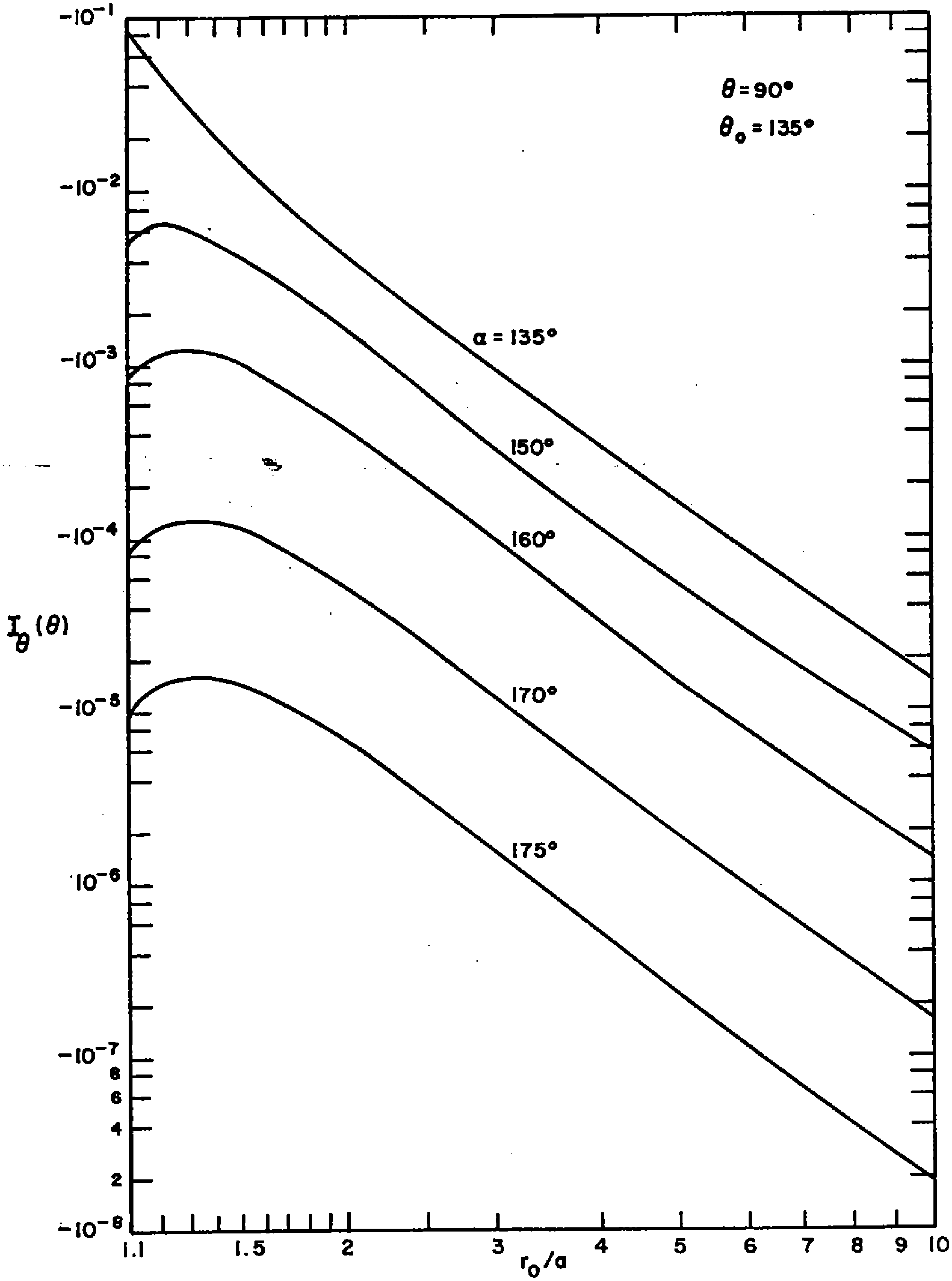
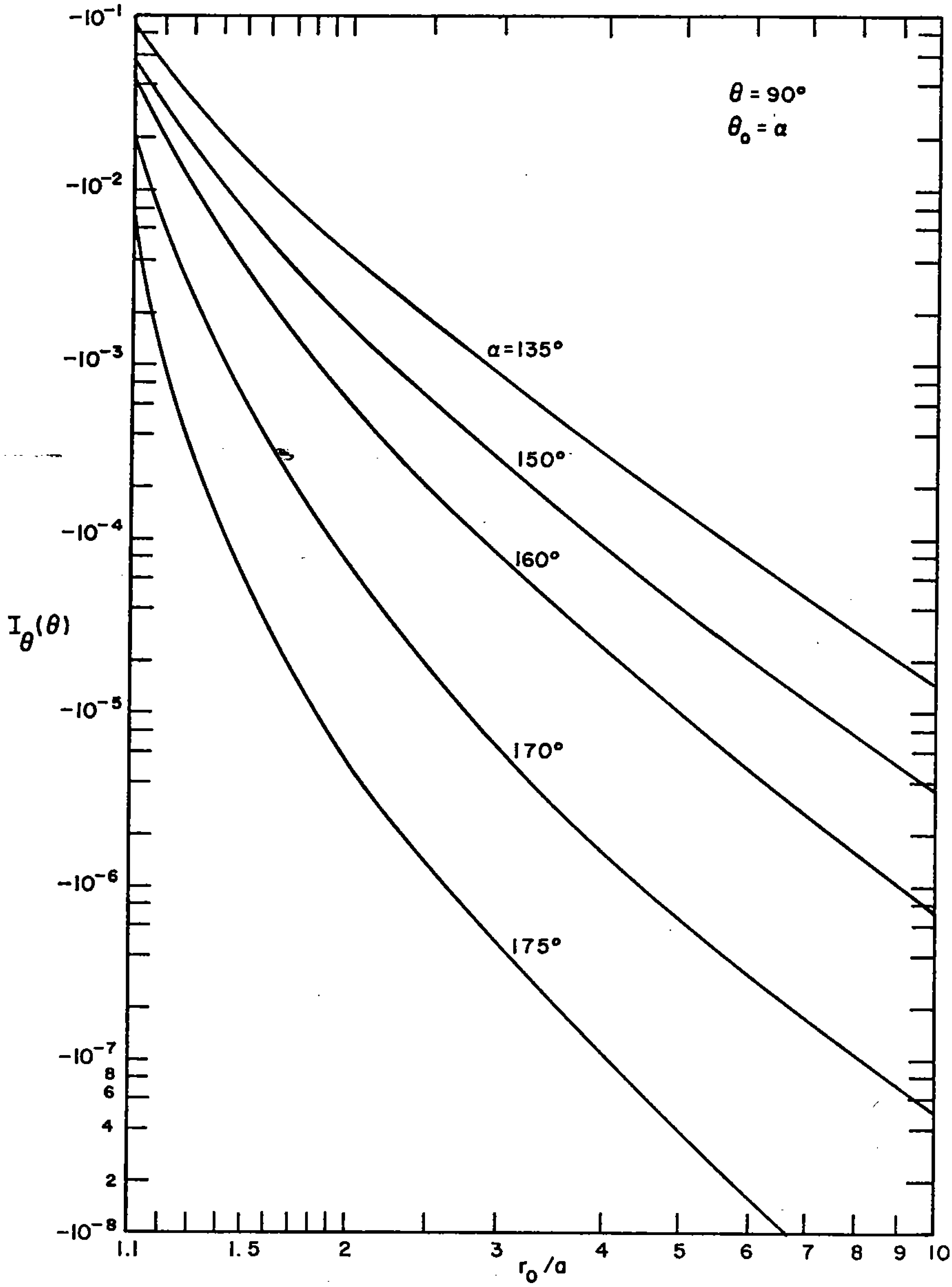

