DNA 3221T

Theoretical Notes
Note 110

DEVELOPMENT OF THE GLANC EMP CODE

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10 December 1973

Topical Report

CONTRACT No. DASA 01-69-C-0058

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Prepared for
Director
DEFENSE NUCLEAR AGENCY
Washington, D. C. 20305

PREFACE

GLANC is a computer code for obtaining the solution, by finite difference methods, of Maxwell's equations in one space dimension and retarded time, for the electromagnetic fields produced by a nuclear burst near or on the ground. The fields are calculated in the air, in the ground, and on the ground-air interface. The approximations in GLANC are such that the field calculations are valid for early times and for distances less than about one kilometer. The yield of the nuclear device determines the maximum distance for which the approximations are valid. The field components considered are B_x , E_z , and E_y in rectangular coordinates. In the ground the electrical conductivity is constant in time and space. The source (Compton) current in the ground can be different from zero or zero. The ground current is always set to zero when the height of the nuclear burst above the ground is zero. The conductivity in the air is found by solving the "air-ion" equations, which take account of gamma-induced ionization, electron attachment to O_2 , and electron-ion and ion-ion recombination. Transport of gamma rays in the air is handled by prescription, using attenuation lengths and build-up factors. The sources in the air for the currents and ionization are determined by any of three methods: (1) injection of Compton electrons and solution of their equations of motion; (2) using the LEMP fits to Compton current and ionization rate as function of the fields; (3) by simply making the Compton current and ionization rate proportional to the gamma flux, without field reaction. Sample results from GLANC are given in a companion (classified) report.

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SECTION 1

THE DIFFERENTIAL EQUATIONS

1.1 MAXWELL'S EQUATIONS

GLANC is a computer code for obtaining the solution, by finite difference methods, of Maxwell's equations in one space dimension and retarded time, for the electromagnetic fields produced by a nuclear burst near or on the ground. We start with the two Maxwell equations that determine how the magnetic field $\vec{B}$ and the electric field $\vec{E}$ change with time,

$$\left.\begin{aligned} \frac{1}{c}\frac{\partial \vec{B}}{\partial t} &= -\nabla \times \vec{E} \\ \frac{\varepsilon}{c}\frac{\partial \vec{E}}{\partial t} + 4\pi\sigma\vec{E} &= \nabla \times \vec{B} - 4\pi\vec{J} \end{aligned}\right\} \tag{1-1}$$

We use cgs Gaussian units in the code: thus charge and electric fields are in esu and currents and magnetic fields are in emu. The output from the code can be designated as Gaussian or as MKS units. The relation between the system of units is given in the Appendix. Since the other two Maxwell equations, not written here, are only initial conditions, we need not consider them further.

In Eqs. (1-1), the medium has been assumed to be nonmagnetic ($\mu = 1$), and the dielectric constant ε has been assumed constant in time. We take $\varepsilon = 1$ in the air and $\varepsilon =$ constant in the ground. The electrical conductivity σ will be constant in the ground. The conductivity in the air and the Compton recoil current density $\vec{J}$ will vary with space and time and, for various cases, with the fields (see Section 3). The velocity of light $c = 3 \times 10^{10}$ cm/sec.

We shall use standard rectangular coordinates x,y,z and let only the field components B_x, E_y, and E_z differ from zero. Then Eqs. (1-1) become

$$\frac{1}{c}\frac{\partial B_x}{\partial t} = \frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} , \tag{1-2}$$

$$\frac{\varepsilon}{c}\frac{\partial E_y}{\partial t} + 4\pi\sigma E_y + 4\pi J_y = \frac{\partial B_x}{\partial z} , \tag{1-3}$$

$$\frac{\varepsilon}{c}\frac{\partial E_z}{\partial t} + 4\pi\sigma E_z + 4\pi J_z = -\frac{\partial B_x}{\partial y} . \tag{1-4}$$

The geometry is shown in Figure (1-1), where the source is shown off the ground. The ground-air interface is the x-y plane. The line from the source to the air-ground interface is given by the vector $\vec{r} = \vec{i}_y y - \vec{i}_z z$. The unit vector in the direction of $\vec{r}$ is $\vec{n}$. We want to calculate the fields along the line in the z-direction (y-z plane) which is drawn at the intersection of the source ray and the ground-air interface. Specifically we make the approximation that all rays from the source which cross this line of calculation are parallel, i.e., the line of calculation is short and far from the source. We now define the retarded time τ' by,

$$\tau' = ct - \vec{n} \cdot \vec{r} \tag{1-5}$$

and note that $\vec{n} \cdot \vec{r} = y \cos\theta - z \sin\theta$. Thus, the retarded time at the ground $(z = 0)$ τ is given by

$$\tau = ct - y\cos\theta \tag{1-6}$$

so that

$$\tau' = \tau + z\sin\theta . \tag{1-7}$$

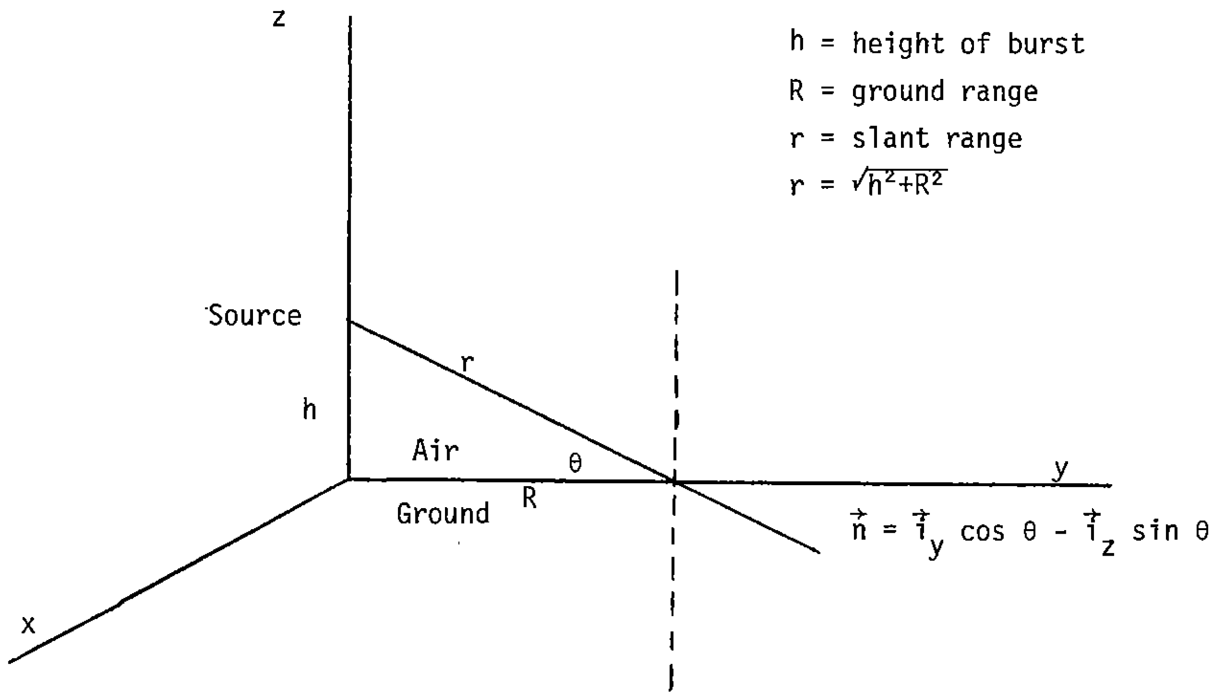

Figure (1-1). Geometry

We note the functional dependence

$$J = J(ct - \vec{n} \cdot \vec{r}, z) = J(ct - y \cos \theta, z) \tag{1-8}$$

which allows us to use the retarded time on the ground. Thus, from

$$\begin{aligned} y' &= y \\ \tau &= ct - y \cos \theta \end{aligned} \tag{1-9}$$

we find that

$$\begin{aligned} &\frac{1}{c} \frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial \tau} \\ &\frac{\partial}{\partial y} \rightarrow - \cos \theta \frac{\partial}{\partial \tau} + \frac{\partial}{\partial y'} \end{aligned} \tag{1-10}$$

so that Eq. (1-2) and Eq. (1-4) may be transformed to retarded time and give,

$$\frac{\partial B_x}{\partial \tau} = \frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y'} + \cos\theta \frac{\partial E_z}{\partial \tau}$$

$$\varepsilon \frac{\partial E_z}{\partial \tau} + 4\pi\sigma E_z + 4\pi J_z = \cos\theta \frac{\partial B_x}{\partial \tau} - \frac{\partial B_x}{\partial y'} \tag{1-11}$$

For our one-dimensional approximation we drop the $\partial/\partial y'$ terms in Eqs. (1-11) so that, with Eq. (1-3), we have

$$\frac{\partial E_y}{\partial \tau} + \frac{4\pi\sigma}{\varepsilon} E_y = -\frac{4\pi}{\varepsilon} J_y + \frac{1}{\varepsilon}\frac{\partial B_x}{\partial z} \tag{1-12}$$

$$\frac{\partial}{\partial \tau}(B_x - \cos\theta E_z) = \frac{\partial E_y}{\partial z} \tag{1-13}$$

$$\frac{\partial}{\partial \tau}(\varepsilon E_z - \cos\theta B_x) + 4\pi\sigma E_z = -4\pi J_z \tag{1-14}$$

or, by multiplying Eq. (1-13) by $\cos\theta$ and adding Eq. (1-14), and by multiplying Eq. (1-13) by ε and Eq. (1-14) by $\cos\theta$ and adding we obtain

$$\frac{\partial E_z}{\partial \tau}(\varepsilon - \cos^2\theta) + 4\pi\sigma E_z = \cos\theta \frac{\partial E_y}{\partial z} - 4\pi J_z \tag{1-15}$$

and

$$\frac{\partial B_x}{\partial \tau}(\varepsilon - \cos^2\theta) + 4\pi\sigma \cos\theta E_z = \varepsilon \frac{\partial E_y}{\partial z} - 4\pi \cos\theta J_z \tag{1-16}$$

When the burst point (source) is on the ground-air interface we have $\theta = 0$ $(\cos\theta = 1)$ so that in the air $(\varepsilon = 1)$ Eqs. (1-15) and (1-16) become redundant. Thus, when the height of burst is zero (ground burst) we cannot use Eqs. (1-15) and (1-16) in the air but must use

$$E_z = \frac{1}{4\pi\sigma}\frac{\partial E_y}{\partial z} - \frac{J_z}{\sigma} \tag{1-17}$$

and

$$\frac{\partial B_x}{\partial \tau} = \frac{\partial E_z}{\partial \tau} + \frac{\partial E_y}{\partial z} \tag{1-18}$$

where Eq. (1-18) was obtained by the use of Eq. (1-13).

To summarize, we use Eq. (1-12) to advance E_y in time and Eqs. (1-15) and (1-16) to advance E_z and B_x if the burst point is above ground and Eqs. (1-17) and (1-18) in the air to advance E_z and B_x if the burst point is on the ground. Eqs. (1-15) and (1-16) are used in the ground $(\varepsilon > 1)$ in either case for E_z and B_x.

1.2 THE AIR-ION EQUATIONS

In order to compute the air conductivity one has to keep accounts of the production and recombination of electrons, positive ions, and negative ions. Electrons, density n_e, and positive ions, density n_+, are made as a result of the absorption of gamma rays. The source of both will be called $\dot{\gamma}$, ion pairs per cm³ per sec. Electrons attach with rate coefficient α to O_2, forming negative ions O_2^-, density n_-. Electrons recombine with positive ions, with rate coefficient β. Positive and negative ions recombine with each other, with rate coefficient γ. The differential equations for n_e, n_-, and n_+ are

$$c\frac{dn_e}{d\tau} + (\alpha + \beta n_+)n_e = \dot{\gamma} \tag{1-19}$$

$$c\frac{dn_-}{d\tau} + (\gamma\, n_+)n_- = \alpha\, n_e \tag{1-20}$$

$$c\frac{dn_+}{\partial\tau} + (\gamma\, n_- + \beta n_e)n_+ = \dot{\gamma} \tag{1-21}$$

The effect of charge transport on the densities is unimportant and is neglected.

It is not necessary to solve all three of these equations, because of the conditions of charge neutrality which follows from them, plus the assumption of initial neutrality,

$$n_+ = n_e + n_- \tag{1-22}$$

In GLANC we carry n_e and n_-. If n_e and n_+ are carried there is an instability of the difference equations when βn_+ becomes comparable with or larger than α.

