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Theory of the Radio Flash - Part VI
High Frequency Signal from a Ground Burst

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Abstract

In this report an approximation scheme is developed which is applicable, in principle, to both ground bursts and to very low altitude air bursts and to all distances from the explosion. This result is achieved by limiting our attention to fields near the ground and to the early phases of such signals, specifically to times $\tau < \lambda/c$ (λ, c respectively gamma ray mean free path and light velocity).

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1. Introduction

In a private conversation with the author, W. J. Karzas pointed out that the high frequency approximation, recently developed for a unified theory of the air burst,[1] could also lead to considerable simplification of the ground burst problem. The essence of this method is to express everything in terms of retarded time, $\tau = t - r/c$, and to neglect $\partial/\partial r$ of any field quantity compared with $\partial/c\partial\tau$ of the same quantity. However, for the ground burst, terms in $\partial/\partial\theta$ must be kept as they will be large near the ground. This scheme leads to equations previously found[1] for E_θ and B_φ but to a complicated equation for E_r.

So far everything is straightforward; it is at this point that Karzas' suggestion becomes significant. His proposal is to simplify the radial equation by a procedure used by Longmire[2] in a different context. Namely one considers two regions, an inner region where conductivity is high and conduction current dominates displacement current, and an outer region where the reverse is true. Following Longmire, we may call the region where conduction current dominates the diffusion zone and the outer region the wave zone.

In this report we have carried out the details of making the approximations outlined above and of applying the results to the calculation of the magnetic field in the near zone. The results are found to compare quite favorably with those of the same problem solved by stepwise numerical integration of Maxwell's equations.[3] This part of the theory, then, consists of an improvement of Longmire's theory of the near field[2] in that: (1) the calculations are easier in the present theory and (2) the present theory extends to larger distances from the explosion. In Section 4 the theory is applied to calculate the distant field. However there exists nothing to which we can compare this result. Thus there is no inherent upper limit to the distance of applicability of the present theory. It also turns out to be very easy to extend the theory to the case of low altitude air bursts. This topic is discussed in Section 5.

2. The High Frequency Approximation

In the variables r and retarded time $\tau = t - r/c$, Maxwell's equations for a ground burst are

$$(2.1)\qquad \left\{ \begin{aligned} &\frac{1}{c}\frac{\partial E}{\partial \tau} + 4\pi\sigma E = \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}(\sin\theta G) - 4\pi j \quad , \\ &\frac{1}{c}\frac{\partial F}{\partial \tau} + 4\pi\sigma F = \frac{1}{c}\frac{\partial G}{\partial \tau} - \frac{\partial G}{\partial r} \quad , \\ &\frac{1}{c}\frac{\partial G}{\partial \tau} = \frac{\partial E}{\partial \theta} + \frac{1}{c}\frac{\partial F}{\partial \tau} - \frac{\partial F}{\partial r} \quad , \end{aligned} \right.$$

where we have set

(2.2) $$E_r \overset{\text{def}}{=} E , \qquad rE_\theta \overset{\text{def}}{=} F , \qquad rB_\varphi \overset{\text{def}}{=} G .$$

It is convenient to define a new function Φ by the relation

(2.3) $$\Phi \overset{\text{def}}{=} \int_{-\infty}^{\tau} cE(\tau')d\tau' .$$

We see from this definition that Φ vanishes at $\tau = -\infty$, or equivalently at $r = +\infty$, or at $\sigma = 0$. Integrating the G-equation yields

(2.4) $$G = F + \frac{\partial\Phi}{\partial\theta} - \int_{-\infty}^{\tau} \frac{\partial F(\tau')}{\partial r} c d\tau' .$$

Setting this into the F equation yields

(2.5) $$2 \frac{\partial F}{\partial r} + 4\pi\sigma F = \frac{\partial E}{\partial\theta} - \frac{\partial^2\Phi}{\partial r\partial\theta} + \int_{-\infty}^{\tau} \frac{\partial^2 F(\tau')}{\partial r^2} c d\tau' .$$

Now for a burst in contact with the ground the operator $\partial/\partial\theta$ cannot be considered small, but $\partial/\partial r$ operating on j or on σ is of order $(1/\lambda + 2/r)$;

we anticipate that it will be of the same order when operating on field quantities. Therefore the approximation

(2.6) $$\frac{1}{c}\frac{\partial}{\partial\tau} \gg \frac{\partial}{\partial r}$$

should have the same range of validity as we found when treating the air burst.[1] When we consistently ignore $\partial/\partial r$ of a quantity compared with $\partial/c\partial\tau$ <u>of the same quantity</u>, Eqs. (2.5) and (2.4) simplify respectively to

(2.7) $$\begin{cases} \dfrac{\partial F}{\partial r} + 2\pi\sigma F = \dfrac{1}{2}\dfrac{\partial E}{\partial \theta} \quad , \\ \\ G = F + \dfrac{\partial \Phi}{\partial \theta} \quad . \end{cases}$$

If E were known these would be soluble by quadratures. In particular we have

(2.8) $$F = e^{X(r)} \int_0^r \frac{1}{2}\frac{\partial E(r')}{\partial\theta} e^{-X(r')} dr' \quad ,$$

where

(2.9)
$$X \overset{\text{def}}{=} \int_{r}^{\infty} 2\pi\sigma(r')dr' \ .$$

However setting the second of Eqs. (2.7) into the first of Eqs. (2.1) does not lead immediately to a decoupled system.

In order to simplify the radial equation we separate space into two regions $r < R_r$ and $r > R_r$, where R_r is the radius of radial saturation defined by

(2.10)
$$\left|\frac{\partial E}{c\partial\tau}\right|_{R_r} = \left|4\pi\sigma E\right|_{R_r} \ .$$

Because σ is a very rapidly varying function of position, we see that as soon as r is appreciably less than R_r we have

(2.11)
$$\frac{\partial}{c\partial\tau} \ll 4\pi\sigma \qquad \text{for } r < R_r \ .$$

It therefore follows that

(2.12)
$$F = \frac{1}{4\pi\sigma}\frac{\partial E}{\partial\theta}$$

to very good precision and that

(2.13) $$|F| \ll |\partial\Phi/\partial\theta| \quad \text{for } r < R_r .$$

Using the inequality (2.13) to approximate G and (2.11) to drop the displacement current the equation for the radial field reduces to the following diffusion equation for Φ:

(2.14) $$\frac{4\pi\sigma}{c}\frac{\partial\Phi}{\partial\tau} = \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\Phi}{\partial\theta}\right) - 4\pi j \quad \text{for } r < R_r .$$

When $r > R_r$ the inequality (2.11) is reversed and we have

(2.15) $$|F| \gg |\partial\Phi/\partial\theta| \quad \text{for } r > R_r .$$

When r is only a bit greater than R_r, it follows from the inequality (2.6) that Eq. (2.12) is still a good approximation to F and (2.15) follows from the fact that $\partial/c\partial\tau \gg 4\pi\sigma$. For larger r we are essentially in a radiation field; therefore F is nearly constant and Φ varies as $1/r^2$. Thus the inequality becomes even stronger with increasing r. Making use of the inequality (2.15) to estimate G and dropping now the conduction current, the first member of Eq. (2.1) simplifies to

(2.16) $$\frac{1}{c}\frac{\partial E}{\partial\tau} = \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}(\sin\theta F) - 4\pi J \quad \text{for } r > R_r .$$

The final step in the simplification of the radial equation would be to use Eq. (2.12) for F, ignoring the fact that it is no longer correct at large distances for which $2\pi\sigma < \partial/\partial r$. Actually, we shall modify Eq. (2.12) slightly, setting

$$(2.17) \qquad F = \frac{A}{4\pi\sigma} \frac{\partial E}{\partial \theta} ,$$

where A is as yet undetermined; we suppose, however, that A is independent of θ, so that we have the following diffusion equation for E

$$(2.18) \qquad \frac{4\pi\sigma}{Ac} \frac{\partial E}{\partial \tau} = \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial E}{\partial \theta}\right) - \frac{4\pi\sigma}{A} 4\pi j \qquad \text{for } r > R_r .$$

In the end we shall find that A can be so chosen that, although Eq. (2.17), and hence E as determined by Eq. (2.18), are wrong at large distances, nevertheless E is accurately determined over the range which contributes significantly to the integral of Eq. (2.8).

Note that our procedure inside the saturation zone leads exactly to Longmire's diffusion phase treatment.[2] For according to the inequality (2.13) we have

$$(2.19) \qquad G = \frac{\partial \Phi}{\partial \theta} \qquad \text{for } r < R_r .$$

Differentiating Eq. (2.14) with respect to θ yields

(2.20) $$\frac{4\pi\sigma}{c}\frac{\partial G}{\partial \tau} = \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial G}{\partial\theta}\right) - 4\pi\frac{\partial j}{\partial\theta} .$$

Noting the discontinuity in j at the ground plane and noting that

(2.21) $$\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right) = \frac{\partial^2}{\partial z^2}$$

within our approximation* we recover exactly Longmire's "diffusion phase" equation(2)

(2.22) $$\frac{4\pi\sigma}{c}\frac{\partial G}{\partial \tau} = \frac{\partial^2 G}{\partial z^2} + 8\pi r j\,\delta(z) .$$

Here z is height above the ground and δ is the usual Dirac δ-function.

The equations developed above will treat adequately the diffusion zone and the wave zone of the problem. There is, as we shall see, a third zone, which we shall call the intermediate zone where saturation is achieved near the gamma ray peak but which later desaturates, owing to the decrease of σ. In this zone the principles are as outlined

*The difference between the two expressions involves derivatives with respect to r which are negligible.

above but the detailed treatment is a bit more complicated, as we shall see in Section 3.

3. The Near Fields

(A) The Conductivity

For a ground burst the Compton current density is well described by the very simple expression

$$(3.1) \qquad j = -\frac{Ye^{-r/\lambda}}{r^2}\begin{cases} e^{\alpha\tau} & \text{for } \tau < 0 \ , \\ e^{-\varkappa\tau} & \text{for } 0 < \tau < \tau_1 \ , \\ e^{-\varkappa\tau_1} & \text{for } \tau > \tau_1 \ . \end{cases}$$

Here Y is a constant proportional to the bomb yield, λ is the effective gamma ray mean free path and α, $\varkappa$, τ, are constants. Typically α is of order 10^8, $\varkappa$ a few times 10^7 and $e^{-\varkappa\tau_1}$ of order 1/200. Numerical calculations have demonstrated[4] that such refinements as rounding off the corners of Eq. (3.1) have little effect on the signals produced.

In calculating the conductivity, it is convenient to multiply the equation for the electron density by ew_0, electron charge times the electron drift velocity evaluated at unit electric field strength. We obtain the equation

$$\frac{dS}{d\tau} + \beta S = -A_e j \tag{3.2}$$

where S is the conductivity at a field strength of one esu, A_e is a constant involving ew_0 and the number of secondaries made per primary, and β is the electron attachment coefficient. The actual conductivity depends on field strength, being proportional, very nearly, to $1/\sqrt{|\vec{E}|}$ at high fields and leveling off at low fields. Using Eq. (3.1), Eq. (3.2) can be readily solved to give

$$S = \begin{cases} -\dfrac{A_e j}{\alpha + \beta} & \text{for } \tau < 0 \,, \\ \\ -\dfrac{A_e j}{\beta - \varkappa}\left[1 - \dfrac{\alpha + \varkappa}{\beta + \varkappa}\, e^{-(\beta - \varkappa)\tau}\right] & \text{for } 0 < \tau < \tau_1 \,. \end{cases} \tag{3.3}$$

If we were to use a conductivity which is truly field dependent, our equations would become nonlinear and their solution in closed analytic form would be hopeless. We therefore use a linear conductivity law but nevertheless take some account of field dependence as follows. At close distances the radial E-field dominates except very near the ground where, however, the vertical field is large. For the purpose of calculating σ, therefore, we assume that the electric field has everywhere the magnitude it would have for a symmetrical burst. Thus inside the saturation region, for example, we set

$$|\vec{E}| = E_s \overset{\text{def}}{=} -j/\sigma \; . \tag{3.4}$$

In this region σ is given by

$$\sigma = S/\sqrt{|\vec{E}|} = S/\sqrt{E_s} \; . \tag{3.5}$$

These combine to give

$$E_s = (-j/S)^2 \tag{3.6}$$

whence we can calculate σ. At large distances the fields are small and σ assumes a value independent of field strength.

The second of Eqs. (3.3) will prove a bit awkward. We see that it may be simplified by noting that at sea level $\beta = 10^8 (\text{sec})^{-1}$ whereas $\varkappa$ is a few times 10^7. Thus after two shakes or so the exponential will have become small and the equation

$$S = - \frac{A_e j}{\beta - \varkappa} \qquad \text{for } \tau > 0 \tag{3.7}$$

will be a good approximation. We shall use Eq. (3.7) rather than the second member of Eq. (3.3). This treats incorrectly a short period following $\tau = 0$, but as only integrals over σ enter into the theory, this contributes but a small error. As an example we have, according to the above scheme,

$$(3.8) \qquad \sigma = \begin{cases} -\left(\dfrac{A_e}{\alpha + \beta}\right)^2 j & \text{for } \tau < 0 \\ \\ -\left(\dfrac{A_e}{\beta - \varkappa}\right)^2 j & \text{for } \tau > 0 \end{cases} \qquad \text{when } r < R_r ,$$

during the saturation phase.

It is now clear that saturation is much more easily produced during the $\varkappa$-phase than during the α-phase. In the first place the ratio of saturation voltages*

$$(3.9) \qquad \frac{E_\varkappa}{E_\alpha} = \left(\frac{\beta - \varkappa}{\beta + \alpha}\right)^2$$

is quite small, typically of order 1/50. Moreover the operator $\partial/c\partial\tau$ is of order α when $\tau < 0$ and only of order $\varkappa$ for $\tau > 0$. This introduces an additional factor of order 10 in favor of $\varkappa$-stage saturation. Thus the

*E_α, $E_\varkappa$ represent the saturation voltage E_s evaluated respectively for $\tau < 0$ and for $\tau > 0$.

situation is as follows: Points sufficiently near the explosion saturate during the α-phase and remain in saturation thereafter. Slightly more distant points saturate at $\tau = 0$, the onset of the ϰ-phase, and thereafter remain in saturation. Still more distant points saturate at $\tau = 0$ but at some later time σ will have dropped to the point that they unsaturate. And finally there is the region which never saturates. The time at which saturation occurs we call τ_s and,

$$\tau_s \leq 0 \,, \tag{3.10}$$

the inequality holding if saturation occurs during the α-phase and the equality if not. Similarly the time at which the observation point comes out of saturation is denoted by τ_u. Clearly

$$0 \leq \tau_u < \tau_1 \,, \tag{3.11}$$

for if unsaturation has not occurred by the time σ levels off, it never will.

(B) The Diffusion Zone

We have seen that there is at any instant a radius $r = R_r$ inside of which saturation has occurred and outside of which it has not. Clearly R_r is a function of τ, increasing during the α-stage and decreasing during the ϰ-stage. At sufficiently early times, any fixed observation point r is outside the saturation zone. Later, however, saturation may occur.

That region which saturates at some time and never subsequently unsaturates we shall call the diffusion zone and this zone is the subject of this subsection.

