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ON QUASI-MONOCHROMATIC SIGNALS PROPAGATED THROUGH DISPERSIVE CHANNELS

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ABSTRACT

A quasi-monochromatic source signal is convolved with an ionospheric impulse response function to obtain a time-domain representation of the dispersed signal. Application of numerical approximation techniques to the convolution integral leads to a characterization of the ionosphere in terms of statistical moments. It is shown that the dispersed signal envelope is given by the sum of the undispersed source signal and a series consisting of products of the moments and the corresponding derivatives of the source envelope function.

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INTRODUCTION

The waveform of a signal may undergo substantial distortion if the refractive index of the medium through which the signal is propagated is dispersive. Because the refractive index changes with frequency, the different frequency components making up the composite received signal travel over different paths to the receiver. These components, therefore, arrive at the receiver at different times so that the reconstituted signal is a distorted reproduction of the one originally sent. Although the particular approach described in this paper should be adaptable to a variety of related propagation phenomena, e.g., seismic or acoustic propagation, the only subject of concern here is the distortion of an electromagnetic signal transmitted from a source on earth through the ionosphere. As the frequencies of interest increase, the effects of dispersion decrease. However, at decametric wavelengths (~ 10 m), the spectral region of interest in the present paper, dispersive effects are still significant.

A number of papers relevant to the subject of dispersive distortion have appeared in the literature. Of these, several merit special attention, for example, the classical studies of Sommerfeld[1] and Brillouin,[2] in which the basic theoretical treatment of propagation of a monochromatic signal in dispersive media was developed. In two papers emphasizing the utility of function theory, Baerwald,[3] in 1930, considered signal spreading in very general dispersive systems. More recent and very comprehensive treatments of the subject have been published in treatises by Budden[4] and Ginzburg.[5] Wait[6] has published a noteworthy survey paper on pulse propagation in dispersive media, while Sollfrey[7] and Inston[8] have both discussed, in different contexts, the effects of ionospheric dispersion. Moreover, at least superficial treatments of the subject of signal modification resulting from propagation through dispersive channels can be found in most standard reference works on propagation.

Rather than invert directly from the frequency domain, as in the references cited above, the convolution theorem is invoked to transform the product of the source and network-transfer functions to the time domain where the convolution integral is amenable to some of the more powerful approximation techniques. In particular, it is possible to characterize the ionosphere in terms of certain functions which are identified as the "ionospheric moments." By using moments, a dispersed quasi-monochromatic signal can be expressed in the form of a sum of the undispersed source signal and distortion consisting of a series of derivatives of the envelope function multiplied by the appropriate moment. Apparently, useful approximations of the dispersed signal can be obtained by truncating the series. In addition, it appears that it might be feasible to provide a measure of "antidispersion filtering" by operating on the post-detected signal in the time domain, thereby achieving some innovation in dispersed signal processing. This particular approach to signal processing has not, to the author's knowledge, been described in the literature.

In the analysis which follows, the same simplifying assumptions often used in ionospheric propagation theory are made. The source is assumed to be quasi-monochromatic. Energy absorption resulting from electron collisions in the ionosphere is not considered. In addition, the ionosphere is considered to be homogeneous and isotropic, the ionospheric phase function is represented by a quadratic, and the amplitude function is regarded as a constant. Considerable simplification in the analysis (while still retaining most of the salient features of dispersion theory) is achieved by neglecting the birefringent effect of the geomagnetic field.

THE DISPERSED QUASI-MONOCHROMATIC PULSE

After propagation through a dispersive medium characterized by the transfer function $R(\omega)$, the field at the receiving antenna, on deleting explicit reference to z (the space variable), is

$$e_R(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_s(\omega)\, R(\omega)\, e^{j\omega t}\, d\omega , \tag{1}$$

where $E_s(\omega)$ is the spectrum of the source field. By definition, a quasi-monochromatic source is one in which component frequencies of significant amplitude extend through only a short range on each side of a carrier frequency, ω_0. In the time domain,

$$e_s(t) = f_m(t)\, e^{j\omega_0 t}, \tag{2}$$

where $f_m(t)$ is the envelope function. The duration of the signal is considered to be long compared with one period of the carrier wave; therefore, E_s is appreciable only for $\omega \approx \omega_0$. In the vicinity of ω_0, the transfer function can be approximated by

$$R(\omega) \approx |R(\omega_0)| \exp\left[-j\phi(\omega)\right]. \tag{3}$$

The amplitude function, $|R(\omega_0)|$, can be assumed to be slowly varying compared with $\exp\left[-j\phi(\omega)\right]$ in the neighborhood of ω_0. For convenience, take $R(\omega_0) = 1$ and expand $\phi(\omega)$ about ω_0, truncating the series beyond the term containing the second derivative,

$$\phi(\omega) \approx \phi(\omega_0) + (\omega - \omega_0)\, \phi'(\omega_0) + \frac{1}{2}(\omega - \omega_0)^2\, \phi''(\omega_0). \tag{4}$$

The primes denote differentiation with respect to ω evaluated at ω_0. Deferring, for the moment, a discussion of the significance of these terms, substitution of the Fourier transform of (2) and (3) into (1) leads to an integral which can be inverted by means of the standard techniques of complex integration to provide a time-dependent expression for $e_R(t)$, usually in the form of tabulated integral functions.

In order to examine the effect of dispersion on signal duration, it is informative to convolve the impulse response.

$$h(t) \approx \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left[j\omega t - j\phi(\omega)\right] d\omega , \tag{5}$$

with the source function $e_s(t)$ to obtain $e_R(t)$;

$$e_R(t) \approx \int_{-\infty}^{\infty} e_s(\lambda)\, h(t - \lambda)\, d\lambda . \tag{6}$$

Changing the variable of integration from ω to Ω, where $\Omega = (\omega - \omega_0)$, and taking the limits of the new variable as $-\infty$ and ∞ because ω is restricted to values near ω_0, (5) can be written

$$h(t) \approx \frac{1}{2\pi}\, e^{j\omega_0 t}\int_{-\infty}^{\infty} e^{-j\left[\phi(\Omega) - \Omega t\right]}\, d\Omega. \tag{7}$$

Writing $\phi(\omega)$ in the form

$$\phi(\omega) \approx \phi(\omega_0) + \phi'(\omega_0)\Omega + \frac{1}{2}\phi''(\omega_0)\Omega^2 \tag{8}$$

and substituting (8) into (7) gives

$$h(t) \approx \frac{1}{2\pi}\, e^{j\left[\omega_0 t - \phi(\omega_0)\right]}\int_{-\infty}^{\infty} e^{-j\left\{\frac{1}{2}\phi''(\omega_0)\Omega^2 + \left[\phi'(\omega_0) - t\right]\Omega\right\}}\, d\Omega, \tag{9}$$

which reduces to the simple closed form,[9]

$$\int_{-\infty}^{\infty} e^{-(\alpha x^2 + \beta x)j}\, dx = \sqrt{\frac{\pi}{\alpha}}\, \exp\left[j\left(\frac{\beta^2}{4\alpha} - \frac{\pi}{4}\right)\right], \quad \alpha > 0 ; \tag{10}$$

$$h(t) \approx \frac{1-j}{2\pi} e^{j\left[\omega_0 t - \phi(\omega_0)\right]} \sqrt{\frac{\pi}{\phi''(\omega_0)}} \exp\left\{ j \frac{\left[\phi'(\omega_0) - t\right]^2}{2\phi''(\omega_0)} \right\} . \qquad (11)$$