Having n_e and n_-, we calculate the conductivity from the equation

$$\sigma = \frac{e}{c}\left[n_e\mu_e + (2n_- + n_e)\mu_i\right] \quad , \tag{1-23}$$

where $e = 4.803 \times 10^{-10}$ esu, μ_e is the electron mobility and μ_i is the ion mobility. Note that we use the symbol e for the absolute value of the electronic charge. We assume, for lack of data, that positive and negative ions have the same mobility. In GLANC we use the following fits* and values:

$$\left.\begin{aligned}
\alpha &= \frac{0.72 \times 10^8}{\sqrt{|E|} + 0.03} + 6.45 \times 10^7 \exp\left(-\frac{12.76\rho}{|E| + 0.01}\right) \ (\text{sec}^{-1}) \\
|E| &= \sqrt{E_y^2 + E_z^2} \quad (\text{esu}) \\
\rho &= \text{air density} \left(\frac{\text{gms}}{\text{liter}} = \frac{\text{milligms}}{\text{cm}^3}\right) \\
\beta &= 2.5 \times 10^{-7} \quad (\text{cm}^3/\text{sec}) \\
\gamma &= 2.3 \times 10^{-6} \quad (\text{cm}^3/\text{sec} \\
\mu_i &= 750 \quad \text{cm/sec per esu} \\
\mu_e &= \frac{3.93\times10^8 \exp(-0.87P)}{\rho\left[\frac{3\times10^4 |E|}{\rho} + 1.4 \times 10^3 + 4\times10^3 P\right]^{(0.61-0.07P)} + 3\times10^6 [0.04 + 0.01P]} \\
P &= \text{percent water vapor in air}
\end{aligned}\right\} \tag{1-24}$$

*The fit for μ_e was done by John S. Malik.

It may be noted that α and μ_e will be functions of z and τ through their dependence on $|E|$. GLANC can also have constant (input) values of α and μ_e.

1.3 THE EQUATIONS OF MOTION

One method of obtaining a source (see Section 3) for $\dot{\gamma}$ and $\vec{J}$ is by running sample Compton electrons (particles). For this procedure we need the correct relativistic equations of motion for the electrons in the retarded time frame. We start with the equations of motion in real time

$$\frac{d\vec{P}}{dt} = -e\left[\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right] - A\,\frac{\vec{P}}{P} \qquad (1\text{-}25)$$

where $\vec{v}/c = \vec{P}/\sqrt{P^2+m^2c^2}$, m is the electron rest mass, $P = |\vec{P}|$, and A is the drag force of the air on the electrons in dynes. A is obtained from the fitted range-energy relation,

$$R_e = \frac{0.312 \times 10^3\, E_e^{\,2}}{\rho(0.3 + E_e)} \qquad (1\text{-}26)$$

where R_e is the range in cm and E_e is the electron kinetic energy in Mev. We calculate $A = dE_e/dR_e$ from Eq. (1-26) and obtain (A in dynes, and E_e in Mev),

$$A = \frac{1.6021\cdot 10^{-6}\rho(0.3+E_e)^2}{312.0\;E_e(0.6+E_e)} \qquad (1\text{-}27)$$

where $E_e = mc^2(\sqrt{1+(P/mc)^2} - 1)$, with mc^2 in Mev. By Eq. (1-6) we calculate

$$d\tau = c\,dt - dy\cos\theta = c\,dt\left(1 - \frac{v_y}{c}\cos\theta\right) \qquad (1\text{-}28)$$

so that Eq. (1-25) may be written as

$$\frac{d\vec{P}}{d\tau} = -\frac{1}{1-\cos\theta\,\frac{v_y}{c}}\,\frac{e}{c}\left[\vec{E} + \frac{\vec{v}}{c} \times \vec{B} + \frac{A}{e}\,\frac{\vec{P}}{P}\right] \quad . \qquad (1\text{-}29)$$

For the field components of GLANC, Eq. (1-29) gives the equations of motion in a retarded time frame as

$$\frac{dP_x}{d\tau} = -\frac{1}{1 - \cos\theta \frac{v_y}{c}} \frac{A}{c} \frac{P_x}{P}$$

$$\frac{dP_y}{d\tau} = -\frac{1}{1 - \cos\theta \frac{v_y}{c}} \frac{e}{c} \left[E_y + \frac{v_z}{c} B_x + \frac{A}{e} \frac{P_y}{P}\right] \qquad (1\text{-}30)$$

$$\frac{dP_z}{d\tau} = -\frac{1}{1 - \cos\theta \frac{v_y}{c}} \frac{e}{c} \left[E_z - \frac{v_y}{c} B_x + \frac{A}{e} \frac{P_z}{P}\right]$$

When electrons (particles) are injected to serve as the source for GLANC, their motion in the z-direction must be followed in retarded time. By Eq. (1-28) we calculate

$$\frac{dz}{d\tau} = \frac{1}{1 - \cos\theta \frac{v_y}{c}} \frac{v_z}{c} \qquad (1\text{-}31)$$

In GLANC we inject a specified number of particles at regular time intervals. At each injection time we inject n_p particles at each space (z) location. Each of the n_p particles is injected with different initial momentum components. A gamma ray of energy E_γ is assumed to make the Compton recoil electrons. In Figure (1-2) we show the geometry of the gamma ray and the relative electron recoil angles, θ_e and ϕ_e. θ_e is the Compton electron scattering angle and ranges from zero to $\pi/2$ in the angular scattering cross section (the Klein-Nishina formula) given by σ_e, i.e.,

$$\sigma_c = \int_0^{\pi/2} \sigma_e \, d\Omega_e$$

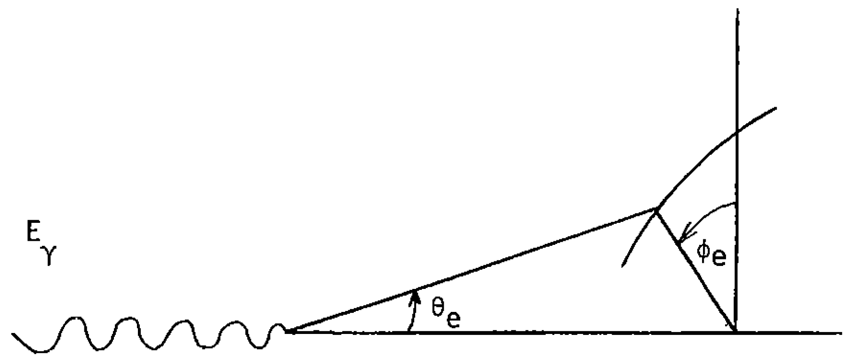

Figure (1-2). Geometry for θ_e and ϕ_e .

is the total Compton cross section when the element of solid angle of the recoil electron is $d\Omega_e = 2\pi \sin\theta_e \, d\theta_e$. To obtain n_θ values of θ_e such that at each and every angle the electron is equally probable we first let $\delta P = \sigma_c/n_\theta$. Then, let $P = \delta P/2$ and integrate on θ_e until

$$\int_0^\theta \sigma_e \, d\Omega_e = P \quad .$$

This determines the first θ_e . Then add δP to P and continue the integration until

$$\int_0^\theta \sigma_e \, d\Omega_e = P \quad .$$

This determines the second desired value of θ_e . Continue this latter procedure until all n_θ values of θ_e have been determined. All values of ϕ_e are equally probable, so we select n_ϕ equal angles ϕ_e . We here let $\delta\phi = \pi/n_\phi$ and set the first value of ϕ_e equal to $\delta\phi/2$. The second value of ϕ_e is obtained by adding $\delta\phi$ to the first value. Adding $\delta\phi$ to the last obtained value of ϕ_e is continued until n_ϕ values are obtained.

We set $n_p = n_\theta \times n_\phi$ and note, for determination of initial values of momentum components, that

$$v_r = v_e \cos\theta_e$$

$$v_\theta = v_e \sin\theta_e \cos\phi_e \qquad (1\text{-}32)$$

$$v_\phi = v_e \sin\theta_e \sin\phi_e$$

where

$$v_e = c\sqrt{1 - \frac{1}{\left[1 + \frac{E_e}{mc^2}\right]^2}} \qquad (1\text{-}33)$$

and

$$E_e = \frac{mc^2\ \gamma^2\xi}{1 + \gamma\xi} \qquad (1\text{-}34)$$

and

$$\xi = \frac{2\cos^2\theta_e}{(1+\gamma)^2 - \gamma(\gamma+2)\cos^2\theta_e} \qquad (1\text{-}35)$$

and

$$\gamma = \frac{E_\gamma}{mc^2} \quad .$$

The velocity components in Eqs. (1-32) determine the spherical components P_r , P_θ , and P_ϕ of the momentum with the r-component in the direction of the gamma ray. For GLANC the rectangular components of the initial momentum are given by

$$P_{x_0} = P_\phi$$

$$P_{y_0} = P_r \cos\theta + P_\theta \sin\theta \qquad (1\text{-}36)$$

$$P_{z_0} = -P_r \sin\theta + P_\theta \cos\theta$$

where θ is the illustrated angle of Figure (1-1).

SECTION 2

THE MESH AND THE DIFFERENCE EQUATIONS

2.1 THE MESH

GLANC uses a nonuniform z-mesh. Near the ground the gradients of the fields are large and a smaller mesh size is required. By examining the differential Eqs. (1-12), (1-15), and (1-16) which are to be differenced, we see that if E_z and B_x are carried at some mesh point z_k , then E_y should be carried at $z_{k+1/2}$. If the mesh is to be nonuniform, then the z_k mesh points should be exactly centered between the $z_{k+1/2}$ mesh points, i.e., the E_z and B_x equations will be centered and only one equation, that for E_y , will then need special attention to make it centered. The z-mesh defined below accommodates these features.

The z-mesh in the air is calculated from four input numbers: z_{oa} is the smallest mesh size in the air, n_{sa} is the number of "split" cells in the air, n_{fa} is the number of "final" cells in the air, and M_a is the magnification factor in the air. Figure (2-1) shows the mesh for a specific set of input numbers. Four similar numbers, z_{og} , n_{sg} , n_{fg} , and M_g are used for the ground mesh. The value of z on the ground is zero and z on the ground is indexed as $z_{n_g+1+1/2}$, where n_g and n_a are defined in Figure (2-1). E_y is indexed as $z_{k+1/2}$ and B_x , E_z , J_y , J_z and σ are indexed as z_k . The mesh always has two uniform mesh points just above the ground-air interface and two uniform mesh points just below the ground-air interface, these mesh increments being z_{oa} and z_{og} . The mesh increments are then (usually) increased by the magnification factors (M_a and/or M_g can be unity) for n_{sa} (or n_{sg}) successive times. One then adds n_{fa} (or n_{fg}) uniform increment