According to the approximation scheme of Section 2 we have

$$G = \begin{cases} \partial\Phi/\partial\theta & \text{for } r < R_r , \\ \\ F & \text{for } r > R_r , \end{cases} \tag{3.12}$$

and moreover, at least for radii and times not too far removed from saturation*

$$F = \frac{1}{4\pi\sigma}\frac{\partial E}{\partial\theta} \qquad \text{for } r \text{ not} \gg R_r . \tag{3.13}$$

We shall make G, as defined by Eqs. (3.12) and (3.13), continuous as we pass from the wave phase into the diffusion phase. To this end it is convenient to define a new field function Ψ by

*Note that here and elsewhere in this Section the constant A of Section 2 is set equal to unity.

$$(3.14) \qquad \Psi \overset{\text{def}}{=} \begin{cases} E/4\pi\sigma_s & \text{for } \tau < \tau_s , \\ \Phi & \text{for } \tau_s < \tau < \tau_u , \end{cases}$$

and a new source function S^* by

$$(3.15) \qquad S^* \overset{\text{def}}{=} \begin{cases} \frac{E_\alpha}{4\pi s}(4\pi\sigma)^2 & \text{for } \tau < \tau_s , \\ E_s(4\pi\sigma) & \text{for } \tau_s < \tau < \tau_u . \end{cases}$$

With these definitions Eqs. (2.14) and (2.18) are both comprised in the single equation

$$(3.16) \qquad \frac{4\pi\sigma}{c}\frac{\partial\Psi}{\partial\tau} = \frac{\partial^2\Psi}{\partial z^2} + S^*(\tau) .$$

If Ψ is a continuous solution to this equation, the magnetic field is given by

$$
(3.17) \qquad G = \begin{cases} \dfrac{\sigma_s}{\sigma}\dfrac{\partial\Psi}{\partial\theta} & \text{for } \tau < \tau_s \text{ (but not} \ll \tau_s) \quad , \\ \\ \dfrac{\partial\Psi}{\partial\theta} & \text{for } \tau_s < \tau < \tau_u \quad , \end{cases}
$$

and is clearly continuous. At points well before saturation we must compute F from Eq. (2.8) and thence G from Eq. (3.12).

If we introduce the new independent variable

$$
(3.18) \qquad \zeta = \int \frac{c\,d\tau}{4\pi\sigma} \; ,
$$

Eq. (3.16) becomes

$$
(3.19) \qquad \frac{\partial\Psi}{\partial\zeta} = \frac{\partial^2\Psi}{\partial z^2} + S^*(\zeta) \quad ,
$$

and is readily solved by the standard Greene's function method to yield*

*We have used the Greene's function appropriate to a perfectly conducting earth.

(3.20) $$\Psi(\zeta,z) = \int_{-\infty}^{\zeta} S^*(\zeta')\ \mathrm{erf}\left(\frac{z}{2\sqrt{\zeta - \zeta'}}\right) d\zeta' ,$$

where "erf" is the error function

(3.21) $$\mathrm{erf}\ (x) \overset{\mathrm{def}}{=} \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2}\, dy .$$

Actually we are more interested in $\partial\Psi/\partial\theta$ than in Ψ itself. Differentiating Eq. (3.20) yields

(3.22) $$\frac{\partial\Psi}{\partial\theta} = -\frac{r}{\sqrt{\pi}} \int_{-\infty}^{\zeta} \frac{S^*(\zeta')d\zeta'}{\sqrt{\zeta - \zeta'}}\, e^{-z^2/4(\zeta-\zeta')} .$$

Using the expressions previously developed for σ and j we find that ζ, S^* may be expressed explicitly as follows:

(3.23) $$\zeta = \begin{cases} -c/4\pi\alpha\sigma & \text{for } \tau < 0 , \\ c/4\pi\kappa\sigma + \zeta_0 - \zeta_+ & \text{for } 0 < \tau < \tau_1 , \\ c(\tau - \tau_1)/4\pi\sigma_1 + \zeta_1 & \text{for } \tau > \tau_1 , \end{cases}$$

where ζ_0, ζ_+, ζ_1 are constants so chosen that ζ is continuous, namely

$$(3.24) \qquad \begin{cases} \zeta_0 = -c/4\pi\alpha\sigma_{-0} \,, \qquad \zeta_+ = c/4\pi\varkappa\sigma_{+0} \,, \\ \\ \zeta_1 = \zeta(\tau_1) \,. \end{cases}$$

Here σ_{-0}, σ_{+0} mean respectively the α-phase and the $\varkappa$-phase values of σ evaluated at $\tau = 0$. (Remember we are using an expression for σ which is discontinuous at $\tau = 0$). In terms of ζ we can express the source function as

$$(3.25) \qquad S^*(\zeta') = \begin{cases} \dfrac{c^2 E_\alpha}{4\pi\alpha^2\sigma_s(\zeta')^2} & \text{for } \tau < \tau_s \,, \\ \\ -\dfrac{cE_\alpha}{\alpha\zeta'} & \text{for } \tau_s < \tau < 0 \,, \\ \\ \dfrac{cE_\varkappa}{\varkappa(\zeta' + \zeta_+ - \zeta_0)} & \text{for } 0 < \tau < \tau \,, \\ \\ E_\varkappa 4\pi\sigma_1 & \text{for } \tau > \tau_1 \,. \end{cases}$$

It is fortunate that the field on the ground is particularly interesting, for in this case the integrals arising from an application of

Eq. (3.22) are all elementary. Limiting ourselves now to the case $z = 0$, it is convenient to break the range of integration at the two points $\tau = 0$ and $\tau = \tau_1$ writing

$$(3.26) \qquad I \equiv I_1 + I_2 + I_3 = \left\{ \int_{-\infty}^{\zeta_0} + \int_{\zeta_0}^{\zeta_1} + \int_{\zeta_1}^{\zeta} \right\} \frac{S^*(\zeta')d\zeta'}{\sqrt{\zeta - \zeta'}} .$$

For the individual integrals we readily obtain the result

$$(3.27) \qquad I_1 = \begin{cases} \dfrac{cE_\alpha}{\alpha\sqrt{-\zeta}} \left\{ \dfrac{\zeta_s}{\zeta}\left(\dfrac{\pi}{2} - \tan^{-1}\sqrt{\dfrac{\zeta_s}{\zeta} - 1}\right) - \sqrt{\dfrac{\zeta_s}{\zeta} - 1} + \right. \\ \qquad \left. 2\left(\tan^{-1}\sqrt{\dfrac{\zeta_s}{\zeta} - 1} - \tan^{-1}\sqrt{\dfrac{\zeta_0}{\zeta} - 1}\right)\right\} & \text{for } \zeta < 0 , \\ \dfrac{cE_\alpha}{\alpha\sqrt{\zeta}} \left\{ \sqrt{1 - \dfrac{\zeta_s}{\zeta}} + \log\dfrac{\sqrt{1 - \zeta_0/\zeta} + 1}{\sqrt{1 - \zeta_0/\zeta} - 1} - \right. \\ \qquad \left. \left(1 - \dfrac{\zeta_s}{2\zeta}\right)\log\dfrac{\sqrt{1 - \zeta_s/\zeta} + 1}{\sqrt{1 - \zeta_s/\zeta} - 1}\right\} & \text{for } \zeta > 0 , \end{cases}$$

$$I_2 = \frac{cE_\varkappa}{\varkappa\sqrt{\xi}}\left\{\log\frac{1 + \sqrt{1 - \xi_0/\xi}}{1 - \sqrt{1 - \xi_0/\xi}}\right\} ,$$

$$I_3 = 2E_\varkappa\sqrt{4\pi c\sigma_1(\tau - \tau_1)} .$$

In the above formulas we have set, for short

$$(3.28) \qquad \xi \overset{\text{def}}{=} \zeta + \zeta_{+} - \zeta_{0} = \frac{c}{4\pi\kappa\sigma} ,$$

and the subscripts s, 0, 1 refer to evaluation of the quantity indexed at respectively $\tau = \tau_s, 0, \tau_1$.

Naturally I_3 is to be taken as zero when $\tau < \tau_1$ and I_2 vanishes for $\tau < 0$. For $\tau < \tau_s$ we have, of course, only I_1 which reduces to the particularly simple form

$$(3.29) \qquad I_1 = \frac{cE_\alpha}{\alpha\sqrt{-\zeta}} \frac{\pi\zeta_\alpha}{2\zeta} \qquad \text{for } \tau < \tau_s .$$

As we have mentioned earlier, saturation is much more readily achieved in the κ-stage than in the α-stage. There will therefore be a considerable range of distances where saturation does not occur during the α-phase but does during the κ-phase. In all such cases one is to set $\tau_s = 0$ and use the simple Eq. (3.29) for I_1.

Our one remaining task is to determine the time of saturation τ_s; clearly it is the dividing line between dominance of displacement and of conduction current. From the form of the solutions obtained we see that any determination of τ_s which lead to $\alpha/2 \leq 4\pi c\sigma_s \leq 3\alpha/2$ would be satisfactory. We choose to define τ_s by

$$(3.30) \qquad 4\pi c\sigma_s \equiv 4\pi c\sigma(\tau_s) = 3\alpha/2 \quad ,$$

because this definition assures the continuity of $\partial E/\partial\theta$, and hence of F, as well as of G.

The equations developed above have been used to calculate a pulse shape, B_φ vs. τ, shown as the solid curve of Fig. I. The pulse has a sharp peak at, or near, $\tau = 0$, and this peak field has been plotted against the distance from the explosion, again as a solid curve, in Fig. II. Unfortunately no data exist for comparison with these results. We therefore chose the constants, yield, α, etc., to be the same as those used for a previously reported numerical integration of Maxwell's equations (Ref. 3, Figs. 8, 9, 10). The results of this numerical integration are plotted as circled points in Figs. I and II in order that comparison might be made to the present theory.

(C) The Intermediate Zone - Desaturation

Outside the diffusion zone is a region where σ is large enough at the peak to cause saturation, but at $\tau = \tau_1$ it has dropped so low that saturation is no longer possible. Thus at some time τ_u, $0 < \tau_u < \tau_1$, desaturation occurs. Thus the period $\tau > \tau_u$ is to be treated in a way which is analogous to, but as we shall quickly see not identical to, the treatment of the wave phase.

The time τ_u is characterized by the fact that at this time the approximation $G = \partial\Phi/\partial\theta$ breaks down, and from the equations already developed we readily see that this time is given by

$$(3.31) \qquad 4\pi\sigma_u \equiv 4\pi\sigma(\tau_u) = \varkappa/2c$$

approximately. We shall define τ_u by Eq. (3.31) and, as before, arrange that G shall be continuous across τ_u.

In order to discuss the transition to desaturation it is convenient to return to the second of Eqs. (2.1) which in our high frequency approximation may be written as

$$(3.32) \qquad \frac{\partial F}{c\partial\tau} + 4\pi\sigma F = \frac{\partial G}{c\partial\tau} \; .$$

Considering first the period $0 < \tau \le \tau_u$ we can write the solution as

$$(3.33) \qquad \begin{cases} F = e^{-\Sigma}\left\{F_0 + \displaystyle\int_0^\tau \left(\frac{1}{4\pi\sigma}\,\frac{\partial G}{c\partial\tau}\right) e^{\Sigma} 4\pi\sigma c d\tau\right\} , \\[2ex] \displaystyle\sum \overset{\text{def}}{=} \int_0^\tau 4\pi\sigma c d\tau = \frac{4\pi c\sigma_{+0}}{\varkappa}\,(1 - \sigma/\sigma_{+0}) \; . \end{cases}$$

Integrating by parts yields

(3.34)
$$F = e^{-\Sigma}\left\{F_0 + \left[\frac{e^{\Sigma}}{4\pi\sigma} \frac{\partial G}{c\partial\tau} \right]_0^{\tau} - \int_0^{\tau} \frac{1}{4\pi\sigma} \frac{\partial}{c\partial\tau}\left[\frac{1}{4\pi\sigma} \frac{\partial G}{c\partial\tau} \right] 4\pi\sigma e^{\Sigma} c\,d\tau \right\},$$

and the process may be continued obtaining an expansion in powers of the operator

(3.35)
$$\mathcal{D} \equiv \frac{1}{4\pi\sigma} \frac{\partial}{c\partial\tau} .$$

During saturation $\mathcal{D}$ is small. Therefore, keeping only lowest order terms we have

(3.36)
$$F = \frac{1}{4\pi\sigma} \frac{\partial G}{c\partial\tau} + e^{-\Sigma}\left\{F_0 - \left[\frac{1}{4\pi\sigma} \frac{\partial G}{c\partial\tau} \right]_{+0} \right\} .$$

We shall be particularly interested in Eq. (3.36) evaluated at $\tau = \tau_u$.

The time interval $\tau > \tau_u$ can be treated in the same fashion and we obtain an equation like Eq. (3.33) but τ_u replaces the time $\tau = 0$. Again successive partial integration produces an expansion of the integral, this time in powers of $\mathcal{D}^{-1}$. The first step of this process yields

$$(3.37)\quad \begin{cases} F = e^{-\tilde{\Sigma}}\left[F_u + e^{\tilde{\Sigma}}G - G_u - \int_{\tau_u}^{\tau} 4\pi\sigma G c d\tau\right], \\ \tilde{\Sigma} = \int_{\tau_u}^{\tau} 4\pi\sigma c d\tau . \end{cases}$$

As the operator $\mathfrak{D}$ is large for $\tau > \tau_u$, we can drop higher order terms and obtain

$$(3.38)\qquad F = G + e^{-\tilde{\Sigma}}[F_u - G_u] .$$

Combining this with Eq. (3.13) we obtain

$$(3.39)\qquad G = \frac{1}{4\pi\sigma}\frac{\partial E}{\partial \theta} + e^{-\tilde{\Sigma}}[G_u - F_u] .$$

Setting this value into the radial E-equation and dropping, as before, the conduction current, we obtain

$$(3.40)\qquad \frac{\partial E}{c\partial \tau} = \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left[\frac{\sin\theta}{4\pi\sigma}\frac{\partial E}{\partial\theta}\right] - 4\pi j + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta e^{-\tilde{\Sigma}}(G_u - F_u)\right] .$$

The essential difference between this phase and the wave phase is seen to be the term $e^{-\tilde{\Sigma}}[G_u - F_u]$ in Eq. (3.39) leading to an additional source term in Eq. (3.40). This added term is merely an integration constant which vanished for the wave phase.