Substitution of (11) and (2) into (6) gives the expression derived by Ginzburg for the dispersed quasi-monochromatic pulse:

$$e_R(t) \approx \frac{1-j}{2\pi} e^{j\left[\omega_0 t - \phi(\omega_0)\right]} \sqrt{\frac{\pi}{\phi''(\omega_0)}} \int_{-\infty}^{\infty} f_m(\lambda)\, e^{\frac{j(\lambda - t + \phi')^2}{2\phi''}} d\lambda . \qquad (12)$$

Then, on introducing the variable u defined by

$$\frac{(\lambda - t + \phi')^2}{2\phi''} = \frac{\pi}{2} u^2 ,$$

(12) can be written in the form*

$$e_R(t) \approx \frac{1}{2}(1-j)\, e^{j\omega_0 t - j\phi(\omega_0)} \int_{-\infty}^{\infty} f_m\left[t - \phi' + (\pi \phi'')^{1/2} u\right] \exp(j\pi u^2/2)\, du. \qquad (13)$$

*See also Wait,[6] who obtains this form by setting $\phi''' = 0$ in his third-order approximation.

DISPERSION DISTORTION

In order to adapt communication-theory models to dispersion analysis, it will be convenient to express the received signal as the sum of the source signal and of a distortion signal which characterizes the dispersion. In a 1963 paper, Prosin[10] obtained a series expression for a dispersed UHF signal by taking only the first two terms of the expansion of the exponential $e^{-j\phi(\omega)}$ but retaining higher order terms in the expansion of $\phi(\omega)$. Then, by use of the method of analytical continuation, the series was inverted term by term to obtain an expression for a received wideband FM signal in the form of the sum of a source function and distortion terms which were called "dispersion noise." In the case of interest here, the desired expression can be obtained directly from the expansion of the convolution integral in (12), i.e., from the integral

$$f(t) = \int_{-\infty}^{\infty} f_m(\lambda) \exp\left[j(t - \phi' - \lambda)^2/2\phi''\right] d\lambda . \tag{14}$$

Now, f(t) can be expressed as a series in terms of the derivatives of $f_m(t)$ and the moments of the exponential term.[11] Thus,

$$f(t) = m_0 f_m(t) - \frac{m_1}{1!} \frac{df_m}{dt} + \dots + (-1)^k \frac{m_k}{k!} \frac{d^k f_m}{dt^k} + \dots , \tag{15}$$

where

$$m_k = \int_{-\infty}^{\infty} t^k \exp\left\{\frac{j(t - \phi')^2}{2\phi''}\right\} dt . \tag{16}$$

Then, on introducing the variable v defined by

$$\frac{(t - \phi')^2}{2\phi''} = \frac{\pi}{2} v^2 ,$$

(16) can be written in the form

$$m_k = \sqrt{\pi \phi''} \int_{-\infty}^{\infty} \left(\phi' + \sqrt{\pi \phi''}\, v\right)^k \exp(j\pi v^2/2)\, dv \,. \tag{17}$$

The zeroth-order term is given by

$$m_0 = 2\sqrt{\pi\phi''}\left[C(\infty) + jS(\infty)\right] = \sqrt{\pi\phi''}\,(1 + j)\,, \tag{18}$$

where C(z) and S(z) are the standard Fresnel integrals:

$$C(z) = \int_0^z \cos \frac{\pi}{2} v^2\, dv\,; \qquad S(z) = \int_0^z \sin \frac{\pi}{2} v^2\, dv\,.$$

Substitution of (18) and (15) into (12) and taking $\phi(\omega_0) = \omega_0 \tau$, where τ is the phase delay of the channel, gives

$$e_R(t) \approx f_m(t)\, e^{j\omega_0(t-\tau)} + \frac{1 - j}{2\sqrt{\pi\phi''}}\, e^{j\omega_0(t-\tau)} \sum_{k=1}^{\infty} (-1)^k \frac{m_k}{k!} \frac{d^k f_m}{dt^k}\,. \tag{19}$$

Evidently, except for a linear phase shift, the received signal is equal to the source signal plus a dispersion distortion, which is

$$e_D(t) \approx \frac{1 - j}{2\sqrt{\pi\phi''}}\, e^{j\omega_0(t-\tau)} \sum_{k=1}^{\infty} (-1)^k \frac{m_k}{k!} \frac{d^k f_m}{dt^k}\,. \tag{20}$$

Rather than evaluating (16), it is easier to use the characteristic function, or moment generating function, $\psi(n)$, given by

$$\psi(n) = \int_{-\infty}^{\infty} dx\, e^{jnx} f(x) = \int_{-\infty}^{\infty} dt\, \exp\left\{j\left[nt + \frac{(t - \phi')^2}{2\phi''}\right]\right\}. \tag{21}$$

On carrying out the integration in (21),

$$\psi(n) = \sqrt{\pi\phi''}\,(1 + j)\, e^{-j\left(\frac{1}{2}n^2\phi'' - n\phi'\right)} . \tag{22}$$

The kth moment is given by

$$m_k = \frac{d^k}{d(jn)^k} \left.\psi(n)\right]_{n=0} . \tag{23}$$

The series representation of dispersion distortion in terms of derivatives of the envelope function is obtained by substituting (23) and (22) into (20). Thus,

$$e_D(t) \approx e^{j\omega_0(t-\tau)} \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \left(\frac{d^k f_m}{dt^k}\right) \left\{ \frac{d^k}{d(jn)^k} \left. e^{-j\left(\frac{1}{2}n^2\phi'' - n\phi'\right)} \right]_{n=0} \right\} . \tag{24}$$

Now let ϕ'' approach zero in (24). Then the received signal (19) reduces to the form

$$e_R(t) \approx e^{j\omega_0(t-\tau)} \left\{ f_m + \sum_{k=1}^{\infty} \frac{(-\phi')^k}{k!} \frac{d^k f_m}{dt^k} \right\} , \tag{25}$$

where the bracket term is simply the Taylor expansion about t of the function $f_m(t - \phi')$. That is, (19) reduces to the well-known result for a nondispersive channel,

$$e_R(t) = f_m\left[t - \phi'(\omega_0)\right] e^{j\omega_0(t-\tau)} , \tag{26}$$

as it should. The carrier phase is determined by $\omega_0\tau$ (τ is the phase delay) and the envelope has not been distorted but has only been delayed by a length of time

$\phi'(\omega_0)$. The quantity $\phi'(\omega_0)$ is the "group delay," τ_g, associated with the channel. With reference to a network representation of the communication channel, the source signal has undergone distortionless transmission.

