Theoretical Notes
Note 115
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Electric Fields Produced by an Electronic Current Emitted Perpendicular to a Surface

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Abstract

Poisson's equation is solved for the charge density, electric field, and potential as a function of distance, due to a surface current of electrons emitted from a flat surface. The current is assumed independent of time and all electrons are returned to the plate due to the space potential of the electron cloud. The development is general in the sense that no restrictions are placed on the distribution of electron energies; however, three particular electron spectra are considered in detail: monoenergetic, uniform, and triangular spectra. Finally, assuming the surface current is produced by a photon source, generalized scaling laws for the electric field and penetration distances as a function of photon fluence, radiation time, and electron energy are obtained.

1. Introduction

A problem of interest in the analysis of system interactions with X and gamma radiation is that of predicting space-charge limited electromagnetic fields associated with energetic photoelectrons and Compton electrons. In general these fields are a complex function of the radiation parameters (intensity, spectrum, impulse length) and the structural parameters (chemical composition, geometry, orientation with respect to incident radiation). In the development which follows, a much simpler problem is solved, that of the electric field produced by electrons emitted normally from a plate and subsequently returned to the plate by the retarding potential of the electron cloud.

Various aspects of space-charge have been studied by a number of workers. The 3/2 power law for the relationship between voltage and transmitted current was obtained by Childs (Ref. 1) in 1911. Later, Fry (Ref. 2) obtained a solution for the voltage distribution and transmitted current for a Maxwell energy distribution of electrons. An exhaustive solution for the voltage distribution, transmitted current, and transit time was obtained by Fay, et al. (Ref. 3) for the case of monoenergetic electrons emitted from plane parallel electrodes at arbitrary potentials. In computing the dipole moment for charge emitted from and returned to a single plate, Karzas and Latter (Ref. 4) obtained the charge and velocity distributions for monoenergetic electrons. This is also a special case of the work by Fay, et al. (Ref. 3). The case of an arbitrary energy distribution of electrons emitted between plane, parallel electrodes at arbitrary potentials was solved by Baum.* This development is similar to but more general than that of Fry (Ref. 2), who treated only the case of a Maxwell electron spectrum.

*Baum, C. E., "Gamma Internally Generated EM Fields: Notes and Calculations," unpublished notes.

2. Solution to Poisson's Equation

The following development was adapted from that of Baum for the case of a single rather than parallel plates. Figure 1 illustrates the geometry, and the restrictions are summarized below:

a. no variations in the plane of the plate; variations occur only perpendicular to the plate
b. steady-state emitted current
c. arbitrary distribution of electron energies
d. all electrons return to the plate; no net current in the x direction
e. the plate potential is arbitrarily set to zero
f. emitted electrons move perpendicular to the plate (this condition can be relaxed if the energy distribution properly accounts for the x component of velocity)

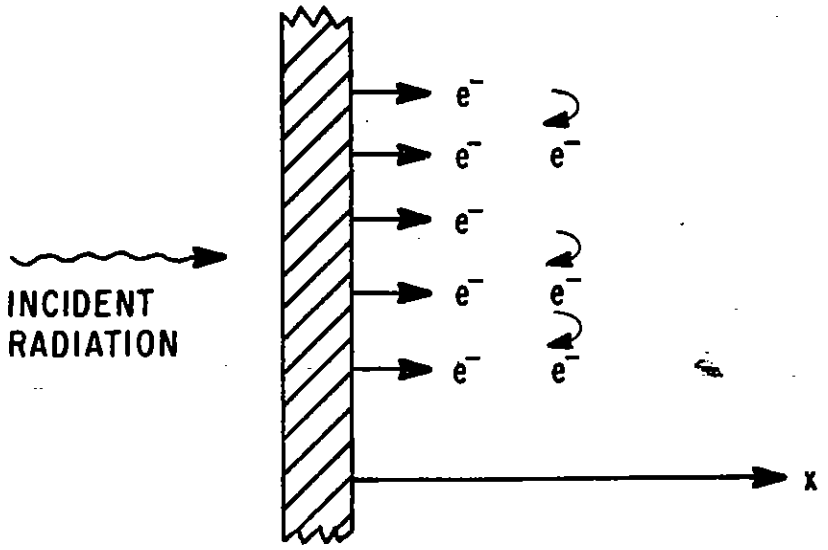


Figure 1. Charge emitted and returned to a single plate

The starting point is Poisson's equation in one-dimensional Cartesian coordinates:

