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Development and Testing of LEMP 1

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DEVELOPMENT AND TESTING OF LEMP 1

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ABSTRACT

LEMP 1 is a computer code for obtaining the solution, by finite difference methods, of Maxwell's equations in two space dimensions and retarded time, for the electromagnetic fields produced by a nuclear burst on the ground. The field components considered are B_φ, E_θ, and E_r in the usual spherical coordinates. These fields are calculated in the air, in the ground, and on the ground-air interface. In the ground the electrical conductivity is constant in time and space, and the source current is zero. The conductivity in the air is found by solving the "air-ion" equations, which take account of gamma-induced ionization, electron attachment to O_2, and electron-ion and ion-ion recombination. The source current in the air is the Compton recoil current produced by gamma rays, the source and transport of which are given by a fairly general and flexible prescription. The back-action of the fields on the air conductivity and the source current is treated. Two problems with known solutions are presented - a wave test problem and a diffusion test problem. The results of these problems show that the differencing scheme used, with the proper selection of the finite mesh, gives better than one percent accuracy in the calculated fields. Without using huge numbers of mesh points, the code gives fields whose accuracy is limited only by the source accuracy. There are two Los Alamos reports which serve as companion reports to this one. These reports are numbered LA-4347 and LA-4348.

I. THE DIFFERENTIAL EQUATIONS

1.1 Maxwell's Equations

LEMP 1 is a computer code for obtaining the solution, by finite difference methods, of Maxwell's equations in two space dimensions and retarded time, for the electromagnetic fields produced by a nuclear burst on the ground. We start with the two Maxwell equations that determine how the magnetic field $\vec{B}$ and the electric field $\vec{E}$ change with time,

$$\left.\begin{aligned} \frac{1}{c}\frac{\partial \vec{B}}{\partial t} &= -\nabla \times \vec{E} \\ \frac{\epsilon}{c}\frac{\partial \vec{E}}{\partial t} + 4\pi\sigma\vec{E} &= \nabla \times \vec{B} - 4\pi\vec{J} \end{aligned}\right\} . \qquad 1.0$$

For simplicity, we use cgs Gaussian units in the code; thus charge and electric fields are in esu and currents and magnetic fields are in emu. The relation between these units and the engineers' MKS units is given in the Appendix. For the convenience of engineers, output of the code is expressed in MKS units.

In Eqs. 1.0, the medium has been assumed to be nonmagnetic ($\mu = 1$), and the dielectric constant ϵ has been assumed constant in time. We shall take $\epsilon = 1$ in the air and ϵ = constant in the ground. The electrical conductivity σ will be constant in the ground, but depend on the fields, space, and time in the air. The Compton recoil current density $\vec{J}$ will be zero in the ground, and will depend on the fields, space, and time in the air. The velocity of light $c = 3 \times 10^{10}$ cm/sec.

The other two Maxwell equations, not written here, are only initial conditions; if they are satisfied initially, they will be satisfied at all times if $\vec{B}$ and $\vec{E}$ are carried forward by Eqs. 1.0. Since we start the problem with all charge and current densities and fields equal to zero, they are satisfied initially, and we need not consider them further.

We shall use the standard spherical coordinates r,θ,φ in the air, and cylindrical coordinates r,z,φ

in the ground. The origin of coordinates is placed at the burst point. The Compton current will be primarily radial; such a current will generate, from Eqs 1.0, field components B_φ, E_r, and E_θ. The field components B_φ and E_θ will in turn cause $\vec{J}$ to acquire a θ-component, but this will not lead to additional field components. (The geomagnetic field is neglected.) All quantities will be independent of φ.

In spherical coordinates, Eqs. 1.0 become

$$\frac{1}{c}\frac{\partial B_\varphi}{\partial t} = -\frac{1}{r}\left[\frac{\partial}{\partial r}(rE_\theta) - \frac{\partial E_r}{\partial \theta}\right] , \qquad 1.1$$

$$\frac{\epsilon}{c}\frac{\partial E_r}{\partial t} + 4\pi\sigma E_r = \frac{1}{r \sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, B_\varphi) - 4\pi J_r , \qquad 1.2$$

$$\frac{\epsilon}{c}\frac{\partial E_\theta}{\partial t} + 4\pi\sigma E_\theta = -\frac{1}{r}\frac{\partial}{\partial r}(rB_\varphi) - 4\pi J_\theta . \qquad 1.3$$

It is convenient, in the EMP problem, to replace B_φ and E_θ by "outgoing" and "ingoing" fields F and G, defined by

$$F = r(\sqrt{\epsilon}\, E_\theta + B_\varphi) , \qquad 1.4$$

$$G = r(\sqrt{\epsilon}\, E_\theta - B_\varphi) , \qquad 1.5$$

or by the inverse of these

$$B_\varphi = \frac{F - G}{2r} , \qquad 1.6$$

$$E_\theta = \frac{F + G}{2r\sqrt{\epsilon}} . \qquad 1.7$$

Equations for F and G are obtained by multiplying Eq. 1.1 by $r\sqrt{\epsilon}$, Eq. 1.3 by r, and taking the sum and the difference of the resulting equations. One finds

$$\frac{\sqrt{\epsilon}}{c}\frac{\partial F}{\partial t} + \frac{\partial F}{\partial r} + \frac{2\pi\sigma}{\sqrt{\epsilon}} F = -4\pi r J_\theta + \sqrt{\epsilon}\frac{\partial E_r}{\partial\theta} - \frac{2\pi\sigma}{\sqrt{\epsilon}} G , \qquad 1.8$$

$$\frac{\sqrt{\epsilon}}{c}\frac{\partial G}{\partial t} - \frac{\partial G}{\partial r} + \frac{2\pi\sigma}{\sqrt{\epsilon}} G = -4\pi r J_\theta - \sqrt{\epsilon}\frac{\partial E_r}{\partial\theta} - \frac{2\pi\sigma}{\sqrt{\epsilon}} F . \qquad 1.9$$

Equation 1.2 for E_r is retained.

It is also convenient to use the retarded time. Replace r and t by

$$\left.\begin{aligned} r' &= r \\ \tau &= ct - r \end{aligned}\right\} . \qquad 1.10$$

Then

$$\left.\begin{aligned} \frac{\partial}{\partial r} &= \frac{\partial}{\partial r'} - \frac{\partial}{\partial\tau} \\ \frac{1}{c}\frac{\partial}{\partial t} &= \frac{\partial}{\partial\tau} \end{aligned}\right\} . \qquad 1.11$$

In the new variables, the field equations become (dropping the prime on r after the transformation has been performed)

$$\epsilon\frac{\partial E_r}{\partial\tau} + 4\pi\sigma E_r = -4\pi J_r + \frac{1}{2r^2 \sin\theta}\frac{\partial}{\partial\theta}[\sin\theta(F - G)] , \qquad 1.12$$

$$(\sqrt{\epsilon} - 1)\frac{\partial F}{\partial\tau} + \frac{\partial F}{\partial r} + \frac{2\pi\sigma}{\sqrt{\epsilon}} F = -4\pi r J_\theta + \sqrt{\epsilon}\frac{\partial E_r}{\partial\theta} - \frac{2\pi\sigma}{\sqrt{\epsilon}} G , \qquad 1.13$$

$$(\sqrt{\epsilon} + 1)\frac{\partial G}{\partial\tau} - \frac{\partial G}{\partial r} + \frac{2\pi\sigma}{\sqrt{\epsilon}} G = -4\pi r J_\theta - \sqrt{\epsilon}\frac{\partial E_r}{\partial\theta} - \frac{2\pi\sigma}{\sqrt{\epsilon}} F . \qquad 1.14$$

In the air, $\epsilon = 1$, and these equations become

$$\text{(AIR)}\begin{cases} \dfrac{\partial E_r}{\partial\tau} + 4\pi\sigma E_r = -4\pi J_r + \dfrac{1}{2r^2 \sin\theta}\dfrac{\partial}{\partial\theta}[\sin\theta(F - G)] , & 1.15 \\[2ex] \dfrac{\partial F}{\partial r} + 2\pi\sigma F = -4\pi r J_\theta + \dfrac{\partial E_r}{\partial\theta} - 2\pi\sigma G , & 1.16 \\[2ex] \dfrac{\partial G}{\partial\tau} + \pi\sigma G = \dfrac{1}{2}\dfrac{\partial G}{\partial r} - 2\pi r J_\theta - \dfrac{1}{2}\dfrac{\partial E_r}{\partial\theta} - \pi\sigma F . & 1.17 \end{cases}$$

Notice that Eq. 1.16 for the outgoing field F contains no retarded time derivatives. The retarded time is used because, while the sources and fields are rapidly varying functions of τ, they are slowly varying functions of r, permitting a coarse r-mesh. This is not true for r,t variables. Once the choice r,τ is made, it is mathematically wise to go to F and G to eliminate time derivatives from one equation. Over much of the problem, and particularly over the rapidly varying part, G is very small compared with F.