The numerical value for the group delay is given by the expression

$$\phi'(\omega_0) = \frac{5 \times 10^{-6}}{\omega_0^2} \int_0^s N \, ds . \qquad \text{(MKS)} \tag{27}$$

The integral represents the integrated electron content along the propagation path. From (4) it can be inferred that the "dispersion" is given by

$$\phi''(\omega_0) = \frac{2}{\omega_0} \phi'(\omega_0) . \tag{28}$$

THE IONOSPHERIC MOMENTS

Before proceeding with a discussion of the evaluation of the "ionospheric moments" it is necessary to define the "bandwidth" of a dispersive channel. According to Papoulis's formulation[11] of the uncertainty principle,* bandwidth and signal duration are canonically related by the inequality†

$$D_\omega \cdot D_t \geq \sqrt{\pi/2} \ . \tag{29}$$

Examination of the expression for the k^{th} moment, (17), indicates that the time of establishment of the signal amplitude is determined by the characteristic time

$$\tau_c = \sqrt{\pi\phi''} \ \text{sec.} \tag{30}$$

The channel bandwidth, W, corresponding to τ_c is the value of D_ω, satisfying (29), i.e.,

$$W \geq 1/\sqrt{2\phi''} \ . \tag{31}$$

Although the equality applies only if the envelope function is Gaussian, for present purposes we choose the equality and take W as the bandwidth of the dispersive channel.

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*Some writers refer to this phenomenon as reciprocal spreading but it is really the uncertainty principle of quantum mechanics. For a complete discussion of the role of the uncertainty principle in engineering see Brillouin.[12]

†Duration is defined here by the second moment of $|f_m(t)|^2$, i.e.,

$$D_t^2 = \int_{-\infty}^{\infty} t^2 \, |f_m(t)|^2 \, dt \ .$$

Similarly, bandwidth is defined by the second moment of $|F_m(\omega)|^2$.

Now, substituting (15) and (30) into (12), an expression for $e_R(t)$ is obtained in terms of the moments m_k:

$$e_R(t) \approx \frac{1-j}{2}\, e^{j\left[\omega_0 t - \phi(\omega_0)\right]} \left[\left(\frac{m_o}{\tau_c}\right) f_m(t) + \ldots + \frac{(-1)^k}{k!}\left(\frac{m_k}{\tau_c}\right)\frac{d^k f_m(t)}{dt^k} + \ldots\right], \tag{32}$$

where the m_k are given by (17). Therefore, it is appropriate to define a new moment, an ionospheric moment, which takes the form

$$\mathfrak{M}_k = \frac{d^k}{d(jn)^k}\, \psi_i(n)\Big|_{n=0}, \tag{33}$$

where $\psi_i(n)$ is a function, characteristic of the ionosphere, defined as follows:

$$\psi_i(n) = (1 + j)\exp\left\{- j\left[\frac{1}{2\pi}(n\tau_c)^2 - n\tau_g\right]\right\}. \tag{34}$$

Noting that $\mathfrak{M}_k = m_k/\tau_c$, in considering the propagation of band-limited signals, it is more convenient to evaluate (17) to obtain the ionospheric moments. That is, $\mathfrak{M}_k$ is given by

$$\mathfrak{M}_k = \int_{v_1}^{v_2} (\tau_g + \tau_c v)^k \exp(j\pi v^2/2)\, dv, \tag{35}$$

where the dummy variable v is

$$v = \frac{t - \tau_g}{\tau_c}. \tag{36}$$

The integration is straightforward, giving as the zeroth moment, for example,

$$\mathfrak{M}_0 = \Big[C(v) + jS(v)\Big]_{v_1}^{v_2}. \tag{37}$$

Significa[illegible] by specifica[illegible] of integration. Let T denote the duration of the source signal. The spectral width of the signal, $D_\omega \sim \frac{1}{T}$, is assumed to be much smaller than ω_0 in order to ensure the quasi-monchromatic nature of the source signal. Time, θ, measured from the arrival of the signal at the observation point is usually defined[5, 6] by

$$\theta = \frac{T}{2} + t - \tau_g . \tag{38}$$

(In the absence of dispersion the signal would arrive at the observation point at $t = -\frac{T}{2} + \tau_g$.) Using θ/τ_c as the independent variable, the limits of integration are

$$v_2 = \frac{T - \theta}{\tau_c} , \quad v_1 = -\frac{\theta}{\tau_c} ,$$

and the real and imaginary parts of the ionospheric moments are, respectively,

$$\mathbf{R}\mathfrak{M}_k = \sum_{r=0}^{k} \binom{k}{r} \tau_g^{k-r} \tau_c^{r} \int_{-\frac{\theta}{\tau_c}}^{\frac{T-\theta}{\tau_c}} x^r \cos \frac{\pi}{2} x^2 \, dx , \tag{39}$$

$$\mathbf{I}\mathfrak{M}_k = \sum_{r=0}^{k} \binom{k}{r} \tau_g^{k-r} \tau_c^{r} \int_{-\frac{\theta}{R_c}}^{\frac{T-\theta}{R_c}} x^r \sin \frac{\pi}{2} x^2 \, dx. \tag{40}$$

It should be noted that the moments approach zero for $\theta >> T$ by virtue of the symmetry relations

$$C(-v) = -C(v), \quad S(-v) = -S(v).$$

If the source signal is indeed quasi-monochromatic, $E_s(\omega)$ is negligible outside an interval $(-\Omega, \Omega)$, and the series expansion for $e_R(t)$ given by (32) can be truncated.

Then, denoting the operator $\frac{d^k}{dt^k}$ by D^k, $e_R(t)$ can be approximated by

$$e_R(t) \approx \frac{1}{\sqrt{2}} e^{j[\omega_0 t - \phi(\omega_0) - \pi/4]} \left[\mathfrak{M}_0 D_0 + \ldots + (-1)^k \frac{\mathfrak{M}_k}{k!} D^k\right] f_m(t). \tag{41}$$

Except for introducing a constant phase shift, evidently the effect of the ionosphere on a propagated quasi-monochromatic source signal can be characterized in the time domain by a matrix (or row vector) operator $\mathcal{O}$, where

$$\mathcal{O} \approx \left[\mathfrak{M}_o D^o + \ldots + (-1)^k \frac{\mathfrak{M}_k}{k!} D^k\right]. \tag{42}$$

By substituting in Eq. (41), $e_R(t)$ can be written

$$e_R(t) \approx \frac{1}{\sqrt{2}} e^{j[\omega_0 t - \phi(\omega_0) - \pi/4]} \mathcal{O} f_m(t) . \tag{43}$$

The form of (41) suggests polar representation of the moments; that is,

$$\mathfrak{M}_k = |\mathfrak{M}_k| \; e^{j\delta_k} , \tag{44}$$

where

$$\delta_k = \tan^{-1} \frac{\mathbf{I}\mathfrak{M}_k}{\mathbf{R}\mathfrak{M}_k} . \tag{45}$$