As we are more interested in $\partial E/\partial\theta$ than in E itself, it is convenient to differentiate Eq. (3.40) by θ, obtaining directly an equation for $\partial E/\partial\theta$. This introduces derivatives of $G_u - F_u$ by θ which are most conveniently eliminated by introducing the new variable

$$V \overset{\text{def}}{=} \frac{\partial E}{\partial \theta} + 4\pi\sigma(G_u - F_u)e^{-\tilde{\Sigma}} . \tag{3.41}$$

The resulting equation for V becomes

$$\frac{4\pi\sigma}{c}\frac{\partial V}{\partial \tau} = \frac{\partial^2 V}{\partial z^2} - 2rE_s(4\pi\sigma)^2\delta(z) - \frac{\varkappa}{c}(4\pi\sigma)^2[G_u - F_u]e^{-\tilde{\Sigma}} . \tag{3.42}$$

It is clear that for that period of time for which Eq. (3.13) is a good approximation we have

$$V = 4\pi\sigma G . \tag{3.43}$$

Thus, to solve continuously across $\tau = \tau_u$ we extend definitions as follows:

(3.44)
$$\begin{cases} V = V_a = 4\pi\sigma_u G & \text{for } \tau < \tau_u , \\ S_a = 4\pi\sigma_u S^* & \text{for } \tau < \tau_u , \\ S_b = 0 & \text{for } \tau < \tau_u , \end{cases}$$

and

(3.45)
$$\begin{cases} S_a = E_s(4\pi\sigma)^2 & \text{for } \tau > \tau_u , \\ S_b = \frac{\varkappa}{c} (4\pi\sigma)^2[G_u - F_u]e^{-\tilde{\Sigma}} & \text{for } \tau > \tau_u . \end{cases}$$

We also set

(3.46)
$$V = V_a + V_b ,$$

V_a being that portion of V arising from the source term, S_a, and given as before by the expression

(3.47)
$$V_a = - \frac{r}{\sqrt{\pi}} \int_{-\infty}^{\zeta} \frac{S_a(\zeta')d\zeta'}{\sqrt{\zeta - \zeta'}} e^{-z^2/4(\zeta-\zeta')} ,$$

and V_b being that part of the solution which vanishes at $\tau = \tau_u$

and thereafter has the source S_b. For V_b the Greene's function solution is

$$(3.48) \qquad V_b = \int_{\zeta_u}^{\zeta} \frac{d\zeta'}{\sqrt{4\pi(\zeta - \zeta')}} \int_{-\infty}^{\infty} S_b(\zeta', z')dz' e^{-(z-z')^2/4(\zeta-\zeta')} ,$$

$$\tau \geq \tau_u .$$

To evaluate the integrals of Eq. (3.48) we first note that it has already been determined that

$$(3.49) \qquad G = - \frac{r}{\sqrt{\pi}} \int_{-\infty}^{\zeta} \frac{S^*(\zeta')d\zeta'}{\sqrt{\zeta - \zeta'}} e^{-z^2/4(\zeta-\zeta')} , \qquad \tau \leq \tau_u .$$

Differentiating gives (see Appendix A),

$$(3.50) \qquad \frac{1}{4\pi\sigma} \frac{\partial G}{c\partial\tau} = \frac{1}{(4\pi\sigma)^2} \frac{\partial G}{\partial\zeta} =$$

$$- \frac{r}{(4\pi\sigma)^2\sqrt{\pi}} \int_{-\infty}^{\zeta} \frac{S^{*\prime}(\zeta')d\zeta'}{\sqrt{\zeta - \zeta'}} e^{-z^2/4(\zeta-\zeta')} , \qquad \tau \leq \tau_u ,$$

where $S^{*\prime}$ means $\partial S^*/\partial \zeta'$. Combining Eqs. (3.50) and (3.36) gives an expression for F; hence, evaluating F and G at τ_u we obtain the following expression for S_b:

$$S_b(\zeta', z') = \frac{r\varkappa(4\pi\sigma)^2}{c\sqrt{\pi}} \int_{-\infty}^{\zeta_u} \frac{S^*(\zeta'') - S^{*\prime}(\zeta'')/(4\pi\sigma_u)^2}{\sqrt{\zeta_u - \zeta''}} d\zeta'' e^{-(z')^2/4(\zeta_u - \zeta'') - \widetilde{\Sigma}}$$

(3.51)

$$- \frac{r\varkappa(4\pi\sigma)^2}{c\sqrt{\pi}} e^{-\Sigma_u} \int_{-\infty}^{\zeta_0} \frac{S^*(\zeta'') - S^{*\prime}(\zeta'')/(4\pi\sigma_0)^2}{\sqrt{\zeta_0 - \zeta''}} d\zeta'' e^{-(z')^2/4(\zeta_0 - \zeta'') - \widetilde{\Sigma}} .$$

When Eq. (3.51) is set into Eq. (3.48), the integration over z′ is readily performed and one obtains

$$V_b = \frac{r\varkappa}{c\sqrt{\pi}} \left\{ \int_{\zeta_u}^{\zeta} [4\pi\sigma(\zeta')]^2 d\zeta' \int_{-\infty}^{\zeta_u} \frac{S^*(\zeta'') - S^{*\prime}(\zeta'')/(4\pi\sigma_u)^2}{\sqrt{\zeta + \zeta_u - \zeta' - \zeta''}} d\zeta'' e^{-z^2/4(\zeta + \zeta_u - \zeta' - \zeta'') - \widetilde{\Sigma}} \right.$$

(3.52)

$$\left. - e^{-\Sigma_u} \int_{\zeta_u}^{\zeta} [4\pi\sigma(\zeta')]^2 d\zeta' \int_{-\infty}^{\zeta_0} \frac{S^*(\zeta'') - S^{*\prime}(\zeta'')/(4\pi\sigma_0)^2}{\sqrt{\zeta + \zeta_0 - \zeta' - \zeta''}} d\zeta'' e^{-z^2/4(\zeta + \zeta_u - \zeta' - \zeta'') - \widetilde{\Sigma}} \right\} .$$

Here Σ_u is a constant (in τ) given by evaluating the second of Eqs. (3.33) at τ_u. Evidently V_b vanishes for $\tau = \tau_u$ and for $\tau_u = 0$.

In general Eq. (3.52) is very complicated, but when $z = 0$ it can be worked out. Note first that the integrations over ζ'' are elementary and similar to those already worked out. Writing

$$(3.53)\quad \begin{cases} I_1 \overset{\text{def}}{=} \displaystyle\int_{-\infty}^{\zeta_0} \frac{s^*(\zeta'')d\zeta''}{\sqrt{\zeta - \zeta''}}, \quad I_2 \overset{\text{def}}{=} \displaystyle\int_{\zeta_0}^{\zeta_u} \frac{s^*(\zeta'')d\zeta''}{\sqrt{\zeta - \zeta''}}, \\ \\ \tilde{I} \overset{\text{def}}{=} \displaystyle\int_{-\infty}^{\zeta_0} \frac{s^{*\prime}(\zeta'')d\zeta''}{\sqrt{\zeta - \zeta''}}, \quad \tilde{I}_2 \overset{\text{def}}{=} \displaystyle\int_{\zeta_0}^{\zeta_u} \frac{s^{*\prime}(\zeta'')d\zeta''}{\sqrt{\zeta - \zeta''}}, \end{cases}$$

we see that I_1 is precisely the I_1 of Eqs. (3.27) and I_2 is a slight generalization of Eqs. (3.27), namely

$$(3.54)\qquad I_2 = \frac{cE_\kappa}{\kappa\sqrt{\xi}}\left[\log\frac{1 + \sqrt{1 - \xi_0/\xi}}{1 - \sqrt{1 - \xi_0/\xi}} - \log\frac{1 + \sqrt{1 - \xi_u/\xi}}{1 - \sqrt{1 - \xi_u/\xi}}\right].$$

The other two integrals are of the same type and yield

$$\tilde{I}_1 = \frac{cE_\alpha e^{-\tilde{\Sigma}}}{\alpha(-\zeta_0)\sqrt{-\zeta}} \left\{ \frac{3}{4}\left(\frac{\zeta_0}{\zeta}\right)^2 \left(\pi - 2\tan^{-1}\sqrt{\frac{\zeta_0}{\zeta} - 1} \right) - \left(1 + \frac{3\zeta_0}{2\zeta}\right)\sqrt{\frac{\zeta_0}{\zeta} - 1} \right\}, \qquad \text{for } \zeta < 0 ,$$

$$\tilde{I}_1 = \frac{cE_\alpha e^{-\tilde{\Sigma}}}{\alpha(-\zeta_0)\sqrt{\zeta}} \left\{ \left(1 + \frac{3\zeta_0}{2\zeta}\right)\sqrt{1 - \frac{\zeta_0}{\zeta}} + \frac{3}{4}\left(\frac{\zeta_0}{\zeta}\right)^2 \log \frac{\sqrt{1 - \zeta_0/\zeta} + 1}{\sqrt{1 - \zeta_0/\zeta} - 1} \right\}, \qquad \text{for } \zeta > 0 , \tag{3.55}$$

$$\tilde{I}_2 = -\frac{cE_\varkappa e^{-\tilde{\Sigma}}}{\varkappa \xi_0 \sqrt{\xi}} \left\{ \sqrt{1 - \frac{\xi_0}{\xi}} - \frac{\xi_0}{\xi_u}\sqrt{1 - \xi_u/\xi} + \frac{1}{2}\left(\frac{\xi_0}{\xi}\right)^2 \left(\log \frac{1 + \sqrt{1 - \xi_0/\xi}}{1 - \sqrt{1 - \xi_0/\xi}} - \log \frac{1 + \sqrt{1 - \xi_u/\xi}}{1 - \sqrt{1 - \xi_u/\xi}} \right) \right\} .$$

When setting these results into Eq. (3.52) we must make the substitutions $\zeta \to \zeta + \zeta_u - \zeta'$, $\xi \to \xi + \xi_u - \xi'$ into the first integral and similar ones in the second. In working out these integrals we have made use of the fact that $\tau_s = 0$. For if saturation occurs during the α-stage, desaturation cannot occur.

The final integration over ζ' has the appearance of being elementary but very complicated. We shall not attempt to carry it out exactly but shall rather treat a particular case, where τ_u is sufficiently great that $e^{-\varkappa\tau_u} \ll 1$. In this case $e^{-\Sigma_u}$ is essentially zero and the second term of Eq. (3.50) may be dropped. In this case $\widetilde{I}_1$ and $\widetilde{I}_2$ enter only as a sum and there is a cancellation of the largest terms in the brackets. We start, therefore, by subtracting this term, defining

$$(3.56) \qquad \widetilde{I}_1 - \frac{S^*(\zeta_0)}{\sqrt{\zeta - \zeta_0}} = \zeta^{-3/2}K_3 , \qquad \widetilde{I}_2 + \frac{S^*(\zeta_0)}{\sqrt{\zeta - \zeta_0}} = \xi^{-1/2}K_4 ,$$

and also defining

$$(3.57) \qquad I_1 = \zeta^{-1/2}K_1 , \qquad I_2 = \xi^{-1/2}K_2 .$$

We find

$$\left\{ \begin{aligned} K_1(\zeta) &= \frac{cE_\alpha}{\alpha}\left\{\sqrt{1 - \frac{\zeta_0}{\zeta}} + \frac{\zeta_0}{2\zeta}\log\frac{\sqrt{1 - \zeta_0/\zeta} + 1}{\sqrt{1 - \zeta_0/\zeta} - 1}\right\} , \\ K_2(\xi) &= \frac{cE_\varkappa}{\varkappa}\left\{\log\frac{1 + \sqrt{1 - \xi_0/\xi}}{1 - \sqrt{1 - \xi_0/\xi}} - \log\frac{1 + \sqrt{1 - \xi_u/\xi}}{1 - \sqrt{1 - \xi_u/\xi}}\right\} , \end{aligned} \right.$$

(continued)

$$(3.58)\quad \begin{cases} K_3(\zeta) = -\dfrac{cE_\alpha}{\alpha}\left\{\dfrac{1 - 3\zeta_0/\zeta}{2\sqrt{1-\zeta_0/\zeta}} + \dfrac{3\zeta_0}{4\zeta}\log\dfrac{\sqrt{1-\zeta_0/\zeta}+1}{\sqrt{1-\zeta_0/\zeta}-1}\right\}, \\[2ex] K_4(\xi) = \dfrac{cE_\varkappa}{\varkappa\xi_u}\left\{\sqrt{1-\xi_u/\xi} + \dfrac{\xi_u/\xi}{\sqrt{1-\xi_0/\xi}} - \right. \\[2ex] \qquad \left. \dfrac{\xi_0\xi_u}{2\xi^2}\left[\log\dfrac{1+\sqrt{1-\xi_0/\xi}}{1-\sqrt{1-\xi_0/\xi}} - \log\dfrac{1+\sqrt{1-\xi_u/\xi}}{1-\sqrt{1-\xi_u/\xi}}\right]\right\}. \end{cases}$$

Now, when $e^{-\varkappa\tau_u}$ is appreciably less than unity, it is readily seen that $K_1(\zeta + \zeta_u - \zeta')$, $K_2(\xi + \xi_u - \xi')$, $K_3(\zeta + \zeta_u - \zeta')$ and $K_4(\xi + \xi_u - \xi')$ are slowly varying functions of ζ' as is also $e^{-\widetilde{\Sigma}}$, whereas $[4\pi\sigma(\zeta')]^2$ is rapidly varying. The latter has its peak at $\tau' = 0$ (i.e. at ζ_0). Thus it is a good approximation to evaluate the K's all at $\zeta' = \zeta_0$ and remove them from the integration obtaining

$$(3.59)\qquad V_b = \frac{r}{\sqrt{\pi}}\left[J_1K_1 + J_2K_2 - \left(\frac{J_3K_3 + J_2K_4}{(4\pi\sigma_u)^2}\right) e^{-\widetilde{\Sigma}}\right],$$

where

$$
(3.60)\quad \begin{cases} J_1 = \dfrac{\varkappa}{c} \displaystyle\int_{\zeta_u}^{\zeta} \dfrac{(4\pi\sigma)^2 d\zeta'}{\sqrt{\zeta + \zeta_u - \zeta'}} , \qquad J_2 = \dfrac{\varkappa}{c} \displaystyle\int_{\xi_u}^{\xi} \dfrac{(4\pi\sigma)^2 d\xi'}{\sqrt{\xi + \xi_u - \xi'}} , \\[2ex] J_3 = \dfrac{\varkappa}{c} \displaystyle\int_{\zeta_u}^{\zeta} \dfrac{(4\pi\sigma)^2 d\zeta'}{(\zeta + \zeta_u - \zeta')^{3/2}} , \text{ and } \widetilde{\sum} = \dfrac{1}{2} [1 - e^{-\varkappa(\tau - \tau_u)}] . \end{cases}
$$

Explicitly, these work out to give

$$
(3.61)\quad \begin{cases} J_1 = \dfrac{c[1 + \xi_u/\zeta]^{-3/2}}{\varkappa \xi_u \sqrt{\zeta}} \left\{ 1 - \dfrac{\xi_u}{\xi} \sqrt{\dfrac{\zeta_u}{\zeta}} + \right. \\[2ex] \qquad \left. \dfrac{\xi_u}{2\zeta} \left[\log \dfrac{\sqrt{1 - \xi_u/\zeta} + 1}{\sqrt{1 + \xi_u/\zeta} - 1} - \log \dfrac{\sqrt{1 + \xi_u/\zeta} + \sqrt{\zeta_u/\zeta}}{\sqrt{1 + \xi_u/\zeta} - \sqrt{\zeta_u/\zeta}} \right] \right\} , \\[2ex] J_2 = \dfrac{c[1 + \xi_u/\xi]^{-3/2}}{\varkappa \xi_u \sqrt{\xi}} \left\{ 1 - \left(\dfrac{\xi_u}{\xi} \right)^{3/2} + \right. \\[2ex] \qquad \left. \dfrac{\xi_u}{2\xi} \left[\log \dfrac{\sqrt{1 + \xi_u/\xi} + 1}{\sqrt{1 + \xi_u/\xi} - 1} - \log \dfrac{\sqrt{1 + \xi_u/\xi} + \sqrt{\xi_u/\xi}}{\sqrt{1 + \xi_u/\xi} - \sqrt{\xi_u/\xi}} \right] \right\} , \end{cases}
$$

(continued)

$$J_3 = \frac{c[1 + \xi_u/\zeta]^{-5/2}}{\varkappa\xi_u(\zeta)^{3/2}}\left\{1 - \left(2 - \left[3 - \frac{\zeta}{\xi}\left(1 + \frac{\xi_u}{\zeta}\right)\right]\sqrt{\frac{\zeta}{\zeta_u}}\right)\frac{\xi_u}{\zeta}\right.$$

$$\left. + \frac{3\xi_u}{2\zeta}\left[\log\frac{\sqrt{1 + \xi_u/\zeta} + 1}{\sqrt{1 + \xi_u/\zeta} - 1} - \log\frac{\sqrt{1 + \xi_u/\zeta} + \sqrt{\zeta_u/\zeta}}{\sqrt{1 + \xi_u/\zeta} - \sqrt{\zeta_u/\zeta}}\right]\right\} .$$

In Eq. (3.59) the K's stand for $K_1(\zeta + \zeta_u - \zeta_0)$, $K_2(\xi + \xi_u - \xi_0)$, $K_3(\zeta + \zeta_u - \zeta_0)$ and $K_4(\xi + \xi_u - \xi_0)$. The J's are precisely given by Eqs. (3.61) and $\widetilde{\Sigma}$ by (3.60).