θ = 90°
θ₀ = 90°
α = 135°
150°
160°
170°
175°
-10⁻²
-10⁻³
-10⁻⁴
-10⁻⁵
-10⁻⁶
-10⁻⁷
8
6
4
2
-10⁻⁸
I_θ(θ)
1.1
1.5
2
3
4
5
6
7
8
9
10
r₀/a

Figure 31



-10⁻¹
-10⁻²
-10⁻³
-10⁻⁴
-10⁻⁵
10⁻⁶
-10⁻⁷
8
6
4
2
-10⁻⁸
$I_\theta(\theta)$
θ = 90°
θ₀ = 135°
α = 135°
150°
160°
170°
175°
1.1
1.5
2
3
4
5
6
7
8
9
10
r₀/a

Figure 32



θ = 90°
θ₀ = α
α=135°
150°
160°
170°
175°
$I_\theta(\theta)$
-10^{-1}
-10^{-2}
-10^{-3}
-10^{-4}
-10^{-5}
-10^{-6}
-10^{-7}
8
6
4
2
-10^{-8}
1.1
1.5
2
3
4
5
6
7
8
9
10
r_0/a

Figure 33

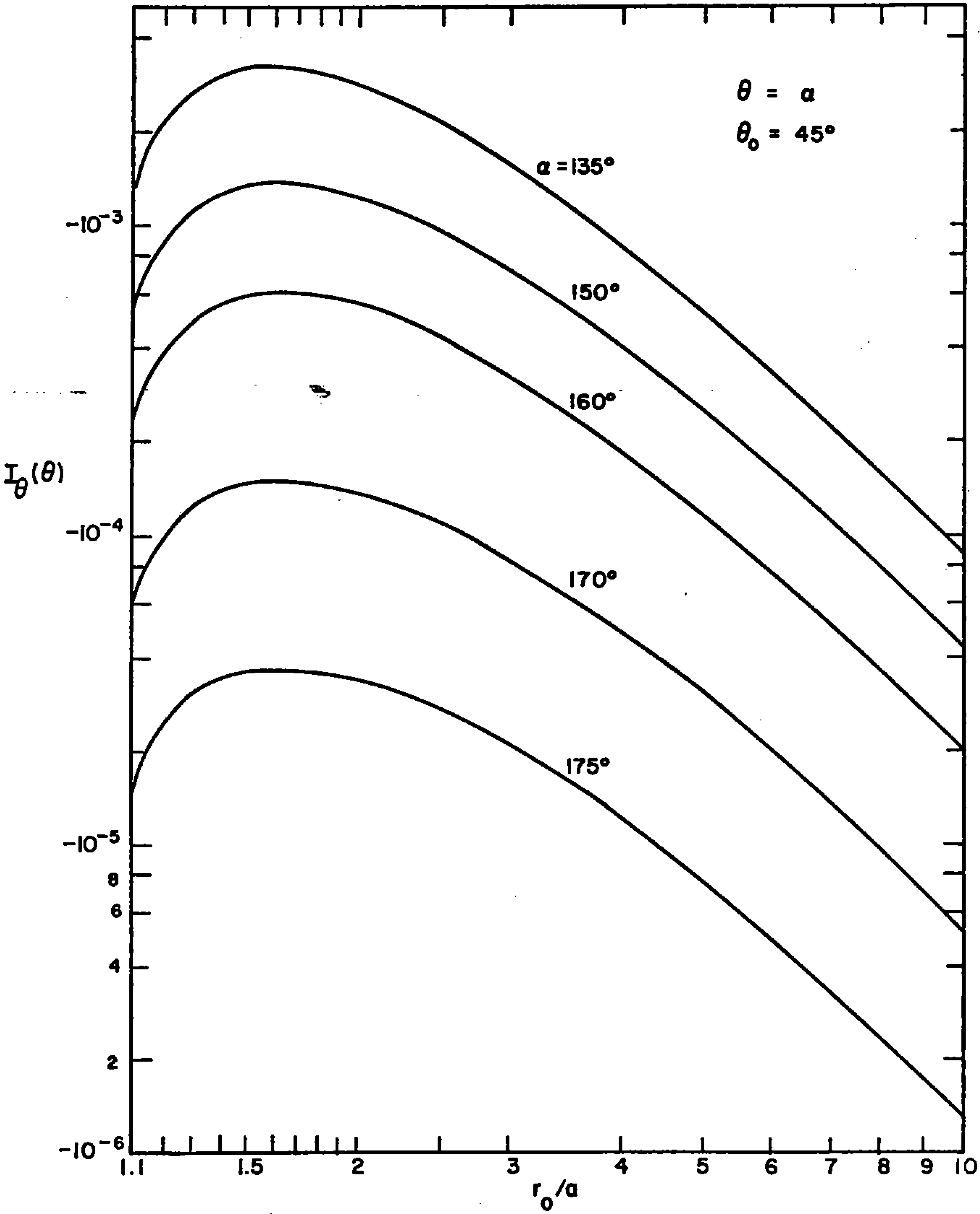

θ = α
θ₀ = 45°
α = 135°
150°
160°
170°
175°
-10^-3
-10^-4
-10^-5
8
6
4
2
-10^-6
I_θ(θ)
1.1
1.5
2
3
4
5
6
7
8
9
10
r₀/a