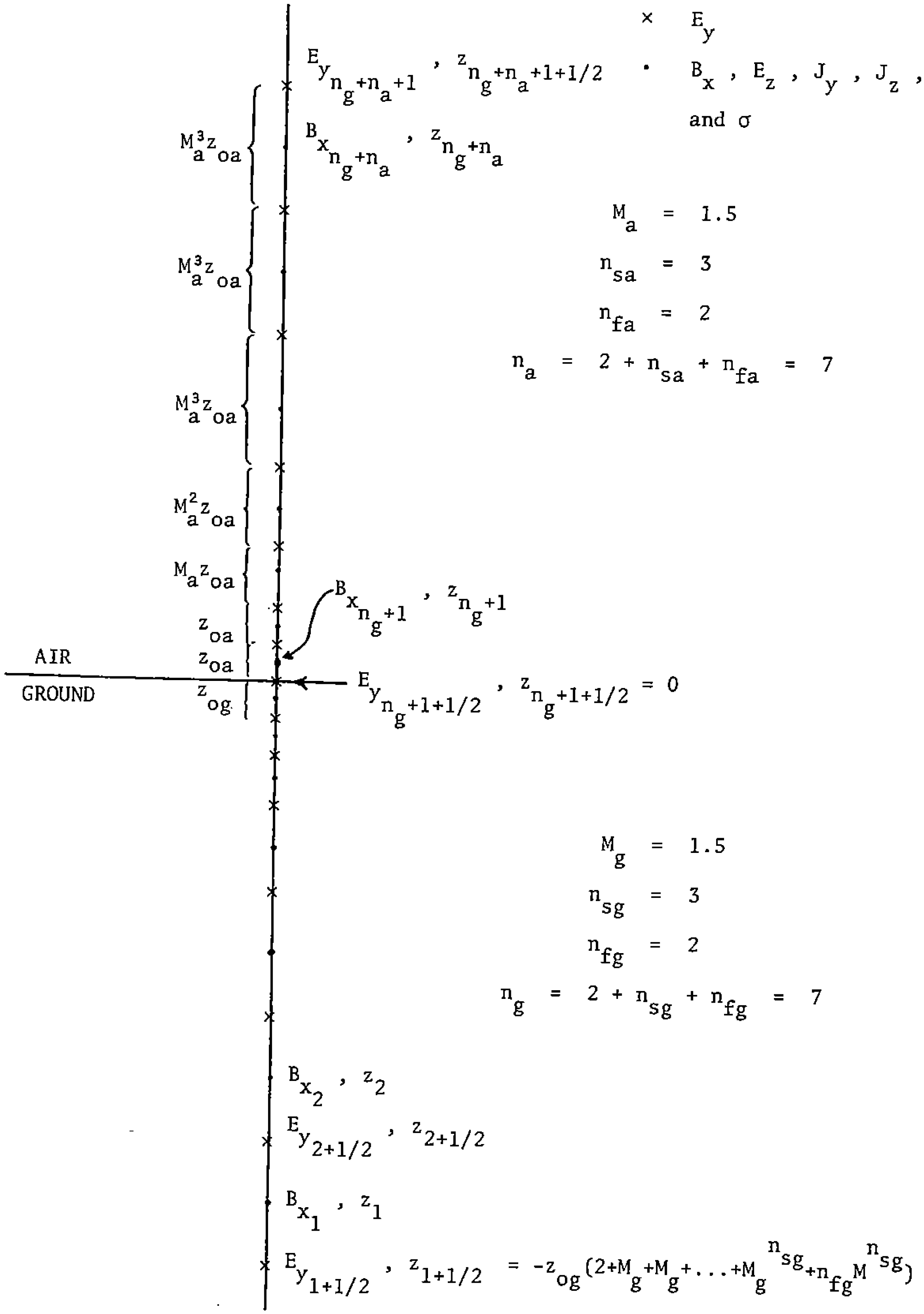

Figure (2-1). The mesh.

cells to the "top" (or "bottom") of the mesh. These last n_{fa} (or n_{fg}) uniform increments are all of width $M_a^{n_{sa}} z_{oa}$ (or $M_g^{n_{sg}} z_{oa}$) . One first calculates the $z_{k+1/2}$, for $k = 1$, n_g+n_a+1 , mesh points. Then set $z_k = 1/2(z_{k+1+1/2} + z_{k+1/2})$ for $k = 1, n_g+n_a$. This last operation ensures that the z_k values are (exactly) centered between the correct $z_{k+1/2}$ values. $z_{k+1/2}$ is not centered between values of z_k in the nonuniform portions of the mesh. Thus the z-derivatives will automatically be centered when differencing the B_x and E_z equations but the E_y equation will require special attention. Conversely, if one forms values of z_k first and lets $z_{k+1/2}$ be centered between these values of z_k , then one must give special attention to centering two equations instead of only one equation as above.

The time (τ) mesh is obtained by advancing forward in time by successive cycles. At the beginning of any time cycle, say τ^n (n = cycle number), it is assumed that the mesh values (E_y , B_x , E_z , etc.) are known at τ^n and the cycle calculation is performed to advance time to τ^{n+1} . The time increment ($\delta\tau$) is constant in GLANC, i.e., $\delta\tau = \tau^{n+1} - \tau^n$ for all n , due to the fact that GLANC is not designed to run to late (large) problem times.

The calculations performed during each cycle are ordered. This ordering is partly dictated by the variety of sources allowed and partly by implicit differencing considerations. In a cycle one first calculates the sources, i.e., J_y , J_z , and $\dot{\gamma}$. ($\dot{\gamma}$ is only calculated in the air.) Then n_e , n_- , and σ are calculated (only in the air). With $\vec{J}$ and σ we then advance the fields E_y , E_z , and B_x in time. There are various kinds of input and output operations before, during, and after the cycling operations. The presentations of the various cycle calculations will not be given in the order described above in order to avoid various complications.

2.2 A GENERAL FORM FOR MOST DIFFERENCE EQUATIONS

Before presenting the difference equations, it may be noted that most of Maxwell's equations and the air-ion equations can be written in a similar form. We note that if one has an equation of the form*

$$\frac{\partial f}{\partial \xi} + \gamma f = \psi \quad , \tag{2-1}$$

then the exact solution is

$$f(\xi) = e^{-\chi(\xi)} \left[f(\xi_0) + \int_{\xi_0}^{\xi} \psi(\xi') e^{\chi(\xi')} d\xi' \right] \quad , \tag{2-2}$$

where $\chi(\xi) \equiv \int_{\xi_0}^{\xi} \gamma\chi(\xi'')d\xi''$. To first order in $\delta\xi = \xi - \xi_0$, the solution to Eq. (2-2) is

$$f(\xi) = f(\xi_0)e^{-\overline{\chi}} + (1-e^{-\overline{\chi}}) \left[\frac{\psi}{\gamma}\right] \text{ at } \overline{\xi} \quad , \tag{2-3}$$

where $\overline{\chi} = \delta\xi \cdot \overline{\gamma}$ and $\overline{\gamma}$ is γ evaluated at $\overline{\xi} = 1/2(\xi+\xi_0)$. For second order accuracy in the integral term of Eq. (2-2), one uses the following procedure. Let $\Phi(\xi') \equiv \psi(\xi')/\gamma(\xi')$ so that $\Phi_1 \equiv \Phi(\xi_0)$ and $\Phi_2 \equiv \Phi(\xi)$; then for $\Phi(\xi')$ assumed to be a linear function of $\chi(\xi')$, we calculate

$$f(\xi) = f(\xi_0)e^{-\overline{\chi}} + \Phi_1 \left[\frac{1}{\overline{\chi}}(1 - e^{-\overline{\chi}}) - e^{-\overline{\chi}}\right] + \Phi_2 \left[1 - \frac{1}{\overline{\chi}}(1 - e^{-\overline{\chi}})\right] \quad . \tag{2-4}$$

The forms presented by Eqs. (2-3) and (2-4) will be evident in the difference equations as they are presented.

2.3 THE AIR-ION EQUATIONS

The air-ion equations (Eqs. (1-19) and (1-20))are differenced as

$$n_{e_k}^{n+1} = n_{e_k}^{n} e^{-\Phi_e} + (1-e^{-\Phi_e}) \left[\frac{\dot{\gamma}}{\alpha+\beta n_+}\right]_k^n \tag{2-5}$$

and

*The γ or other symbols used in this section have nothing to do with the same symbols used in the air-ion equations or elsewhere.

$$n_{-k}^{n+1} = n_{-k}^{n} e^{-\Phi_-} + (1 - e^{-\Phi_-}) \left[\frac{\alpha n_e}{\gamma n_+}\right]_k^n \qquad (2\text{-}6)$$

where

$$\Phi_e \equiv [\alpha + \beta n_+]_k^n \frac{\delta\tau}{c} ,$$

$$\Phi_- \equiv [\gamma n_+]_k^n \frac{\delta\tau}{c} , \quad \text{and}$$

$n_+ = n_e + n_-$. The values of n_e and n_- are then used to calculate

$$\sigma_k^{n+1} = \frac{e}{c} [n_e\mu_e + (2n_- + n_e)\mu_i]_k^{n+1} . \qquad (2\text{-}7)$$

When the various parameters in Eqs. (2-5), (2-6) and (2-7) are field dependent (see Eqs. (1-24)) and should use the electric field magnitude ($|E|^{n+1}$) which is not known at τ^{n+1} in this portion of a cycle, one uses $|E|^n$. Also, one could have integrated the air-ion equation from τ^{n-1} to τ^{n+1} with a time interval of $2\delta\tau$ by back storing n_e^{n-1} and n_-^{n-1} and thus formed equations like Eqs. (2-5) and (2-6) which would be time centered. This last procedure was not considered necessary in GLANC. This is because essentially no error is introduced and the added storage requirements were not desirable. We do, however, have J_y, J_z, and σ at two successive times for centering Maxwell's equations.

2.4 MAXWELL'S EQUATIONS

We will first difference the E_y equation (Eq. (1-12) so that it will be centered in the nonuniform (and uniform) portion of the mesh. We will use the form indicated by Eq. (2-4) and with the indexing indicated in Figure (2-1). In the air ($\varepsilon = 1$) we write (for $k = n_g + 2$ to $n_g + n_a$),

$$E^{n+1}_{y_{k+1/2}}\left(\frac{1}{1+\Delta_{a_k}}\right)+\left(\frac{\Delta_{a_k}}{1+\Delta_{a_k}}\right)E^{n+1}_{y_{k+1+1/2}} = e^{-Y}\left[E^{n}_{y_{k+1/2}}\left(\frac{1}{1+\Delta_{a_k}}\right)+\left(\frac{\Delta_{a_k}}{1+\Delta_{a_k}}\right)E^{n}_{y_{k+1+1/2}}\right]$$

$$+\left[\frac{1}{Y}\left(1-e^{-Y}\right)-e^{-Y}\right]\left[-\frac{J^n_{y_{k+1/2}}}{\sigma^n_{k+1/2}}+\frac{B^n_{x_k}-B^n_{x_{k-1}}}{4\pi\sigma^n_{k+1/2}\left(z_k-z_{k-1}\right)}\right]$$

$$+\left[1-\frac{1}{Y}\left(1-e^{-Y}\right)\right]\left[-\frac{J^{n+1}_{y_{k+1/2}}}{\sigma^{n+1}_{k+1/2}}+\frac{B^{n+1}_{x_k}-B^{n+1}_{x_{k-1}}}{4\pi\sigma^{n+1}_{k+1/2}\left(z_k-z_{k-1}\right)}\right] \qquad (2\text{-}8)$$

where

$$\Delta_{a_k} = \frac{z_{c_k}-z_{k+1/2}}{z_{k+1+1/2}-z_{c_k}} ,$$

$$z_{c_k} = \frac{1}{2}(z_k+z_{k-1}) ,$$

$$Y = 2\pi\delta\tau\left(\sigma^{n+1/2}_k + \sigma^{n+1/2}_{k-1}\right) ,$$

$$\sigma^{n+1/2}_k = \frac{1}{2}\left(\sigma^n_k + \sigma^{n+1}_k\right) ,$$

$$\sigma^n_{k+1/2} = \frac{1}{2}\left(\sigma^n_k + \sigma^n_{k-1}\right) \quad \text{(or replace } n \text{ by } n+1\text{)} ,$$

and

$$J^{n+1}_{k+1/2} = \frac{1}{2}\left(J^{n+1}_{y_k} + J^{n+1}_{y_{k-1}}\right) \quad \text{(or, replace } n+1 \text{ by } n\text{)} .$$

Eq. (2-8) is centered at z_{c_k} . If the mesh is uniform, then $z_{c_k} = z_{k+1/2}$ and Δ_{a_k} is zero. For our nonuniform mesh Δ_{a_k} is normally small and depends on the magnification factor in the air. The left-hand side of Eq. (2-8) is the interpolated (not extrapolated) value of E_y at z_{c_k} and at τ^{n+1} . The first bracket on the right side of this equation is E_y at z_{c_k} and at τ^n . The rest of the terms on the right side of Eq. (2-8) are centered at z_{c_k} by the definition of z_{c_k} and the positions in the mesh of B_x , J_y , and σ . To make the implicit character more emphatic and to facilitate later operations, we write Eq. (2-8) (for $k = n_g+2$ to n_g+n_a) as,

$$E^{n+1}_{y_{k+1/2}}\left(\frac{1}{1+\Delta_{a_k}}\right)+\left(\frac{\Delta_{a_k}}{1+\Delta_{a_k}}\right)E^{n+1}_{y_{k+1+1/2}} = D_{1_k}+D_{2_k}\left(B^{n+1}_{x_k}-B^{n+1}_{x_{k-1}}\right) \quad (2\text{-}9)$$

where

$$D_{1_k} \equiv e^{-Y}\left[E^n_{y_{k+1/2}}\left(\frac{1}{1+\Delta_{a_k}}\right)+\left(\frac{\Delta_a}{1+\Delta_a}\right)E^n_{y_{k+1+1/2}}\right]$$

$$+\left[\frac{1}{Y}(1-e^{-Y})-e^{-Y}\right]\left[-\frac{J^n_{y_{k+1/2}}}{\sigma^n_{k+1/2}}+\frac{B^n_{x_k}-B^n_{x_{k-1}}}{4\pi\sigma^n_{k+1/2}(z_k=z_{k-1})}\right]$$

$$-\frac{J^{n+1}_{y_{k+1/2}}}{\sigma^n_{k+1/2}}\left[1-\frac{1-e^{-Y}}{Y}\right] ,$$

and

$$D_{2_k} \equiv \left[1-\frac{1-e^{-Y}}{Y}\right]\frac{1}{4\pi\sigma^{n+1}_{k+1/2}(z_k-z_{k-1})}$$