$$\frac{d^2V}{dx^2} = -\frac{\rho}{\varepsilon} \qquad (1)$$

where V is the potential at position x (volts)

x is the distance from plate (meters)

ρ is the charge density at position x (coulombs/m^3)

ε is the vacuum dielectric constant ($10^{-9}/36\pi$ farad/meter)

The expression for charge density must account for variable electron velocities and the fact that at any point x, electrons with energies less than -eV(x) have turned back. The correct expression is

$$\rho(x) = -2J\int_{V_c}^{V(x)} \frac{n(V_o)dV_o}{v(V_o,V)} \tag{2}$$

where J is the emitted current density, assumed positive (amp/m^2)

V_c is the potential corresponding to maximum electron energy (volts)

$n(V_o)$ is the normalized emitted electron spectrum (volts^{-1})

v is the x component of electron velocity (meters/sec)

The factor 2 enters Equation (2) due to the fact all electrons return to the plate, doubling the charge density. The sign convention follows from the fact that all potentials are negative.

If non-relativistic electronic energies are assumed, the velocity can be written

$$v(V_o,V) = \left[-\frac{2e}{m}(V_o-V)\right]^{1/2} \tag{3}$$

in terms of the initial and final potentials, where e/m is the electronic charge-to-mass ratio (1.759×10^{11} coulomb/kg). The term "normalized" in the definition of $n(V_o)$ means simply that

$$\int_{V_c}^{o} n(V_o)dV_o = 1 \tag{4}$$

Clearly, Equation (1) is nonlinear due to the square root dependence of charge densities on potential. However, the first integration may be accomplished by application of the property

$$\frac{d^2V}{dx^2} = \frac{d}{dV}\left[\frac{1}{2}\left(\frac{dV}{dx}\right)^2\right] \tag{5}$$

The result is

$$\frac{dV}{dx} = \pm\left[\frac{4J}{\varepsilon}\left(\frac{2e}{m}\right)^{-1/2}\int_{V_c}^{V} dV_1 \int_{V_c}^{V_1} \frac{n(V_o)\,dV_o}{(V_1-V_o)^{1/2}}\right]^{1/2} \tag{6}$$

Equation (6) also represents the electric field E, since

$$E = -\frac{dV}{dx} \tag{7}$$

A subsequent integration of Equation (6) yields

$$x = -\frac{1}{2}\left(\frac{J}{\varepsilon}\right)^{-1/2}\left(\frac{2e}{m}\right)^{1/4}\int_{o}^{V} dV_2 \left[\int_{V_c}^{V_2} dV_1 \int_{V_c}^{V_1} \frac{n(V_o)\,dV_o}{(V_1-V_o)^{1/2}}\right]^{-1/2} \tag{8}$$

The negative sign before the radical in Equation (6) was chosen on the physical observation that the electric field E must be everywhere positive.

Another quantity of interest is the stored electric field energy per unit area, defined

$$U = \int_{o}^{x_c} \frac{1}{2}\,\varepsilon E^2 dx \tag{9}$$

which can also be written

$$U = -\frac{1}{2}\,\varepsilon \int_{o}^{V_c} E\,dV \tag{10}$$

or

$$U = \int_{V_c}^{o} dV_2\ \frac{1}{2}\,\varepsilon \left[\frac{4J}{\varepsilon}\left(\frac{2e}{m}\right)^{-1/2} \int_{V_c}^{V_2} dV_1 \int_{V_c}^{V_1} \frac{n(V_o)\,dV_o}{(V_1 - V_o)^{1/2}}\right]^{1/2} \tag{11}$$

after inserting the right-hand side of Equation (6) for E. The quantity x_c in Equation (9) is the turn-around point for the most energetic electrons.