Figure 34

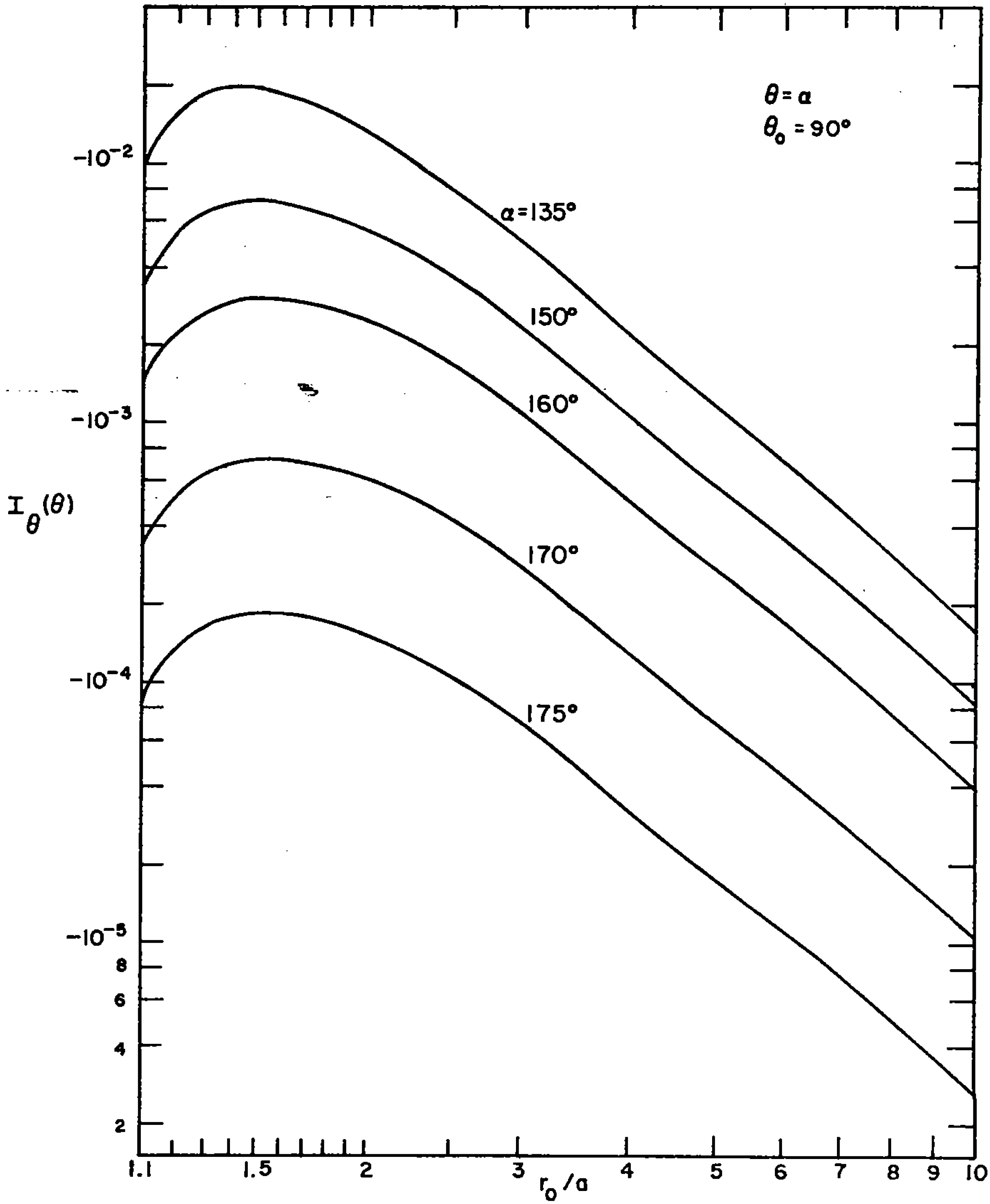

θ = α
θ₀ = 90°
α=135°
150°
160°
170°
175°
-10⁻²
-10⁻³
-10⁻⁴
-10⁻⁵
8
6
4
2
I_θ(θ)
1.1
1.5
2
3
4
5
6
7
8
9
10
r₀/a