In the ground we have $\varepsilon > 1$ and $\sigma = \sigma_0$ so that, by similar techniques as used above, the E_y equation (for $k=2$ to n_g) is written as

$$E^{n+1}_{y_{k+1/2}}\left(\frac{1}{1+\Delta_{g_k}}\right)+\left(\frac{\Delta_{g_k}}{1+\Delta g_k}\right)E^{n+1}_{y_{k-1+1/2}} = D_{1_k}+D_{2_k}\left(B^{n+1}_{x}-B^{n+1}_{x_{k-1}}\right) \quad (2\text{-}10)$$

where

$$z_{c_k} = \frac{1}{2}(z_k+z_{k-1}) ,$$

$$\Delta_{g_k} = \frac{z_{k+1/2}-z_{c_k}}{z_{c_k}-z_{k-1-1/2}} ,$$

$$Y_0 = \frac{4\pi\sigma_0}{\varepsilon}\,\delta\tau ,$$

$$J^n_{k+1/2} = \frac{1}{2}\left(J^n_{y_k}+J^n_{y_{k-1}}\right) \quad \text{(or, replace } n \text{ by } n+1\text{)} ,$$

$$D_{1_k} \equiv e^{-Y_0}\left[E^n_{y_{k+1/2}}\left(\frac{1}{1+\Delta_{g_k}}\right)+\left(\frac{\Delta_{g_k}}{1+\Delta_{g_k}}\right)E^n_{y_{k-1+1/2}}\right]$$

$$+\left[\frac{1}{Y_0}\left(1-e^{-Y_0}\right)-e^{-Y_0}\right]\left[-\frac{J^n_{y_{k+1/2}}}{\sigma_0}+\frac{B^n_{x_k}-B^n_{x_{k-1}}}{4\pi\sigma_0\left(z_k-z_{k-1}\right)}\right]$$

$$-\left[1-\frac{1-e^{-Y_0}}{Y_0}\right]\frac{J^{n+1}_{y_{k+1/2}}}{\sigma_0} ,$$

and

$$D_{2_k} \equiv \left[1-\frac{1}{Y_0}\left(1-e^{-Y_0}\right)\right]\frac{1\ 1}{4\pi\sigma_0\left(z_k-z_{k-1}\right)} .$$

Note that Δ_{g_k} is zero in a uniform mesh and that we have again interpolated (not extrapolated) to center the equation. The symbols D_{1_k} and D_{2_k} are again used though the index now runs from two to n_g .

The solution for E_y on the ground-air interface $(k = n_g+1 \equiv L)$ is a little more involved. E_y and B_x are continuous at the interface and E_z is discontinuous. E_y, B_x, and E_z all have discontinuous derivatives at the interface. We thus write the E_y equation twice, once centered just above the interface and once centered just below the interface, and use these two equations to eliminate B_x on the ground $(B^{n+1}_{x_g})$. Thus we write the E_y equation centered 1/4th of z_{oa} above the interface as (for $k = n_g+1 \equiv L$ and, for simplicity, dropping the 1/2 on the E_y index),

$$\frac{3}{4}E^{n+1}_{y_L}+\frac{1}{4}E^{n+1}_{y_{L+1}} = D_{1_L}+D_{2_L}\left(B^{n+1}_{x_L}-B^{n+1}_{x_g}\right) \qquad (2\text{-}11)$$

and the E_y equation centered 1/4th of z_{og} below the interface as,

$$\frac{3}{4}E^{n+1}_{y_L}+\frac{1}{4}E^{n+1}_{y_{L-1}} = \overline{D}_{1_L}+\overline{D}_{2_L}\left(B^{n+1}_{x_g}-B^{n+1}_{x_{L-1}}\right) \qquad (2\text{-}12)$$

where

$$\overline{Y} = 4\pi\sigma^{n+1/2}_{L+1/4}\,\delta\tau ,$$

$$Y_0 = \frac{4\pi\sigma_0}{\varepsilon}\,\delta\tau \qquad ,$$

$$\left.\begin{aligned}
\sigma_{L+1/4} &= \frac{5}{4}\,\sigma_L - \frac{1}{4}\,\sigma_{L+1} \\
J_{y_{L+1/4}} &= \frac{5}{4}\,J_{y_L} - \frac{1}{4}\,J_{y_{L+1}} \quad , \\
J_{y_{L-1/4}} &= \frac{5}{4}\,J_{y_{L-1}} - \frac{1}{4}\,J_{y_{L-2}}
\end{aligned}\right\} \text{(all at } n,\ n+1, \text{ or } n+1/2)$$

$$D_{1_L} \equiv e^{-\overline{Y}}\left[\frac{3}{4}\,E^n_{y_L} + \frac{1}{4}\,E^n_{y_{L+1}}\right] + \left[\frac{1}{\overline{Y}}\left(1-e^{-\overline{Y}}\right) - e^{-\overline{Y}}\right]\left[-\frac{J^n_{y_{L+1/4}}}{\sigma^n_{L+1/4}} + \frac{B^n_{x_L} - B^n_{x_g}}{4\pi\sigma^n_{L+1/4}\left(\frac{1}{2}\,z_{oa}\right)}\right]$$

$$- \frac{J^{n+1}_{y_{L+1/4}}}{\sigma^{n+1}_{L+1/4}}\left[1 - \frac{1-e^{-\overline{Y}}}{\overline{Y}}\right] \qquad ,$$

$$D_{2_L} \equiv \left[1 - \frac{1}{\overline{Y}}\left(1-e^{-\overline{Y}}\right)\right]\frac{1}{4\pi\sigma^{n+1}_{L+1/4}\left(\frac{1}{2}\,z_{oa}\right)} \qquad ,$$

$$\overline{D}_{1_L} = e^{-Y_0}\left[\frac{3}{4}\,E^n_{y_L} + \frac{1}{4}\,E^n_{y_{L-1}}\right] + \left[\frac{1}{Y_0}\left(1-e^{-Y_0}\right) - e^{-Y_0}\right]\left[-\frac{J^n_{y_{L-1/4}}}{\sigma_0} + \frac{B^n_{x_g} - B^n_{x_{L-1}}}{4\pi\sigma_0\left(\frac{1}{2}\,z_{og}\right)}\right]$$

$$- \frac{J^{n+1}_{y_{L-1/4}}}{\sigma_0}\left[1 - \frac{1}{Y_0}\left(1-e^{-Y_0}\right)\right] \qquad ,$$

and

$$\overline{D}_{2_L} = \left[1 - \frac{1}{Y_0}\left(1-e^{-Y_0}\right)\right]\frac{1}{4\pi\sigma_0\left(\frac{1}{2}\,z_{og}\right)} \qquad .$$

$\overline{D}_{1_L}$ and $\overline{D}_{2_L}$ are two numbers which are calculated and stored separately from the D_{1_k} and D_{2_k} meshes. Though one is always worried by having to use extrapolations, the extrapolations for $\sigma_{L+1/4}$ and $J_{y_{L\pm1/4}}$ in GLANC give no problems, probably due to the fact that the curvature of these source functions is small near the ground and due to the small mesh size near the ground. We first solve Eq. (2-11) and (2-12) for $B^{n+1}_{x_g}$ and find

$$\left.\begin{aligned} B_{x_g}^{n+1} &= \frac{1}{D_{2_L}}\left[-\frac{3}{4}E_{y_L}^{n+1}-\frac{1}{4}E_{y_{L+1}}^{n+1}+D_{1_L}+D_{2_L}B_{x_L}^{n+1}\right] \\ -B_{x_g}^{n+1} &= \frac{1}{\overline{D}_{2_L}}\left[-\frac{3}{4}E_{y_L}^{n+1}-\frac{1}{4}E_{y_{L-1}}^{n+1}+\overline{D}_{1_L}-\overline{D}_{2_L}B_{x_{L-1}}^{n+1}\right] \end{aligned}\right\} . \qquad (2\text{-}13)$$

Adding these two equations gives one equation in which $B_{x_g}^{n+1}$ is eliminated. This one equation, with the terms collected and multiplied by D_{2_L}, gives

$$E_{y_L}^{n+1}\frac{3}{4}\left(1+\frac{D_{2_L}}{\overline{D}_{2_L}}\right)+\frac{1}{4}E_{y_{L+1}}^{n+1}+\frac{1}{4}\frac{D_{2_L}}{\overline{D}_{2_L}}E_{y_{L-1}}^{n+1} = \left(D_{1_L}+\overline{D}_{1_L}\frac{D_{2_L}}{\overline{D}_{2_L}}\right)+D_{2_L}\left(B_{x_L}^{n+1}-B_{x_{L-1}}^{n+1}\right)$$

(2-14)

It should be noted that $B_{x_g}^{n}$ is needed to calculate D_{1_L} and to calculate $\overline{D}_{1_L}$ and therefore must have been back stored before the cycle began. One thus needs a value of $B_{x_g}^{n}$ initially (zero time solution), which will be given later in this report. Also, at the end of a cycle when E_y and B_x are known at τ^{n+1}, one must calculate $B_{x_g}^{n+1}$ from (one of) the Eqs. (2-13) and store this number so it can be used for $B_{x_g}^{n}$ in the next cycle. Since the calculation of the fields is implicit we must have all the difference equations to solve for the fields. After differencing the B_x equation and the E_z equation we will then return to the use of Eqs. (2-11), (2-12) and (2-14).

The B_x Eq. (1-16) is not of the form of Eq. (2-1) and is differenced with space and time centering in the usual differencing manner in the air as,

$$B_{x_k}^{n+1} = B_{x_k}^{n}-\frac{\delta\tau}{1-\cos^2\theta}\left[2\pi\cos\theta\left(\sigma_k^n E_{z_k}^n+\sigma_k^{n+1}E_{z_k}^{n+1}\right)+4\pi\cos\theta\, J_{z_k}^{n+1/2} - \frac{E_{y_{k+1+1/2}}^{n+1}+E_{y_{k+1+1/2}}^{n}-E_{y_{k+1/2}}^{n+1}-E_{y_{k+1/2}}^{n}}{2\{z_{k+1+1/2}-z_{k+1/2}\}}\right] . \qquad (2\text{-}15)$$

The only pitfall one will encounter in differencing this equation is that poor (inaccurate) results will occur if the nonlinear term $(\sigma E_z)_k^{n+1/2}$ is differenced as $\sigma_k^{n+1/2} E_{z_k}^{n+1/2}$ instead of $\frac{1}{2}\left(\sigma_k^n E_{z_k}^n + \sigma_k^{n+1} E_{z_k}^{n+1}\right)$ as is shown in Eq. (2-15). We write Eq. (2-15) in the air for a burst point above the ground-air interface as (for $k = n_g+1$ to n_g+n_a)

$$B_{x_k}^{n+1} = C_{1_k} + C_{2_k} E_{z_k}^{n+1} + C_{3_k}\left(E_{y_{k+1+1/2}}^{n+1} - E_{y_{k+1/2}}^{n+1}\right) \qquad (2\text{-}16)$$

where (for $k = n_g+1$ to n_g+n_a)

$$J_{z_k}^{n+1/2} = \frac{1}{2}\left(J_{z_k}^{n+1} + J_{z_k}^{n}\right) ,$$

$$C_{1_k} \equiv B_{x_k}^n - \frac{\delta\tau}{1-\cos^2\theta}\left\{2\pi \cos\theta\ \sigma_k^n E_{z_k}^n + 4\pi \cos\theta\ J_{z_k}^{n+1/2} - \frac{E_{y_{k+1+1/2}}^n - E_{y_{k+1/2}}^n}{2\left(z_{k+1+1/2} - z_{k+1/2}\right)}\right\}$$

$$C_{2_k} \equiv -\frac{\delta\tau}{1-\cos^2\theta}\, 2\pi \cos\theta\ \sigma_k^{n+1} ,$$

$$C_{3_k} \equiv \frac{\delta\tau}{2(1-\cos^2\theta)(z_{k+1+1/2} - z_{k+1/2})} .$$