In order to present results which apply to general classes of problems, Equations (2), (6), (8), and (11) are normalized, using the following definitions:

$$u = V/V_c \tag{12}$$

$$y = x/x_n \tag{13}$$

$$\xi = E/E_n \tag{14}$$

$$p = \rho/\rho_n \tag{15}$$

$$\bar{n}(u) = -V_c n(uV_c) \tag{16}$$

$$\bar{U} = U/U_n \tag{17}$$

where

$$x_n = \frac{1}{2}(-V_c)^{3/4}\left(\frac{2e}{m}\right)^{1/4}\left(\frac{J}{\varepsilon}\right)^{-1/2} \tag{18}$$

$$E_n = 2\left(\frac{2e}{m}\right)^{-1/4}\left(\frac{J}{\varepsilon}\right)^{1/2}(-V_c)^{1/4} \tag{19}$$

$$\rho_n = -2J\left(-\frac{2eV_c}{m}\right)^{-1/2} \tag{20}$$

$$U_n = (-V_c)^{5/4}(\varepsilon J)^{1/2}\left(\frac{2e}{m}\right)^{-1/4} \tag{21}$$

After change of variables, the charge density, electric field, potential, and stored energy equations take on the following simple forms:

$$p(u) = \int_u^1 \frac{\bar{n}(u_o)\,du_o}{(u_o-u)^{1/2}} \tag{22}$$

$$\xi(u) = \left[\int_u^1 du_1 \int_{u_1}^1 \frac{\bar{n}(u_o)\,du_o}{(u_o-u_1)^{1/2}}\right]^{1/2} \tag{23}$$

$$y(u) = \int_o^u du_2 \left[\int_{u_2}^1 du_1 \int_{u_1}^1 \frac{\bar{n}(u_o)\,du_o}{(u_o-u_1)^{1/2}}\right]^{-1/2} \tag{24}$$

$$\bar{U} = \int_o^1 du_2 \left[\int_{u_2}^1 du_1 \int_{u_1}^1 \frac{\bar{n}(u_o)\,du_o}{(u_o-u_1)^{1/2}}\right]^{1/2} \tag{25}$$

3. Applications to Particular Electron Spectra

Three electron distributions are examined in this section to determine the effect on charge density, electric field, potential, and electrostatic energy of different spectra. It was possible to perform the integrations analytically for the two cases, monoenergetic and uniform spectra; however one numerical

integration was required for the triangular spectrum. Solutions for the three cases are summarized below.

a. Monoenergetic Spectrum

$$\bar{n}(u) = \delta(u-1) \tag{26}$$

$$p(y) = \left(1 - \frac{3y}{2\sqrt{2}}\right)^{-2/3} \tag{27}$$

$$\xi(y) = \sqrt{2}\left(1 - \frac{3y}{2\sqrt{2}}\right)^{1/3} \tag{28}$$

$$u(y) = 1 - \left(1 - \frac{3y}{2\sqrt{2}}\right)^{4/3} \tag{29}$$

$$\bar{U} = \frac{4}{5}\sqrt{2} \tag{30}$$

b. Uniform Spectrum

$$\bar{n}(u) = \begin{cases} 1 & 0 \leq u \leq 1 \\ 0 & u > 1 \end{cases} \tag{31}$$

$$p(y) = 2\left(1 - \frac{y}{2\sqrt{3}}\right)^{2} \tag{32}$$

$$\xi(y) = \frac{2}{\sqrt{3}}\left(1 - \frac{y}{2\sqrt{3}}\right)^{3} \tag{33}$$

$$u(y) = 1 - \left(1 - \frac{y}{2\sqrt{3}}\right)^{4} \tag{34}$$

$$\bar{U} = \frac{8}{7\sqrt{3}} \tag{35}$$

c. Triangular Spectrum

$$\bar{n}(u) = \begin{cases} 2u & 0 \leq u \leq 1 \\ 0 & u > 1 \end{cases} \tag{36}$$