In the ground the equations do not have such nice properties. But here σ is large and ϵ is of the order of 10 at least, so that signals propagate slowly compared with the speed of light. The fields are driven into the ground from the surface, and are also rapid functions of τ but slow functions of r. The equations for cylindrical coordinates can be obtained by putting $\sin\theta = 1$ and replacing $1/r\ \partial/\partial\theta$ by $\partial/\partial z$ and E_θ by E_z. One finds, dropping the source current,

(GROUND)

$$\frac{\partial E_r}{\partial\tau} + \frac{4\pi\sigma_0}{\epsilon_0} E_r = \frac{1}{2\epsilon_0 r}\frac{\partial}{\partial z}(F - G) , \qquad 1.18$$

$$\frac{\partial F}{\partial\tau} + \frac{2\pi\sigma_0}{\sqrt{\epsilon_0}(\sqrt{\epsilon_0} - 1)} F = -\frac{1}{(\sqrt{\epsilon_0} - 1)}\frac{\partial F}{\partial r} + \frac{\sqrt{\epsilon_0}\, r}{(\sqrt{\epsilon_0} - 1)}\frac{\partial E_r}{\partial z} - \frac{2\pi\sigma_0}{\sqrt{\epsilon_0}(\sqrt{\epsilon_0} - 1)} G , \qquad 1.19$$

$$\frac{\partial G}{\partial\tau} + \frac{2\pi\sigma_0}{\sqrt{\epsilon_0}(\sqrt{\epsilon_0} + 1)} G = \frac{1}{(\sqrt{\epsilon_0} + 1)}\frac{\partial G}{\partial r} - \frac{\sqrt{\epsilon_0}\, r}{(\sqrt{\epsilon_0} + 1)}\frac{\partial E_r}{\partial z} - \frac{2\pi\sigma_0}{\sqrt{\epsilon_0}(\sqrt{\epsilon_0} + 1)} F . \qquad 1.20$$

We use ϵ_0 and σ_0 to denote the values of ϵ and σ in the ground.

1.2 The Air-Ion Equations

In order to compute the air conductivity one has to keep accounts of the production and recombination of electrons, positive ions, and negative ions. Electrons, density n_e, and positive ions, density n_+, are made as a result of the absorption of gamma rays. The source of both will be called $\dot{\gamma}$, ion pairs per cm^3 per sec. Electrons attach with rate coefficient α to O_2, forming negative ions O_2^-, density n_-. Electrons recombine with positive ions, with rate coefficient β. Positive and negative ions recombine with each other, with rate coefficient γ. The differential equations for n_e, n_-, and n_+ are

$$c\,\frac{dn_e}{d\tau} + (\alpha + \beta n_+)n_e = \dot{\gamma} , \qquad 1.21$$

$$c\,\frac{dn_-}{d\tau} + (\gamma n_+)n_- = \alpha n_e , \qquad 1.22$$

$$c\,\frac{dn_+}{d\tau} + (\gamma n_- + \beta n_e)n_+ = \dot{\gamma} . \qquad 1.23$$

The effect of charge transport on the densities is unimportant and is neglected.

It is not necessary to solve all three of these equations, because of the condition of charge neutrality which follows from them plus the assumption of initial neutrality,

$$n_+ = n_e + n_- . \qquad 1.24$$

In LEMP 1 we carry n_e and n_-. (If n_e and n_+ are carried there is an instability of the difference equations when βn_+ becomes comparable with or larger than α.)

Having n_e and n_-, we calculate the conductivity from the equation

$$\sigma = \frac{e}{c}\left[n_e\mu_e + (2n_- + n_e)\mu_i\right] , \qquad 1.25$$

where $e = 4.803 \times 10^{-10}$ esu, μ_e is the electron mobility, and μ_i is the ion mobility. We assume, for lack of data, that positive and negative ions have the same mobility. In LEMP 1 we use the following fits* and values:

*The fit for μ_e was done by John S. Malik.

$$\left.\begin{aligned}
&\alpha = \frac{0.72 \times 10^8}{\sqrt{|E| + 0.03}} + 6.45 \times 10^7 \exp\left(- \frac{12.76\,\rho}{|E| + 0.01}\right) \quad (\mathrm{sec}^{-1}) \\
&|E| = \sqrt{E_\theta^2 + E_r^2} \quad (\mathrm{esu}) \\
&\rho = \text{air density} \quad \left(\frac{\mathrm{gms}}{\mathrm{liter}} = \frac{\mathrm{milligms}}{\mathrm{cm}^3}\right) \\
&\beta = 2.5 \times 10^{-7} \quad (\mathrm{cm}^3/\mathrm{sec}) \\
&\gamma = 2.3 \times 10^{-6} \quad (\mathrm{cm}^3/\mathrm{sec}) \\
&\mu_i = 150 \quad \mathrm{cm/sec\ per\ esu} \\
&\mu_e = \frac{3.93 \times 10^8 \exp(-0.87P)}{\rho\left[\frac{3 \times 10^4 |E|}{\rho} + 1.4 \times 10^3 + 4 \times 10^3 P\right]^{(0.61-0.07P)}} + 3 \times 10^6 [0.04 + 0.01P] \\
&P = \text{percent water vapor in air}
\end{aligned}\right\} \quad 1.26$$

It may be noted that α and μ_e will be functions of r, θ, and τ through their dependence on $|E|$.

1.3 Gamma Transport and the Compton Current

In LEMP 1 the transport of gamma rays is not treated by differential equations. Rather, the results of transport calculations have been fitted by fairly general formulae, which are discussed in Chapter 4.

The Compton recoil current has to be determined by solving Newton's law for a Klein-Nishina distribution of recoil electrons, taking into account the slowing-down and the electric and magnetic fields. Again, this is not done in LEMP 1. Rather, such calculations have been done for a large number of values of the fields and original gamma energy, and the results fitted by formulae which are used in the code (see Chapter 4 and LA-4348). In the calculations, the fields were assumed constant over the range of the electrons. In the resulting fits, the mean forward range in the absence of fields is called R, the average radial displacement of the Compton electron is called DX, and the average displacement in the θ direction is called DY. R was fitted as a function of the initial gamma energy, and dX/R and dY/R were fitted as functions of the fields for each of the several gamma energies used in the code.

1.4 The Inner Boundary Condition

The Compton current has a $1/r^2$ singularity at the origin, which, of course, would disappear if one took account of the actual size of the bomb. Since the fields at distances of a few meters are not of practical concern, we use a simplified boundary condition near the origin. We imagine that a superconducting hemisphere (center at the origin) lies on the ground, attached to a superconducting cylinder that extends downward into the ground. The electric field component (E_θ or E_Z) parallel to the surface of this hemisphere or cylinder is set equal to zero. The radii of the hemisphere and cylinder are usually taken to be 30 meters.

In the air, the high conductivity near the burst means that the assumed hemisphere can have no practical effect on the fields at larger distances of interest. Fields in the ground directly under the burst are modified by the assumption of the superconducting cylinder.