Figure 35

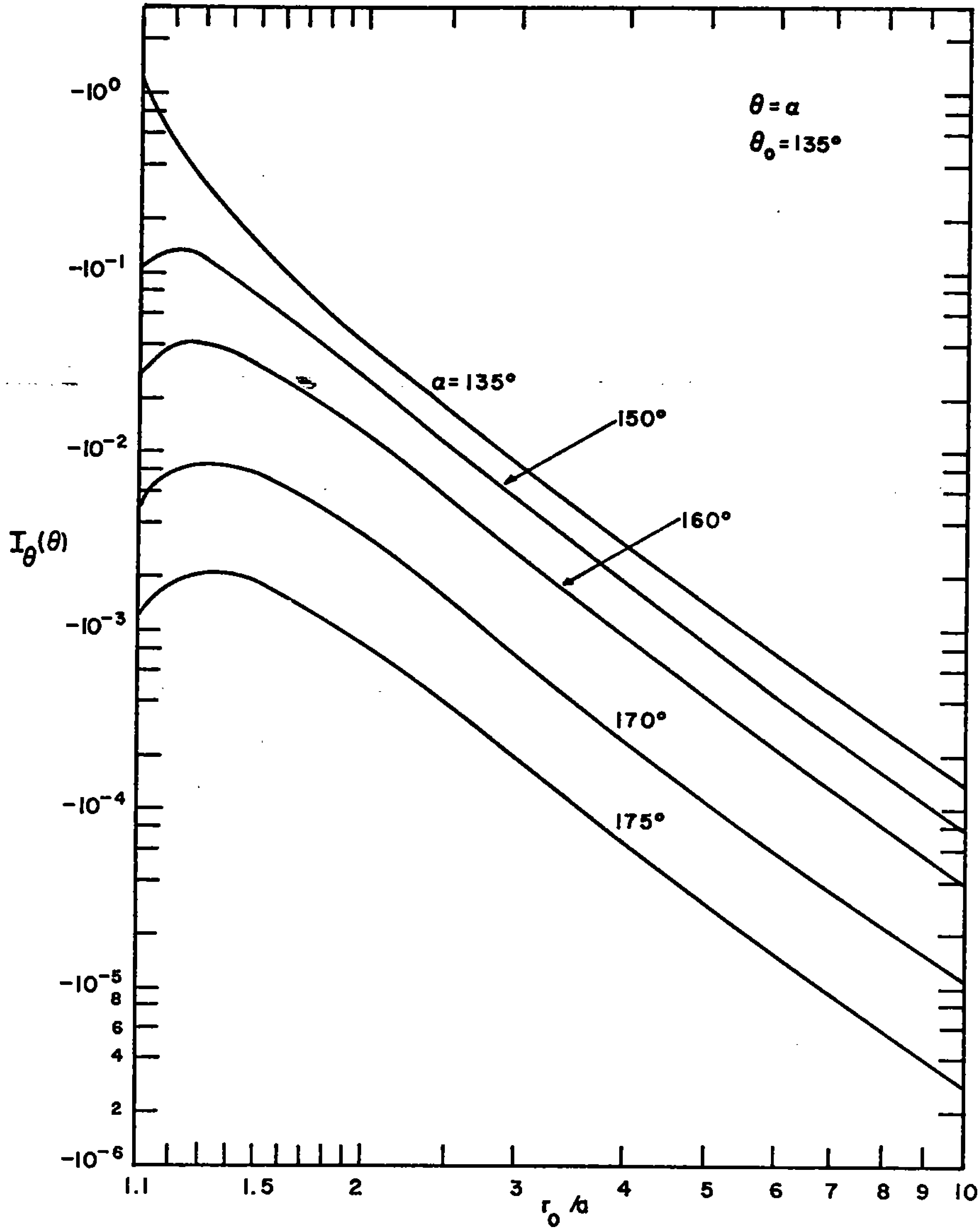
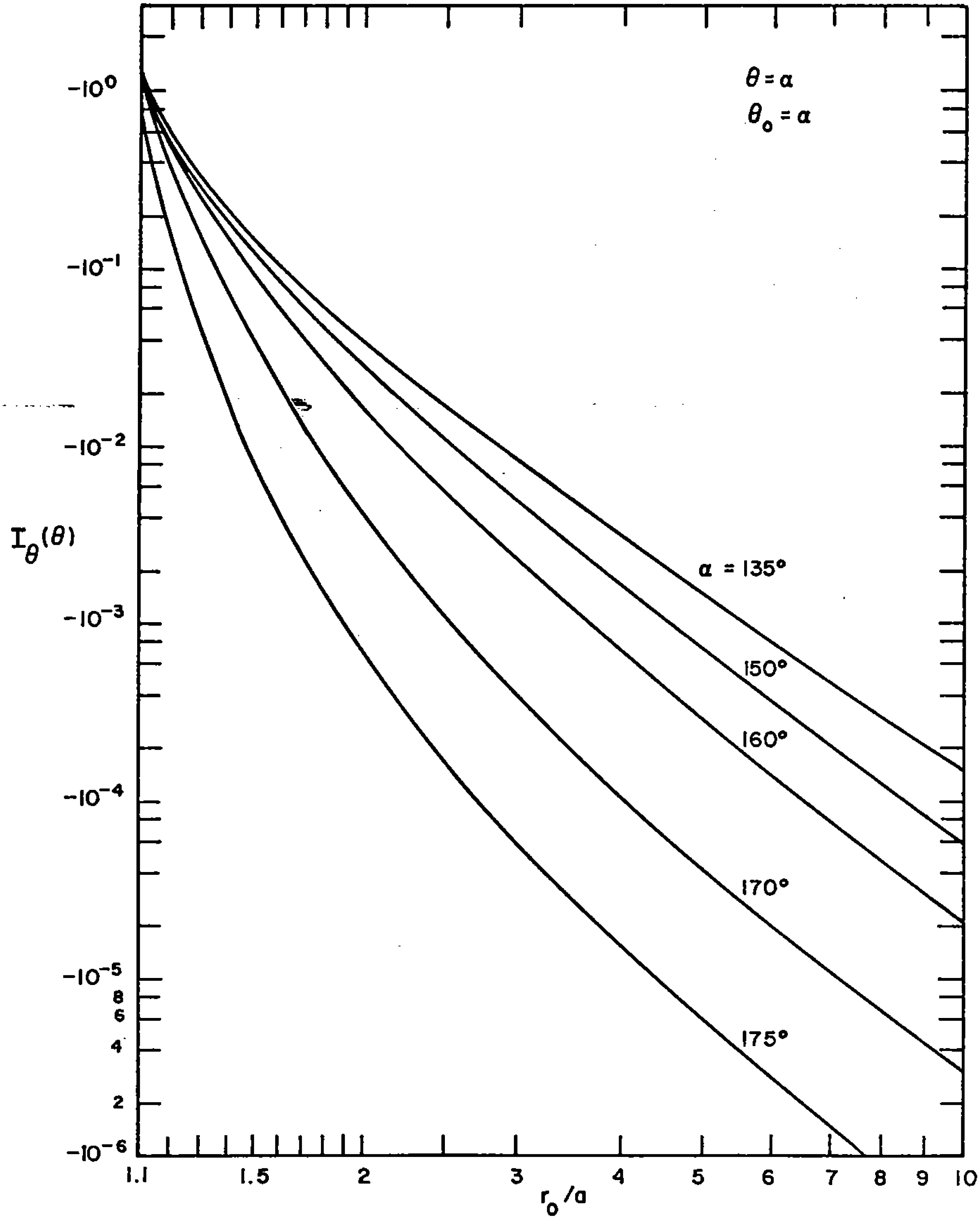
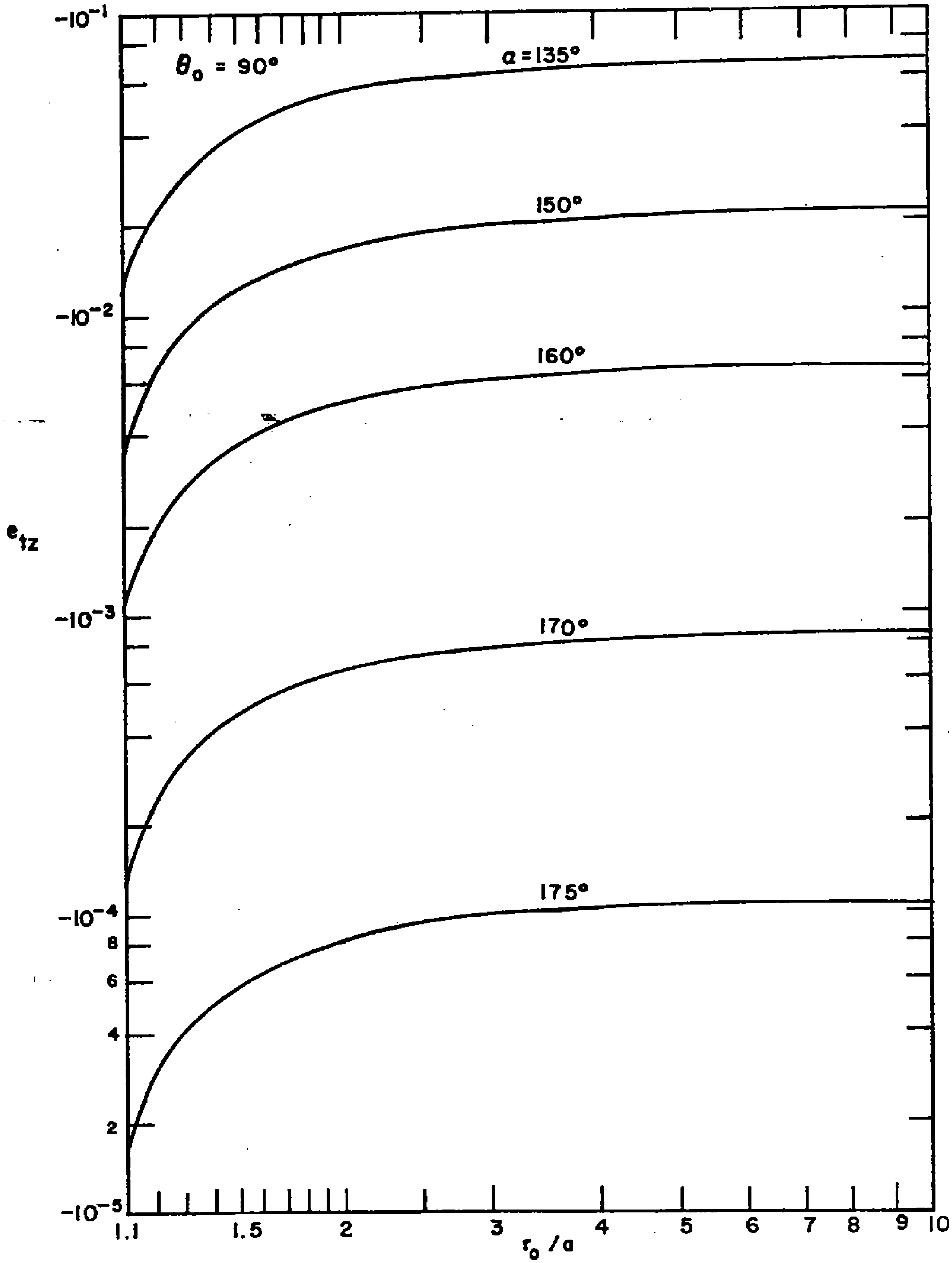

θ = α
θ₀ = 135°
α = 135°
150°
160°
170°
175°
$I_\theta(\theta)$
-10^0
-10^{-1}
-10^{-2}
-10^{-3}
-10^{-4}
-10^{-5}
8
6
4
2
-10^{-6}
1.1
1.5
2
3
4
5
6
7
8
9
10
r_0/a

Figure 36



θ = α
θ0 = α
-10^0
-10^-1
-10^-2
-10^-3
-10^-4
-10^-5
8
6
4
2
-10^-6
I_θ(θ)
α = 135°
150°
160°
170°
175°
1.1
1.5
2
3
4
5
6
7
8
9
10
r0/a