$$p(u) = \frac{4}{3}(1-u)^{1/2}(1+2u) \tag{37}$$

$$\xi(u) = 2\sqrt{\frac{2}{15}}(1-u)^{3/4}(2u+3)^{1/2} \tag{38}$$

$$y = \frac{1}{2}\sqrt{\frac{15}{2}}\int_o^u (1-u_1)^{-3/4}(2u_1+3)^{-1/2}du_1 \tag{39}$$

$$\bar{U} = 2\sqrt{\frac{2}{15}}\int_o^1 (1-u)^{3/4}(2u+3)^{1/2}du \tag{40}$$

The normalized spectrum for each of the three cases is shown graphically in Figure 2, while solutions for charge density, electric field, and potential are plotted in Figures 3, 4, and 5. Since the same normalization factors are used for all spectra, the graphs portray a true linear comparison of the effects of the different spectra investigated. Absolute or true values may be obtained by application of the formulas shown on each graph. The small arrows along the y-axis on the graphs indicate the point of intersection of nearby curves with the y-axis. These intersection points are otherwise unclear due to the zero slope of the curves at intersection.

An unexpected result of this investigation is that the peak electric field, which occurs at the emitting surface, appears to be weakly dependent upon electron spectrum (see Figure 4). For the spectra investigated, variations are within ± 10 percent of the average of the three values. An additional observation is that as the proportion of low-energy electrons is increased, the range of the electric field from the plate