Similarly, the B_x equation in the ground is given by Eq. (2-16) when we define (for $k = 1$ to n_g)

$$C_{1_k} \equiv B_{x_k}^n - \frac{\delta\tau}{\varepsilon-\cos^2}\left\{2\pi\sigma_0 \cos\theta\ E_{z_k}^n + 4\pi \cos\theta\ J_{z_k}^{n+1/2} - \frac{\varepsilon\left(E_{y_{k+1+1/2}}^n - E_{y_{k+1/2}}^n\right)}{2\left(z_{k+1+1/2} - z_{k+1/2}\right)}\right\}$$

$$C_{2_k} \equiv -\frac{\delta\tau}{\varepsilon-\cos^2\theta}\, 2\pi\sigma_0 \cos\theta ,$$

$$C_{3_k} \equiv \frac{\delta\tau}{2(\varepsilon-\cos^2\theta)\left(z_{k+1+1/2} - z_{k+1/2}\right)} .$$

When the burst is on the ground $(\cos\theta = 1)$ we use Eq. (1-18) in the air and difference it as,

$$B_{x_k}^{n+1} = B_{x_k}^n + E_{z_k}^{n+1} - E_k^n + \frac{\delta\tau}{2}\, \frac{E_{y_{k+1+1/2}}^{n+1} + E_{y_{k+1+1/2}}^{n} - E_{y_{k+1/2}}^{n+1} - E_{y_{k+1/2}}^{n}}{z_{k+1+1/2} - z_{k+1/2}} \qquad (2\text{-}17)$$

or, we can write (2-17) as (2-16) if we define (for $k = n_g+1$ to n_g+n_a) ,

$$C_{1_k} \equiv B^n_{x_k} - E^n_{z_k} + \frac{\delta\tau}{2} \frac{E^n_{y_{k+1+1/2}} - E^n_{y_{k+1/2}}}{z_{k+1+1/2} - z_{k+1/2}} ,$$

$$C_{2_k} \equiv 1.0 ,$$

and

$$C_{3_k} \equiv \frac{\delta\tau}{2} \frac{1}{z_{k+1+1/2} - z_{k+1/2}} .$$

The E_z equation (Eq. (1-15)) is of the form of Eq. (2-1) and can be differenced in the form of Eq. (2-4), i.e., in the air ($k = n_g+1$ to n_g+n_a) with the burst point above the ground-air interface we have

$$E^{n+1}_{z_k} = e^{-Z}E^n_{z_k} + \left[\frac{1-e^{-Z}}{Z} - e^{-Z}\right]\left[-\frac{J^n_{z_k}}{\sigma^n_k} + \frac{\cos\theta\left(E^n_{y_{k+1+1/2}} - E^n_{y_{k+1/2}}\right)}{4\pi\sigma^n_k\left(z_{k+1+1/2} - z_{k+1/2}\right)}\right]$$

$$+ \left[1 - \frac{1-e^{-Z}}{Z}\right]\left[-\frac{J^{n+1}_{z_k}}{\sigma^{n+1}_k} + \frac{\cos\theta\left(E^{n+1}_{y_{k+1+1/2}} - E^{n+1}_{y_{k+1/2}}\right)}{4\pi\sigma^{n+1}_k\left(z_{k+1+1/2} - z_{k+1/2}\right)}\right] \quad (2\text{-}18)$$

where

$$Z = \frac{2\pi\delta\tau\left(\sigma^{n+1}_k + \sigma^n_k\right)}{1-\cos^2\theta} .$$

Write (2-18) as

$$E^{n+1}_{z_k} = G_{1_k} + G_{2_k}\left(E^{n+1}_{y_{k+1+1/2}} - E^{n+1}_{y_{k+1/2}}\right) \quad (2\text{-}19)$$

where we define (for $k = n_g+1$ to n_g+n_a)

$$G_{1_k} \equiv e^{-Z}E^n_{z_k} + \left[\frac{1-e^{-Z}}{Z} - e^{-Z}\right]\left[-\frac{J^n_{z_k}}{\sigma^n_k} + \frac{\cos\theta\left(E^n_{y_{k+1+1/2}} - E^n_{y_{k+1/2}}\right)}{4\pi\sigma^n_k\left(z_{k+1+1/2} - z_{k+1/2}\right)}\right]$$

$$- \frac{J^{n+1}_{z_k}}{\sigma^{n+1}_k}\left[1 - \frac{1-e^{-Z}}{Z}\right] ,$$

and

$$G_{2_k} \equiv \left[1-\frac{1-e^{-Z}}{Z}\right]\frac{\cos\theta}{4\pi\sigma_k^{n+1}\left(z_{k+1+1/2}-z_{k+1/2}\right)} .$$

In the ground the E_z equation is also given by Eq. (2-19) when we define (for $k = 1$ to n_g) ,

$$G_{1_k} \equiv e^{-Z_0}E^n_{z_k}+\left[\frac{1-e^{-Z_0}}{Z_0}-e^{-Z_0}\right]\left[-\frac{J^n_{z_k}}{\sigma_0}+\frac{\cos\theta\left(E^n_{y_{k+1+1/2}}-E^n_{y_{k+1/2}}\right)}{4\pi\sigma_0\left(z_{k+1+1/2}-z_{k+1/2}\right)}\right]$$
$$-\frac{J^{n+1}_{z_k}}{\sigma_0}\left[1-\frac{1-e^{-Z_0}}{Z_0}\right] ,$$

and

$$G_{2_k} \equiv \left[1-\frac{1-e^{-Z_0}}{Z_0}\right]\frac{\cos\theta}{4\pi\sigma_0\left(z_{k+1+1/2}-z_{k+1/2}\right)} ,$$

where

$$Z_0 \equiv \frac{4\pi\sigma_0\delta\tau}{\varepsilon-\cos^2\theta} .$$

If the burst is on the ground, then in the air we use Eq. (1-17) for E_z and difference it as

$$E^{n+1}_{z_k} = \frac{E^{n+1}_{y_{k+1+1/2}}-E^{n+1}_{y_{k+1/2}}}{4\pi\sigma_k^{n+1}\left(z_{k+1+1/2}-z_{k+1/2}\right)}-\frac{J^{n+1}_{z_k}}{\sigma_k^{n+1}} \qquad (2\text{-}20)$$

or, to make the E_z equation have the same difference form as Eq. (2-19), we define (for $k = n_g+1$ to n_g+n_a) ,

$$G_{1_k} \equiv -\frac{J^{n+1}_{z_k}}{\sigma^{n+1}_k} ,$$

and

$$G_{2_k} \equiv \frac{1}{4\pi\sigma_k^{n+1}\left(z_{k+1+1/2}-z_{k+1/2}\right)} .$$

The implicit solution to Maxwell's equations in difference form can now be given. One first eliminates $E^{n+1}_{z_k}$ from the B_x equation (2-16) by substitution of $E^{n+1}_{z_k}$ from Eq. (2-19). Then one forms the difference $(k = 2$ to $n_g+n_a)$,

$$B_{x_k}^{n+1} - B_{x_{k-1}}^{n+1} = \chi_{1_k} + \chi_{2_k} E_{y_{k+1+1/2}}^{n+1} - \chi_{3_k} E_{y_{k+1/2}}^{n+1} + \chi_{4_k} E_{y_{k-1+1/2}}^{n+1} \qquad (2\text{-}21)$$

where

$$\chi_{1_k} \equiv C_{1_k} - C_{1_{k-1}} + C_{2_k} G_{1_k} - C_{2_{k-1}} G_{1_{k-1}} ,$$

$$\chi_{2_k} \equiv C_{2_k} G_{2_k} + C_{3_k} ,$$

$$\chi_{3_k} \equiv C_{2_k} G_{2_k} + C_{3_k} + C_{2_{k-1}} G_{2_{k-1}} + C_{3_{k-1}} ,$$

and

$$\chi_{4_k} \equiv C_{2_{k-1}} G_{2_{k-1}} + C_{3_{k-1}} .$$

With Eq. (2-21) inserted into the E_y equations (Eqs. (2-9), (2-10) and (2-14)) we obtain a three-value difference equation for E_y and write the result as (for $k = 2$ to n_g+n_a)

$$E_{y_{k+1/2}}^{n+1} A_{1_k} + E_{y_{k+1+1/2}}^{n+1} A_{2_k} + E_{y_{k-1+1/2}}^{n+1} A_{3_k} = A_{4_k} \qquad (2\text{-}22)$$

where the coefficients are given by the matrix

	$k = 2$ to n_g	$k = n_g+1 = L$	$k = n_g+2$ to n_g+n_a
$A_{1_k} =$	$\frac{1}{1+\Delta_{gk}} + D_{2_k}\chi_{3_k}$	$\frac{3}{4}\left(1+\frac{D_{2_k}}{\overline{D}_{2_k}}\right) + D_{2_k}\chi_{3_k}$	$\frac{1}{1+\Delta_{ak}} + D_{2_k}\chi_{3_k}$
$A_{2_k} =$	$-D_{2_k}\chi_{2_k}$	$\frac{1}{4} - D_{2_k}\chi_{2_k}$	$\frac{\Delta_{ak}}{1+\Delta_{ak}} - D_{2_k}\chi_{2_k}$
$A_{3_k} =$	$\frac{\Delta_{gk}}{1+\Delta_{gk}} - D_{2_k}\chi_{4_k}$	$\frac{1}{4}\frac{D_{2_k}}{\overline{D}_{2_k}} - D_{2_k}\chi_{4_k}$	$-D_{2_k}\chi_{4_k}$
$A_{4_k} =$	$D_{1_k} + D_{2_k}\chi_{1_k}$	$D_{1_k} + \overline{D}_{1_k}\frac{D_{2_k}}{\overline{D}_{2_k}} + D_{2_k}\chi_{1_k}$	$D_{1_k} + D_{2_k}\chi_{1_k}$

To solve Eq. (2-22) one needs a boundary condition so that $E_{y_{1+1/2}}^{n+1}$ can be obtained and a boundary condition so that $E_{y_{n_g+n_a+1+1/2}}^{n+1}$

can be obtained. We let $E^{n+1}_{y_{1+1/2}}$ (for all n) be zero, i.e., we place an infinite conductor at our deepest mesh point in the ground. Thus, by selecting the four input parameters which define the ground mesh in an appropriate manner, one can either make the mesh deep enough in the ground that no electromagnetic signal reaches the infinite conductor or make the mesh a desired depth which will show the reflection of an electromagnetic pulse. At the largest value of z in the air we solve Eq. (2-8) for $E^{n+1}_{y_{N+1+1/2}}$ by assuming that $B^{n+1}_{x_{N+1}} = -\eta B^{n+1}_{N}$, where $N \equiv n_g+n_a$. This "upper" boundary condition has the effect of stabilizing the field calculations at the "open end" of our calculation. Before giving the details of this upper boundary condition we must first solve Eq. (2-22).

To solve Eq. (2-22) we let (for $k = 2$ to n_g+n_a),

$$E^{n+1}_{y_{k+1/2}} = e_k E^{n+1}_{y_{k+1+1/2}} + f_k \tag{2-23}$$

define e_k and f_k. Then

$$E^{n+1}_{y_{k-1+1/2}} = e_{k-1} E^{n+1}_{y_{k+1/2}} + f_{k-1}$$

substituted into Eq. (2-22) gives

$$E^{n+1}_{y_{k+1/2}} = E^{n+1}_{y_{k+1+1/2}}\left(\frac{-A_{2_k}}{A_{1_k}+A_{3_k}e_{k-1}}\right)+\left(\frac{A_{4_k}-A_{3_k}f_{k-1}}{A_{1_k}+A_{3_k}e_{k-1}}\right)$$

or, by Eq. (2-23),

$$\left.\begin{aligned} e_k &= \frac{-A_{2_k}}{A_{1_k}+A_{3_k}e_{k-1}}, \\ \text{and} \quad f_k &= \frac{A_{4_k}-A_{3_k}f_{k-1}}{A_{1_k}+A_{3_k}e_{k-1}}. \end{aligned}\right\} \tag{2-24}$$

which are the recursion relations for e_k and f_k ($k = 2$ to $N = n_g+n_a$). With $k = 2$ and $E^{n+1}_{y_{1+1/2}} = 0$ (lower boundary condition) Eq. (2-22) gives

$$E^{n+1}_{y_{2+1/2}} = E^{n+1}_{y_{3+1/2}} \frac{-A_{2_2}}{A_{1_2}} + \frac{A_{4_2}}{A_{1_2}}$$

or, by Eq. (2-23)

$$e_2 = \frac{-A_{2_2}}{A_{1_2}}$$

and (2-25)

$$f_2 = \frac{A_{4_2}}{A_{1_2}}$$

With the recursion relations (Eqs. (2-24) and (2-25)) one solves for e_k and f_k for $k = 2$ to N. With the quantity $E^{n+1}_{y_{N+1+1/2}}$ known (as will be given below) and the values of e_k and f_k known, one solves Eq. (2-23) by (backward) recursion for all values of $E^{n+1}_{y_{k+1/2}}$ for $k = N$ to $k = 2$. With all mesh values of E_y known at τ^{n+1}, one solves for all mesh values of E_z at τ^{n+1} by using Eq. (2-19). Then the values of B_x at τ^{n+1} can be found from Eq. (2-16) and Maxwell's equations will have been advanced one time step.