The integrals for V_a have been already worked out and we have

$$(3.62)\quad \begin{cases} \displaystyle\int_{-\infty}^{\zeta_0}\frac{S_a d\zeta'}{\sqrt{\zeta - \zeta'}} = \frac{c^2E_\alpha}{\alpha\varkappa\xi_u\sqrt{\zeta}}\left[\sqrt{1 - \frac{\zeta_0}{2\zeta}} + \right. \\ \displaystyle\qquad\qquad \left. \frac{\zeta_0}{2\zeta}\log\frac{\sqrt{1 - \zeta_0/\zeta} + 1}{\sqrt{1 - \zeta_0/\zeta} - 1}\right] \quad \text{for } \zeta > 0 \quad , \\ \displaystyle\int_{\zeta_0}^{\zeta}\frac{S_a d\zeta'}{\sqrt{\zeta - \zeta'}} = \frac{c^2E_\varkappa}{\varkappa^2\xi_u\sqrt{\xi}}\left[\sqrt{1 - \frac{\xi_u}{\xi}} + \log\frac{1 + \sqrt{1 - \xi_0/\xi}}{1 - \sqrt{1 - \xi_0/\xi}} - \right. \\ \displaystyle\qquad\qquad \left. \left(1 - \frac{\xi_u}{2\xi}\right)\log\frac{1 + \sqrt{1 - \xi_u/\xi}}{1 - \sqrt{1 - \xi_u/\xi}}\right] . \end{cases}$$

For our numerical example we chose the desaturation time τ_u to be half way between the peak, $\tau = 0$, and the shoulder $\tau = \tau_1$. The above formulas were employed to compute the solid curve of Fig. III and for comparison the results of a numerical solution of Maxwell's equations[3] are shown as circles. For times near τ_1 it was found that the terms V_a and V_b very nearly cancel one another. Thus had the term V_b not been included, i.e. had the integration constant $F_u - G_u$ been omitted, quite a different result would have been found. In fact, it may readily be seen from Eq. (3.62) that the term V_a alone gives a G which increases in magnitude during the interval $\tau_u < \tau < \tau_1$.

4. The Distant Region

By definition, the distant zone is that in which saturation never occurs. Thus the radial field is determined for all time by Eq. (2.18) which may be written in the form

$$\frac{\partial E}{\partial \zeta} = \frac{\partial^2 E}{\partial z^2} + S(\zeta) , \tag{4.1}$$

where

$$\zeta \overset{\text{def}}{=} \int \frac{Acd\tau}{4\pi\sigma} , \qquad S \overset{\text{def}}{=} \frac{E_s}{A} (4\pi\sigma)^2 . \tag{4.2}$$

Note that now the A of Eq. (2.17) is not assumed to be unity, which was an appropriate value for the near field. It will be our task to show that $A = 1$ is also a good approximation for the distant field. Eq. (4.1) is formally identical with Eq. (3.19) and Eq. (3.22) gives the solution on simply replacing $\psi \to E$ and $S^* \to S$. On the ground plane $z = 0$ and all the integrations are elementary. We limit our attention henceforth to this plane and write

$$\frac{\partial E}{\partial \theta} = - \frac{rA}{\sqrt{\pi}} [I_a + I_b + I_c] , \tag{4.3}$$

where

$$I_a = \frac{c^2 E_\alpha}{\alpha^2 (-\zeta)^{3/2}} \frac{\pi}{2} , \qquad I_b = I_c = 0 \qquad \text{for } \tau < 0 , \tag{4.4}$$

and

$$I_a = \begin{cases} \dfrac{c^2 E_\alpha}{\alpha^2(-\zeta)^{3/2}} \left[\dfrac{\pi}{2} - \tan^{-1} \sqrt{\dfrac{\zeta_0}{\zeta} - 1} - \dfrac{\zeta}{\zeta_0} \sqrt{\dfrac{\zeta_0}{\zeta} - 1} \right] & \text{for } \zeta < 0 , \\[2ex] \dfrac{c^2 E_\alpha}{\alpha^2 \zeta^{3/2}} \left[- \dfrac{\zeta}{\zeta_0} \sqrt{1 - \dfrac{\zeta_0}{\zeta}} - \dfrac{1}{2} \log \dfrac{\sqrt{1 - \zeta_0/\zeta} + 1}{\sqrt{1 - \zeta_0/\zeta} - 1} \right] & \text{for } \zeta > 0 , \end{cases}$$

(continued)

$$(4.5)\quad \left\{\begin{aligned} I_b &= \frac{c^2 E_\varkappa}{\varkappa^2 \xi^{3/2}}\left[\frac{\xi}{\xi_0}\sqrt{1-\frac{\xi_0}{\xi}}+\frac{1}{2}\log\frac{1+\sqrt{1-\xi_0/\xi}}{1-\sqrt{1-\xi_0/\xi}}\right], \\ I_c &= 0, \end{aligned}\right\} ;\ 0 < \tau < \tau_1$$

for the period $0 < \tau < \tau_1$. For $\tau > \tau_1$, I_a is as in Eq. (4.5),

$$(4.6)\quad \left\{\begin{aligned} I_b &= \frac{c^2 E_\varkappa}{\varkappa^2 \xi^{3/2}}\left[\frac{\xi}{\xi_0}\sqrt{1-\frac{\xi_0}{\xi}}-\frac{\xi}{\xi_1}\sqrt{1-\frac{\xi_1}{\xi}}\right. \\ &\quad \left.+\frac{1}{2}\log\frac{1+\sqrt{1-\xi_0/\xi}}{1-\sqrt{1-\xi_0/\xi}}-\frac{1}{2}\log\frac{1+\sqrt{1-\xi_1/\xi}}{1-\sqrt{1-\xi_1/\xi}}\right], \\ I_c &= \frac{8\pi\sigma_1}{A^{3/2}} E_\varkappa\sqrt{4\pi\sigma_1 c(\tau-\tau_1)}, \end{aligned}\right\} ;\ \tau > \tau_1 .$$

It is seen that, during this period, $\partial E/\partial\theta$ is of the form $r\sigma^{3/2}$ times a function of τ alone, and also that $\partial E/\partial\theta$ is proportional to $1/\sqrt{A}$.

As the region now under consideration has never saturated, we can use the approximation

$$(4.7)\qquad F = G$$

with confidence. For the calculation of F, however, we must use Eq. (2.8). The complication arising from this is that the range of integration includes regions which were previously saturated for varying portions of their history.

Let R_m be the maximum range at which $\varkappa$-phase saturation occurs. Then for all $r > R_m$ we can apply Eqs. (4.3) through (4.6) to the calculation of $\partial E/\partial\theta$. An examination of these equations shows that $\partial E/\partial\theta$ can be expressed in the form

$$\frac{\partial E}{\partial \theta} = -\frac{\alpha r}{c} E_\alpha \left(\frac{4\pi c\sigma}{\alpha}\right)^{3/2} f_0(\tau) , \quad r \geq R_m . \tag{4.8}$$

For $r < R_m$ we have the more complicated expression

$$\frac{\partial E}{\partial \theta} = V_a + V_b - 4\pi\sigma e^{-\widetilde{\Sigma}}[G_u - F_u] , \qquad r < R_m . \tag{4.9}$$

The symbols on the right-hand side of Eq. (4.9) are all defined in Section 3C where analytic expressions for them will be found. From Eq. (3.62) we see that V_a may be written as

$$
(4.10)\begin{cases}
V_a = -\frac{r\varkappa E_\alpha}{2c\sqrt{\pi}}\sqrt{\frac{4\pi c\sigma}{\alpha}}\left\{\frac{\sqrt{1-\frac{\zeta_0}{\zeta}}+\frac{\zeta_0}{2\zeta}\log\frac{\sqrt{1-\zeta_0/\zeta}+1}{\sqrt{1-\zeta_0/\zeta}-1}}{\sqrt{1-(1+A)e^{-\varkappa\tau}}}+\right. \\
\qquad \frac{1}{A}\sqrt{\frac{\varkappa}{\alpha}}\left[\sqrt{1-\frac{\xi_u}{\xi}}+\log\frac{1+\sqrt{1-\xi_0/\xi}}{1-\sqrt{1-\xi_u/\xi}}-\right. \\
\qquad \left.\left.\left(1-\frac{\xi_u}{2\xi}\right)\log\frac{1+\sqrt{1-\xi_u/\xi}}{1-\sqrt{1-\xi_u/\xi}}\right]\right\}, \\
A \overset{\text{def}}{=} \frac{\varkappa E_\alpha}{\alpha E_\varkappa};
\end{cases}
$$

V_b is given by Eq. (3.59) in conjunction with Eqs. (3.58) and (3.61), and $4\pi\sigma(G_u - F_u)e^{-\tilde{\Sigma}}$ is obtained by multiplying Eq. (3.51) by $-c/(4\pi\varkappa\sigma)$ and then using Eqs. (3.58). Quantities of the type ζ_0/ζ, ξ_0/ξ etc. are all functions of τ alone. Quantities of the type ζ_u/ζ are functions of τ_u and hence depend on r, for we have

$$
(4.11) \qquad \varkappa\tau_u = \frac{1}{\lambda}(R_m - r) + 2\log(R_m/r) .
$$

However, so long as τ is appreciably greater than τ_u terms like ζ_u/ζ are small. It is thus seen that all of the expressions in curly brackets in Eqs. (3.58), (3.61), and (4.10) are slowly varying functions of r so long as τ is not too close to τ_u. Thus we see that the functions V_a, and the part $(r/\sqrt{\pi})[J_1\varkappa_1 + J_2\varkappa_2 - J_2\varkappa_4/(4\pi\sigma_u)^2]$ are of the form $r\sqrt{4\pi c\sigma/\alpha}$ times a slowly varying function of r; the term $(r/\sqrt{\pi})[-J_3\varkappa_3/(4\pi\sigma_u)^2]$ is of the form $r(4\pi c\sigma/\alpha)^{3/2}$ times a slowly varying function of r, and $(G_u - F_u)e^{-\tilde{\Sigma}}$ is r times a slowly varying function of r. We can therefore write $\partial E/\partial\theta$ in the form

$$\frac{\partial E}{\partial \theta} = -\frac{\alpha r}{c} E_\alpha\left\{\left(\frac{4\pi c\sigma}{\alpha}\right)^{1/2} f_1(\tau,\tau_u) + \left(\frac{4\pi c\sigma}{\alpha}\right)^{3/2} f_3(\tau,\tau_u)\right\} - 4\pi\sigma(G_u - F_u)e^{-\tilde{\Sigma}} , \qquad r < R_m . \tag{4.12}$$

We have stated that f_1, f_2 and $(G_u - F_u)e^{-\tilde{\Sigma}}$ are slowly varying functions of r; so also is r. This statement we now make precise. Note that, according to Eqs. (4.8) and (4.12), all of the integrals entering into Eq. (2.8) are of the form

$$\vartheta_n = \frac{\alpha}{2c}\int\left(\frac{4\pi c\sigma}{\alpha}\right)^{n/2} e^{-X} r\psi_n(r)dr , \qquad n = 1, 2, 3 , \tag{4.13}$$

the limits of integration being (R_m, r) for Eq. (4.8) or $(0, R_m)$ for Eq. (4.12). Note that the factor $(4\pi c\sigma/\alpha)^{n/2} e^{-X}$ has a peak at $r = R_n$, given by

$$(4.14) \qquad 2\pi\sigma(R_n) = \frac{n}{2\lambda} (1 + 2\lambda/R_n) \quad ,$$

and that this peak is very sharp, so that most of the contribution to the integral comes from the range $(R_n - \lambda, R_n + \lambda)$. As R_n normally turns out to be of order 5 or 10 times λ, the factor r varies little over $(R_n - \lambda, R_n + \lambda)$, as also does $\psi_n(r)$. Therefore Eq. (4.13) can be readily approximated by the expression

$$(4.15) \qquad \mathcal{I}_n = \frac{\alpha}{2c} R_n \psi_n(R_n) \int \left(\frac{4\pi c\sigma}{\alpha}\right)^{n/2} e^{-X} dr \ , \qquad n = 1,\ 2,\ 3 \ ,$$

provided R_n lies inside the range of integration. If R_n lies outside the range, we are to substitute for R_n in Eq. (4.15) that limit nearest to R_n. Note, moreover, that the three R_n given by Eq. (4.14) differ from the mean by less than λ. Thus the three R_n of Eq. (4.14) may be identified with this mean, setting

$$(4.16) \qquad \begin{cases} R_1 = R_2 = R_3 \equiv R_s \ , \\ 2\pi\sigma(R_s) = \frac{1}{\lambda} (1 + 2\lambda/R_s) \ . \end{cases}$$

We see now that Eq. (2.8) may be written in one of two forms, depending on whether R_s is greater than or less than R_m. In the former case we have

$$e^{-X_F} = -\frac{\alpha E_\alpha}{2c} R_s f_0(\tau) \int_{R_m}^{r} \left(\frac{4\pi c\sigma}{\alpha}\right)^{3/2} e^{-X} dr$$

$$(4.17) \qquad - \frac{\alpha E_\alpha}{2c} R_m \left\{ f_1(\tau, R_m) \int_0^{R_m} \left(\frac{4\pi c\sigma}{\alpha}\right)^{1/2} e^{-X} dr + f_3(\tau, R_m) \int_0^{R_m} \left(\frac{4\pi c\sigma}{\alpha}\right)^{3/2} e^{-X} dr \right\}$$

$$- \left[(G_u - F_u) e^{-\tilde{\Sigma}} \right]_{R_m} \int_0^{R_m} 2\pi\sigma e^{-X} dr , \qquad \text{for } R_s > R_m .$$