Consider now the magnitudes of the zeroth and first moments expressed in dimensionless form:

$$|\mathfrak{M}_o| = \left\{\left[C\left(\frac{T - \theta}{\tau_c}\right) + C\left(\frac{\theta}{\tau_c}\right)\right]^2 + \left[S\left(\frac{T - \theta}{\tau_c}\right) + S\left(\frac{\theta}{\tau_c}\right)\right]^2\right\}^{1/2} \tag{46}$$

$$\left|\frac{\mathfrak{M}_1}{\tau_c}\right| = \left\{\left[\frac{\tau_g}{\tau_c} C\left(\frac{T-\theta}{\tau_c}\right) + \frac{\tau_g}{\tau_c} C\left(\frac{\theta}{\tau_c}\right) + \frac{1}{\pi} \sin\left(\frac{\pi}{2} \cdot \frac{T-\theta}{\tau_c}\right)^2 - \frac{1}{\pi} \sin\left(\frac{\pi}{2} \frac{\theta}{\tau_c}\right)^2\right]^2\right.$$

$$\left. + \left[\frac{\tau_g}{\tau_c} S\left(\frac{T-\theta}{\tau_c}\right) + \frac{\tau_g}{\tau_c} S\left(\frac{\theta}{\tau_c}\right) - \frac{1}{\pi} \cos\left(\frac{\pi}{2} \cdot \frac{T-\theta}{\tau_c}\right)^2 + \frac{1}{\pi} \cos\left(\frac{\pi}{2} \frac{\theta}{\tau_c}\right)^2\right]^2\right\}^{1/2} . \quad (47)$$

By combining (27) and (28), an estimate of the magnitude of the coefficient of the Fresnel terms in (47) is obtained:

$$\frac{\tau_g}{\tau_c} \approx \left(\frac{10^{-6}\int N\,dh}{\omega_o}\right)^{1/2} . \quad (48)$$

For transionospheric propagation of decametric signals ($10 > \lambda > 1$ meter) through the range of ionospheres usually encountered, e.g., 5×10^{17} to 5×10^{16} electrons m^{-2} (total electron content, or TEC), it is estimated that $50 > (\tau_g/\tau_c) > 5$. Clearly, except for values of θ in the neighborhood of zero, the trigonometric terms are negligible compared to the Fresnel terms in (47). Even for small θ, the error incurred by deleting the trigonometric terms is not serious, so that, to a first approximation,

$$\mathfrak{M}_1 \approx \tau_g \mathfrak{M}_o . \quad (49)$$

Unfortunately, it appears that the series (41) may converge only very slowly. Therefore, the utility of the ionospheric moments should be assessed on the basis of the given source function and the particular ionosphere of interest. However, because the zeroth-order moment is independent of τ_g/τ_c, it is shown in Fig. 1. The magnitude and phase angle were numerically calculated on a CDC 6600 computer; plots were made with a Stromberg Carlson Model SC 4020 plotter.

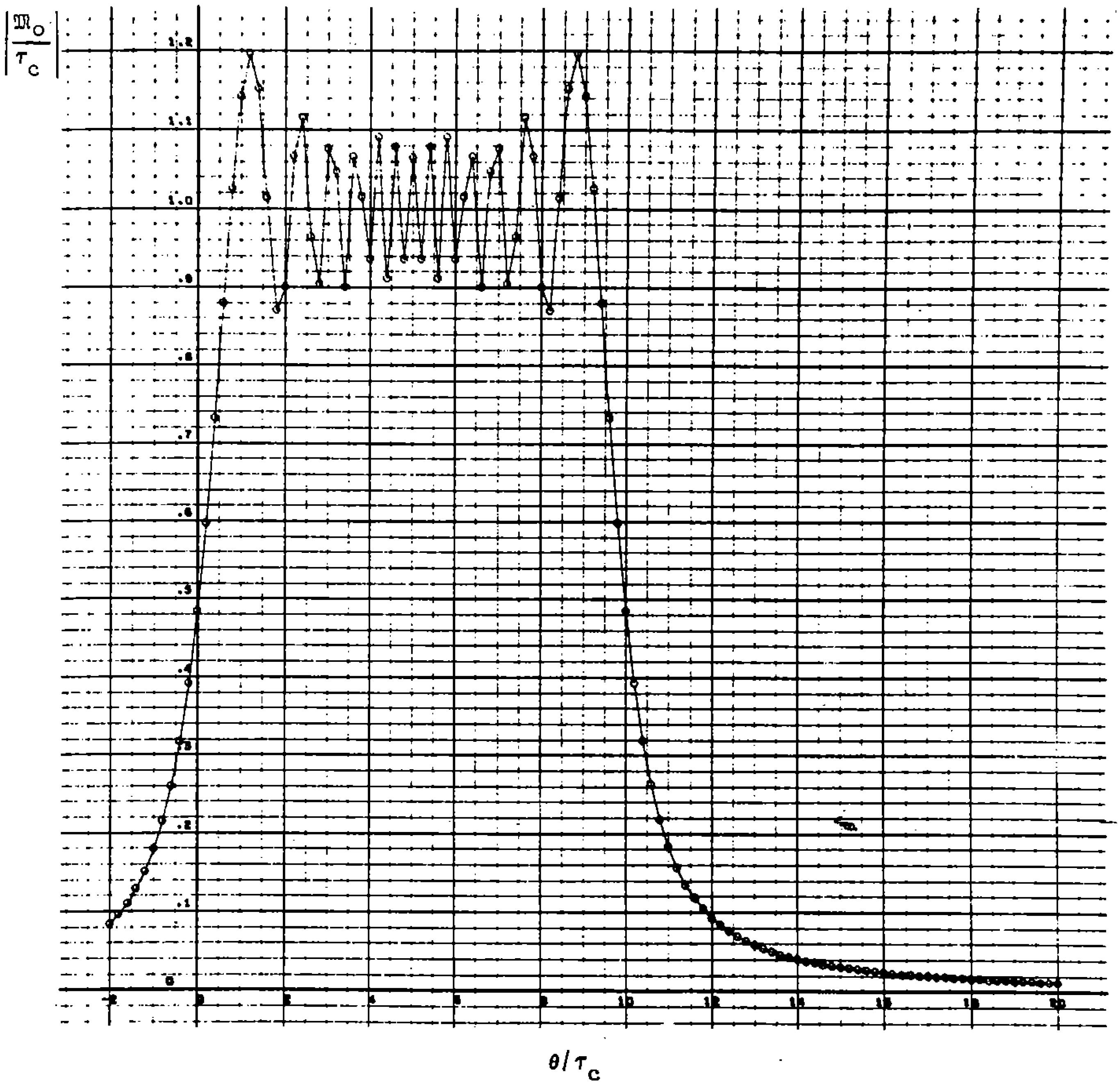

Fig. 1a Magnitude of the zeroth-order ionospheric moment for $T/\tau_c = 10$.