1.5 The Outer Boundary Condition

From Eq. 1.15, it is clear that no outer boundary condition is needed for E_r in the air, since no radial derivatives of E_r occur in the equation. Since Eq. 1.16 for F is integrated outwards in r, no outer boundary condition is needed for F. However in Eq. 1.17 for G, $\partial G/\partial r$ occurs on the right hand side and an outer boundary condition is needed for G.

For the high frequency parts of the fields, G is very nearly the ingoing waves. If the outer radius is chosen large enough that the Compton current and air conductivity are negligible beyond this distance, G = 0 will be a suitable boundary condition for these high frequency parts. However, for the low frequency parts, G is not nearly equal to the ingoing waves, and a more detailed boundary condition is needed.

In deriving such a boundary condition, we assume that Compton current and air conductivity vanish beyond the boundary, and that the ground conductivity is infinite. Thus, in the ground, G = 0 = F at the outer boundary. In the air beyond the boundary, the fields satisfy the vacuum equations; we therefore take them to be a superposition of outgoing spherical multipole TM waves. With the assumption that the ground conductivity is infinite, we need take only odd spherical harmonics.

We start with the usual multipole expansions in the frequency domain, with solutions that go like $e^{-i\omega\tau} = e^{i\omega(r-ct)}$ at large r (outgoing waves). (Actually ω is the free space wave number.) Let the expansion of $rB_\varphi(\omega)$ be

$$rB_\varphi(\omega) = e^{-i\omega\tau} \sum_{\text{odd } \ell} b_\ell(\omega)\varphi_\ell(\omega r)P'_\ell(\cos\theta) \quad . \qquad 1.27$$

Here $b_\ell(\omega)$ is the expansion coefficient, $P'_\ell(\cos\theta)$ is the associated Legendre polynomial (note minus sign!),

$$P'_\ell(\cos\theta) \equiv -\frac{\partial}{\partial\theta} P_\ell(\cos\theta) \ , \qquad 1.28$$

and $\varphi_\ell(\omega r)$ is a polynomial in ωr. For ℓ = 1,3,5,

$$\varphi_1(\omega r) = 1 + \frac{1}{-i\omega r} \ , \qquad 1.29$$

$$\varphi_3(\omega r) = 1 + \frac{6}{-i\omega r} + \frac{15}{(-i\omega r)^2} + \frac{15}{(-i\omega r)^3} \ , \qquad 1.30$$

$$\varphi_5(\omega r) = 1 + \frac{15}{-i\omega r} + \frac{105}{(-i\omega r)^2} + \frac{420}{(-i\omega r)^3} + \frac{945}{(-i\omega r)^4} + \frac{945}{(-i\omega r)^5} \ . \qquad 1.31$$

The expansions for rE_θ and rE_r are then

$$rE_\theta(\omega) = e^{-i\omega\tau} \sum_{\text{odd } \ell} b_\ell(\omega)\left[\varphi_\ell(\omega r) - \frac{\partial}{\partial(-i\omega r)}\varphi_\ell(\omega r)\right]P'_\ell(\cos\theta) \ , \qquad 1.32$$

$$rE_r(\omega) = e^{-i\omega\tau} \sum_{\text{odd } \ell} b_\ell(\omega)\left[\frac{\ell(\ell+1)}{(-i\omega r)}\varphi_\ell(\omega r)\right]P_\ell(\cos\theta) \ . \qquad 1.33$$

For the functions F and G introduced above, the expansions are

$$F(\omega) = e^{-i\omega\tau} \sum_{\text{odd } \ell} b_\ell(\omega)\left[2\varphi_\ell(\omega r) - \frac{\partial}{\partial(-i\omega r)}\varphi_\ell(\omega r)\right]P'_\ell(\cos\theta) \ , \qquad 1.34$$

$$G(\omega) = e^{-i\omega\tau} \sum_{\text{odd } \ell} b_\ell(\omega)\left[-\frac{\partial}{\partial(-i\omega r)}\varphi_\ell(\omega r)\right]P'_\ell(\cos\theta) \ . \qquad 1.35$$

We transform these expansions back to the retarded time domain. Let

$$rE_r(\tau) = \sum_{\text{odd } \ell} e_\ell(r,\tau)P_\ell(\cos\theta) \ , \qquad 1.36$$

$$F(\tau) = \sum_{\text{odd } \ell} f_\ell(r,\tau)P'_\ell(\cos\theta) \ , \qquad 1.37$$

$$G(\tau) = \sum_{\text{odd } \ell} g_\ell(r,\tau)P'_\ell(\cos\theta) \ . \qquad 1.38$$

Then, letting $b_\ell(\tau)$ be the inverse transform of $b_\ell(\omega)$, one finds, for ℓ = 1

$$f_1(r,\tau) = 2b_1(\tau) + \frac{2}{r}\int b_1(\tau)d\tau + \frac{1}{r^2}\iint b_1(\tau)d\tau d\tau \ , \qquad 1.39$$

$$g_1(r,\tau) = \frac{1}{r^2}\iint b_1(\tau)d\tau d\tau \ , \qquad 1.40$$

$$e_1(r,\tau) = \frac{2}{r}\int b_1(\tau)d\tau + \frac{1}{r^2}\iint b_1(\tau)d\tau d\tau \ . \qquad 1.41$$

Here all integrals are from $-\infty$ to τ. It is seen, for example, that the electric dipole part of G must extrapolate like $1/r^2$, and this would be a sufficient boundary condition for this part. Note that $g_1(r_1,\tau)$ can be found at the outer boundary r_1 by expanding $G(r_1,\tau,\theta)$ in spherical harmonics, and this determines $\iint b_1(\tau)d\tau d\tau$. The terms in f_1 and e_1 can then be found by differentiation. The dipole part can then be found for any $r > r_1$. This procedure is used in LEMP for extrapolating the fields to large distances.

For $\ell = 3$, one finds

$$f_3(r,\tau) = 2b_3(\tau) + \frac{12}{r}\int b_3 d\tau + \frac{36}{r^2}\iint b_3 d\tau^2 + \frac{60}{r^3}\iiint b_3 d\tau^3 + \frac{45}{r^4}\iiiint b_3 d\tau^4 \ , \qquad 1.42$$

$$g_3(r,\tau) = \frac{6}{r^2}\iint b_3 d\tau^2 + \frac{30}{r^3}\iiint b_3 d\tau^3 + \frac{45}{r^4}\iiiint b_3 d\tau^4 \ , \qquad 1.43$$

$$e_3(r,\tau) = \frac{12}{r}\int b_3 d\tau + \frac{72}{r^2}\iint b_3 d\tau^2 + \frac{180}{r^3}\iiint b_3 d\tau^3 + \frac{180}{r^4}\iiiint b_3 d\tau^4 \ . \qquad 1.44$$

Again $g_3(r_1,\tau)$ can be found from the expansion of $G(r_1,\tau,\theta)$ in spherical harmonics. Then $\iint b_3(\tau)d\tau^2$ can be found by solving Eq. 1.43, and all the other coefficients can be found by integration or differentiation. Thus the electric octopole part of the fields can be found for any $r > r_1$. In particular, the value of g_3 at the next mesh point beyond r_1 can be found, which amounts to providing the boundary condition for g_3.

Equation 1.43 is easily solved. Let

$$I_2(b_3) \equiv \iint b_3 d\tau^2 \ . \qquad 1.45$$

Then write Eq. 1.43 as

$$I_2(b_3) = \frac{r_1^2}{6}\left[g_3(r_1,\tau) - \frac{30}{r_1^3}\int I_2 d\tau - \frac{45}{r_1^4}\iint I_2 d\tau^2\right] . \qquad 1.46$$

By an iterative technique, this equation can be carried forward one time step at a time.