In the other case

$$e^{-X_F} = -\frac{\alpha E_\alpha}{2c} R_m f_0(\tau) \int_{R_m}^{r} \left(\frac{4\pi c\sigma}{\alpha}\right)^{3/2} e^{-X} dr$$

$$(4.18) \qquad - \frac{\alpha E_\alpha}{2c} R_s \left\{ f_1(\tau, R_s) \int_0^{R_m} \left(\frac{4\pi c\sigma}{\alpha}\right)^{1/2} e^{-X} dr + f_3(\tau, R_s) \int_0^{R_m} \left(\frac{4\pi c\sigma}{\alpha}\right)^{3/2} e^{-X} dr \right\} -$$

(continued)

$$- \left[(G_u - F_u) e^{-\tilde{\Sigma}} \right]_{R_s} \int_0^{R_m} 2\pi\sigma e^{-X} dr \, , \qquad \text{for } R_s < R_m \, .$$

The integrals can all be evaluated. First we note that

$$(4.19) \qquad \int_0^{R_m} e^{-X} 2\pi\sigma dr = e^{-X_m}$$

(X_m means X at $r = R_m$) exactly. Next we note that an obvious change of variable yields

$$(4.20) \qquad \frac{\alpha}{2c} \int_a^b \left(\frac{4\pi c\sigma}{\alpha} \right)^{1/2} e^{-X} dr = \int_{X_b}^{X_a} \left(\frac{4\pi c\sigma}{\alpha} \right)^{-1/2} e^{-X} dX \, .$$

Now

$$(4.21) \qquad X \overset{\text{def}}{=} \int_r^{\infty} 2\pi\sigma dr \doteq \frac{2\pi\lambda\sigma}{1 + 2\lambda/r}$$

to a good approximation so long as r is two or three times λ or greater. Thus the approximation of Eq. (4.21) is valid over the important part of the range of integration and Eq. (4.20) becomes

$$\text{(4.22)} \qquad \frac{\alpha}{2c}\int_a^b \left(\frac{4\pi c\sigma}{\alpha}\right)^{1/2} e^{-X} dr \doteq \frac{1}{\sqrt{\frac{2c}{\alpha\lambda}\left(1 + \frac{2\lambda}{R_s}\right)}} \int_{X_b}^{X_a} \frac{e^{-X}}{\sqrt{X}}\, dX \ ,$$

use having been made of the fact that $1 + 2\lambda/r$ is a slowly varying function. The remaining integral can be expressed as an error function. The integral of $(4\pi c\sigma/\alpha)^{3/2}$ is done in the same way, and we obtain

$$\text{(4.23)} \quad \begin{cases} \dfrac{\alpha}{2c}\displaystyle\int_0^{R_m} \left(\frac{4\pi\sigma c}{\alpha}\right)^{1/2} e^{-X} dr = \left[\frac{2c}{\alpha\lambda}\left(1 + \frac{2\lambda}{R}\right)\right]^{-1/2} \sqrt{\pi}\left[1 - \text{erf}\sqrt{X_m}\right] , \\[2ex] \dfrac{\alpha}{2c}\displaystyle\int_0^{R_m} \left(\frac{4\pi c\sigma}{\alpha}\right)^{3/2} e^{-X} dr = \left[\frac{2c}{\alpha\lambda}\left(1 + \frac{2\lambda}{R}\right)\right]^{1/2} \left\{\sqrt{X_m}\, e^{-X_m} + \frac{\sqrt{\pi}}{2}\left[1 - \text{erf}\sqrt{X_m}\right]\right\} , \\[2ex] \dfrac{\alpha}{2c}\displaystyle\int_{R_m}^{R} \left(\frac{4\pi c\sigma}{\alpha}\right)^{3/2} e^{-X} dr = \left[\frac{2c}{\alpha\lambda}\left(1 + \frac{2\lambda}{R}\right)\right]^{1/2} \left\{\sqrt{X}e^{-X} - \sqrt{X_m}\, e^{-X_m} + \right. \\[2ex] \qquad\qquad \left. + \dfrac{\sqrt{\pi}}{2}\left[\text{erf}\sqrt{X_m} - \text{erf}\sqrt{X}\right]\right\} . \end{cases}$$

Here R means R_s or R_m according to which is appropriate, and erf is the error function defined by Eq. (3.21).

Equations (4.17), (4.18), and (4.23) combine to give an analytic expression for the distant field. The functions f_0, f_1, and f_3 are complicated but can be constructed from expressions already given. Our formulas were all developed under the assumption that $e^{-\varkappa(\tau-\tau_u)} \ll 1$. When this is no longer true, functions which we supposed to be slowly varying no longer are indeed so. This, however, does not affect our results. For $e^{-\varkappa(\tau-\tau_u)}$ not $\ll 1$ can only happen at radii r approaching the saturation radius, where $4\pi\sigma$ is approaching the value $\varkappa/2c$. But as $\varkappa/2c$ is $\gg 1/\lambda$, it follows that the factor e^{-X} effectively cuts off this portion of the range of integration to a negligible contribution.

We have found an expression for the distant field components F and G under the assumption that Eq. (2.17) is a good approximation when used in the equation for the radial E-field. According to our above considerations the approximation is adequate if it is good over the range $(R_s - \lambda, R_s + \lambda)$ which contributes significantly to the integral of Eq. (2.8). We are now in a position to check this.

The full expression for F is much too complicated to work with. We shall rather restrict our attention to such early time that R_s is noticeably larger than R_m. In this case Eq. (4.17) applies and only the first term is important. We can, in fact, write

(4.24) $$F = -R_s E_\alpha f_0(\tau) \sqrt{\frac{2c}{\alpha\lambda}\left(1 + 2\lambda/R_s\right)} \; I e^{X} ,$$

where

(4.25) $$\mathcal{J} = \int_0^r \sqrt{X} e^{-X} 2\pi\sigma dr = \sqrt{X} e^{-X} + \frac{\sqrt{\pi}}{2}\left[1 - \text{erf}\,(\sqrt{X})\right] .$$

Combining this with Eq. (4.8) yields the result

(4.26) $$F = \frac{1}{4\pi\sigma}\frac{\partial E}{\partial \theta}\left\{\frac{R_3}{r}\frac{\sqrt{1 + 2\lambda/R_3}}{\sqrt{1 + 2\lambda/r}}\left[1 + \frac{\sqrt{\pi}}{2}\frac{1 - \text{erf}\,(\sqrt{X})}{\sqrt{X}e^{-X}}\right]\right\} .$$

In Fig. IV we have plotted the quantity $\mathcal{J}(X)/\mathcal{J}(0)$ and from this we can see that 80 percent of the infinite integral comes from the range $0.3 < X < 3.4$. In Fig. V we have plotted the bracketted expression of Eq. (4.26) versus X, and over the same range of X it varies 1.14 to 1.48. Accordingly, identifying, for the moment, this bracketted expression with A, A is slowly varying and can be removed from the integration when it has been evaluated at $r = R_3$, the peak of the integrand. This procedure gives

(4.27) $$A = \left[1 + \frac{\sqrt{\pi}}{2}\frac{1 - \text{erf}\,\sqrt{X}}{\sqrt{X}e^{-X}}\right]_{r=R_3} = 1.2524$$

as our best estimate of A. As the resulting fields $\partial E/\partial\theta$, F and G are all proportional to $A^{-1/2}$, we see that setting A = 1 makes an error of at most 12 percent in the distant field and becomes extremely accurate in the near field.

In considering only the first term of Eq. (4.17) we restricted ourselves to early times, such that R_s is appreciably larger than R_m. Later on R_m will exceed R_s appreciably and the appropriate equation is Eq. (4.18). It is now found that the dominant term is the second, the term proportional to $f_1(\tau,R_s)$. As now $\sigma(R_m) \ll \sigma(R_s)$, the upper limit of integration can be replaced by infinity. Repeating the foregoing analysis leads to an analogous result. It thus appears that setting A = 1 in Eq. (2.17) is very accurate in the near and intermediate zones; in the far zone it leads to errors of ten to twenty percent.

As an illustration of the foregoing theory, we have calculated a pulse shape for the field radiated to large distances. For this we set $X = 0$ ($r = \infty$) into Eqs. (4.17), (4.18), and (4.23), we computed f_0, f_1, f_3 and $(G_u - F_u)e^{-\tilde{\Sigma}}$ from the appropriate equations, and combined results. Anticipating that the fields would be large at $r = R_s$, the region from which most of the signal comes, we used our "large field" conductivity, i.e., we computed conductivity as though the field were E_α, as in Eq. (3.8). The resulting fields were indeed large, but not as large as E_α. An attempt was made to take into account the field dependence of the conductivity by scaling the calculated value of F_∞* with the correction factor

*The index ∞ means F at $r = \infty$.

$$\text{Corr. Factor} = \frac{1}{2R_s E_\alpha} \quad (4.28)$$

This factor is explained as follows: The factor $1/R_s$ converts F to E_θ; the factor 1/2 accounts for the fact that half the integral for F_∞ is accumulated for $0 < r < R_s$, the other half in the range (R_s, ∞); the E_α was the electric field originally assumed in calculating σ. The result of all this is shown in Fig. VI as a curve of rE_θ (or rB_φ) vs. τ. The constants α, β, $\varkappa$, λ and yield of our previous examples apply. Unfortunately there exists neither data nor a numerical solution of Maxwell's equations for comparison. Probably the greatest uncertainty in Fig. VI results from the scaling according to Eq. (4.28). This doubtless gives a first step toward a self-consistent conductivity, but, as E_α enters into the computation in other ways than as a pure multiplier, one really ought to repeat the calculation using the fields given by Fig. VI to compute our new values of σ. This would be a rather lengthy process for hand computation.

5. Dependence on Angle and on Height of Burst

The high frequency spike, shown in Figs. I, III, and VI is confined to the vicinity of the ground plane. At close distances the layer thickness δ within which the spike amplitude is appreciable is readily estimated from the relationship

(5.1) $$\delta = \left[\frac{1}{B} \frac{\partial B}{\partial z} \right]^{-1} .$$

But on the ground $E_r = 0$ (perfect conductivity is assumed) and Maxwell's equation for the radial field enables us to evaluate $\partial B/\partial z$, obtaining

(5.2) $$\delta = |4\pi j/B|^{-1} .$$

In the near zone δ will vary from a few centimeters to a few meters at the peak of B, depending on the distance and on the yield. As we have seen, B is roughly proportional to $\sigma^{1/2}$, thus so also is δ^{-1}.

When it comes to calculating the angular dependence of the radiated field, it turns out that identical formulas, properly interpreted, yield peak field vs. colatitude θ, or peak field vs. height of burst. For this reason, we first consider the problem of what we shall call a <u>semiair burst</u>, that is, a burst off the ground, but sufficiently near that the ionized region is considerably larger than the burst height. This will be made precise later.

It is simplest to develop the approximation scheme ab initio, starting with Maxwell's equations written in terms of the cylinder coordinate system (ρ,φ,z); z is measured from the ground plane, positive upward. The distance r from the point of the explosion is

(5.3) $$r = \sqrt{\rho^2 + (z - h)^2} ,$$

h being the burst height. Introducing the retarded time

$$\tau = t - r/c \ . \tag{5.4}$$

Maxwell's equations take the form

$$\begin{cases} \dfrac{\partial E_\rho}{c\partial\tau} + 4\pi\sigma E_\rho + 4\pi j_\rho = -\dfrac{\partial B_\varphi}{\partial z} + \dfrac{z-h}{r}\dfrac{\partial B_\varphi}{c\partial\tau} \ , \\[2ex] \dfrac{\partial E_z}{c\partial\tau} + 4\pi\sigma E_z + 4\pi j_z = \dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}(\rho B_\varphi) - \dfrac{\rho}{r}\dfrac{\partial B_\varphi}{c\partial\tau} \ , \\[2ex] \dfrac{\partial B_\varphi}{c\partial\tau} = \dfrac{\partial E_z}{\partial\rho} - \dfrac{\partial E_\rho}{\partial z} + \dfrac{\partial}{c\partial\tau}\left[\dfrac{z-h}{r}E_\rho - \dfrac{\rho}{r}E_z\right] \ . \end{cases} \tag{5.5}$$

It is convenient to introduce the longitudinal and transverse components of E in place of E_ρ and E_z. Using the same symbols as previously, we define

$$\begin{cases} E \overset{\text{def}}{=} \dfrac{\rho}{r}E_\rho + \dfrac{z-h}{r}E_z \ , \\[2ex] F \overset{\text{def}}{=} \rho\left[\dfrac{z-h}{r}E_\rho - \dfrac{\rho}{r}E_z\right] \ . \end{cases} \tag{5.6}$$

With this notation, the result of formally integrating the third of Eqs. (5.5) is

$$
(5.7)\qquad \begin{cases} G \overset{\text{def}}{=} \rho B_\varphi = F + \Phi \,, \\[2ex] \Phi \overset{\text{def}}{=} \displaystyle\int_{-\infty}^{T} \rho\left(\frac{\partial E_z}{\partial \rho} - \frac{\partial E_\rho}{\partial z}\right) c\,d\tau \;. \end{cases}
$$

The next step is to eliminate B_φ by setting Eq. (5.7) into the first two members of Eqs. (5.5). If we define the differential operators

$$
(5.8)\qquad \begin{cases} \nabla_{||} = \dfrac{\rho}{r}\dfrac{\partial}{\partial \rho} + \dfrac{z - h}{r}\dfrac{\partial}{\partial z} \,, \\[2ex] \nabla_{\perp} = \dfrac{z - h}{r}\dfrac{\partial}{\partial \rho} - \dfrac{\rho}{r}\dfrac{\partial}{\partial z} \,, \end{cases}
$$

and if we construct appropriate linear combinations of the two E-equations, these can be expressed as*

* $\vec{j}$ is presumed to have only a longitudinal component.

$$(5.9)\quad \begin{cases} \dfrac{\partial E}{c\partial\tau} + 4\pi\sigma E = \dfrac{1}{\rho}\nabla_{\perp} G - 4\pi j \,, \\[2ex] \nabla_{\parallel} F + 2\pi\sigma F = \dfrac{1}{2}\rho\nabla_{\perp} E - \dfrac{1}{2}\nabla_{\parallel}\Phi \,, \\[2ex] G = F + \Phi \,, \end{cases}$$

and the expression for Φ may be written

$$(5.10)\quad \Phi = \int_{-\infty}^{\tau} \left[\rho\nabla_{\perp} E - \nabla_{\parallel} F \right] c\, d\tau \,.$$

So far everything is exact; Eqs. (5.9) and (5.10) are equivalent to Eqs. (5.5).

To reduce our equations to a manageable set we must make approximations. As before, we start by dividing space into two zones, $r < R_r$, the diffusion zone where conduction current dominates displacement current, and $r > R_r$ where displacement current dominates. The standard series of approximations in the diffusion zone lead us exactly to Longmire's theory of the diffusion phase;[2] therefore we shall consider only the region where the conduction current is negligible compared with displacement current. Thus our first approximation is to set

(5.11) $$\frac{\partial}{c\partial\tau} \gg 4\pi\sigma \ .$$

To this we add an analogue of our high frequency approximation, namely, we shall assume that, in operator form,

(5.12) $$\frac{\partial}{c\partial\tau} \gg \nabla_{\parallel} \ .$$

When h = 0, Eq. (5.12) is exactly our previous high frequency approximation. However, when $h \neq 0$ there is an additional term on the right-hand side of Eq. (5.12) which is proportional to $\partial/\partial z$, so that Eq. (5.12) is equivalent to the condition

(5.13) $$\frac{\partial}{c\partial\tau} \gg \mathrm{Max}\left[\frac{1}{\lambda}, \frac{h}{r}\frac{\partial}{\partial z}\right] \ .$$

As $\partial/\partial z$ can be large, we see that we are restricted to very modest burst heights, just how modest we shall see after obtaining a solution to our equations.