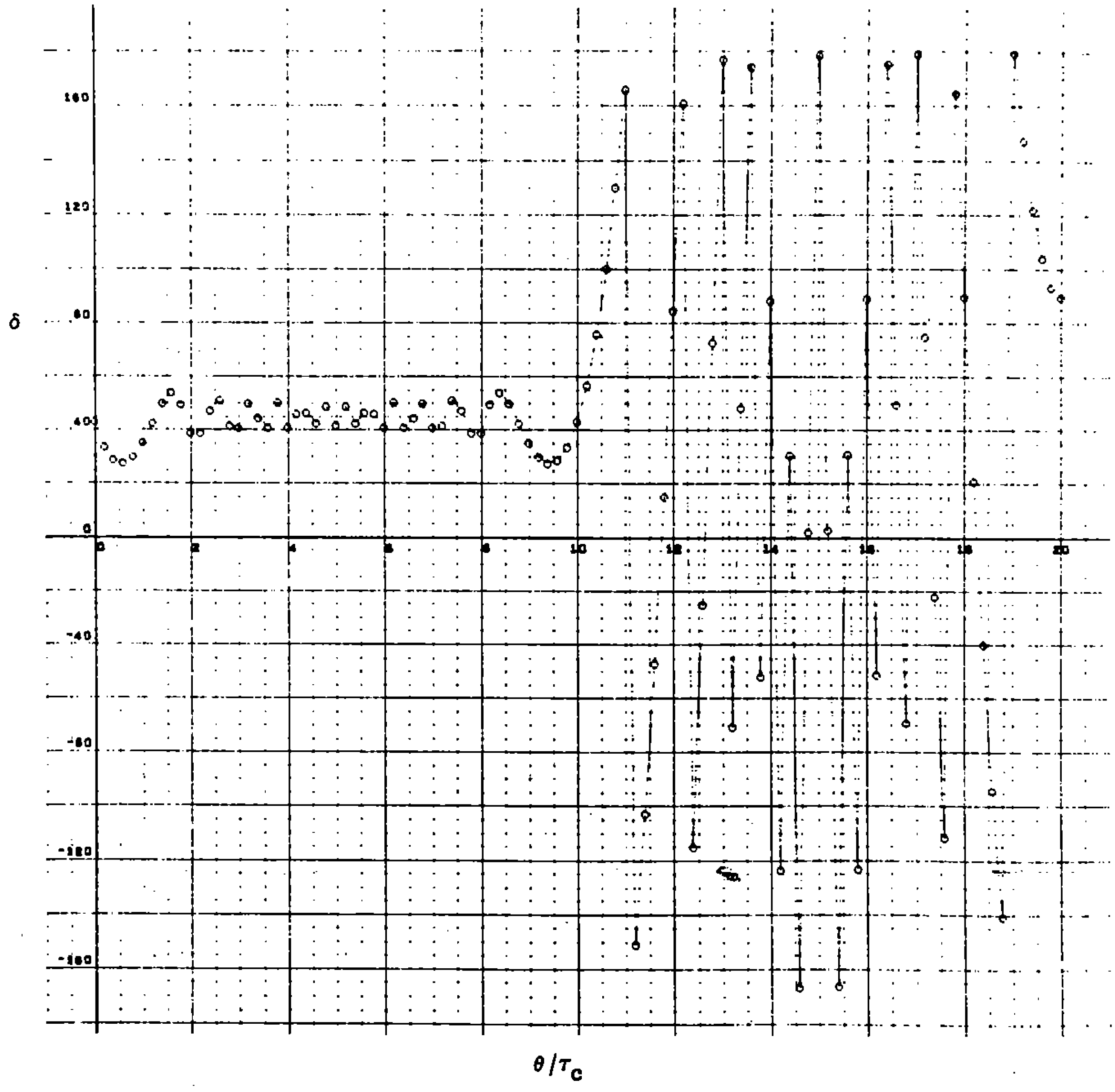

Fig. 1b Phase angle of the zeroth-order ionospheric moment for T/τ_c = 10.

At this point it is informative to consider as an illustrative example the dispersion of a pulsed sinusoidal source signal with a carrier frequency ω_o and an envelope function consisting of a cosine wave of half-period duration, π/ω_m, as shown in Fig. 2. The source function is defined by

$$e_s(t) = \begin{cases} \cos \omega_m t \cos \omega_o t\,, & -\frac{T}{2} < t < \frac{T}{2} \\ 0\,, & |t| > T/2\,, \end{cases} \tag{50}$$

where $T = \pi/\omega_m$. The spectrum is obtained as

$$F(\omega) = \int_{\pi/2\omega_m}^{\pi/2\omega_m} \cos \omega_m t \cos \omega_o t \cos \omega t \; dt$$

$$= \cos\left[(\omega - \omega_o)(\pi/2\omega_m)\right] \frac{\omega_m}{\omega_m^2 - (\omega - \omega_o)^2}$$

$$+ \cos\left[(\omega + \omega_o)(\pi/2\omega_m)\right] \frac{\omega_m}{\omega_m^2 - (\omega + \omega_o)^2}\,. \tag{51}$$

For $\omega_o \gg \omega_m$, the second term in the last equation is very small for positive ω. The approximate spectrum of the transmitted signal can then be written as

$$F(x) \approx \frac{1}{\omega_m} \frac{1}{1 - x^2} \cos \frac{1}{2} \pi x\,, \tag{52}$$

where

$$x = \frac{\omega - \omega_o}{\omega_m} = \frac{\Delta\omega}{\omega_m}\,.$$

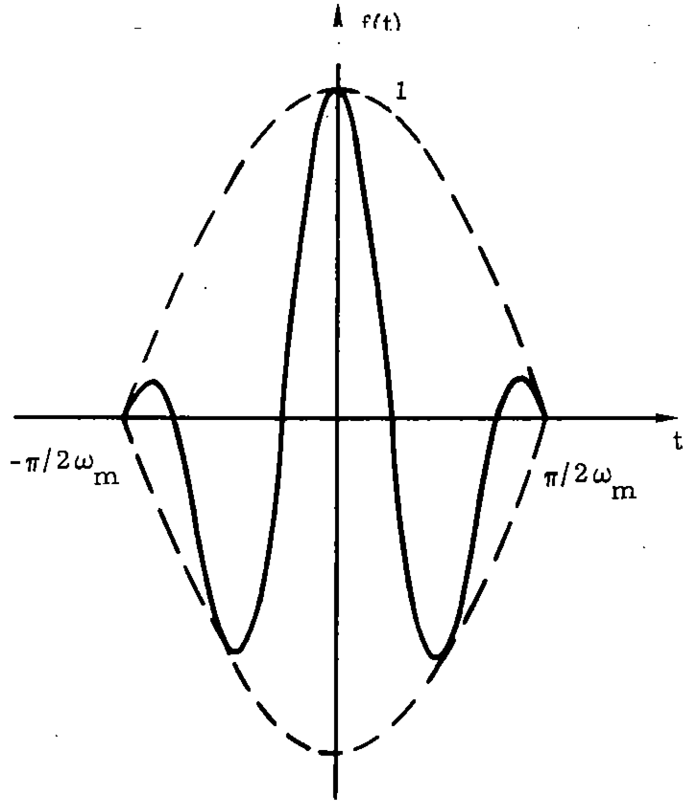

Fig. 2 Cosine wave of duration π/ω_m.