The following procedures are used to obtain the value of $E^{n+1}_{y_{N+1+1/2}}$. The E_y equation is differenced as in Eq. (2-8) except: $\Delta_{a_k} = 0$ in the uniform mesh, J_y and σ at $k = N$ and $k = N-1$ are used to extrapolate to $z_{N+1+1/2}$, and $B^{n+1}_{x_{N+1}} = -\eta B^{n+1}_{x_N}$ as discussed above. One obtains an equation

$$E^{n+1}_{y_{N+1+1/2}} = D_{1_{N+1}} - D_{2_{N+1}} B^{n+1}_{x_N} (\eta+1) \qquad (2\text{-}26)$$

where $D_{1_{N+1}}$ and $D_{2_{N+1}}$ are essentially calculated as in Eq. (2-9), except for the extrapolated values of J_y and σ. The gradients of J_y and σ are small at the upper boundary so these extrapolations do not cause any problems. The input constant η was set to unity in most problems and very little effect was observed with its reasonable variation from unity. Substituting in Eq. (2-26) for $B^{n+1}_{x_N}$ by Eq. (2-16) and then for $E^{n+1}_{z_N}$ by Eq. (2-19), we find

$$E^{n+1}_{y_{N+1/2}} = E^{n+1}_{y_{N+1+1/2}} e'_N + f'_N \qquad (2\text{-}27)$$

where

$$e'_N = \frac{1 + D_{2_{N+1}}(\eta+1)\left(C_{2_N} G_{2_N} + C_{3_N}\right)}{D_{2_{N+1}}(\eta+1)\left(C_{2_N} G_{2_N} + C_{3_N}\right)} ,$$

and

$$f'_N = \frac{-D_{1_{N+1}} + D_{2_{N+1}}(\eta+1)\left(C_{1_N} + C_{2_N} G_{1_N}\right)}{D_{2_{N+1}}(\eta+1)\left(C_{2_N} G_{2_N} + C_{3_N}\right)} .$$

Eq. (2-27) and Eq. (2-23) with $k = N$ are then used to eliminate $E^{n+1}_{y_{N+1/2}}$ and find

$$E^{n+1}_{y_{N+1+1/2}} = -\frac{f_N - f'_N}{e_N - e'_N} . \qquad (2\text{-}28)$$

2.5 THE EQUATIONS OF MOTION

The semi-implicit method used to difference the equations of motion (Eqs. (1-30)) in GLANC, which will now be given, was selected on the basis of simplicity, of calculational speed, and of accuracy. Differencing Eqs. (1-30), we have

$$P^{n+1}_x = P^n_x - \frac{\delta\tau\ A^n\left(P^{n+1}_x + P^n_x\right)}{2\left(1 - \cos\theta\ \frac{v^n_y}{c}\right) cP^n}$$

$$P^{n+1}_y = P^n_y - \frac{\frac{e}{c}\,\delta\tau}{1 - \cos\theta\ \frac{v^n_y}{c}}\left(E^n_y + \left(\frac{v_z}{c}\right)^n B^n_x\right) - \frac{\delta\tau\ A^n\left(P^{n+1}_y + P^n_y\right)}{2\left(1 - \cos\theta\ \frac{v^n_y}{c}\right) cP^n} \qquad (2\text{-}29)$$

$$P^{n+1}_z = P^n_z - \frac{\frac{e}{c}\,\delta\tau}{1 - \cos\theta\ \frac{v^n_y}{c}}\left(E^n_z - \left(\frac{v_y}{c}\right)^n B^n_x\right) - \frac{\delta\tau\ A^n\left(P^{n+1}_z + P^n_z\right)}{2\left(1 - \cos\theta\ \frac{v^n_y}{c}\right) cP^n} .$$

If one defines

$$\xi = \frac{A^n \, \delta\tau}{2\left(1-\cos\theta \, \frac{v_y^n}{c}\right) cP^n} , \qquad (2\text{-}30)$$

then Eqs. (2-29) simplify to

$$P_x^{n+1} = P_x^n \, \frac{1-\xi}{1+\xi}$$

$$P_y^{n+1} = \frac{P_y^n(1-\xi) - \dfrac{\frac{e}{c}\,\delta\tau}{\left(1-\cos\theta \, \frac{v_y^n}{c}\right)}\left[E_y^n + \frac{v_z^n}{c} B_x^n\right]}{1+\xi} \qquad (2\text{-}31)$$

$$P_z^{n+1} = \frac{P_z^n(1-\xi) - \dfrac{\frac{e}{c}\,\delta\tau}{\left(1-\cos\theta \, \frac{v_y^n}{c}\right)}\left[E_z^n - \left(\frac{v_y}{c}\right)^n B_x^n\right]}{1+\xi} .$$

The location in the z-mesh of the particle with momentum $\vec{P}$ is known by the use of Eq. (1-31). The fields used in Eqs. (2-31) are evaluated at the particle location by interpolation of the field mesh values.

There are six numbers carried in the computing machine for each particle that is injected. Besides the three momentum components and the position in the z-mesh, a "weight" is carried and an index (K_p) is carried. The weight is discussed in a later section. The index K_p is zero if the particle has been "turned off." The index K_p is equal to k if the particle is in cell z_k , i.e., if the particle location z_p satisfies $z_{k+1/2} \leq z_p < z_{k+1+1/2}$, and if the particle is "turned on" or active. It may be noted that the index K_p is not necessary, i.e., the weight of the particle can be zero to indicate if the particle is "off" and greater than zero to indicate the particle is "on" and one can locate the cell the particle is in from the number z_p . However, the nonuniform mesh would make the search for the cell containing z_p rather time consuming and, thus, we use the index K_p .

The drag force (Eq. (1-27) becomes infinite as the electron energy approaches zero. Thus ξ as given by Eq. (2-30) is used as an indicator to turn off an active particle when its energy is so small that the particle no longer contributes in the source calculation. The particle is turned off if $\xi \geq 0.5$.

SECTION 3

SOURCES

3.1 INTRODUCTION

In order to solve Maxwell's equations we must have J_y , J_z , and σ as a function of space and time for the GLANC geometry. σ is obtained from the air-ion equations by knowing $\dot{\gamma}$. The source calculations must therefore provide a method of obtaining J_y , J_z , and $\dot{\gamma}$ as functions of space and time.

The transport of the gamma rays from the burst point to the line of calculation (z-mesh) will be characterized by a fractional loss of gamma flux

$$f_r = \frac{e^{-\mu_{t_\gamma} \cdot 10^{-3} \rho r}}{4\pi r^2} \tag{3-1}$$

where r is the slant range (see Figure (1-1)), ρ is the density in gms/liter, μ_{t_γ} is the total absorption coefficient in cm^2/gm.

The number of gamma rays of energy E (Mev) per square centimeter per second gives the gamma flux at the line of calculation:

$$f_\gamma = \frac{Y \varepsilon_\gamma K}{E_\gamma} c \, F(\tau') \, f_r \tag{3-2}$$

where Y is the yield in kilotons of the nuclear device, ε_γ is the fraction of the yield that appears in the form of gamma rays, $K = 2.613 \cdot 10^{25}$ is the number of Mev per kiloton (kt), and $F(\tau')$ gives the time history of the gamma rays at the source, as described in the next section, in units of per centimeter.

It may be noted that the above prescription for the gamma ray source at the line of calculation does not account for any space or time build-up factors. Since GLANC is only designed for close-in and short-time calculations, the buildup factors are easily simulated by selecting the input parameters which determine $F(\tau')$ and by changing the magnitude of, say, ε_γ , i.e., with the shape in time and the peak magnitude controllable by input, the gamma flux input is flexible.

3.2 THE TIME HISTORY

In all source calculations the time history of the gamma ray pulse is prescribed by an analytic function, $F(\tau')$, where $\tau' = \tau + z \sin\theta$ as in Eq. (1-7). We define

$$F(\tau') \begin{cases} = A_1 A_0 e^{\beta_1 \tau'} & \text{for } 0 \leq \tau \leq \tau_a \\ = \dfrac{A_2 A_0 \left(1 + \dfrac{\beta_2}{\beta_3}\right) e^{\beta_2(\tau' - \tau_0)}}{1 + \dfrac{\beta_2}{\beta_3} e^{(\beta_2 + \beta_3)(\tau' - \tau_0)}} & \text{for } \tau_a < \tau < \tau_b \\ = A_3 A_0 e^{-\beta_4 \tau'} & \text{for } \tau_b \leq \tau \leq \tau_{max} \end{cases} \qquad (3\text{-}3)$$

where β_1, β_2, β_3, β_4, τ_a, τ_0, τ_b, τ_{max}, and A_0 are input numbers. The z-dependence of the retarded time and of the gamma ray attenuation in the ground have been included. $A_0 = e^{\gamma_0 z/\sin\theta}$ in the ground when θ is greater than zero and $A_0 = 1$ in the air. If the burst point is on the ground $(\theta=0)$ the sources are zero in the ground so that A_0 is not used in the ground. γ_0 is the reciprocal of the gamma ray attenuation length in the ground. The values of A_1, A_2, and A_3 are determined in the following manner. We want the functions to be continuous at τ_a and τ_b, the integral

$$\int_{-\infty}^{\infty} F(\tau) d\tau = 1 \qquad (3\text{-}4)$$

and τ_0 (the value of τ at the peak) to be selected reasonably so that $\tau_a < \tau_0 < \tau_b$. Set $A_2 = 1$ at, say $z = 0$. Then find A_1 and A_3 such that $F(\tau)$ is continuous at τ_a and τ_b , respectively. Defining I_1 and I_2 by

$$I_1 \equiv \int_{-\infty}^{\tau_a} F(\tau)d\tau = \frac{A_1}{\beta_1} e^{\beta_1 \tau_a}$$

and

$$I_2 \equiv \int_{\tau_b}^{\infty} F(\tau)d\tau = \frac{A_3}{\beta_4} e^{-\beta_4 \tau_b} ,$$

one calculates by use of the computing machine the value

$$I = I_1 + I_2 + \int_{\tau_a}^{\tau_b} F(\tau)d\tau ,$$

so that Eq. (3-4) can be satisfied by the normalization of the initially assumed constants, i.e., one replaces A_1 by A_1/I , A_2 by A_2/I , and A_3 by A_3/I . Since all the z-dependence of the source has been included in $F(\tau')$ and since GLANC carries only the retarded time on the ground $(z = 0)$, we will sometimes write $F(\tau') = F_k(\tau)$ for the time history at z_k .

3.3 THE PRESCRIBED SOURCE

The "prescribed" source, as in most EMP codes, is determined by making it proportional to the gamma ray flux. Thus, we have,

$$\vec{J} = -\vec{n}\, f_\gamma \mu_c R_{Mf} \frac{e}{c} \qquad (3\text{-}5)$$

where $\vec{J}$ is the current density in abamps per cm^2, $\vec{n}$ is as in Figure (1-1), μ_c is the Compton scattering coefficient in cm^2 per gm, and

R_{Mf} is the mean forward range of an electron in gms per cm². Similarly,

$$\dot{\gamma} = f_\gamma \rho \cdot 10^{-3} \mu_a \frac{E_\gamma}{34 \cdot 10^{-6}} \qquad (3\text{-}6)$$

where $\dot{\gamma}$ is in ion-pairs per cm³ per sec, μ_a is the absorption coefficient in cm² per gm, and 34 ev is the energy needed to form each ion pair. It is more convenient to write Eqs. (3-5) and (3-6) as,

$$\vec{J} = \vec{n}\, J(\tau') ,$$

and

$$\dot{\gamma} = \dot{\gamma}_0\, F_k(\tau') \qquad (3\text{-}7)$$

where we define

$$J(\tau') = -J_0 F_k(\tau) ,$$

$$J_0 = \frac{Y\varepsilon_\gamma\, K}{E_\gamma} \mu_c R_{Mf} e\, f_r , \qquad (3\text{-}8)$$

and

$$\dot{\gamma}_0 = \frac{Y\varepsilon_\gamma K c \mu_a \rho \cdot 10^3}{34} f_r .$$

Thus, in GLANC, the "prescribed" value for $\dot{\gamma}$ is given by Eq. (3-7) and the "prescribed" values of the current components are given by

$$J_y = J(\tau') \cos\theta$$

$$J_z = -J(\tau') \sin\theta \qquad (3\text{-}9)$$

3.4 THE LEMP METHOD

The LEMP geometry for DX and DY is shown in Figure 3-1. In LEMP* the field reaction on the electrons was (nonlinearly) fitted as a

*H. J. Longley, "Compton Current in Presence of Fields for LEMP 1," Los Alamos Scientific Laboratory report LA-4348. See also LA-4347 (Secret-RD) and LA-4346.