The approximation implied by Eq. (5.12) clearly allows us to simplify the second of Eqs. (5.9) to

(5.14) $$\nabla_{\parallel} F + 2\pi\sigma F = \frac{1}{2}\rho\nabla_{\perp} E \ .$$

Setting this into Eq. (5.10) and the result into the third of Eqs. (5.9) gives

$$G = \int_{-\infty}^{T} \left\{ \frac{\partial F}{c\partial \tau} + \nabla_{\parallel} F + 4\pi\sigma F \right\} c d\tau \; . \tag{5.15}$$

which, by virtue of Eqs. (5.11) and (5.12) reduces to*

$$G = F \; . \tag{5.16}$$

Using these results and dropping the conduction current, the first of Eqs. (5.9) now simplifies to

$$\frac{\partial E}{c\partial \tau} = \frac{1}{\rho} \nabla_{\perp} F - 4\pi j \; . \tag{5.17}$$

We use Eqs. (5.14), (5.16), and (5.17) to describe the high frequency portion of a semiground burst. The equations bear a close resemblance to those for the ground burst but are complicated by the fact that the operator $\nabla_{\parallel}$ contains an admixture of $\partial/\partial z$.

---

*It should be noted that we are here treating the wave zone proper. For the intermediate zone, after desaturation has occurred, other integration limits must be used as in Section 3C.

Our final task is to decouple Eqs. (5.14) and (5.17). In coordinate form Eq. (5.14) is

$$\frac{\rho}{r}\frac{\partial F}{\partial \rho} + \frac{z - h}{r}\frac{\partial F}{\partial z} + 2\pi\sigma F = -\frac{\rho^2}{2r}\frac{\partial E}{\partial z}, \tag{5.18}$$

the term in $\partial E/\partial \rho$ being small both by virtue of the high frequency approximation and by virtue of small h/r. We now borrow a result from our ground burst results, namely, that for purposes of estimating E, $\partial F/\partial \rho$ may be dropped from Eq. (5.18). Thus we have

$$\frac{z - h}{r}\frac{\partial F}{\partial z} + 2\pi\sigma F = -\frac{\rho^2}{2r}\frac{\partial E}{\partial z}. \tag{5.19}$$

Equation (5.17) takes the form

$$\frac{\partial E}{c\partial \tau} = -\frac{1}{\rho}\frac{\partial F}{\partial z} - 4\pi j. \tag{5.20}$$

The second term on the left-hand side of Eq. (5.19) is rapidly varying. Thus, except for a narrow range in r, it will either dominate or be negligible. In the former case, i.e. when

$$\frac{z - h}{r}\frac{\partial}{\partial z} \ll 2\pi\sigma, \tag{5.21}$$

we have

$$F = -\frac{\rho^2}{r}\left(\frac{1}{4\pi\sigma}\frac{\partial E}{\partial z}\right) , \tag{5.22}$$

and Eq. (5.20) reduces to the diffusion equation

$$\frac{4\pi r^2\sigma}{c\rho^2}\frac{\partial E}{\partial \tau} = \frac{\partial^2 E}{\partial z^2} + \frac{r^2}{\rho^2} E_s(4\pi\sigma)^2 . \tag{5.23}$$

In this case we have recovered exactly the wave phase theory of the ground burst, for r/ρ is simply a constant within the limits of our approximation (among other things the skin thickness must be $\ll r$).

When the reverse of Eq. (5.21) is true, we have

$$\frac{z - h}{r}\frac{\partial F}{\partial z} = -\frac{\rho^2}{2r}\frac{\partial E}{\partial z} , \tag{5.24}$$

which, when set into Eq. (5.20), yields

$$\frac{\partial E}{c\partial \tau} = \frac{\rho}{2(z - h)}\frac{\partial E}{\partial z} - 4\pi j , \tag{5.25}$$

a first order wave equation. Its domain is the region for which

(5.26) $$\frac{z - h}{r} \frac{\partial}{\partial z} \gg 2\pi\sigma ,$$

and therefore does not include the singularity at $z = h$.

A full scale attack on the semiground burst problem would require solving Eqs. (5.23) and (5.25) in the appropriate regions, subject to appropriate boundary conditions, and joining these solutions smoothly. We shall not attempt to carry out such a program here, but shall rather restrict our attention to the region where Eq. (5.21) holds, that is to the region governed by the diffusion equation, Eq. (5.23). The extent of this region is readily seen. The skin depth δz, during the $\varkappa$-phase for example, is given by

(5.27) $$(\delta z)^2 \pi\sigma\varkappa r^2/c\rho^2 = 1 ,$$

whence Eq. (5.21) becomes

(5.28) $$\left[\frac{z - h}{\rho}\right]^2 < \frac{4\pi c\sigma}{\varkappa} .$$

If we are interested in the radiated field, we wish to satisfy this inequality in the neighborhood of $r = R_s$ [see Eq. (4.15)] which gives

(5.29) $$\left[\frac{h}{\rho}\right]^2 < \frac{2c}{\varkappa\lambda} (1 + 2\lambda/R_s) .$$

Typically this will limit us to burst heights of two hundred meters or less.

Formally, the identification $A = \rho^2/r^2$ reduces Eq. (5.23) to an equation already studied. There is, however, the question of a boundary condition at z = 0. Note first that Eqs. (5.6) yield

(5.30) $$E_\rho = \frac{\rho}{r} E + \frac{z - h}{r\rho} F \ ,$$

and that

(5.31) $$\frac{z - h}{r\rho} F = - \frac{\rho}{2r}\left[\frac{z - h}{2\pi\sigma r}\frac{\partial E}{\partial z}\right] .$$

Thus the inequality Eq. (5.21) yields

(5.32) $$\frac{z - h}{r\rho} F \ll \frac{\rho}{r} E \ .$$

It follows that the true boundary condition $E_r(z = 0) = 0$ may be replaced by

(5.33) $$E = 0 \qquad \text{at } z = 0 \ ,$$

within the accuracy of our approximation. Thus our previous results all apply and we may write

$$\nabla_{\perp} E = -\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\zeta} \frac{S(\zeta')d\zeta'}{\sqrt{\zeta - \zeta'}} e^{-z^2/4(\zeta-\zeta')} , \tag{5.34}$$

where

(5.35)

$$S \overset{\text{def}}{=} E_s (r/\rho)^2 (4\pi\sigma)^2 ,$$

$$\zeta \overset{\text{def}}{=} \int^{\tau} \frac{c\rho^2 d\tau}{4\pi\sigma r^2} .$$

The only difference from our previous results is that we can no longer set $e^{-z^2/4(\zeta-\zeta')}$ equal to unity.

Evaluating the integral of Eq. (5.34) for general z solves both the problem of variation of field with height of burst and the problem of variation of field with angle above the horizon. Confining our attention to the radiated field, we see that in the case of the ground burst at an angle θ above the horizon, we must integrate Eq. (2.8) along the line $z = r \cos \theta$. For the semiground burst observed in the horizontal plane Eq. (2.8) is integrated along the line $z = h$.

We saw in the last section that the radiated signal was sharply peaked, with the peak occurring late in the $\varkappa$-phase, where $e^{-\varkappa\tau} \ll 1$.

We shall now compute the variation of this peak with angle θ, or with burst height h. At the time of the peak, it is readily seen that the function $S(\zeta')/\sqrt{\zeta - \zeta'}$ is sharply peaked at $\tau' = 0$, the gamma ray peak, and that, were $z = 0$ in Eq. (5.34), practically all of the integral comes from the neighborhood of $\tau' = 0$. But when $e^{-\varkappa\tau} \ll 0$, the function $e^{-z^2/4(\zeta-\zeta')}$ is slowly varying in τ' near $\tau' = 0$. Thus this factor can be removed from the integration to a good approximation, yielding

$$(5.36) \qquad \nabla_{\perp} E = - \frac{1}{\sqrt{\pi}} e^{-\left(\frac{\varkappa\pi\sigma r^2}{c\rho^2}\right) z^2} \int_{-\infty}^{\zeta} \frac{S(\zeta')d\zeta'}{\sqrt{\zeta - \zeta'}} .$$

The remaining integration has been treated in detail in earlier sections; our previous formulae now simply carry an added exponential factor.

In carrying out the integration of Eq. (2.8) the quantity $(rz/\rho)^2$ is slowly varying near $r = R_s$ and may simply be replaced by its value at $r = R_s$. The quantity σ in the exponent is rapidly varying and, when the integration variable has been changed to X, Eq. (4.20) is used to replace σ. In principle, we would have to work out separately each of the terms of Eq. (4.17). In fact it is sufficient to consider only the term involving $f_1(\tau,R_s)$, for this term is considerably larger than the others at the peak and also contributes the broadest distribution in θ or in h. Thus, instead of the first of the integrals of Eq. (4.22), we have to consider the integral

$$(5.37)\qquad \vartheta(z_0) = \frac{\alpha}{2c}\int_0^{\infty}\left(\frac{4\pi c\sigma}{\alpha}\right)^{1/2}\exp\left\{-X - \frac{\varkappa\pi z_0^2 R_s^2}{c(R_s^2 - z_0^2)}\,\sigma\right\}dr\ .$$

At the time of the peak the upper limits R_m or ∞ are equivalent. The above integral is readily worked out by changing the variable to X and using the approximations of Eqs. (4.20) and (4.21). The result is

$$(5.38)\qquad \frac{\vartheta(z_0)}{\vartheta(0)} = \left[1 + \frac{\varkappa R_s^2 z_0^2(1 + 2\lambda/R_s)}{2\lambda c(R_s^2 - z_0^2)}\right]^{-1/2} .$$

Here z_0 is h in the case of a semiair burst or is $R_s \cos\theta$ in the case of a ground burst, and all other symbols have their previous meanings.

Using the calculation leading to Fig. VI we have calculated the peak signal as a function of burst height h and show the normalized result in Fig. VII. In doing this account has been taken of the fact that the conductivity is field dependent when the fields are strong. This means, essentially, that Eq. (5.38) holds for weak fields, but the field ratio is more nearly the square of Eq. (5.38) when the fields are strong. The same data which make up Fig. VII reappear in Fig. VIII, now plotted as peak field of a ground burst versus θ. All that is required is to make the conversion $h \rightarrow R_s \cos\theta$. In making the calculations we took the "skin thickness" Δ,

(5.39)

$$\Delta = \sqrt{\frac{2\lambda c}{\kappa(1 + 2\lambda/R_s)}} ,$$

to be 60 meters, a rather typical value. In converting h to θ for our angular distribution curve a value for R_s must be assumed; we choose 1000 meters as typical.

6. Conclusions

The main results for the near field of a ground burst as shown in Figs. I, II, and III indicate that the theory of the early part of the radio flash herein developed gives very reliable results in the near zone. In the radiation zone we have, unfortunately, nothing to test our theory against. However, the success of the theory in near and intermediate zones lends confidence to our distant field result, for the principal source of the distant field at the time of the peak, is the intermediate zone, just about the position corresponding with Fig. III.

The major uncertainty centers on the air conductivity. For our analytic theory is linear whereas in fact the air conductivity depends on the field present, provided that field exceeds about 0.04 esu. We have attempted to take this into account by using a linear conductivity law with a conductivity consistent with the resulting field, a sort of a "self consistent conductivity" approach. Comparisons with a numerical solution of Maxwell's equations which employed the full nonlinear conductivity seem to indicate that this approach leads to valid results.

There is much more work which could be done in the development of the theory here given. In the first place, a program was mentioned in Section 5 which would extend the range of validity of the semiground burst theory to altitudes of perhaps a kilometer. Secondly, it at present looks possible to extend the theory to include the effects of finite ground conductivity. Thirdly, it should be mentioned that sufficient progress has been made on a nonlinear theory to make one hopeful that the effects of field dependence of the conductivity might better be taken into account.

References

1. B. R. Suydam, "Theory of the Radio Flash - Part V" Los Alamos Scientific Laboratory Report LA-3408-MS, September 1965.
2. C. L. Longmire, "Lecture Notes on Close-In E.M. Effects," Lectures IV, V, and VI, Los Alamos Scientific Laboratory Report LA-3072-MS, April 1964.
3. B. R. Suydam, "Theory of the Radio Flash - Part III" Los Alamos Scientific Laboratory Report LA-3218-MS, May 1964.
4. B. R. Suydam, "Theory of the Radio Flash - Part I" Los Alamos Scientific Laboratory Report LAMS-2855, November 1962.

Appendix

In the solution of the diffusion equations studied in this report, integrals of the type

$$\text{(A.1)} \qquad I(z) = \int_{\zeta_a}^{\zeta} \frac{S(\zeta')d\zeta'}{\sqrt{\zeta - \zeta'}} e^{-z^2/4(\zeta-\zeta')} ,$$

where ζ_a is a constant, are frequently encountered, and it is convenient to know how to manipulate them. Differentiation with respect to z is straightforward, but that with respect to ζ presents an apparent difficulty. For the usual rule gives

$$\text{(A.2)} \quad \frac{\partial I}{\partial \zeta} = \lim_{\zeta' \to \zeta} \left[\frac{S(\zeta')e^{-z^2/4(\zeta-\zeta')}}{\sqrt{\zeta - \zeta'}} \right] + \int_{\zeta_a}^{\zeta} S(\zeta')d\zeta' \frac{\partial}{\partial \zeta} \left[\frac{e^{-z^2/4(\zeta-\zeta')}}{\sqrt{\zeta - \zeta'}} \right] ,$$

and the first term is clearly singular. Note, however, that

$$\text{(A.3)} \qquad \frac{\partial}{\partial \zeta} \left[\frac{e^{-z^2/4(\zeta-\zeta')}}{\sqrt{\zeta - \zeta'}} \right] = - \frac{\partial}{\partial \zeta'} \left[\frac{e^{-z^2/4(\zeta-\zeta')}}{\sqrt{\zeta - \zeta'}} \right] ,$$

and that if we make this substitution, the second term of Eq. (A.2) can be transformed by partial integration so that we have

$$\text{(A.4)} \qquad \frac{\partial I}{\partial \zeta} = \lim_{\zeta' \to \zeta} \left[\frac{S(\zeta')e^{-z^2/4(\zeta-\zeta')}}{\sqrt{\zeta - \zeta'}} \right] - \left[\frac{S(\zeta')e^{-z^2/4(\zeta-\zeta')}}{\sqrt{\zeta - \zeta'}} \right]_{\zeta_a}^{\zeta} + \int_{\zeta_a}^{\zeta} \frac{\partial S}{\partial \zeta'} \frac{d\zeta'}{\sqrt{\zeta - \zeta'}} e^{-z^2/4(\zeta-\zeta')} ,$$

and the singular term is in fact cancelled out by another. Thus we have the result

$$\text{(A.5)} \qquad \frac{\partial I}{\partial z} = \frac{S(\zeta_a)e^{-z^2/4(\zeta-\zeta_a)}}{\sqrt{\zeta - \zeta_a}} + \int_{\zeta_a}^{\zeta} \frac{S'(\zeta')d\zeta'}{\sqrt{\zeta - \zeta'}} e^{-z^2/4(\zeta-\zeta')} ,$$

where a prime has been used to denote the derivative of S. Often in our work ζ_a is $-\infty$ so that the first term vanishes.