Figure 37



θ₀ = 90°
α = 135°
150°
160°
170°
175°
-10⁻¹
-10⁻²
-10⁻³
-10⁻⁴
8
6
4
2
-10⁻⁵
e_tz
1.1
1.5
2
3
4
5
6
7
8
9
10
r₀ / a

Figure 40

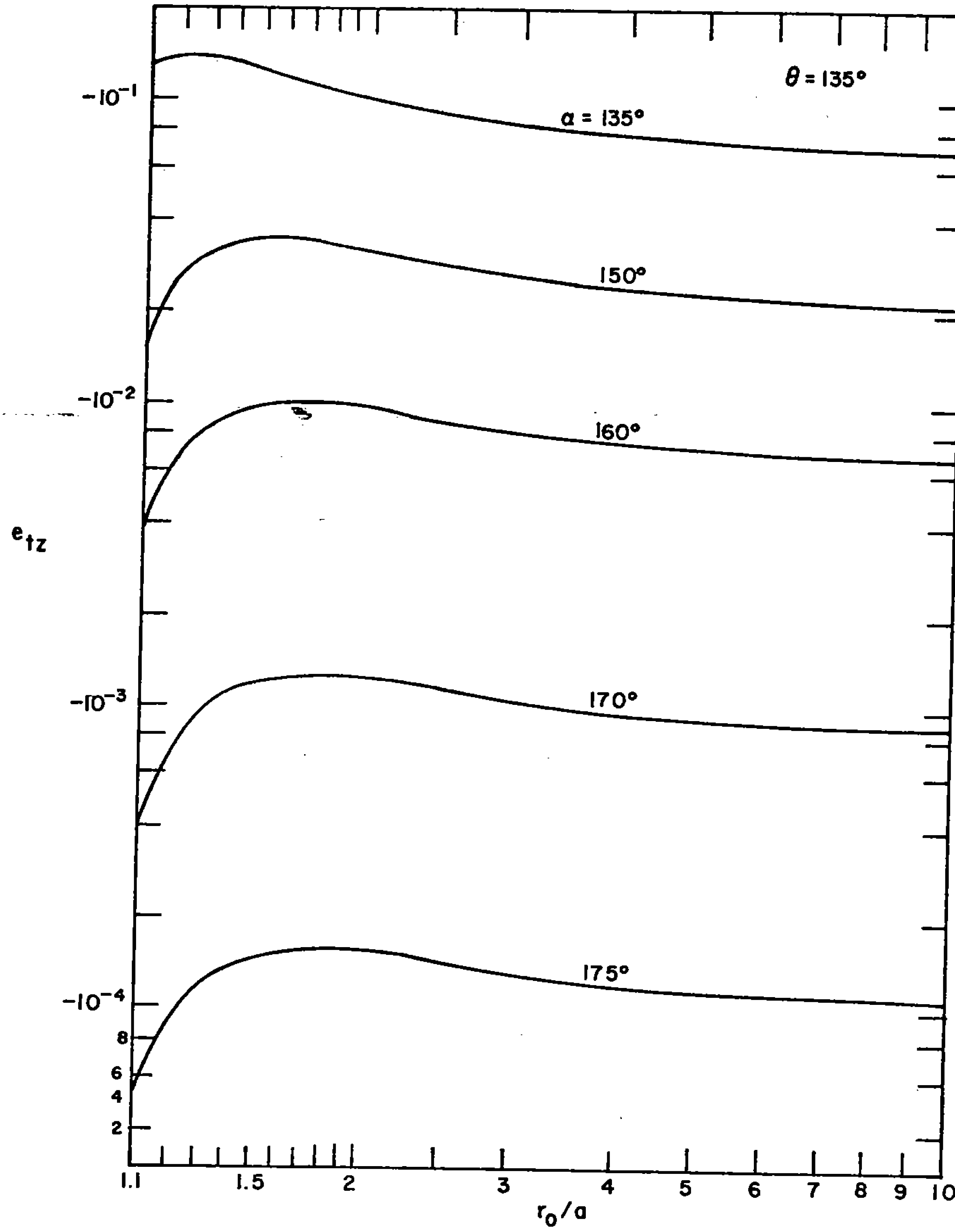
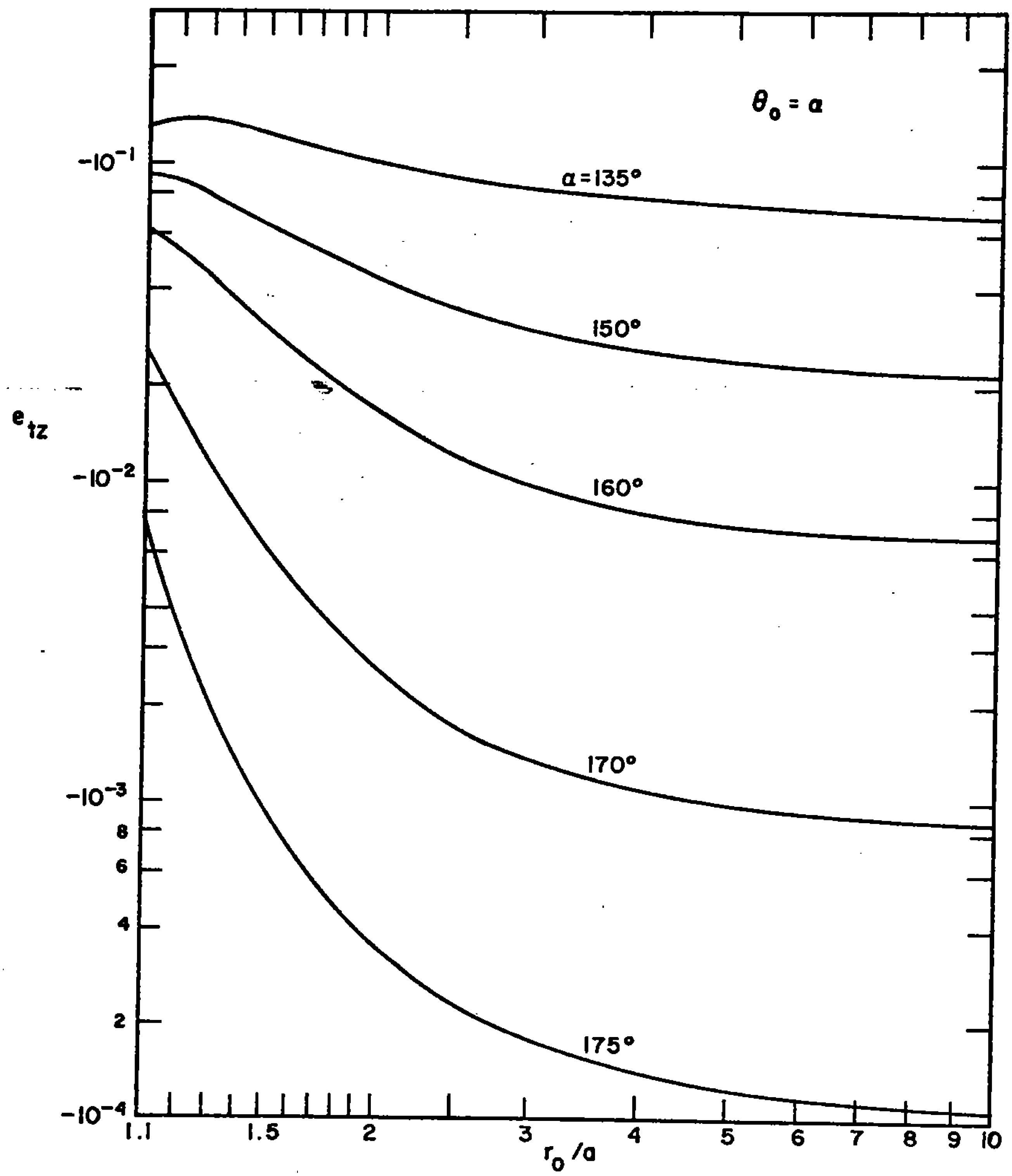
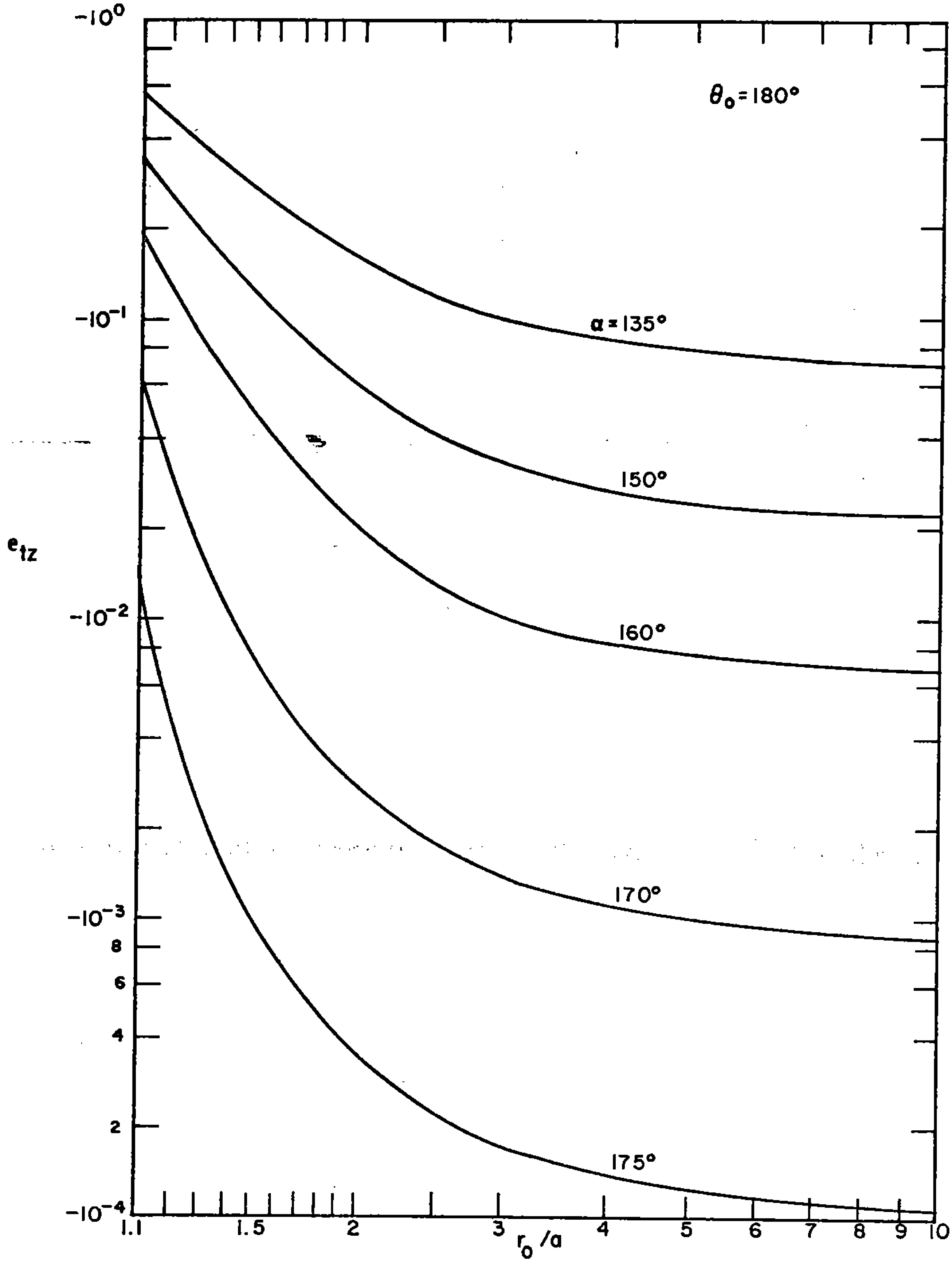