For $\ell = 5$, to save writing integral signs, we define

$$I_n(b_5) \equiv \underbrace{\int\cdots\int}_{n} b_5 d\tau^n \ . \qquad 1.47$$

With this notation, one finds

$$f_5(r,\tau) = 2b_5(\tau) + \frac{30}{r} I_1(b_5) + \frac{225}{r^2} I_2(b_5) + \frac{1050}{r^3} I_3(b_5)$$

$$+ \frac{3150}{r^4} I_4(b_5) + \frac{5670}{r^5} I_5(b_5) + \frac{4725}{r^6} I_6(b_5) \ , \qquad 1.48$$

$$g_5(r,\tau) = \frac{15}{r^2} I_2(b_5) + \frac{210}{r^3} I_3(b_5) + \frac{1260}{r^4} I_4(b_5)$$

$$+ \frac{3780}{r^5} I_5(b_5) + \frac{4725}{r^6} I_6(b_5) \ , \qquad 1.49$$

$$e_5(r,\tau) = \frac{30}{r} I_1(b_5) + \frac{450}{r^2} I_2(b_5) + \frac{3150}{r^3} I_3(b_5)$$

$$+ \frac{12600}{r^4} I_4(b_5) + \frac{28350}{r^5} I_5(b_5) + \frac{28350}{r^6} I_6(b_5) \ . \qquad 1.50$$

Again, $g_5(r_1,\tau)$ can be determined by expanding $G(r_1,\tau,\theta)$ in spherical harmonics, and $I_2(b_5)$ can then be found by solving Eq. 1.49 by an iterative method similar to Eq. 1.46. Then g_5 can be extrapolated to the next mesh point beyond r_1, which provides the boundary condition for g_5. Also, all the other coefficients in Eqs. 1.48 to 1.50 can be found by differentiation or integration, and the electric 2^{32} pole part of the fields can be found for any $r > r_1$.

In LEMP 1 we have chosen to fit G at the boundary and calculate the coefficients $I_n(b_\ell)$ from the spherical harmonic expansion of G. This seems appropriate since only G needs a boundary condition. However, for purposes of extrapolating the fields to large distances, it might have been more appropriate to use the spherical harmonic expansion of F at the boundary, since F is larger than G at early times.

We notice that the first term in $g_\ell(r,\tau)$ is always proportional to $1/r^2$. For all $\ell > 5$ we assume in LEMP 1 that $g_\ell(r,\tau) \sim 1/r^2$, neglecting the higher powers. This approximation is based on the assumption that by the time the higher (negative) powers of r in g_ℓ become important the electric 2^ℓ pole fields are negligible; the fields tend to become smooth functions of θ at late time.

In applying the outer boundary condition, only g_3 and g_5 have powers of r different from $1/r^2$. Thus we need only to separate these two harmonics from G, extrapolate them correctly, and extrapolate the rest of G as $1/r^2$.

In extrapolating the other fields F and E_r to large distances, harmonics 1, 3, and 5 are treated correctly and the remaining parts are extrapolated as constant for F and as $1/r^2$ for E_r.

II. THE MESH AND THE DIFFERENCE EQUATIONS

2.1 The Mesh

LEMP 1 uses nonuniform meshes in r, θ, Z, and τ, for the following reasons. The source and fields change rapidly with τ at early τ, but slowly at late τ. Since we wish to cover times from 10^{-9} seconds to 10^{-4} seconds, a variable τ mesh is necessary. At small r, the source and fields have $1/r^2$ and $1/r$ dependence, but at large r, the source is exponential (with absorption length ~200 meters) while $F \to$ constant, $G \sim 1/r^2$, $E_r \sim 1/r^2$. Thus a smaller r mesh is needed near the origin than at large r. Smaller θ and Z meshes are needed near the ground-air interface because the fields have large θ and Z gradients there.

The radial mesh, both in the air and in the ground, is obtained from four input numbers: r_0 (typically 3×10^3 cm) is the smallest value of r in the mesh, r_{max} (typically 3×10^5 cm) is the largest value of r in the mesh, δr_0 (typically 2×10^3 cm) is the first radial interval, and n_r is the number of radial mesh points. The radius of the k^{th} mesh point ($k = 1, 2, \cdots, n_r$) is

$$r_k = r_0 + \delta r_0(k-1) + \frac{[r_{max} - r_0 - \delta r_0(n_r - 1)]}{(n_r - 1)(n_r - 2)}(k-1)(k-2)\,. \tag{2.1}$$

For a mesh in which the interval increases with r, $r_{max} \geq r_0 + \delta r_0(n_r - 1)$.

The θ-mesh is calculated from two input numbers $n_{\theta f}$ and $n_{\theta s}$. Approximately (but not exactly), the interval $0 < \theta < \pi/2$ is first divided into $n_{\theta f}$ (f for final) equal intervals, and then the resulting interval next to the ground ($\theta = \pi/2$) is split $n_{\theta s}$ (s for split) times. Indicating mesh values by θ_ℓ, we see from Eqs. 1.15, 1.16, and 1.17 that if F and G are carried at θ_ℓ, E_r should be carried at $\theta_{\ell+\frac{1}{2}}$. Boundary conditions require F = G = 0 at $\theta = 0$ (but not E_r), and E_θ must be allowed to be discontinuous at the ground. The θ-mesh defined below accommodates these features. We define L by

$$L = n_{\theta s} + n_{\theta f}\,. \tag{2.2}$$

We first set up the $\theta_{\ell+\frac{1}{2}}$ mesh. Let $\delta\theta_a \equiv (\pi/2)/(n_{\theta f} - \frac{1}{2})$. Set $\theta_{1+\frac{1}{2}} = \delta\theta_a/2$. Calculate $\theta_{\ell+\frac{1}{2}} = \theta_{\ell-1+\frac{1}{2}} + \delta\theta_a$ for $\ell = 2$ to $\ell = n_{\theta f} - 1$. Then calculate $\theta_{n_{\theta f}-1+q+\frac{1}{2}} = \theta_{n_{\theta f}-2+q+\frac{1}{2}} + \delta\theta_a/2^q$ for q = 1 to $q = n_{\theta s}$. $\delta\theta_0 \equiv \delta\theta_a/2^{n_{\theta s}}$ is the smallest θ-mesh increment. Then set $\theta_{L+\frac{1}{2}} = \theta_{L-1+\frac{1}{2}} + \delta\theta_0 = \pi/2$. Now set $\theta_1 = 0$ and $\theta_\ell = \frac{1}{2}(\theta_{\ell-1+\frac{1}{2}} + \theta_{\ell+\frac{1}{2}})$ for $\ell = 2$ to L. This last operation ensures that the θ_ℓ values are (exactly) centered between the correct $\theta_{\ell+\frac{1}{2}}$ values. $\theta_{\ell+\frac{1}{2}}$ is not centered between values of θ_ℓ. Thus the θ-derivatives will automatically be centered when differencing the F and G equations but the E_r equation will require special attention due to the θ-derivatives. Conversely, if one forms θ_ℓ first and lets $\theta_{\ell+\frac{1}{2}}$ be centered between these values of θ_ℓ, then one must give special attention to centering two equations instead of only one equation as above.

The Z-mesh (gnd) requires three input constants: n_{Zs} is the number of split cells in the

Z-mesh, n_{Zf} is the number of final cells in the Z-mesh, and Z_0 is the depth in the ground where the first E_r mesh point is located. The Z-mesh is formed in the ground by reasoning similar to that used in the air. We let $Z_{L+\frac{1}{2}} = 0$ (on the ground), $Z_{L+1+\frac{1}{2}} = Z_0$, and $Z_{L+2+\frac{1}{2}} = 2Z_0$. Then set $Z_{L+q+1+\frac{1}{2}} = Z_{L+q+\frac{1}{2}} + 2^q Z_0$ for q = 2 to q = n_{Zs}. Finally, $Z_{\ell+\frac{1}{2}} = Z_{\ell-1+\frac{1}{2}} + Z_0 2^{n_{Zs}+1}$ for $\ell = L + n_{Zs} + 2$ to $\ell = L + n_{Zs} + n_{Zf} + 1$. The Z_ℓ mesh is then centered; i.e., $Z_{\ell+1} = \frac{1}{2}(Z_{\ell+1+\frac{1}{2}} + Z_{\ell+\frac{1}{2}})$ for ℓ = L to $\ell = L + n_{Zs} + n_{Zf}$.