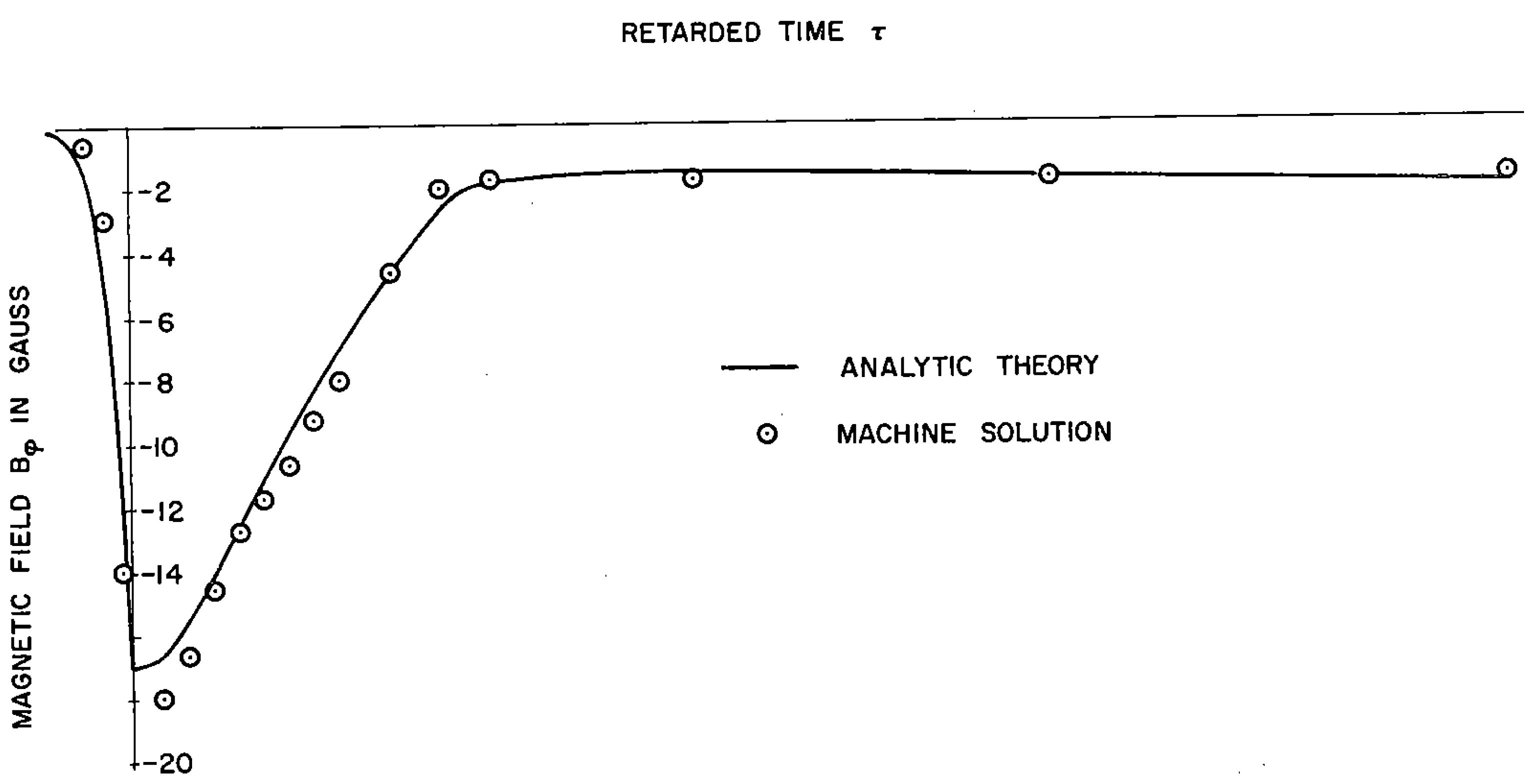

Fig. I. Early Magnetic Field vs. Time Near Field.

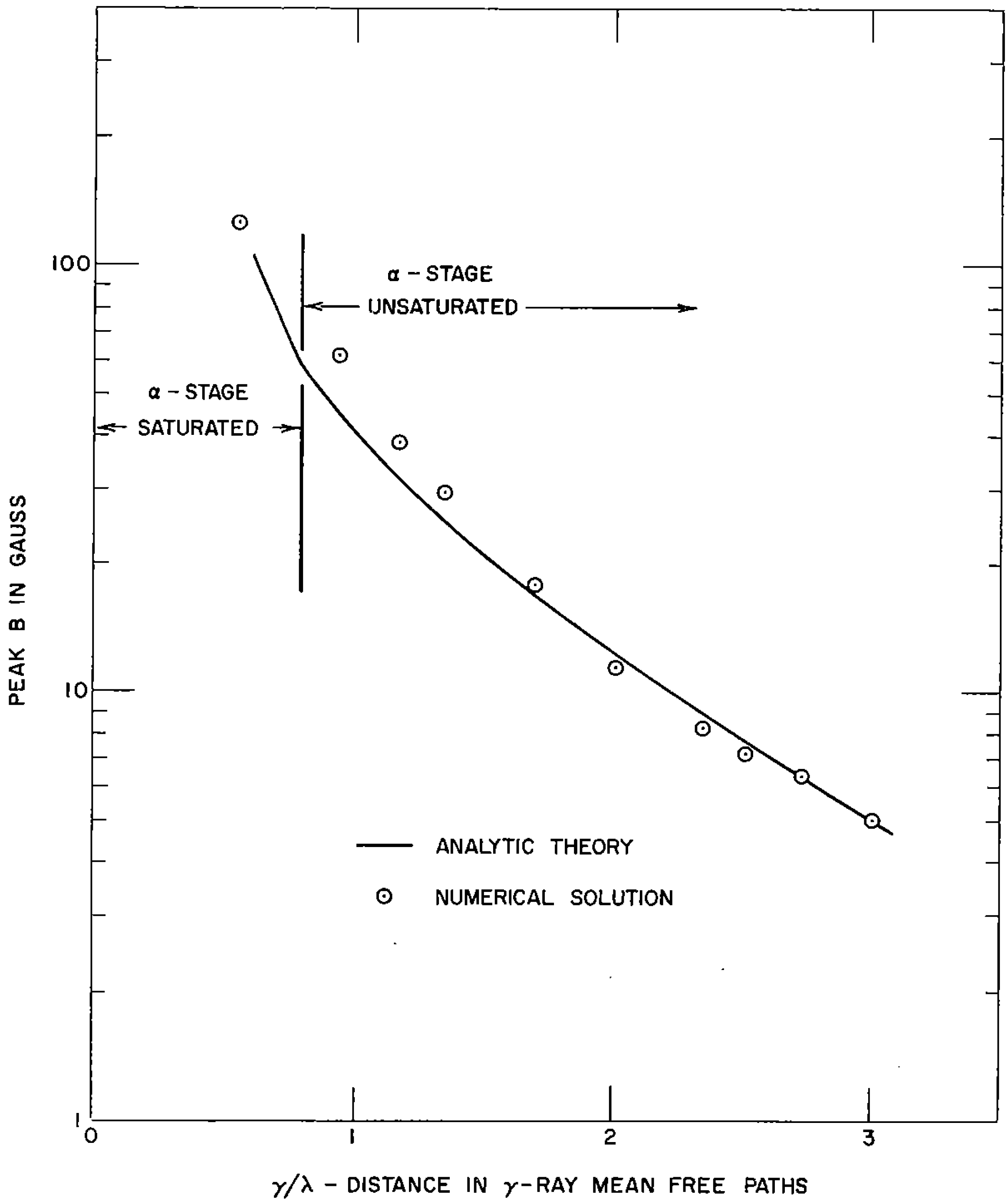

Fig. II. First Peak of B-Field vs. Distance.

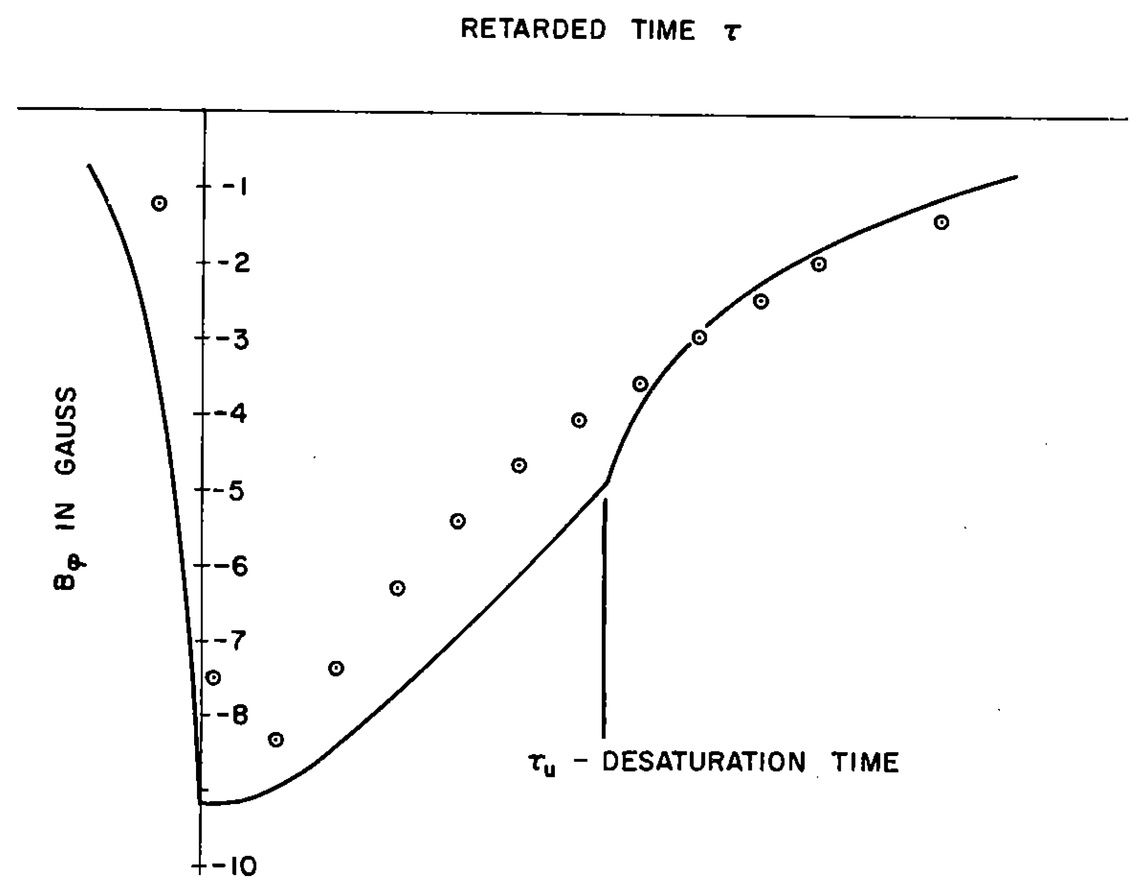

Fig. III. Early Magnetic Field vs. Time - Intermediate Distance.

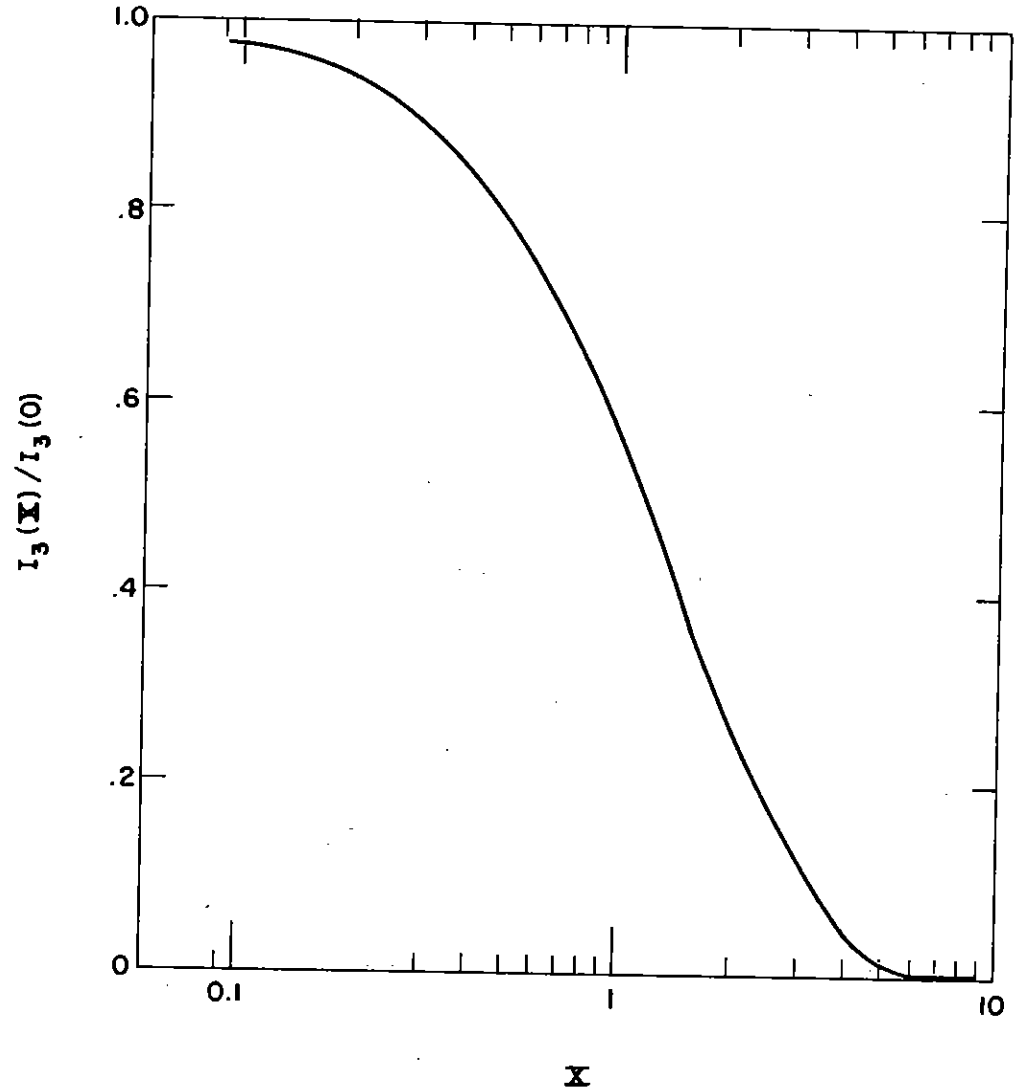

Fig. IV. The Integral $\mathcal{J}$ of Equation (4.24).