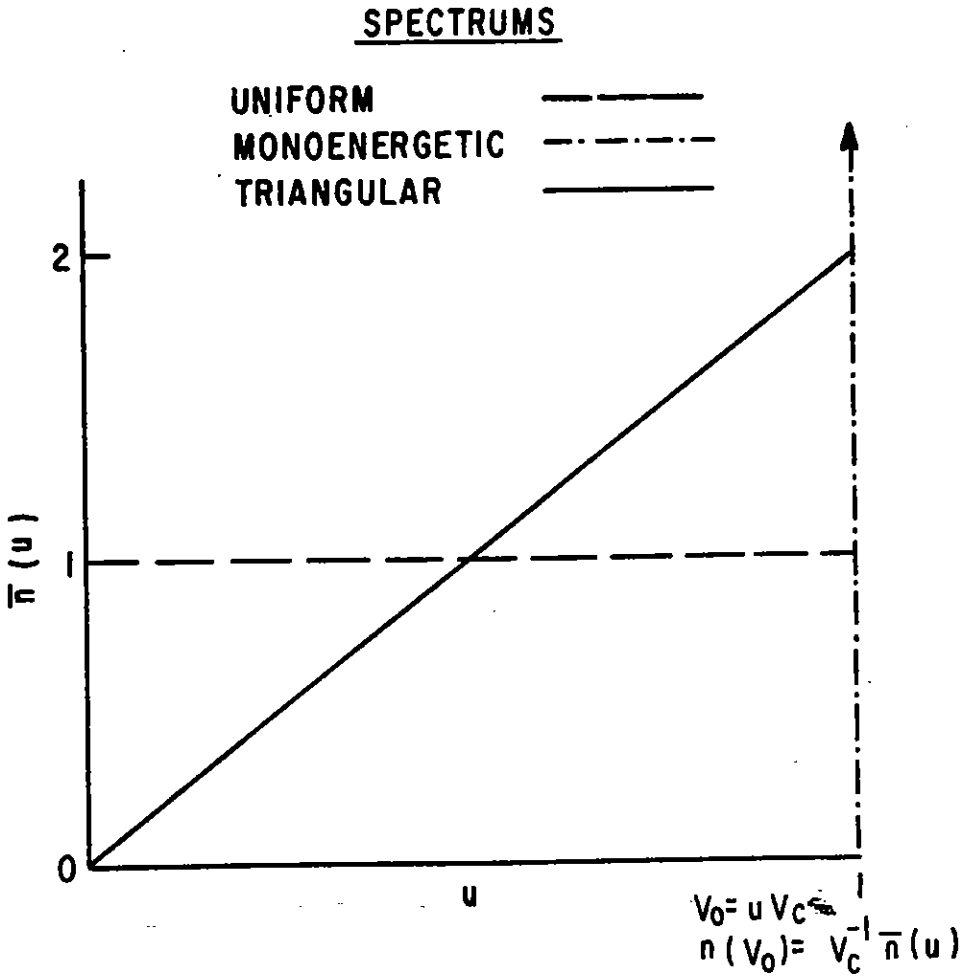


Figure 2. Electron spectra for example problems