θ = 135°
α = 135°
150°
160°
170°
175°
-10^-1
-10^-2
-10^-3
-10^-4
8
6
4
2
e_tz
1.1
1.5
2
3
4
5
6
7
8
9
10
r_o/a

Figure 41



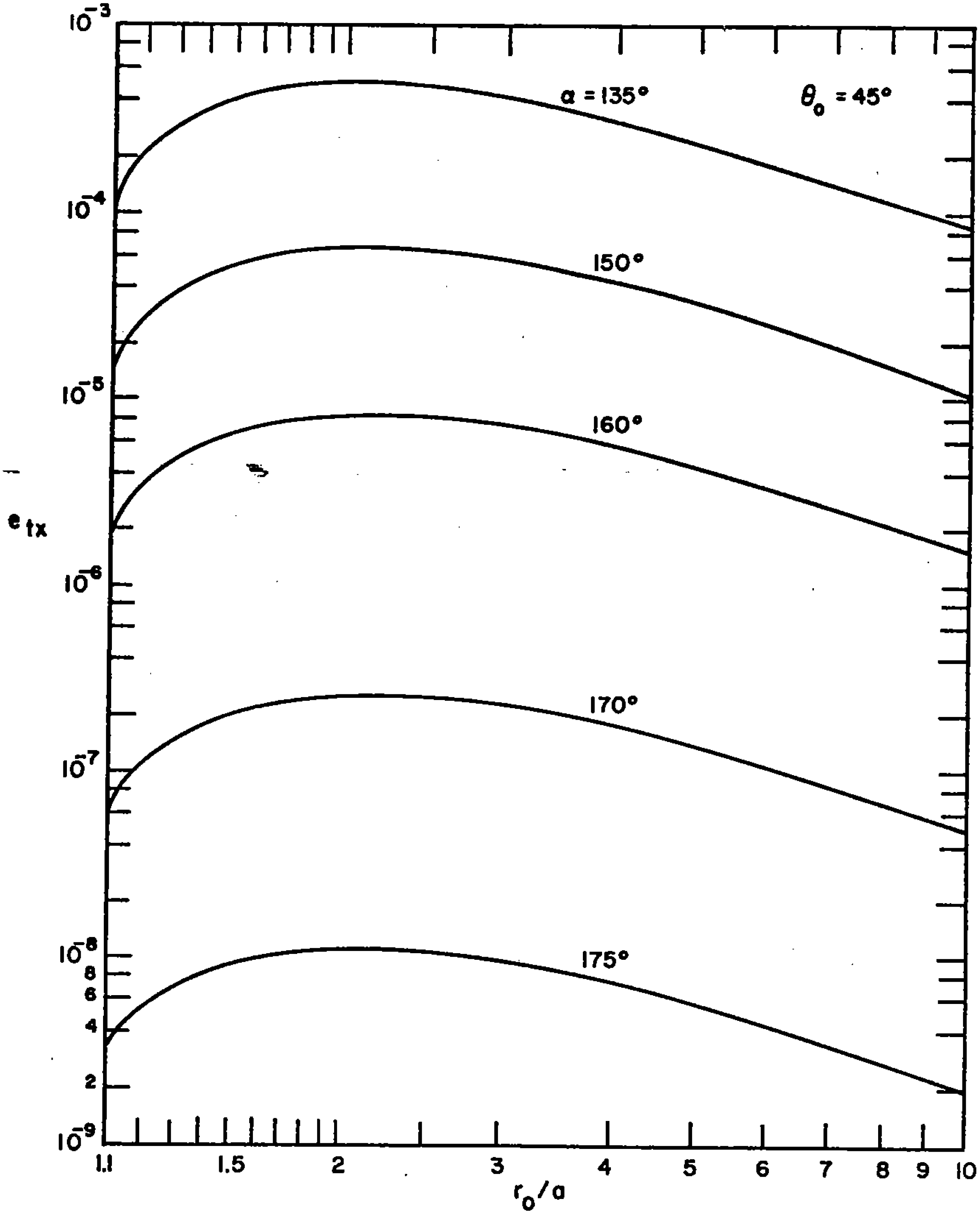
$\theta_0 = \alpha$
α=135°
150°
160°
170°
175°
e_{tz}
-10^{-1}
-10^{-2}
-10^{-3}
8
6
4
2
-10^{-4}
1.1
1.5
2
3
4
5
6
7
8
9
10
r_0/a