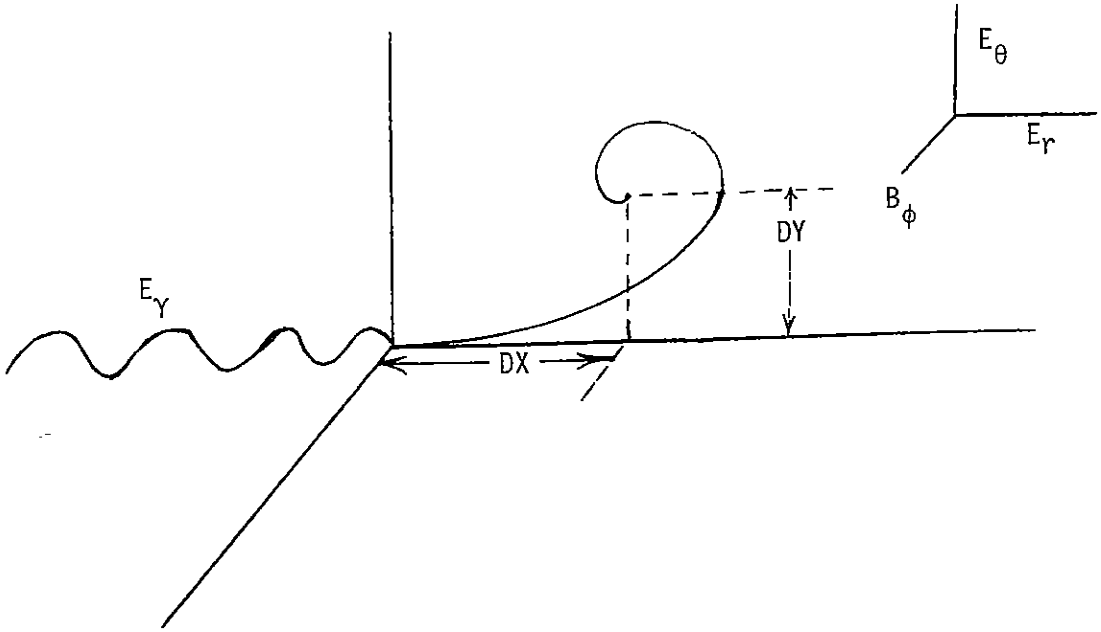

Figure 3-1. LEMP geometry for DX and DY and the fields.

function of E_γ , B_ϕ , E_r , and E_θ . DX is the average distance the electron travels in the direction of the gamma ray and DY is the average distance the electron travels perpendicular to the direction of the gamma ray. E_r , E_θ , and B_ϕ in LEMP form a right-handed spherical coordinate system and, in terms of the GLANC fields, are given by

$$
\begin{aligned}
B_\phi &= -B_x \\
E_r &= E_y \cos\theta - E_z \sin\theta \\
E_\theta &= -E_y \sin\theta - E_z \cos\theta
\end{aligned}
\qquad (3\text{-}10)
$$

If the fields are zero, then DX = R and DY = 0. (Here, and in this section, R is the mean forward range as given by R_{Mf} above.) The LEMP fields as calculated by Eqs. (3-10) were used in the LEMP subroutine to calculate DX and DY . The sources for GLANC, by the LEMP method, are given by

$$J_y = J(\tau') \left[-\frac{DY}{R} \sin\theta + \frac{DX}{R} \cos\theta \right] ,$$

$$J_z = -J(\tau') \left[\frac{DY}{R} \cos\theta + \frac{DX}{R} \sin\theta \right] , \quad (3\text{-}11)$$

and

$$\dot{\gamma} = \dot{\gamma}_0 F_k(\tau) \left[\frac{\overline{E}_e + e(E_r DX + E_\theta DY)}{\overline{E}_e} \right]$$

where $\overline{E}_e$ is the mean electron energy. The factor in the bracket on the right side of this $\dot{\gamma}$ equation accounts for the energy loss (or gain) of the electrons due to the fields. This factor was not allowed to be smaller than 0.2 in LEMP and in GLANC.

3.5 THE PARTICLE SOURCE

The particle source is obtained by injecting sample Compton electrons at each z_k of the mesh in the air at periodic times and then following these electrons by solving their equations of motion. The fields and the particles are advanced in time in a completely self-consistent manner.

At the beginning of a problem the fields are generally small. Thus, the injection of particles is not started until $\sqrt{E_y^2+E_z^2+B_x^2} > E_{00}$, where E_{00} is an input number of about 0.5 (esu).

The initial values of the momentum of the particles were given in section 1.3 and the equation of motion of the particle and its position were discussed in section 2.4.

The number of Compton electrons born per cm^3 per unit of retarded time is given by

$$\frac{dN_e}{d\tau} = \frac{1}{c} 10^{-3} \rho \mu_c f_\gamma \quad (3\text{-}12)$$

and the total number of Compton electrons born per cm^3 is given by

$$N_0 = \int_{-\infty}^{\infty} \frac{dN_e}{d\tau}\, d\tau = \frac{Y \varepsilon_\gamma K}{E_\gamma} 10^{-3} \rho \mu_c \, f_r \quad , \tag{3-13}$$

when one uses the results of Eq. (3-4). The Compton current density (abamps/cm^2) for a particle is given by

$$\vec{J}_p = -WT(\tau,z) e \frac{\vec{v}}{c} \frac{1}{1-\cos\theta \frac{v_y}{c}} \tag{3-14}$$

where

$$WT(\tau,z) = \frac{N_0 F_k(\tau)\delta\tau}{n_\theta n_\phi} \tag{3-15}$$

is the weight assigned to the particle at its injection time for its space (z_k) position. Note that the weight becomes N_0 when summed over all cycles (all τ) and over all $n_\theta \cdot n_\phi = n_p$. The value of $\dot{\gamma}$ (ion-pairs/cm^3 sec) for a particle is given by

$$\dot{\gamma}_p = \frac{c}{34 \cdot 1.60 \cdot 10^{-12}} \, WT(\tau,z) \frac{A\left(\frac{\vec{p}}{p} \cdot \frac{\vec{v}}{c}\right)}{\left(1-\cos\theta \frac{v_y}{c}\right)} \tag{3-16}$$

where

$$\left(\frac{\vec{p}}{p} \cdot \frac{\vec{v}}{c}\right) = \frac{\vec{p}}{p} \cdot \frac{\vec{p}}{\sqrt{p^2+m^2c^2}} = \frac{p}{\sqrt{p^2+m^2c^2}} \quad .$$

In GLANC, the current or $\dot{\gamma}$ at the desired mesh point, z_k , is found by adding the contribution of all particles in the cell surrounding z_k at the desired time.

Since the mesh in GLANC is nonuniform and since the area or volume is not included as a consideration in the weight of the particle (Eq. (3-15)), we change the weight of a particle when it changes cells by the ratio of cell sizes. When the particle changes to the z_{k+1} cell from the z_k cell, its weight is changed by the factor $(z_{k+1+1/2}-z_{k+1/2})/(z_{k+2+1/2}-z_{k+1+1/2})$. Similarly, a particle changing to the z_{k-1} cell from the z_k cell has its weight changed by the factor $(z_{k+1+1/2}-z_{k+1/2})/(z_{k+1/2}-z_{k-1+1/2})$.

The boundary conditions on the particles in the ground cell and in the uppermost (large z) cell are accounted for by the following prescriptions. When a particle has a value of z less than zero, it is turned off. When a particle leaves the uppermost cell and is to enter the cell below, a new particle is injected at the top of the uppermost cell with the parameters of the particle (except for position) which is leaving the bottom of this uppermost cell. A particle leaving the top of the uppermost cell is turned off. At the ground there are other physical effects which can be included in GLANC, such as the electron multiplication effect and gamma-ray scattering effects. However, these effects have not been included in GLANC at this writing due mainly to their complications.

To save computational time in processing particles a method of time averaging was devised to allow one to inject particles only every N cycles, i.e., for injection time intervals $\delta\tau_i = N\delta\tau$. The symbol N used here is an integer input number and should not be confused with the same symbol used above for the last cell in the z_k mesh. The need for such a scheme is due to the small values of Z_{oa} (smallest mesh size in the air) required to resolve the field gradients near the ground and the consequent small values of $\delta\tau$ required, in the solution of the equations of motion of the particles because of the retarded time factor $1/\left(1-\cos\theta \frac{v_y}{c}\right)$. This factor is more than ten for some of the particles at injection time. Also, the small values of z_{oa} require small values of $\delta\tau$ be used in the solutions of Maxwell's equations. For example, we ran problems with GLANC where, for $z_{oa} = 4$ cm , $\delta\tau = 2$ cm in a problem which does not inject particles and $\delta\tau = 1$ cm in a problem where particles are injected. A $\delta\tau$ of one centimeter is (1/300)th of a shake in retarded time.

At each mesh point z_k in the air one calculates an average value for J_y , J_z , and $\dot{\gamma}$ at any given time by an equation of the form

$$\overline{J} \quad = \quad J_1 f_1 + J_2 f_2 + \ldots + J_N f_N \tag{3-17}$$

where J_1 here represents either J_y, J_z, or $\dot{\gamma}$ at the latest time, J_N is the same at a time $N\delta\tau$ earlier, f_i is the ratio of $F_k(\tau_i)$ to F', the time of $F_k(\tau_i)$ is given in Figure 3-2, and F' is $F_k(\tau)$ at the latest injection time. Figure 3-2 shows graphically the quantities used in the averaging scheme at a time of injection with $N = 8$. The sum on the right side of Eq. (3-17) is not divided by N since the $WT(\tau,z)$ function in Eq. (3-15) contains $\delta\tau$ and not $\delta\tau_i = N\delta\tau$. The values of f_i follow the time history of the gamma rays $(F_k(\tau))$ and are normalized by $F_k(\tau)$ at the last injection time (F'). Figure 3-2 is an example of the various functions as they might appear on a log-linear plot at early times when $F_k(\tau)$ is increasing as $e^{\beta_1\tau}$. The saw-tooth curve simulates the effects of the new injection of particles and the decay of all the particles. The line through J_1 represents the source (J_y, J_z, or $\dot{\gamma}$) as might be given by an analytic function or, here, as the average which would result from an assumed continuous injection of particles. A number $q = 1$ to N is carried in the calculating machine to indicate the time cycle within a given injection cycle. Thus, $q = 1$ in Figure 3-2 and it is injection time. At this time one must back store the f_i and J_i values that exist in storage at the start of the cycle except for f_1 and J_1. The back storage is done by setting $f_1 = 1$ and $J_N = J_{N-1}$, (then) $J_{N-1} = J_{N-2}$, etc. until we set $J_2 = J_1$. A new value of J_1 is found by adding all the contributions to J_1 from all previously injected particles (that have not been turned off) plus one half the contributions to J_1 from all the newly injected particles. When $q > 1$ the back storage is done by setting $f_q = F_k(\tau_q)/F'$ and then setting J_N to the value of J_{N-1}, J_{N-1} to the value of J_{N-2}, etc. until J_2 is set to the value of J_1. The new J_1 is then set equal to the sum of all "live" particle contributions (each from the Eqs. (3-14) or the Eqs. (3-16)).