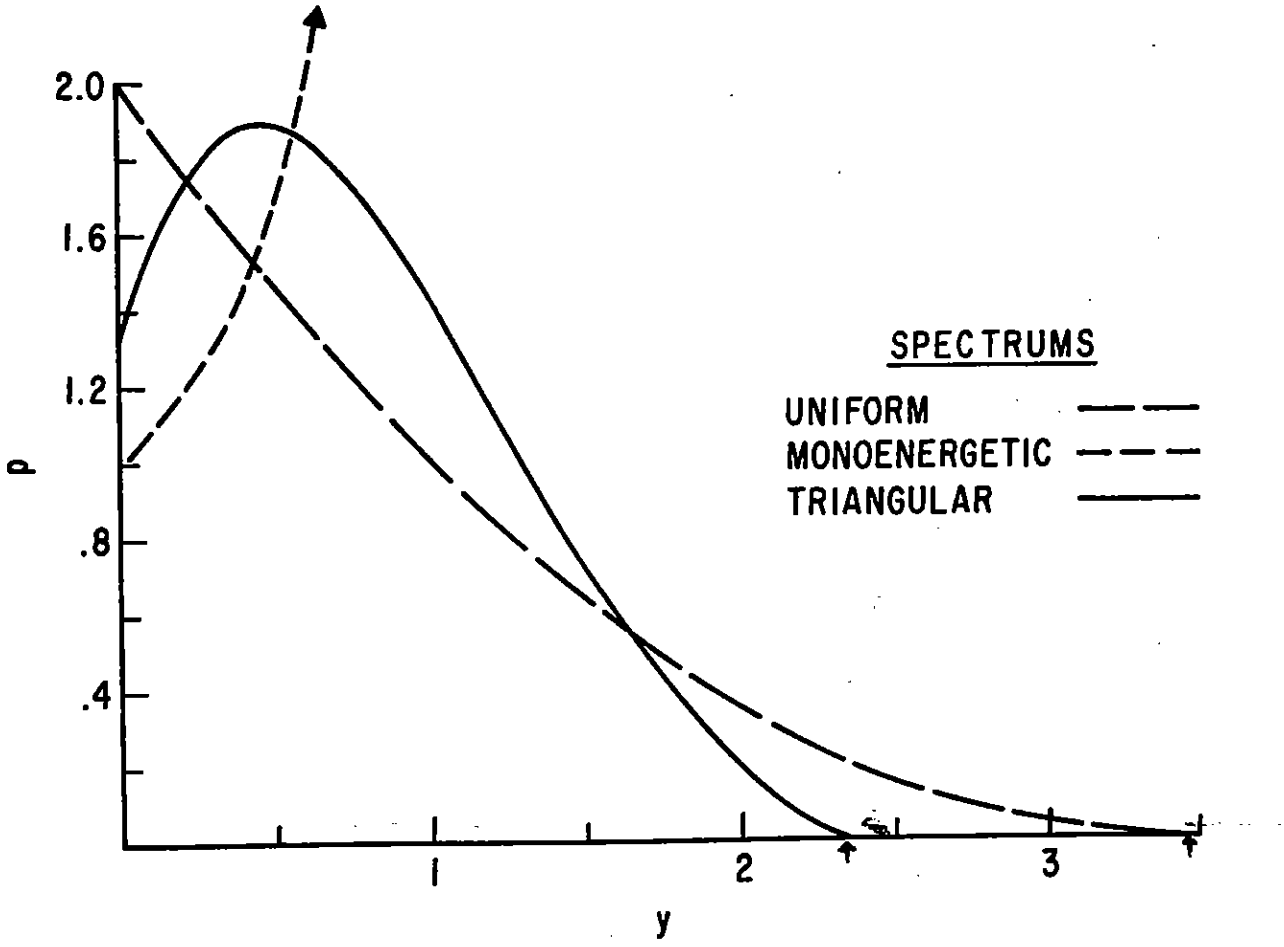


Figure 3. Normalized charge density for example problems