The expression for F(x) has zeros where

$$\Delta\omega = (2n + 1)\,\omega_m\,, \quad n \neq 0\,.$$

From Fig. 3, a plot of Eq. (52), it can be observed that most of the energy is contained in that part of the spectrum from $\omega_o - 3\omega_m$ to $\omega_o + 3\omega_m$. This is indicative of the extent to which the source signal described by (50) can be considered as monochromatic.

By substituting $\cos \omega_m \lambda$ for $f_m(\lambda)$ in (12) and suppressing the carrier term, the following expression is obtained for the dispersed signal at the receiver:

$$e_R(t) \approx \frac{1 - j}{4}\left\{\left[F\left(\frac{T - \theta}{\tau_c} + \frac{\tau_c}{T}\right) + F\left(\frac{\theta}{\tau_c} - \frac{\tau_c}{T}\right)\right] e^{j\left(\frac{\pi\theta}{T} - \frac{\pi}{2}\right)} + \left[F\left(\frac{T - \theta}{\tau_c} - \frac{\tau_c}{T}\right) + F\left(\frac{\theta}{\tau_c} + \frac{\tau_c}{T}\right)\right] e^{-j\left(\frac{\pi\theta}{T} - \frac{\pi}{2}\right)}\right\}. \tag{53}$$

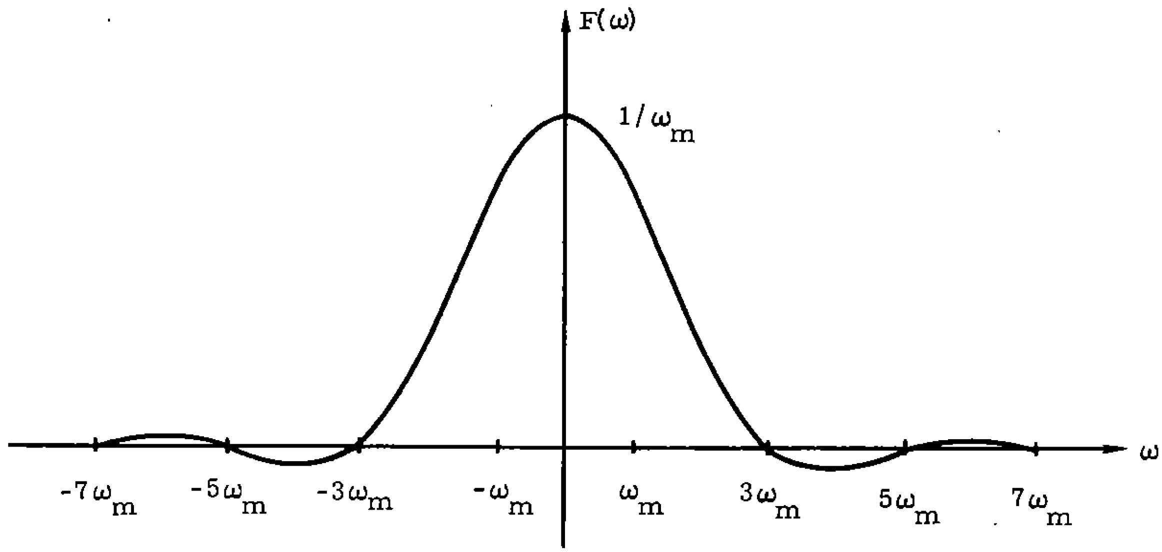

Fig. 3 Spectrum of cosine wave of Fig. 2.

As expected, in the limit as τ_c becomes vanishingly small, $e_R(t)$ is given by

$$\lim_{\tau_c \to 0} e_R(t) \approx \cos \omega_m (t - \tau_g) \ . \tag{54}$$

The envelope, although delayed by τ_g, has undergone distortionless transmission and is not dispersed.

If, on the other hand, $T \gg \tau_c$, then Eq. (53) reduces to the following form:

$$e_R(t) \approx \frac{1 - j}{2} \left[F\left(\frac{T - \theta}{\tau_c}\right) + F\left(\frac{\theta}{\tau_c}\right)\right] \sin \frac{\pi \theta}{T} \ , \tag{55}$$

or

$$e_R(t) \approx \frac{1 - j}{2} \mathfrak{M}_o \cos \omega_m (t - \tau_g) \ ; \tag{56}$$

i.e., a first approximation to the dispersed signal at the receiver for $T \gg \tau_c$ is given by the product of the zeroth-order ionospheric moment and the envelope function delayed by τ_g. From (55), the envelope of the dispersed signal is given by

$$e_R\left(\frac{\theta}{\tau_c}\right) = \frac{1}{\sqrt{2}}\left|\sin \pi\left(\frac{\tau_c}{T}\right)\left(\frac{\theta}{\tau_c}\right)\right|\left\{\left[C\left(\frac{T-\theta}{\tau_c}\right) + C\left(\frac{\theta}{\tau_c}\right)\right]^2 + \left[S\left(\frac{T-\theta}{\tau_c}\right) + S\left(\frac{\theta}{\tau_c}\right)\right]^2\right\}^{1/2} . \tag{57}$$

Figure 4 shows a plot of Eq. (57) for $\left(\frac{T}{\tau_c}\right) = 10$. A comparison of Figs. 4 and 1 illustrates the simple relationship given by (56).

Figure 4 shows the spreading or "smearing out" of the signal characteristic of dispersion. The effect becomes more pronounced as the TEC, and therefore as τ_c, increases. Figure 5 shows a dispersed cutoff sinusoid, i.e., a signal whose envelope is a rectangular pulse for $(T/\tau_c) = 1.6$. The envelope of the signal at the receiver is given by

$$f_R\left(\frac{\theta}{\tau_c}\right) = \frac{1}{\sqrt{2}}\left|F\left(\frac{T-\theta}{\tau_c}\right) + F\left(\frac{\theta}{\tau_c}\right)\right| . \tag{58}$$

The particular value of T/τ_c used was selected to emphasize the initial portion of the wavefront. Apparently the cutoff sinusoid the spectral content of which is

$$F(\omega) = \frac{\sin \omega T}{\omega T} \tag{59}$$

is "less quasi-monochromatic" than that of the sinusoidal pulse (see Eq. (51)).