At the first injection time, i.e., when it has been determined by the criterion given above that the fields are large enough to turn on the particles, one must set up the back storages in the time averaging

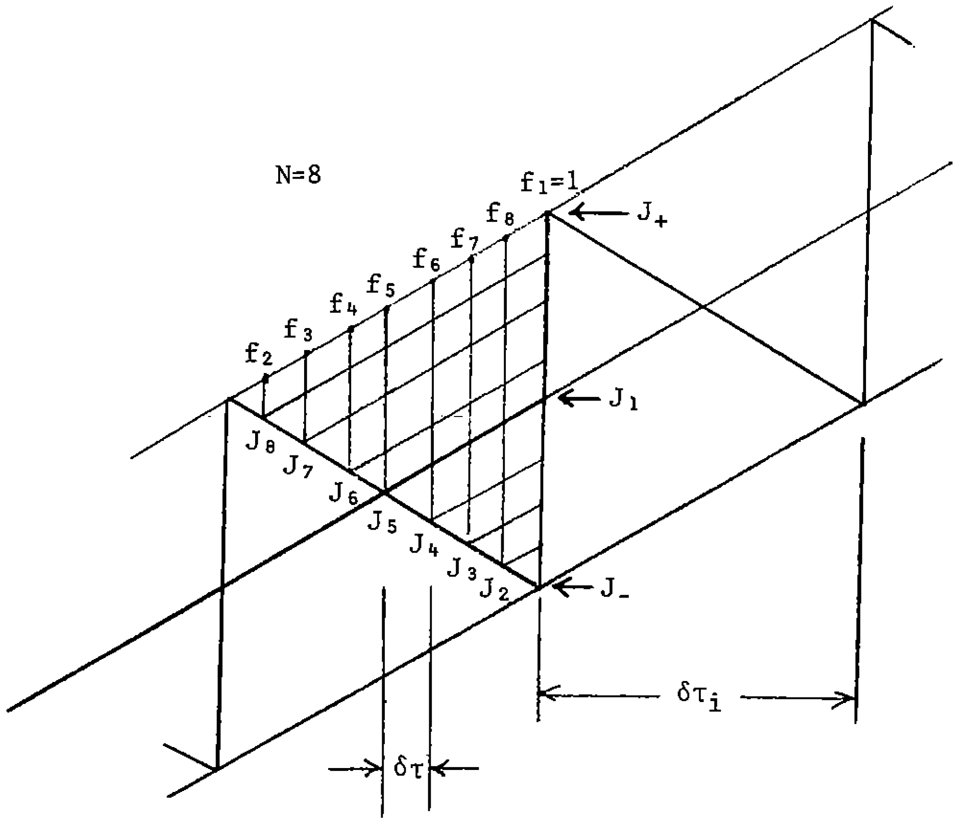

Figure 3-2. Particle Time Average Scheme.

scheme. Figure 3-2 is also appropriate in time for this first injection and the initial back storage calculation. This initial back storage, as given below, is done in such a manner as to make the particle sources fit smoothly onto the prescribed source. We thus set J_1 equal to the prescribed current (which exists at the time of this first injection) divided by N . We inject the first batch of particles and calculate J_{p1} as the source from these particles with their usual standard weights. A weight correction $W_c = \frac{J_{p1}+2J_1}{2J_{p1}}$ is calculated and the weight of each of the initially injected particles is multiplied by W_c . The current of the first injected particles with corrected weights, J_+ , is then calculated. We then set $J_- = J_1-(J_+-J_1) = 2J_1-J_+$. Then $J_2 = J_- e^{\beta_p \delta\tau}$ where $\beta_p = 0.0417\rho$ is an empirically determined decay rate per cm for the particles. β_p was determined by running a batch of particles with zero fields. Then $J_i = J_{i-1}e^{\beta_p \delta\tau}$ for $i = 3, 4, \ldots, N$. The initial values of f_i are determined by knowing the initial rise constant, β_1, of $F_k(\tau)$. We set $f_1 = 1.0$ and $f_i = f_{i-1}e^{\beta_1 \delta\tau}$ for $i = 2, 3, \ldots, N$.

We will state again that the average source values of J_y, J_z, and $\dot{\gamma}$ are all calculated by the methods given above and characterized by Eq. (3-17). These calculations and back storages are done each cycle for each cell in the air to find the sources for Maxwell's equations by the particle method.

SECTION 4

ZERO TIME SOLUTIONS

4.1 AIR CONDUCTIVITY

The differencing of Maxwell's equations as given in section 2.4 assume that σ is greater than zero. The conductivity in the ground, σ_0, is an input number greater than zero. In the air we must find a zero time solution for σ. By Eqs. (1-19), (3-3), and (3-7) for $\tau \simeq 0$, we have in the air for $n_- \simeq 0$,

$$c\,\frac{dn_e}{d\tau} + \alpha n_e \quad = \quad \dot{\gamma} \quad = \quad \dot{\gamma}_0 A_1 e^{\beta_1(\tau + z\sin\theta)} \tag{4-1}$$

Letting n_e be of the form

$$n_e \quad = \quad n_0 e^{\beta_1 \tau} \tag{4-2}$$

one finds

$$n_0 \quad = \quad \frac{\dot{\gamma}_0 A_1 e^{\beta_1 z \sin\theta}}{c\beta_1 + \alpha} \tag{4-3}$$

Thus σ at zero time is given by

$$\sigma \quad = \quad \frac{e}{c}\, n_e \mu_e \quad = \quad \frac{e}{c}\, n_0 \mu_e \tag{4-4}$$

where $|E| = 0$ in the calculation of μ_e and α.

4.2 MAXWELL'S EQUATIONS

When the burst height is zero there is no appropriate zero time solution for Maxwell's equations (Eqs. (1-12), (1-17), and (1-18)). This is due to the fact that the radial derivatives are not present. This difficulty is eliminated by selecting τ_0 (τ at the peak on the ground) such that $4\pi\sigma >> \frac{1}{\lambda}$, where λ is the gamma ray mean free path in centimeters. For this ground burst we set $E_y = E_z = B_x = 0$ at zero time.

When the burst is above the ground an appropriate zero time solution does exist. Here we let σ be zero in the air. By Eq. (3-9) we have for $\tau \simeq 0$,

$$\left.\begin{aligned} J_y &= J_0 A_1 e^{\beta_1\tau+kz} \cos\theta \\ J_z &= J_0 A_1 e^{\beta_1\tau+kz} \sin\theta \end{aligned}\right\} \qquad (4\text{-}5)$$

where, here, $k \equiv \beta_1 \sin\theta + \frac{\gamma_0}{\sin r}$ and γ_0 is given after Eqs. (3-3) and is zero in the air. Letting E_y, E_z, and B_x vary as $e^{\beta_1\tau+kz}$ the Eqs. (1-12), (1-14), and (1-13) give

$$\left.\begin{aligned} (\varepsilon\beta_1+4\pi\sigma)E_{y_0} - kB_{x_0} &= +4\pi J_0 A_1 \cos\theta \\ (\varepsilon\beta_1+4\pi\sigma)E_{z_0} - \beta\cos\theta\, B_{x_0} &= -4\pi J_0 A_1 \sin\theta \\ \beta_1 B_{x_0} - \beta_1 \cos E_{z_0} &= k\,E_{y_0} \end{aligned}\right\} \qquad (4\text{-}6)$$

or, using the last of these equations to eliminate B_{x_0} from the other two, we have

$$\left.\begin{aligned} \left(\eta - \frac{k^2}{\beta_1}\right)E_{y_0} + (-k\cos\theta)E_{z_0} &= 4\pi J_0 A_1 \cos\theta \\ (-k\cos\theta)E_{y_0} + (\eta - \beta_1\cos^2\theta)E_{z_0} &= -4\pi J_0 A_1 \sin\theta \end{aligned}\right\} \qquad (4\text{-}7)$$

where, here, $\eta \equiv \varepsilon\beta_1 + 4\pi\sigma$ and the determinate is given by

$$D = \eta\left(\eta - \beta_1\cos^2\theta - \frac{k^2}{\beta_1}\right) \quad . \tag{4-8}$$

A particular solution for the inhomogeneous equations (Eq. (4-7)) gives,

$$\left.\begin{aligned} E_{y_{0_a}} &= \frac{4\pi J_0 A_1 \cos\theta}{\beta_1} \\ E_{z_{0_a}} &= -\frac{4\pi J_0 A_1 \sin\theta}{\beta_1} \end{aligned}\right. \tag{4-9}$$

and

$$B_{x_{0_a}} = 0$$

in the air and

$$\left.\begin{aligned} E_{y_{0_g}} &= \frac{4\pi J_0 A_1 \cos\theta}{D_g}\left[\eta_g - \beta_1\cos^2\theta - k_g\sin\theta\right] \\ E_{z_{0_g}} &= -\frac{4\pi J_0 A_1}{D_g}\left[\sin\theta\left(\eta_g - \frac{k_g^2}{\beta_1}\right) - k_g\cos^2\theta\right] \\ \text{and} \quad B_{x_{0_g}} &= \frac{4\pi J_0 A_1}{D_g}\left[\frac{k_g}{\beta_1} - \sin\theta\right] \end{aligned}\right\} \tag{4-10}$$

in the ground where $k_g = \beta_1\sin\theta + \dfrac{\gamma_0}{\sin\theta}$, $\eta_g = \varepsilon\beta_1 + 4\pi\sigma_0$, and $D_g = \eta_g(\eta_g - \beta_1\cos^2\theta - k_g^2/\beta_1)$.

The solution to the homogeneous equations (Eqs. (4-7)) is given when $D = 0$, i.e., when

$$k_h^2 = \beta_1\left[4\pi\sigma + \beta_1(\varepsilon - \cos^2\theta)\right] \tag{4-11}$$

and

$$\left.\begin{aligned} E_{y_h} &= \frac{k_h}{\eta} B_{x_h} \\ E_{z_h} &= \frac{\beta_1\cos\theta}{\eta} B_{x_h} \end{aligned}\right\} \tag{4-12}$$

In the air the negative sign in Eqs. (4-11) is used so that B_x in the air is less above ground, i.e., $k_{h_a} = -\beta_1 \sin\theta$. Similarly, in the ground where z is less than zero, we select $k_{h_g} = +\sqrt{\beta_1\{4\pi\sigma_0+\beta_1(\varepsilon-\cos^2\theta)\}}$.

Thus, the general solution at early times in the air is given by,

$$
\begin{aligned}
E_y &= e^{\beta_1\tau}\left[E_{y_{0_a}} e^{\beta_1 z\sin\theta} - B_{x_{h_a}} \sin\theta \; e^{-\beta_1 z\sin\theta}\right] \\
E_z &= e^{\beta\tau}\left[E_{z_{0_a}} e^{\beta_1 z\sin\theta} + B_{x_{h_a}} \cos\theta \; e^{-\beta_1 z\sin\theta}\right] \qquad (4\text{-}13) \\
B_x &= e^{\beta_1\tau}\left[B_{x_{h_a}} e^{-\beta_1 z\sin\theta}\right]
\end{aligned}
$$

In the ground we have,

$$
\begin{aligned}
E_y &= e^{\beta_1\tau}\left[E_{y_{0_g}} e^{z\left(\beta_1\sin\theta+\frac{\gamma_0}{\sin\theta}\right)} + B_{x_{h_g}} \frac{k_{h_g}}{\eta_g} e^{z\,k_{n_g}}\right] \\
E_z &= e^{\beta_1\tau}\left[E_{z_{0_g}} e^{z\left(\beta_1\sin\theta+\frac{\gamma_0}{\sin\theta}\right)} + B_{x_{h_g}} \frac{\beta_1\cos\theta}{\eta_g} e^{z\,k_{n_g}}\right] \qquad (4\text{-}14) \\
B_x &= e^{\beta_1\tau}\left[B_{x_{0_g}} e^{z\left(\beta_1\sin\theta+\frac{\gamma_0}{\sin\theta}\right)} + B_{x_{h_g}} e^{z\,k_{h_g}}\right]
\end{aligned}
$$

The two unknowns in Eqs. (4-13) and (4-14) are $B_{x_{h_a}}$ and $B_{x_{h_g}}$. Since E_y and B_x are to be continuous at $z = 0$, two equations result from these continuity requirements and the two unknowns can be found.

APPENDIX

UNITS

$$E(\text{volts/meter}) = 3 \cdot 10^4 \, E \quad (\text{esu})$$

$$B(\text{webers/m}^2) = 10^{-4} \, B \quad (\text{emu})$$

$$J(\text{amps/m}^2) = 10^5 \, J \quad (\text{abamps/cm}^2)$$

$$\sigma(\text{mho/m}) = \frac{10}{3} \sigma \quad (\text{cm}^{-1})$$

$$\varepsilon(\text{MKS}) = \varepsilon_0 \varepsilon \quad (\text{Gaussian})$$

Here ε_0 [$(8.85_{415} \cdot 10^{-12})$ Farads/meter] is the dielectric constant of free space, which is well known by users of MKS units.