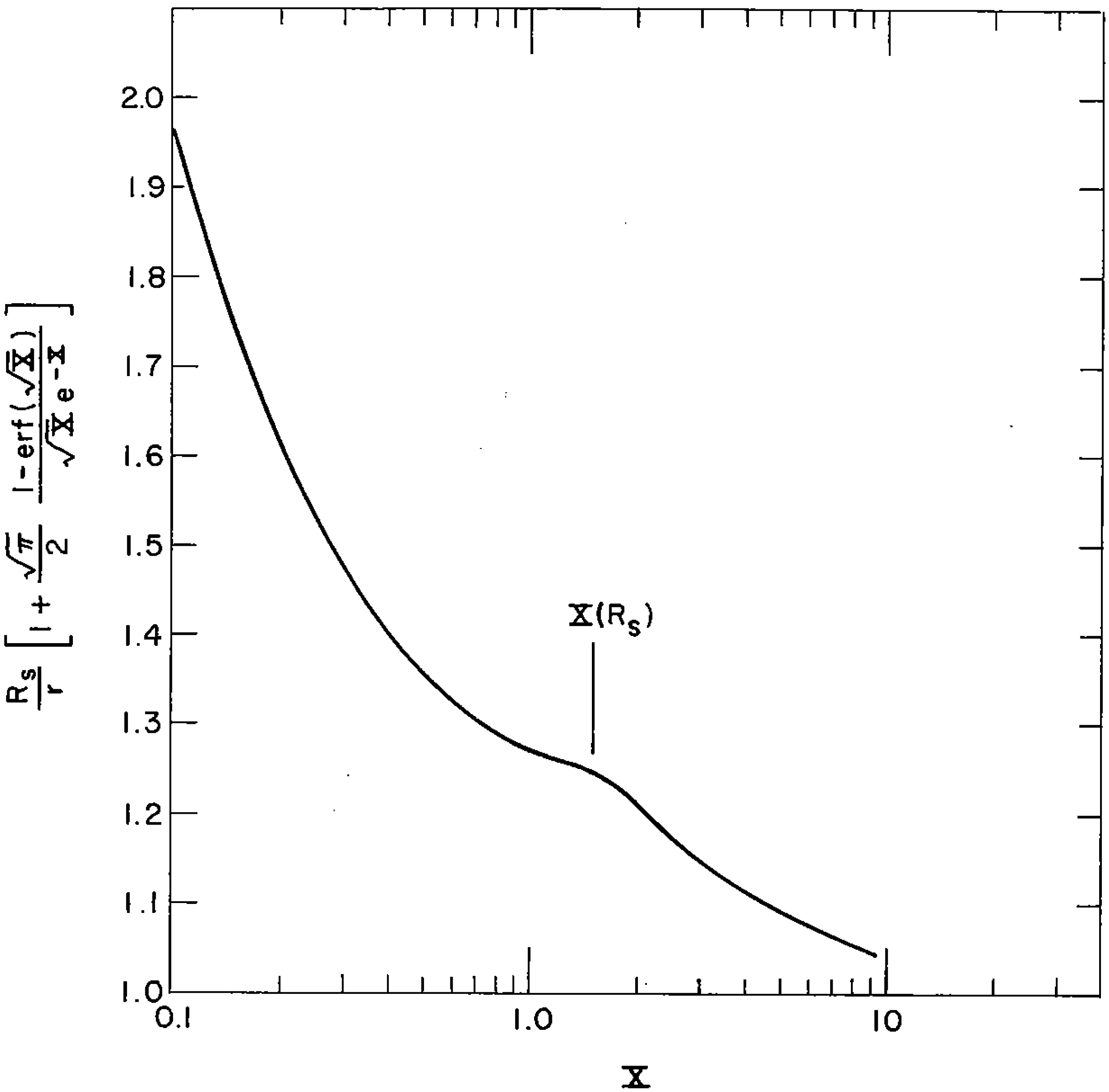

Fig. V. The Ratio $4\pi\sigma F/\left(\frac{\partial E}{\partial \theta}\right)$ of Equation (4.25).

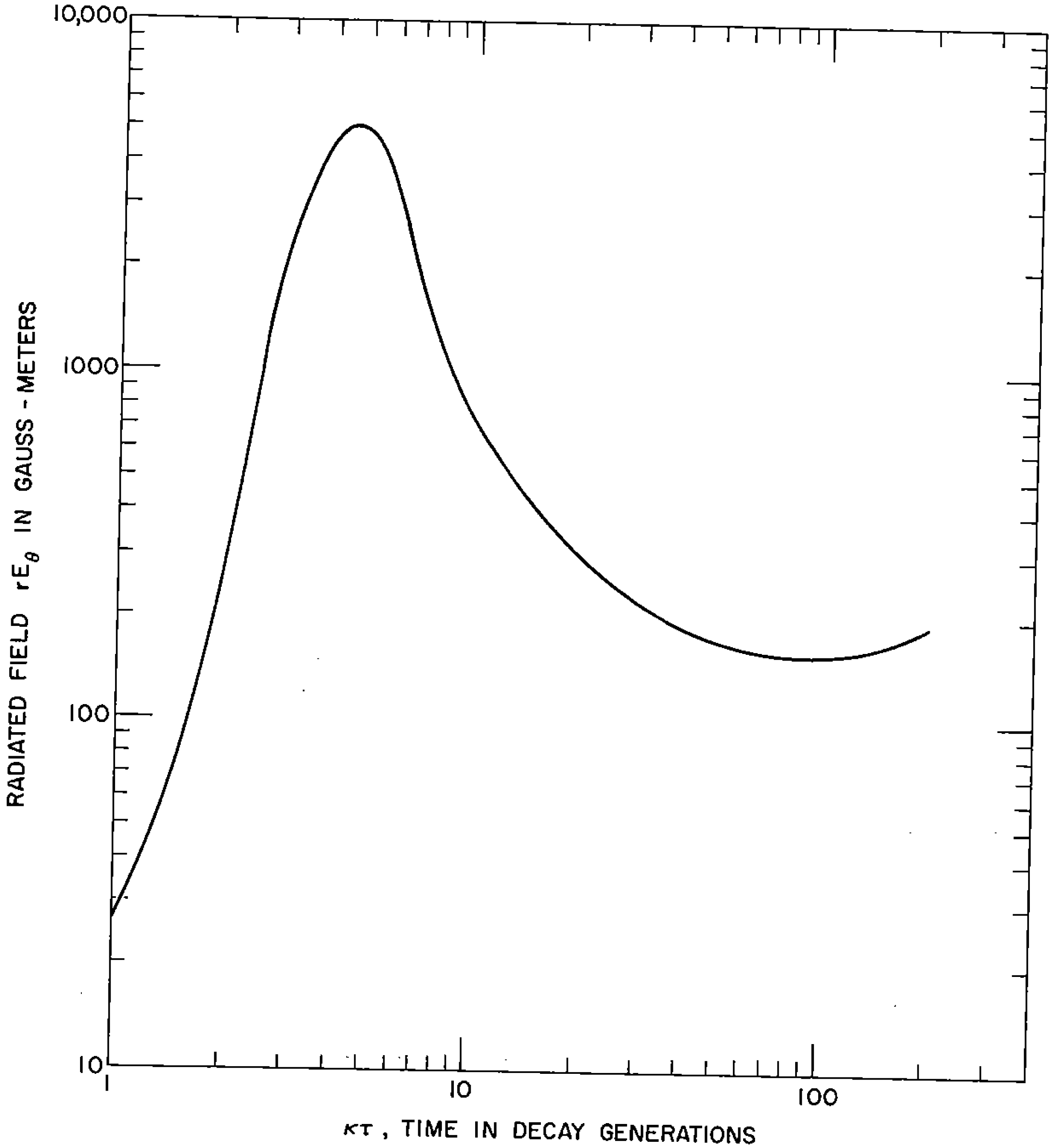

Fig. VI. The Radiated Field in the Equatorial Plane.

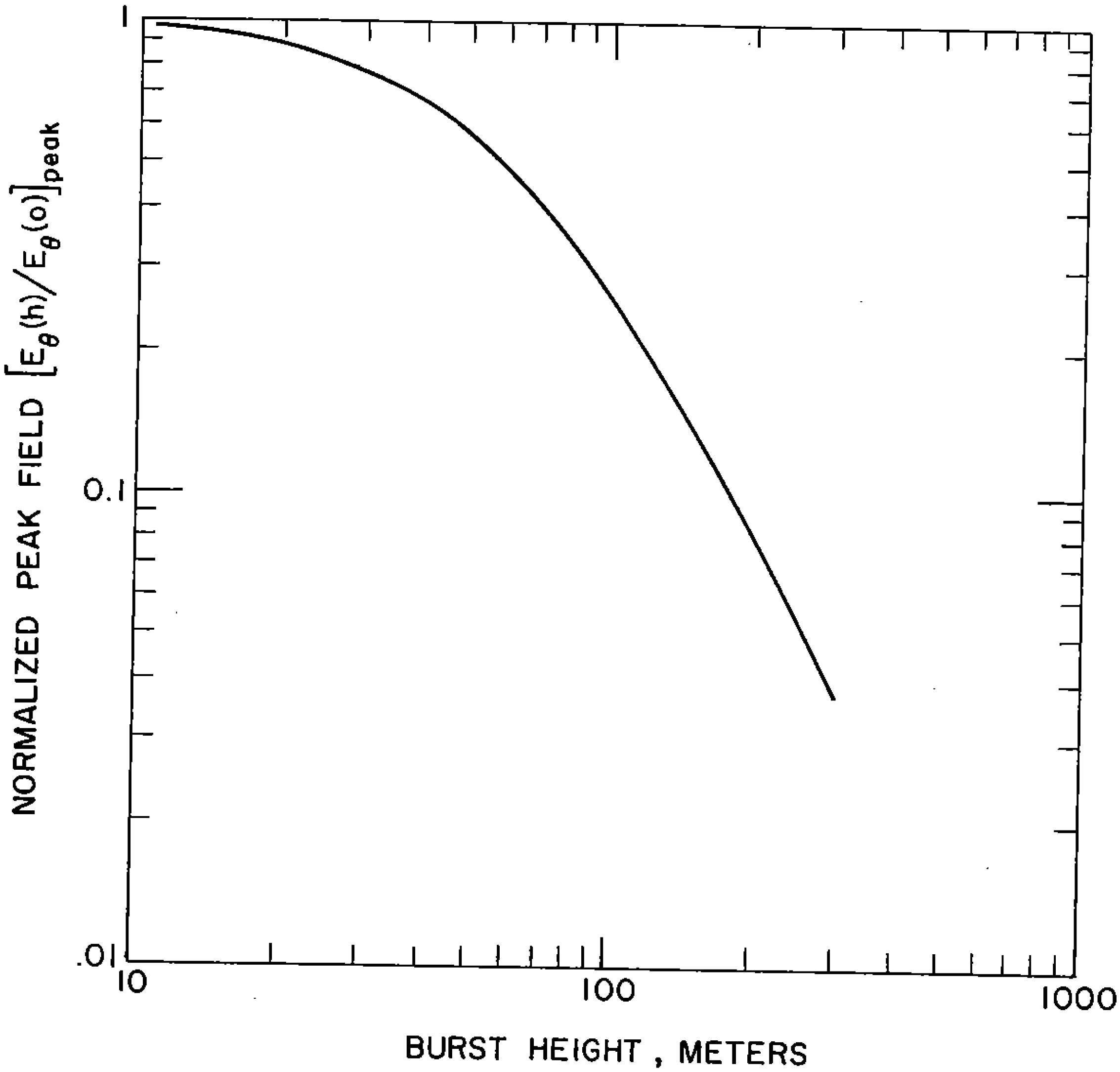

Fig. VII. Normalized "Spike" of Radiated Field vs. Burst Height - Semi-air Burst.

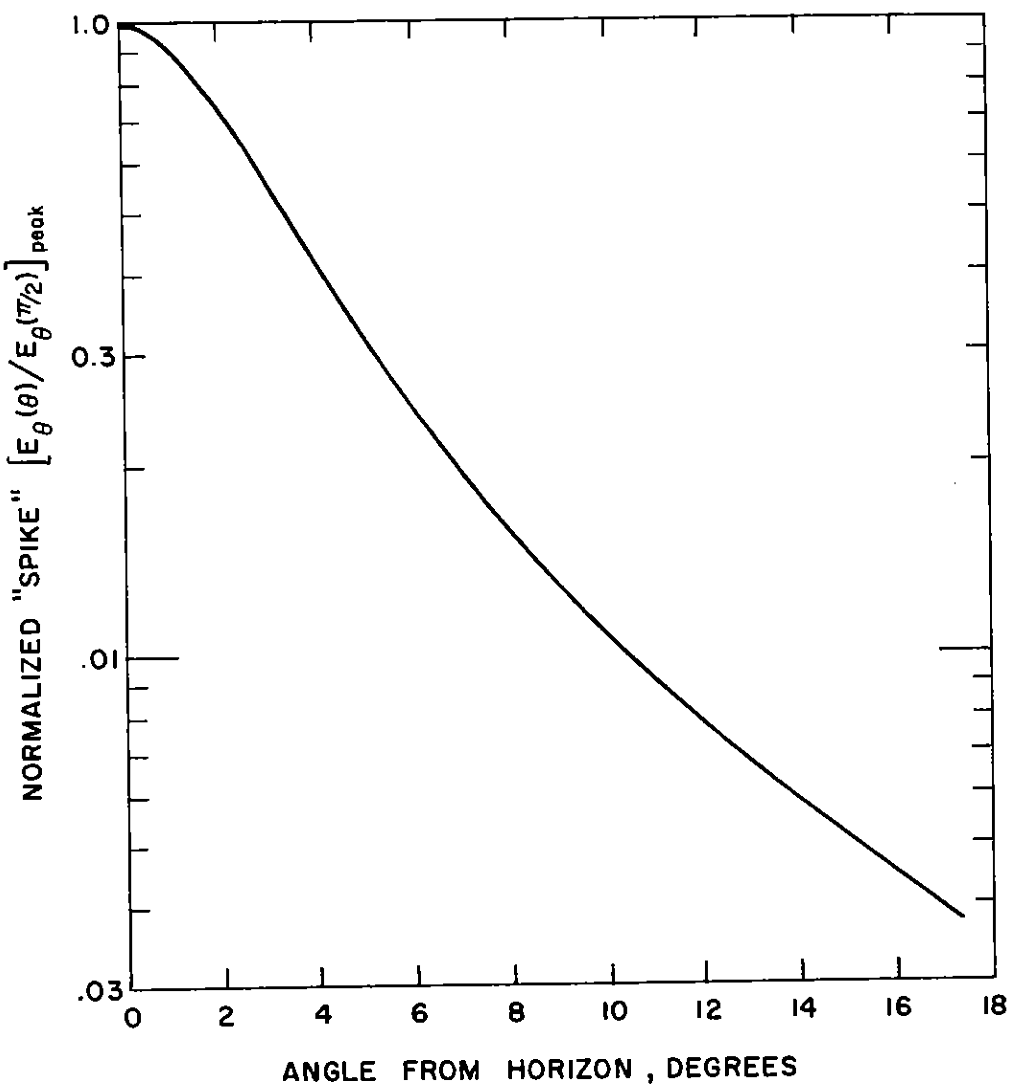

Fig. VIII. Ground Burst - Normalized "Spike" of Radiated Signal vs. Angle from Horizon.