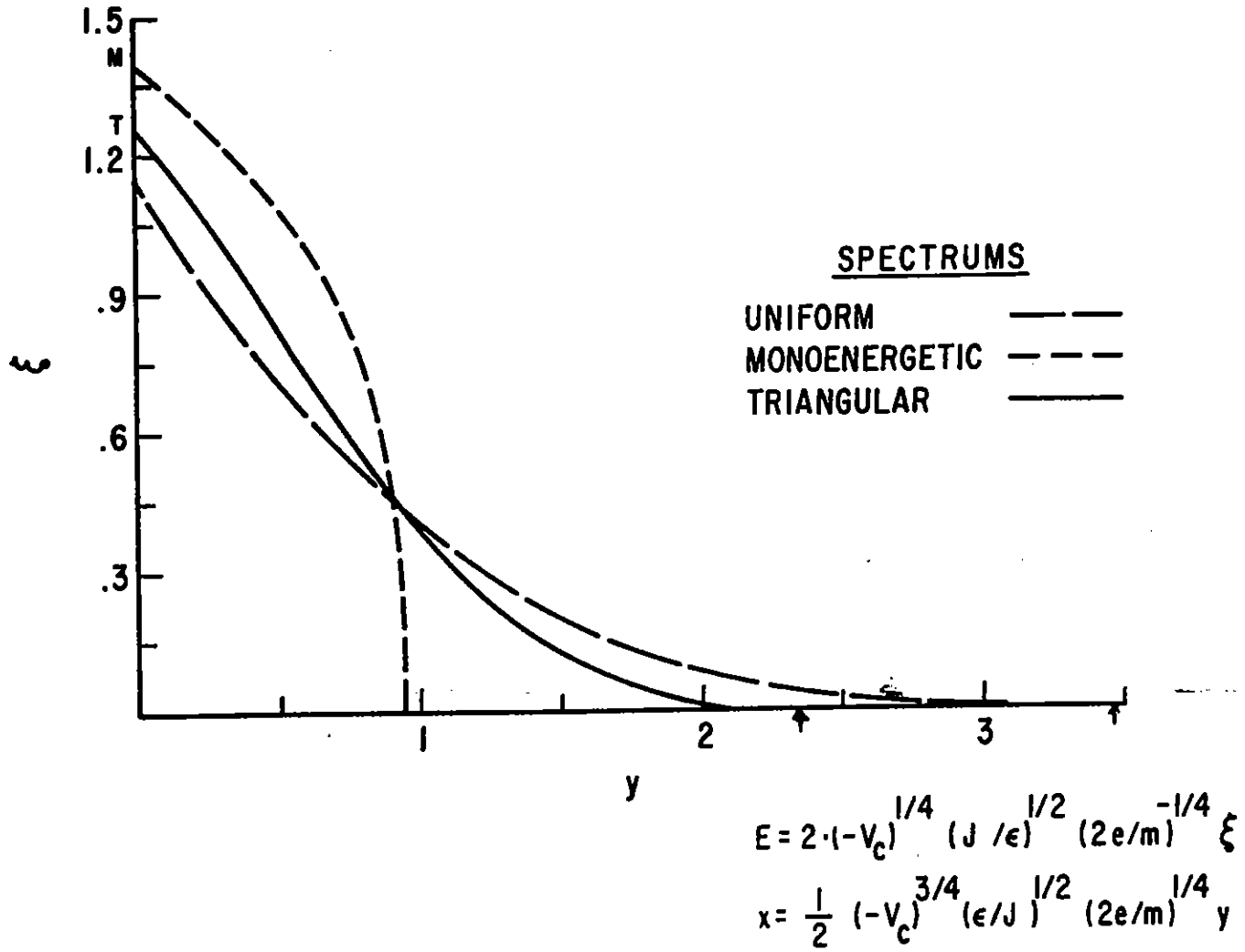
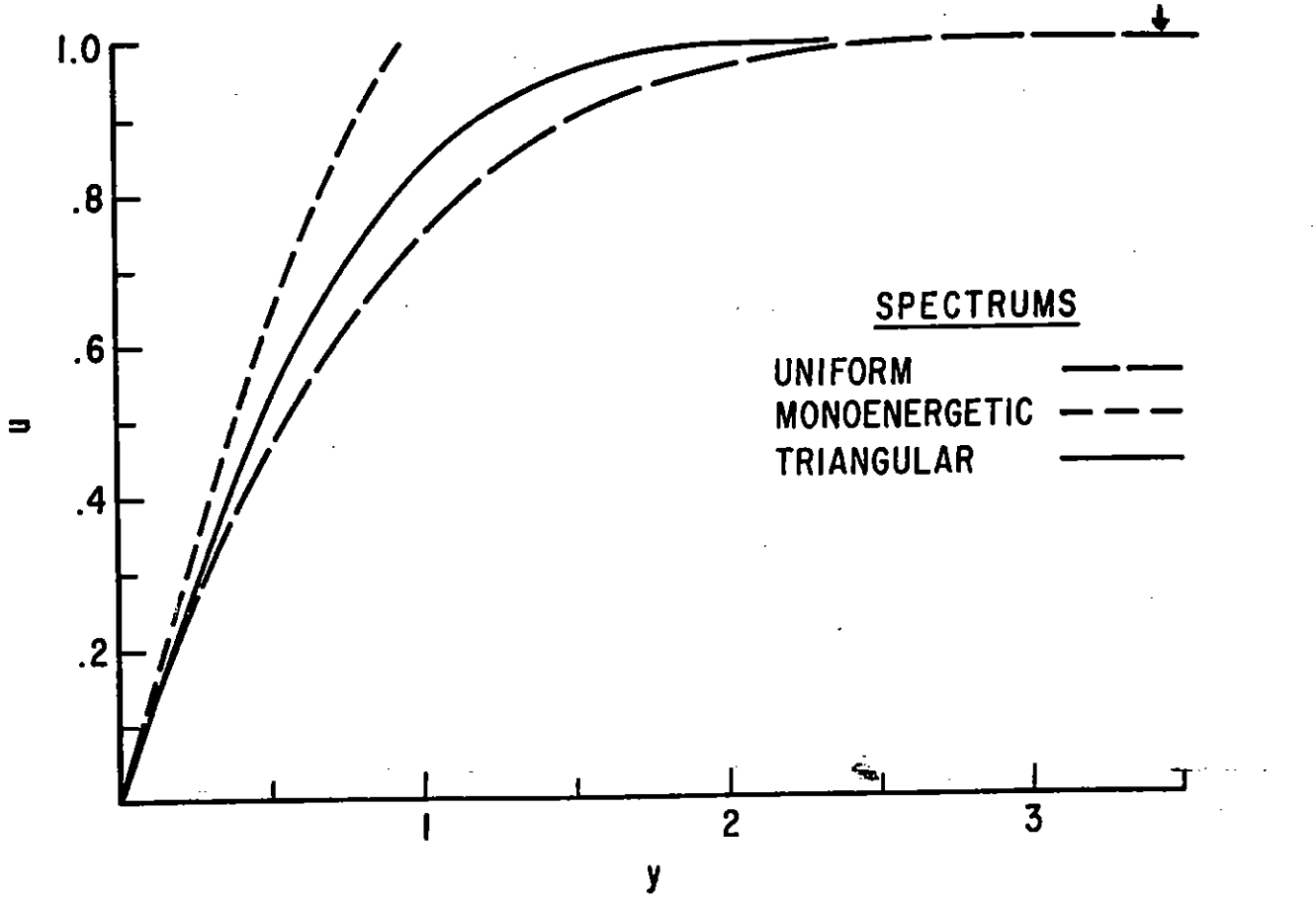