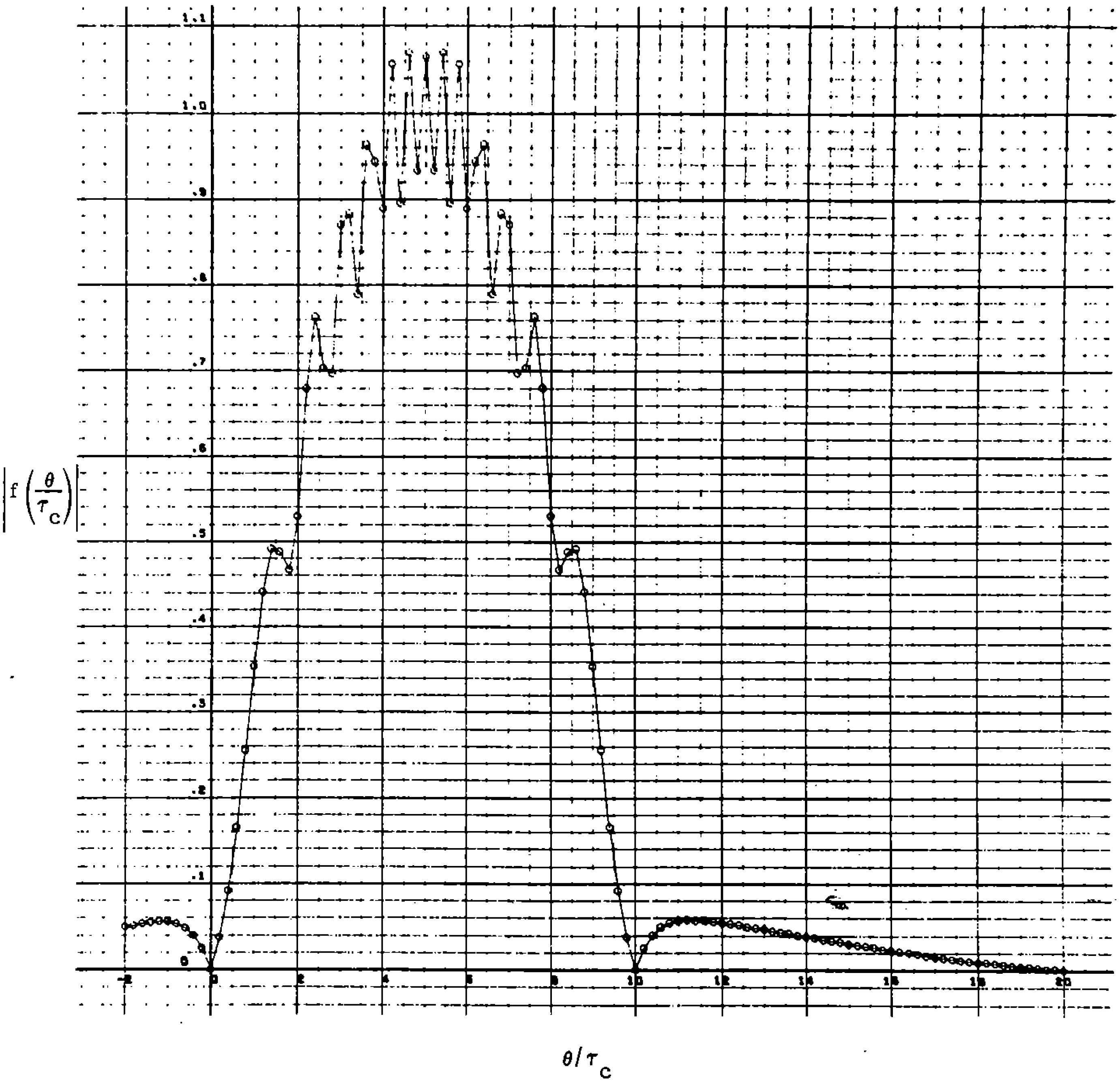

Fig. 4 Dispersed sinusoidal pulse for $\frac{T}{\tau_c} = 10$,

$$f\left(\frac{\theta}{\tau_c}\right) = \frac{1}{\sqrt{2}} \left|\sin \frac{\pi\theta}{T}\right| \cdot \left|F\left(\frac{T-\theta}{\tau_c}\right) + F\left(\frac{\theta}{\tau_c}\right)\right| .$$