Figure 1 shows the mesh for a particular set of input parameters. The locations of $E_{r_{k,\ell+\frac{1}{2}}}$, $F_{k,\ell}$, $G_{k,\ell}$, and $\sigma_{k,\ell}$ are shown. Of course, $\sigma_{k,\ell}$ is only in the air, and $n_{e_{k,\ell}}$ and $n_{-_{k,\ell}}$ are located at the same point as $\sigma_{k,\ell}$. Mesh values are located, for each angular value, at all r_k (k = 1, n_r). Note that B_φ is positive "into the paper" in Fig. 1 for the LEMP 1 coordinate system.

The time (τ) mesh is obtained by advancing forward in time by successive cycles. At the beginning of any time cycle, say τ^n(n = cycle number), it is assumed that the mesh values (E_r, F, G, etc.) are known at times τ^n and τ^{n-1} and the cycle calculation is performed to advance time to τ^{n+1}. The time increment ($\delta\tau$) is variable from cycle to cycle. Since the mesh values are only saved at two times, back storage is utilized each cycle. We define $\delta\tau^n = \tau^{n+1} - \tau^n$.

2.2 Order of Solving Equations

To avoid confusion, we will give an ordered outline at this time of the calculations performed during each cycle. The eight steps of a cycle are:

1. Advance n_e, n_-, and σ to τ^{n+1} ($k = 1, n_r$; ℓ = 1,L) (also, calculate J_r and J_θ).
2. Calculate $\delta\tau$ for the next cycle.
3. Advance F, G, and E_r to τ^{n+1} (k = 1 ; $\ell = 1, L + n_{Zs} + n_{Zf}$).
4. Advance G to τ^{n+1} ($k = 2, n_r - 1$; $\ell = 1, L + n_{Zs} + n_{Zf}$).
5. Set outside boundary conditions ($G^{n+1}_{n_r,\ell}$; $\ell = 1, L + n_{Zs} + n_{Zf}$).
6. Advance F and E_r to τ^{n+1} ($k = 2, n_r$; $\ell = 1, L + n_{Zs} + n_{Zf}$).
7. Mesh check (change).
8. Output (dump and time storage).

We will not follow this outline in the order given due to the complications in presenting calculations 4 and 6. After 4 and 6 have been presented, then 2, 3, and 5 will be explained.

2.3 General Form of Difference Equations

Before presenting the difference equations, it may be noted that Maxwell's equations and the air-ion equations as written in the last chapter are all written in a similar form. We note that if one has an equation of the form*

$$\frac{\partial f}{\partial \xi} + \gamma f = \psi , \qquad 2.3$$

then the exact solution is

$$f(\xi) = e^{-X(\xi)}\left[f(\xi_0) + \int_{\xi_0}^{\xi} \psi(\xi')e^{X(\xi')}\, d\xi' \right] , \qquad 2.4$$

where $X(\xi) \equiv \int_{\xi_0}^{\xi} \gamma(\xi'')\, d\xi''$. To first order in $\delta\xi \equiv \xi - \xi_0$, the solution of 2.3 is

*The γ or other symbols used in Eq. 2.3 have nothing to do with the same symbol used in the air-ion equations or elsewhere.

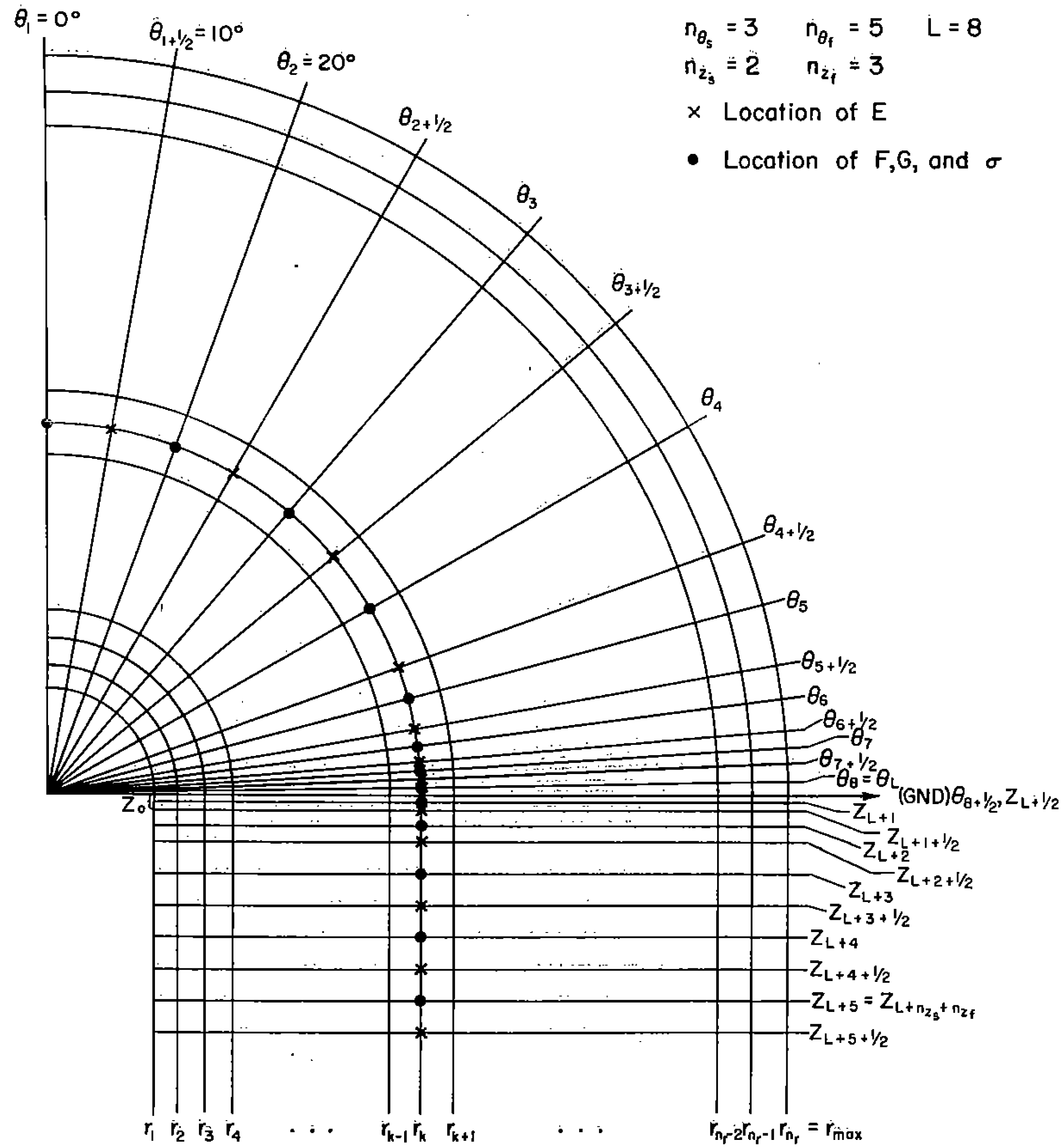

Fig. 1. The mesh.

$$f(\xi) = f(\xi_0)e^{-\overline{X}} + (1 - e^{-\overline{X}})\left[\frac{\psi}{\gamma}\right]_{\text{at } \overline{\xi}} , \qquad 2.5$$

where $\overline{X} = \delta\xi \cdot \overline{\gamma}$ and $\overline{\gamma}$ is γ evaluated at $\overline{\xi} = \frac{1}{2}(\xi + \xi_0)$. For second order accuracy in the integral term of Eq. 2.4, one uses the following procedure. Let $\Phi(\xi') \equiv \psi(\xi')/\gamma(\xi')$ so that $\Phi_1 \equiv \Phi(\xi_0)$ and $\Phi_2 \equiv \Phi(\xi)$; then for $\Phi(\xi')$ assumed to be a linear function of $X(\xi')$, we have

$$f(\xi) = f(\xi_0)e^{-\overline{X}} + \Phi_1\left[\frac{1}{\overline{X}}(1 - e^{-\overline{X}}) - e^{-\overline{X}}\right]$$

$$+ \Phi_2\left[1 - \frac{1}{\overline{X}}(1 - e^{-\overline{X}})\right] . \qquad 2.6$$