Figure 42



θo=180°
α=135°
150°
160°
170°
175°
etz
-10^0
-10^-1
-10^-2
-10^-3
8
6
4
2
-10^-4
1.1
1.5
2
3
4
5
6
7
8
9
10
ro/a

Figure 43



10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
8
6
4
2
10^{-9}
e_{tx}
$\alpha = 135°$
$\theta_o = 45°$
150°
160°
170°
175°
1.1
1.5
2
3
4
5
6
7
8
9
10
r_o/a

Figure 44

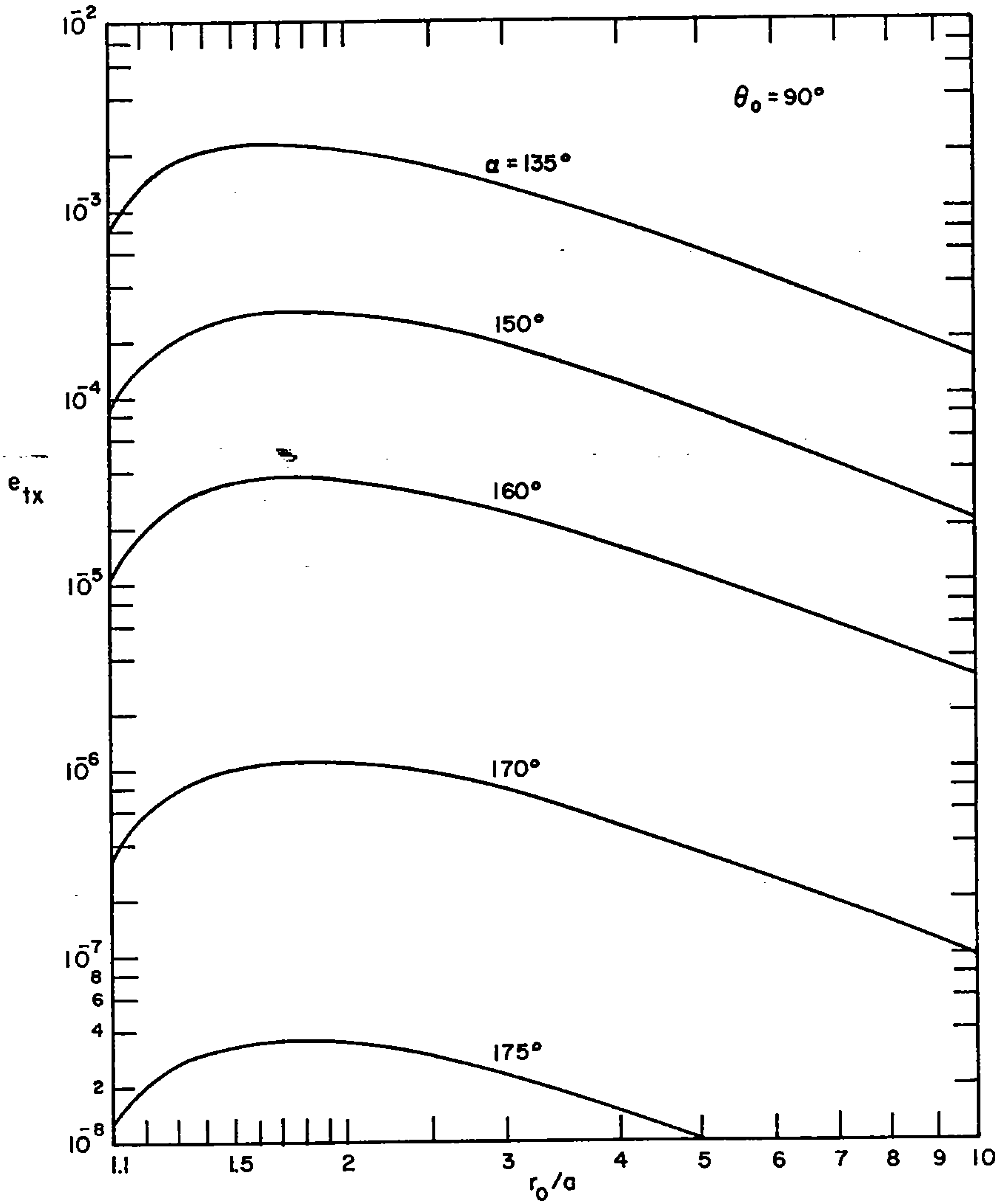

10^{-2}
$\theta_o = 90°$
$\alpha = 135°$
10^{-3}
150°
10^{-4}
e_{tx}
160°
10^{-5}
10^{-6}
170°
10^{-7}
8
6
4
175°
2
10^{-8}
1.1
1.5
2
3
4
5
6
7
8
9
10
r_o/a