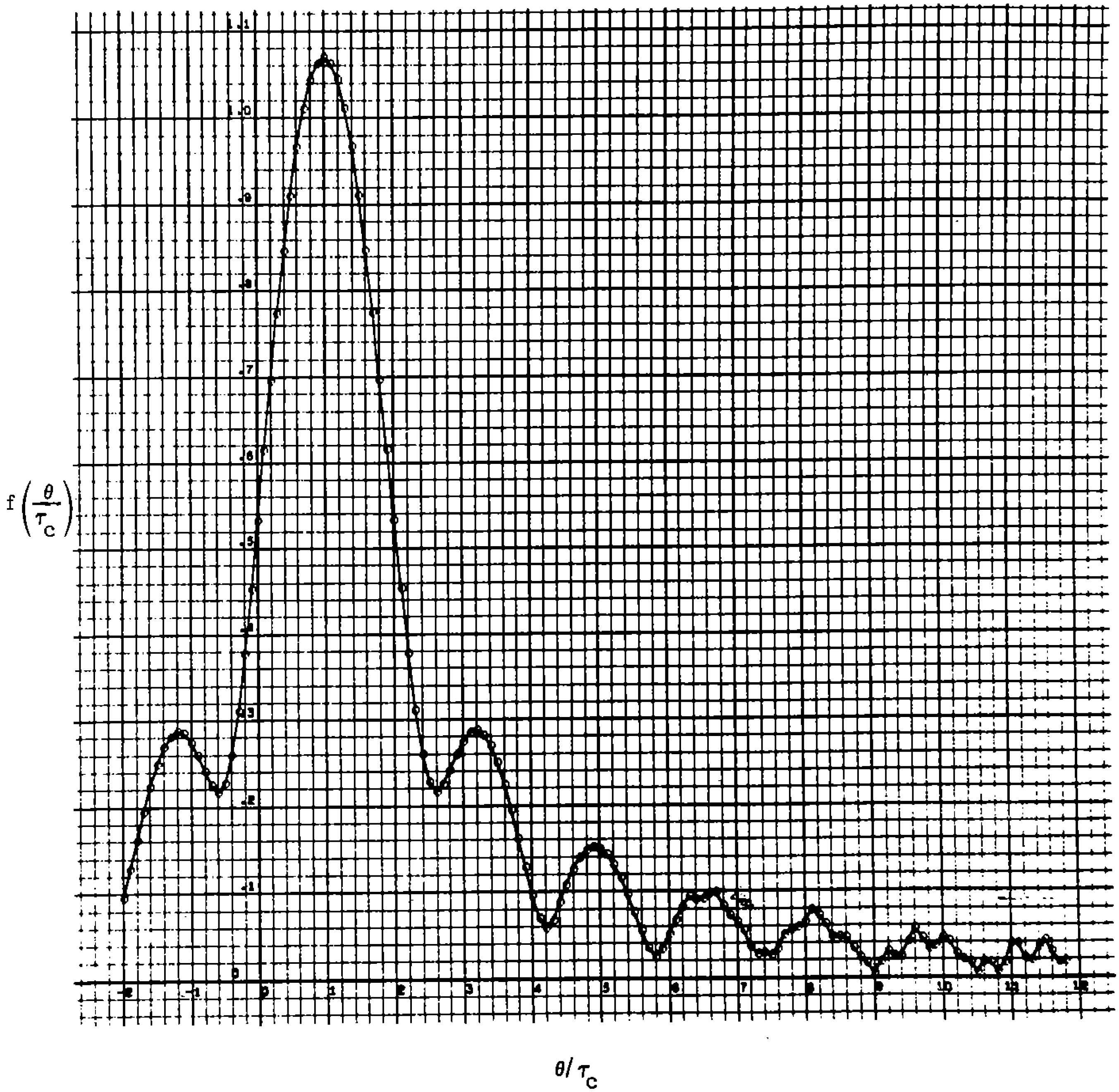

Fig. 5 Dispersed cutoff sinusoid for T/τ_c = 1.6,

$$f\left(\frac{\theta}{\tau_c}\right) = \frac{1}{\sqrt{2}} \left| F\left(\frac{T - \theta}{\tau_c}\right) + F\left(\frac{\theta}{\tau_c}\right) \right| .$$

In the spectral decomposition of signals of finite duration, frequencies as high as desired can be found for which the group velocity approaches the velocity of light. Therefore, some signal energy propagates almost at the speed of light. This portion of the signal is known as a "precursor" or "forerunner." However, that portion of the signal observed in Fig. 5 (and to a lesser degree in Fig. 4) is not a precursor but, rather, an apparent breakdown in causality. This occurred because of liberties taken with the ionospheric transfer function, Eq. (3), in order to simplify the discussion.* With reference to (3), if a system is to be causal, $|R(\omega_o)|$ and $\phi(\omega)$ cannot be assigned values independently of each other. It should be noted that the ionospheric transfer function should be written

$$R(\omega) = |R(\omega)|\ e^{-j\phi(\omega)} = A(\omega)\ e^{-j\phi(\omega)}\ , \qquad (3a)$$

with

$$\phi(\omega) = \tan^{-1} \frac{\mathcal{I}R(\omega)}{\mathcal{R}e R(\omega)}\ .$$

A necessary and sufficient condition for a square-integrable function $A(\omega) \geq 0$ to be the Fourier spectrum of a causal function is that the Paley-Wiener condition be satisfied, i.e., that the form of $A(\omega)$ be so specified as to insure that

$$\int_{-\infty}^{\infty} \frac{|\log A(\omega)|}{1 + \omega^2}\ d\omega < \infty\ . \qquad (60)$$

The causality requirement was abandoned in order not to obscure relationships between time responses and the various frequency characteristics.

*Although often used, this artifice usually leads to some confusion. Therefore, the clarification inserted in the discussion at this point is deemed appropriate.

AN APPLICATION

The simplicity of the form of Eq. (41) suggests an approach to "antidispersive" filtering. A detailed discussion of this topic is beyond the scope of the present paper. However, the derivation of a post-detection filter to obtain a first approximation to the source envelope is at least conceptually elementary if not realizeable. A simple example should suffice.

On taking $t' = t - \tau_g$ and truncating (41) after the first derivative, a first estimate of the envelope $f_m(t')$ of a field component is obtained from

$$f(t') \sim \left[\mathfrak{M}_o f_m(t') - \mathfrak{M}_1 \frac{d}{dt} f_m(t') \right] . \tag{61}$$

Solving for $\frac{d}{dt} f_m(t')$, one obtains

$$\frac{d}{dt} f_m(t') \sim \frac{\mathfrak{M}_o}{\mathfrak{M}_1} f_m(t') - \frac{1}{\mathfrak{M}_1} f(t') . \tag{62}$$

Figure 6 shows one possible analog implementation for obtaining $f_m(t')$ given $f(t')$ as an input. Unfortunately, the illustrated program is deceptively simple because the coefficients $(1/\mathfrak{M}_1)$ and $(\mathfrak{M}_0/\mathfrak{M}_1)$ not only vary with time (see Fig. 1a) but also with the TEC (total electron content). However, if the moments $\mathfrak{M}_0$, $\mathfrak{M}_1$, ... can be generated with any degree of confidence, computation of the estimates is quite straightforward.

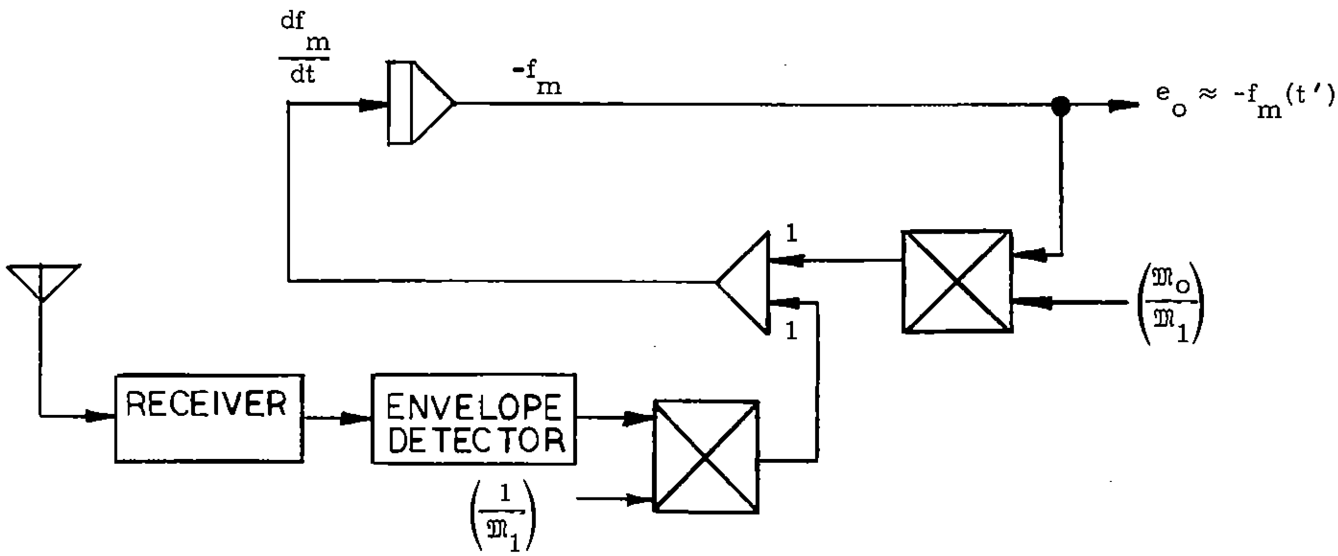

Fig. 6 Postdetector signal processing for implementing first-order approximation to a source signal.

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