The forms presented by Eqs. 2.5 and 2.6 will be evident in the difference equations as they are presented. Various features of Eq. 2.3 were brought to the authors' attention by Suydam.*

2.4 The Air-Ion Equations

The air-ion equations (Eqs. 1.21 and 1.23) are differenced as

$$n_{e_{k,\ell}}^{n+1} = n_{e_{k,\ell}}^{n-1} e^{-\Phi_e} + (1 - e^{-\Phi_e})\left[\frac{\dot{\gamma}}{(\alpha + \beta n_+)}\right]_{k,\ell}^{n} \qquad 2.7$$

and

$$n_{-_{k,\ell}}^{n+1} = n_{-_{k,\ell}}^{n-1} e^{-\Phi_-} + (1 - e^{-\Phi_-})\left[\frac{\alpha n_e}{\gamma n_+}\right]_{k,\ell}^{n} , \qquad 2.8$$

where

$$\Phi_e \equiv [\alpha + \beta n_+]_{k,\ell}^{n} \frac{(\delta\tau^n + \delta\tau^{n-1})}{c} ,$$

$$\Phi_- \equiv [\gamma n_+]_{k,\ell}^{n} \frac{(\delta\tau^n + \delta\tau^{n-1})}{c} , \quad \text{and}$$

$n_+ = n_e + n_-$. The fields used in calculation of the various parameters (see Eqs. 1.26) are obtained from the mesh for the appropriate values of n, k, and ℓ. The values of n_e and n_- are then used to calculate σ; i.e.,

*B. R. Suydam, private communications.

$$\sigma_{k,\ell}^{n+1} = \frac{e}{c}\left[n_e\mu_e + (2n_- + n_e)\mu_i\right]_{k,\ell}^{n+1} . \qquad 2.9$$

In Eq. 2.9 the electric field magnitude ($|E|$) is not known at τ^{n+1} for the calculation of μ_e (see Eq. 1.26) at this part of the cycle so that one uses $|E|^n$.

It should be noted that Eqs. 2.7 and 2.8 are not centered in time unless $\delta\tau^n = \delta\tau^{n-1}$. Other difference equations to be presented will have this same difficulty. Though there are ways* to avoid this difficulty which involve more calculational time, in LEMP 1 we essentially avoid this trouble by allowing $\delta\tau$ to change by no more than ±2% in any one time cycle.

2.5 The Field Equations

Parts 4 and 6 of a cycle need a differencing scheme for Eqs. 1.15 to 1.20 (Maxwell's equations). The basic work on the stability of the differencing scheme for Maxwell's equations as used in LEMP 1 was done by Richtmyer.** Since LEMP 1 uses nonuniform meshes in r, θ, and Z, we modify the equations as written by Richtmyer. Mostly the equations are modified to improve the centering in our nonuniform mesh. The G equations (1.17 and 1.20) are differenced as, for k = 2 to $n_r - 1$,

$$(\text{air}) \quad G_{k,\ell}^{n+1} = G_{k,\ell}^{n-1}e^{-S}$$

$$+ (1 - e^{-S})\left[\frac{1}{2\pi\sigma}\left\{\frac{\partial G}{\partial r} - \frac{\partial E}{\partial \theta}\right\} - \frac{2r}{\sigma} J_\theta - F\right]_{k,\ell}^{n} , \qquad 2.10$$

where $S \equiv \pi\sigma_{k,\ell}^{n}(\delta\tau^n + \delta\tau^{n-1})$ and ℓ = 2 to L, and

$$(\text{gnd}) \quad G_{k,\ell}^{n+1} = G_{k,\ell}^{n-1}e^{-S_0}$$

$$+ (1 - e^{-S_0})\left[\frac{\sqrt{\epsilon_0}}{2\pi\sigma_0}\left\{\frac{\partial G}{\partial r} - \sqrt{\epsilon_0}\, r\, \frac{\partial E}{\partial Z}\right\} - F\right]_{k,\ell}^{n} , \qquad 2.11$$

where

$$S_0 \equiv \frac{2\pi\sigma_0}{\sqrt{\epsilon_0}(\sqrt{\epsilon_0} + 1)}(\delta\tau^n + \delta\tau^{n-1}) ,$$

*R. D. Richtmyer, private communications.

**R. D. Richtmyer, "Stability of the New Radio Flash Code," Los Alamos Scientific Laboratory report LA-3864-MS, 1968.

and $\ell = L + 1, L + 2, \cdots, L + n_{Zs} + n_{Zf}$. (Note: In the difference equations we delete the subscript r from E_r.)

The F equations (1.16 and 1.19) are differenced as, for $k = 2, n_r$,

$$\text{(air)}\quad F^{n+1}_{k,\ell} = F^{n+1}_{k-1,\ell}e^{-X} + \left[\frac{1}{X}(1 - e^{-X}) - e^{-X}\right]\alpha^{n+1}_{k-1,\ell} + \left[1 - \frac{1}{X}(1 - e^{-X})\right]\alpha^{n+1}_{k,\ell} , \qquad 2.12$$

where

$$X \equiv 2\pi\sigma^{n+1}_{k-\frac{1}{2},\ell}(r_k - r_{k-1})$$

and

$$\alpha^{n+1}_{k,\ell} \equiv \frac{E^{n+1}_{k,\ell+\frac{1}{2}} - E^{n+1}_{k,\ell-1+\frac{1}{2}}}{2\pi\sigma^{n+1}_{k,\ell}(\theta_{\ell+\frac{1}{2}} - \theta_{\ell-\frac{1}{2}})} - \frac{2r_k J_{\theta k,\ell}}{\sigma^{n+1}_{k,\ell}} - G^{n+1}_{k,\ell} ,$$

for $\ell = 2$ to L, and

$$\text{(gnd)}\quad F^{n+1}_{k,\ell} = F^{n-1}_{k,\ell}e^{-X_0} + (1 - e^{-X_0})\left\{\frac{-\sqrt{\epsilon_0}}{2\pi\sigma_0}\left.\frac{\partial F}{\partial r}\right)^n_{k,\ell} + \frac{\epsilon_0 r_k}{4\pi\sigma_0}\frac{\left[E^{n+1}_{k,\ell+\frac{1}{2}} + E^{n-1}_{k,\ell+\frac{1}{2}} - E^{n+1}_{k,\ell-1+\frac{1}{2}} - E^{n-1}_{k,\ell-1+\frac{1}{2}}\right]}{(Z_{\ell+\frac{1}{2}} - Z_{\ell-\frac{1}{2}})} - G^n_{k,\ell}\right\} , \qquad 2.13$$

where

$$X_0 \equiv \frac{2\pi\sigma_0}{\sqrt{\epsilon_0}(\sqrt{\epsilon_0} - 1)}(\delta\tau^n + \delta\tau^{n-1})$$

for $\ell = L + 1, L + 2, \cdots, L + n_{Zs} + n_{Zf}$.