Figure 4. Normalized electric field for example problems



$$V = uV_c$$

$$x = \frac{1}{2}(-V_c)^{3/4}(\epsilon/J)^{1/2}(2e/m)^{1/4}y$$

Figure 5. Normalized potential distribution for example problems

increases. This fact follows from the larger electric field intensity near the plate as the proportion of high-energy electrons is increased.

4. Summary and Discussion

The development through Equation (25) placed no restriction on the emitted electron spectrum, except to place an upper bound on electron energy. Hence, expressions (6) through (25) are applicable to an arbitrary electron spectrum. In addition, explicit expressions and graphical results were obtained for three electron distributions: monoenergetic, uniform, and triangular.

The normalization factors x_n and E_n defined by Equations (18) and (19) can also be viewed as general scaling laws. The emitted current J is induced by photoelectric and Compton collisions within the plate and is therefore proportional to the photon flux measured at the emitting surface. Hence, J is proportional to the photon fluence ϕ, and inversely proportional to the radiation time T. The scaling laws can then be written

$$x \sim |V_c|^{3/4} \phi^{-1/2} T^{1/2}$$

$$E \sim |V_c|^{1/4} \phi^{1/2} T^{-1/2}$$

after removing all constant factors. In a vacuum medium, the fluence ϕ varies as r^{-2} where r is the distance from source to the irradiated object; hence, the electric fields vary as r^{-1} and the effective distance fields penetrate, measured from the plate, vary linearly with r.

The reader is cautioned in applying these results to time-dependent problems where other effects occur, such as radiated and oscillatory fields. Oscillatory fields have been predicted by Birdsall and Bridges (Ref. 5), Karzas and Latter (Ref. 4),

and by Chadsey (Ref. 6) for monoenergetic or very narrow energy spectra. However, the latter two papers indicate that the oscillatory behavior is small if not negligible for wideband electron spectra. In addition, one must assure that the radiation time T is long compared to the maximum electron time of flight from the time of emission to the time of return to the plate. Furthermore, it may be difficult to select the correct parameter values for application of the scaling laws to realistic electron spectra.

Acknowledgement

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References

1. Child, C. D.; "Discharge from Hot CaO," Physical Review, 32, pp. 492-511, 1911.

2. Fry, T. C.; "Thermionic Current Between Parallel Plane Electrodes; Velocities of Emission Distributed According to Maxwell's Law," Physical Review, 17, pp. 441-452, April 1921.

3. Fay, C. E.; Samuel, A. L.; Shockley, W.; "On the Theory of Space Charge Between Parallel Plane Electrodes," Bell System Technical Journal, 49, pp. 49-79, 1938.

4. Karzas, W. J.; Latter, R.; "Electromagnetic Radiation From a Nuclear Explosion in Space," Physical Review, 126, pp. 1919-1926, June 1962 (Theoretical Note 40).

5. Birdsall, C. K.; Bridges, W. B.; "Space-Charge Instabilities in Electron Diodes and Plasma Converters," Journal of Applied Physics, 32, pp. 2611-2617, December 1961.

6. Chadsey, W. L.; "A One-Dimensional Time-Dependent, Self-Consistent Space-Charge Model," internal report, General Electric Missile and Space Division, Philadelphia, PA, May, 1969.