Figure 45

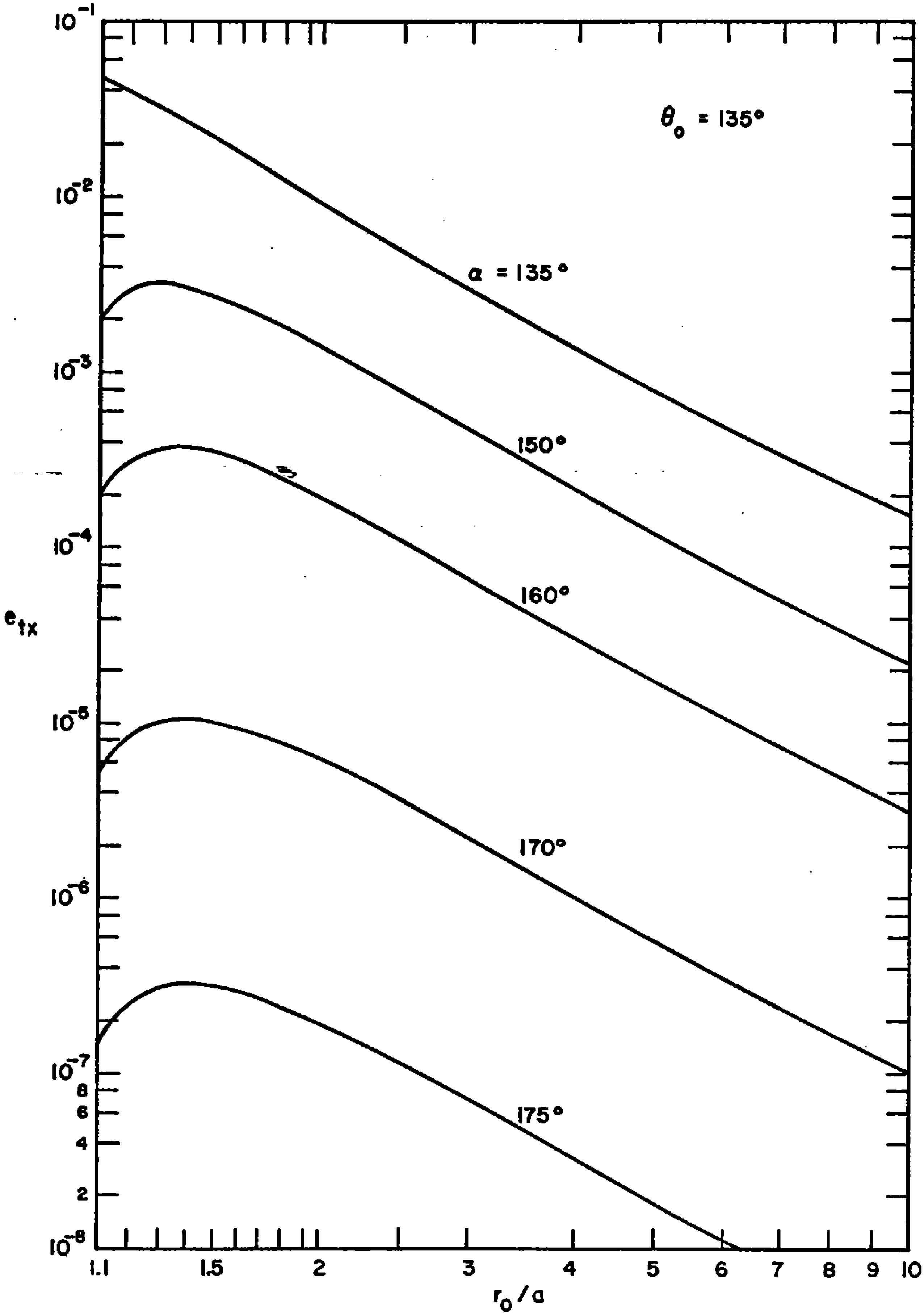
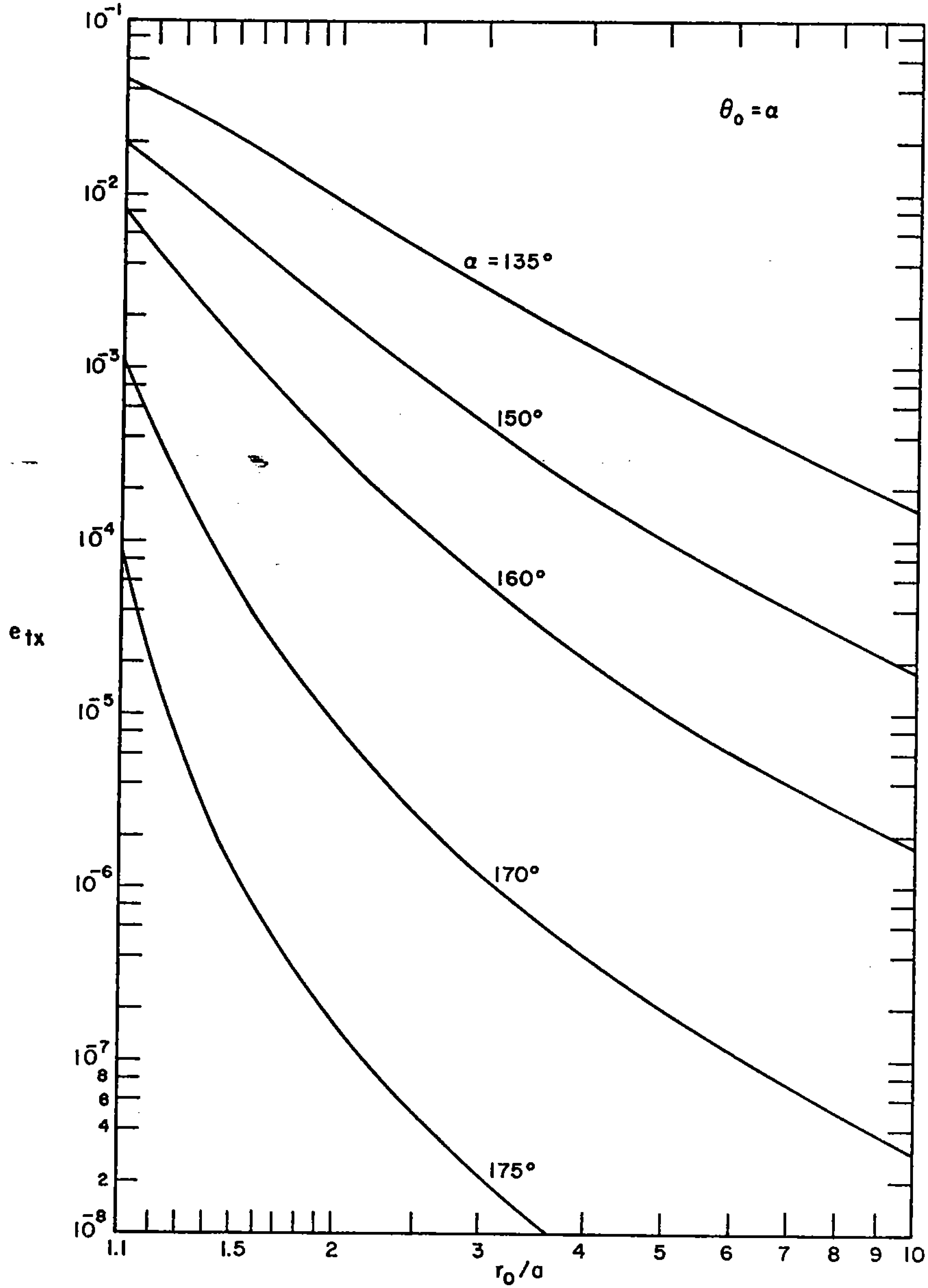

θ_o = 135°
α = 135°
150°
160°
170°
175°
e_{tx}
r_o/a
10⁻¹
10⁻²
10⁻³
10⁻⁴
10⁻⁵
10⁻⁶
10⁻⁷
8
6
4
2
10⁻⁸
1.1
1.5
2
3
4
5
6
7
8
9
10

Figure 46



$\theta_o = \alpha$
$\alpha = 135°$
150°
160°
170°
175°
e_{tx}
r_o/a
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
8
6
4
2
10^{-8}
1.1
1.5
2
3
4
5
6
7
8
9
10

Figure 47

ACKNOWLEDGMENT

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2. Baum, C. E., "Electromagnetic pulse interaction close to nuclear bursts and associated EMP environment specification," Interaction Notes, Note 76, July 1971.

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