In Eqs. 2.10, 2.11, and 2.13 we have written only

$$\left.\frac{\partial G}{\partial r}\right)^n_{k,\ell} \quad\text{and}\quad \left.\frac{\partial F}{\partial r}\right)^n_{k,\ell}$$

for the differenced expressions of

$$\frac{\partial G}{\partial r} \quad\text{and}\quad \frac{\partial F}{\partial r} .$$

To make these expressions accurate to second order in δr in a nonuniform r-mesh, we use three mesh values and expand about r_k. Let f represent either F or G. Then

$$\left.\frac{\partial f}{\partial r}\right)_k = a_1 f_{k-1} + a_2 f_k + a_3 f_{k+1} ,$$

$$= a_1\left\{f_k - \delta_1\left.\frac{\partial f}{\partial r}\right)_k + \frac{\delta_1^2}{2}\left.\frac{\partial^2 f}{\partial r^2}\right)_k - \cdots\right\} + a_2 f_k$$

$$+ a_3\left\{f_k + \delta_2\left.\frac{\partial f}{\partial r}\right)_k + \frac{\delta_2^2}{2}\left.\frac{\partial^2 f}{\partial r^2}\right)_k + \cdots\right\} ,$$

$$= f_k(a_1 + a_2 + a_3) + (a_3\delta_2 - a_1\delta_1)\left.\frac{\partial f}{\partial r}\right)_k$$

$$+ \frac{1}{2}(a_1\delta_1^2 + a_3\delta_2^2)\left.\frac{\partial^2 f}{\partial r^2}\right)_k + \cdots ,$$

where $\delta_1 \equiv r_k - r_{k-1}$ and $\delta_2 \equiv r_{k+1} - r_k$. For second order, we want

$$(a_1 + a_2 + a_3) = 0 ,$$

$$a_3\delta_2 - a_1\delta_1 = 1 ,$$

and

$$a_1\delta_1^2 + a_3\delta_2^2 = 0 ;$$

i.e.,

$$a_1 = \frac{-\frac{\delta_2}{\delta_1}}{\delta_1 + \delta_2} ,$$

$$a_2 = \frac{\frac{\delta_2}{\delta_1} - \frac{\delta_1}{\delta_2}}{\delta_1 + \delta_2} ,$$

and

$$a_3 = \frac{\frac{\delta_1}{\delta_2}}{\delta_1 + \delta_2} ,$$

so that

$$\left.\frac{\partial f}{\partial r}\right)_k = (f_{k+1} - f_k)\,\frac{\frac{\delta_1}{\delta_2}}{\delta_1 + \delta_2} - \frac{\frac{\delta_2}{\delta_1}}{\delta_1 + \delta_2}\,(f_{k-1} - f_k) \;. \qquad 2.14$$

Note that for a uniform mesh we have $\delta_1 = \delta_2$ and

$$\left.\frac{\partial f}{\partial r}\right)_k = \frac{f_{k+1} - f_{k-1}}{r_{k+1} - r_{k-1}} \;,$$

as usual. Since the θ-mesh and Z-mesh are centered in Eqs. 2.10 and 2.11, the θ and Z derivatives are differenced in the usual manner; i.e.,

$$\left.\frac{\partial E}{\partial \theta}\right)^n_{k,\ell} = \frac{E^n_{k,\ell+\frac{1}{2}} - E^n_{k,\ell-\frac{1}{2}}}{\theta_{\ell+\frac{1}{2}} - \theta_{\ell-\frac{1}{2}}}$$

and

$$\left.\frac{\partial E}{\partial Z}\right)^n_{k,\ell} = \frac{E^n_{k,\ell+\frac{1}{2}} - E^n_{k,\ell-\frac{1}{2}}}{Z_{\ell+\frac{1}{2}} - Z_{\ell-\frac{1}{2}}} \;. \qquad 2.15$$

It should be noted that had one differenced Eq. 1.16 by the "form" shown by Eqs. 2.5, instead of using the "form" shown by Eq. 2.6 to obtain Eq. 2.12, the fields calculated in air would have been very inaccurate in the diffusion phase.

The E_r equations (1.15 and 1.18) contain θ (and Z) derivatives which cannot be centered at $\theta_{\ell+\frac{1}{2}}$ (and $Z_{\ell+\frac{1}{2}}$) in the nonuniform portions of the mesh. In the air the derivatives are centered at $\theta_c \equiv \frac{1}{2}(\theta_{\ell+1} + \theta_\ell)$ and in the ground they are centered at $Z_c = \frac{1}{2}(Z_{\ell+1} + Z_\ell)$. In Figure 2 we show an exaggerated portion of the nonuniform mesh. By linear interpolation (not extrapolation) we define $E^n_{k,c}$ in the air as

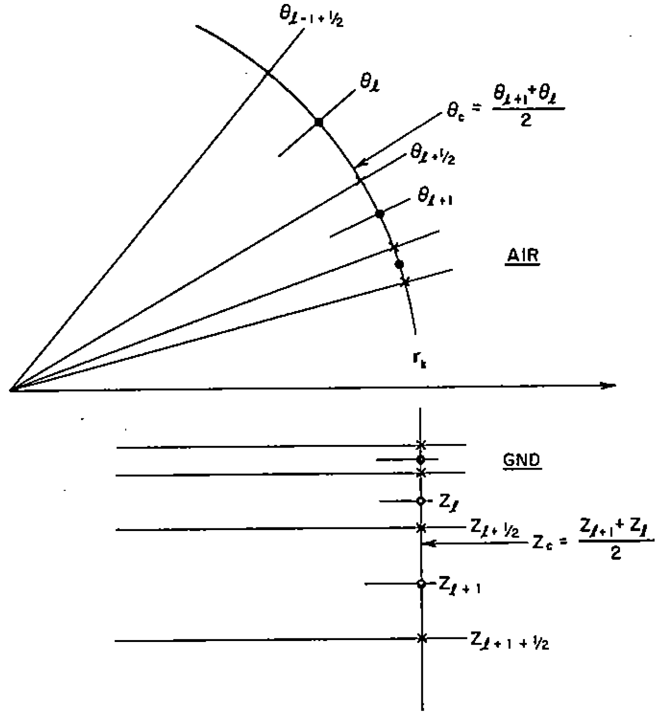

Fig. 2. The mesh for centering the E_r equation in the air and in the ground.

$$E^n_{k,c} = E^n_{k,\ell+\frac{1}{2}}\Delta_\theta + E^n_{k,\ell-1+\frac{1}{2}}(1 - \Delta_\theta) \;, \qquad 2.16$$

where

$$\Delta_\theta \equiv \frac{\theta_c - \theta_{\ell-1+\frac{1}{2}}}{\theta_{\ell+\frac{1}{2}} - \theta_{\ell-1+\frac{1}{2}}} \;.$$

Similarly, in the ground,

$$E^n_{k,c} = E^n_{k,\ell+\frac{1}{2}}(1 - \Delta_Z) + \Delta_Z E_{\ell+1+\frac{1}{2}} \;, \qquad 2.17$$

where

$$\Delta_Z \equiv \frac{Z_c - Z_{\ell+\frac{1}{2}}}{Z_{\ell+1+\frac{1}{2}} - Z_{\ell+\frac{1}{2}}} \;.$$

Note that $\Delta_\theta = 1$ and $\Delta_Z = 0$ in a uniform mesh. We then write the E_r difference equations as, for k = 2 to n_r:

$$\text{(air)} \quad E^{n+1}_{k,c} = E^{n-1}_{k,c}e^{-Y} + \frac{(1 - e^{-Y})}{4\pi\sigma}\left[-4\pi J_r + \frac{\sin\theta_{\ell+1}\left\{F^{n+1}_{k,\ell+1} + F^{n-1}_{k,\ell+1} - 2G^n_{k,\ell+1}\right\} - \sin\theta_\ell\left\{F^{n+1}_{k,\ell} + F^{n-1}_{k,\ell} - 2G^n_{k,\ell}\right\}}{4r_k^2 \sin\theta_c(\theta_{\ell+1} - \theta_\ell)}\right] , \qquad 2.18$$

where $Y \equiv 4\pi\sigma(\delta\tau^n + \delta\tau^{n-1})$, $\sigma = \sigma^n_{k,c}$, and $J_r = J^n_{r_{k,c}}$

for ℓ = 1 to L - 1 and

$$\text{(gnd)}\quad E^{n+1}_{k,c} = E^{n-1}_{k,c}e^{-Y_0} + \frac{(1 - e^{-Y_0})}{4\pi\sigma_0}\left[\frac{F^{n+1}_{k,\ell+1} + F^{n-1}_{k,\ell+1} - 2G^n_{k,\ell+1} - F^{n+1}_{k,\ell} - F^{n-1}_{k,\ell} + 2G^n_{k,\ell}}{4r_k(Z_{\ell+1} - Z_\ell)}\right], \qquad 2.19$$

where $Y_0 = 4\pi\sigma_0/\varepsilon_0(\delta\tau^n + \delta\tau^{n-1})$ for ℓ = L + 1, L + 2, ···, L + n_{Zs} + n_{Zf} - 1.

To calculate E_r on the ground ($E^{n+1}_{r_{k,L+\frac{1}{2}}}$) we difference the E_r equation twice, once in the air and once in the ground, and eliminate B_g ($\equiv B_\varphi$ on the ground) from the two resulting centered equations. Both E_r and B_φ are continuous at the ground, though their θ (or Z) derivatives are not continuous. See Figure 3 for the mesh near the ground. The two equations are, for k = 2 to n_r.

$$E^{n+1}_{k,L+\frac{1}{4}} = E^{n-1}_{k,L+\frac{1}{4}}e^{-Y_a} + \frac{1 - e^{-Y_a}}{4\pi\sigma_a}\left\{-4\pi J^n_{r_{k,L+\frac{1}{4}}} + \frac{B_g - \frac{\sin\theta_L}{4r_k}\left(F^{n+1}_{k,L} + F^{n-1}_{k,L} - 2G^n_{k,L}\right)}{r_k \sin\theta_{L+\frac{1}{4}}(\theta_{L+\frac{1}{2}} - \theta_L)}\right\}, \qquad 2.20$$

where $Y_a = 4\pi\sigma_a(\delta\tau^n + \delta\tau^{n-1})$ and $\sigma_a = \sigma^n_{k,L+\frac{1}{4}}$, and

$$E^{n+1}_{k,L+\frac{3}{4}} = E^{n-1}_{k,L+\frac{3}{4}}e^{-Y_0} + \frac{1 - e^{-Y_0}}{4\pi\sigma_0}\left\{\frac{\frac{1}{4r_k}\left(F^{n+1}_{k,L+1} + F^{n-1}_{k,L+1} - 2G^n_{k,L+1}\right) - B_g}{Z_{L+1} - Z_{L+\frac{1}{2}}}\right\}, \qquad 2.21$$

where Y_0 is as above. Since mesh values of E_r are not carried at L + $\frac{1}{4}$ and L + $\frac{3}{4}$, the values are eliminated by linear interpolation in the mesh, i.e.

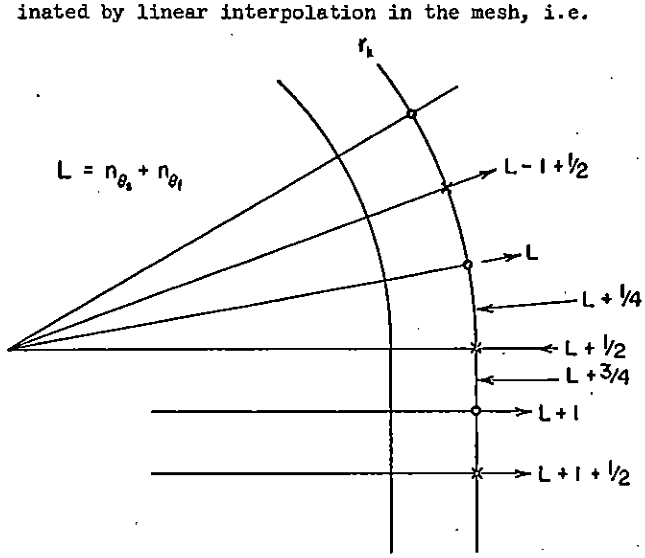

Fig. 3. The mesh for calculation of E_r on the ground.

$$E^n_{k,L+\frac{1}{4}} = E^n_{k,L-1+\frac{1}{2}} + \Delta_a\left(E^n_{k,L+\frac{1}{2}} - E^n_{k,L-1+\frac{1}{2}}\right), \qquad 2.22$$

where

$$\Delta_a \equiv \frac{\theta_{L+\frac{1}{4}} - \theta_{L-1+\frac{1}{2}}}{\theta_{L+\frac{1}{2}} - \theta_{L-1+\frac{1}{2}}},$$

and

$$E^n_{k,L+\frac{3}{4}} = E^n_{k,L+1+\frac{1}{2}} + \Delta_g\left(E^n_{k,L+\frac{1}{2}} - E^n_{k,L+1+\frac{1}{2}}\right), \qquad 2.23$$

where

$$\Delta_g \equiv \frac{Z_{L+1+\frac{1}{2}} - Z_{L+\frac{3}{4}}}{Z_{L+1+\frac{1}{2}} - Z_{L+\frac{1}{2}}}.$$

Solving each of the equations (2.20 and 2.21) for B_g, using Eqs. 2.22 and 2.23 and omitting the subscript k, we have,

$$B_g = \frac{\sin\theta_L}{4r_k}\left(F^{n+1}_L + F^{n-1}_L - 2G^n_L\right) + r_k \sin\theta_{L+\frac{1}{4}}(\theta_{L+\frac{1}{2}} - \theta_L)\left[4\pi J^n_{r_{L+\frac{1}{4}}} + \frac{4\pi\sigma_a}{1 - e^{-Y_a}}\left\{E^{n+1}_{L-1+\frac{1}{2}}(1 - \Delta_a) + \Delta_a E^{n+1}_{L+\frac{1}{2}} - e^{-Y_a}E^{n-1}_{L-1+\frac{1}{2}}(1 - \Delta_a) - e^{-Y_a}\Delta_a E^{n-1}_{L+\frac{1}{2}}\right\}\right]; \qquad 2.24$$

and

$$B_g = \frac{1}{4r_k}\left(F^{n+1}_{L+1} + F^{n-1}_{L+1} - 2G^{n}_{L+1}\right) - \frac{4\pi\sigma_0(Z_{L+1} - Z_{L+\frac{1}{2}})}{1 - e^{-Y_0}}\left[E^{n+1}_{L+1+\frac{1}{2}}(1-\Delta_g) + \Delta_g E^{n+1}_{L+\frac{1}{2}} - e^{-Y_0}E^{n-1}_{L+1+\frac{1}{2}}(1-\Delta_g) - e^{-Y_0}\Delta_g E^{n-1}_{L+\frac{1}{2}}\right] . \quad 2.25$$

Equating Eqs. 2.24 and 2.25 we obtain an equation for each k,

$$H_{L-\frac{1}{2}}E^{n+1}_{L-1+\frac{1}{2}} + H_L F^{n+1}_L + H_{L+\frac{1}{2}}E^{n+1}_{L+\frac{1}{2}} + H_{L+1}F^{n+1}_{L+1} + H_{L+\frac{3}{2}}E^{n+1}_{L+1+\frac{1}{2}} = \overline{H} , \quad 2.26$$

where we have defined

$$\left.\begin{aligned}
H_{L-\frac{1}{2}} &= \frac{(1-\Delta_a)4\pi\sigma_a r_k \sin\theta_{L+\frac{1}{2}}(\theta_{L+\frac{1}{2}} - \theta_L)}{1 - e^{-Y_a}} \\
H_L &= \frac{\sin\theta_L}{4r_k} \\
H_{L+\frac{1}{2}} &= \frac{\Delta_a 4\pi\sigma_a r_k \sin\theta_{L+\frac{1}{2}}(\theta_{L+\frac{1}{2}} - \theta_L)}{1 - e^{-Y_a}} + \frac{\Delta_g 4\pi\sigma_0(Z_{L+1} - Z_{L+\frac{1}{2}})}{1 - e^{-Y_0}} \\
H_{L+1} &= -\frac{1}{4r_k} \\
H_{L+\frac{3}{2}} &= \frac{(1-\Delta_g)4\pi\sigma_0(Z_{L+1} - Z_{L+\frac{1}{2}})}{1 - e^{-Y_0}} \\
\overline{H} &= -\frac{\sin\theta_L}{4r_k}\left(F^{n-1}_{k,L} - 2G^{n}_{k,L}\right) + \frac{1}{4r_k}\left(F^{n-1}_{k,L+1} - 2G^{n}_{k,L+1}\right) - \sin\theta_{L+\frac{1}{2}}r_k(\theta_{L+\frac{1}{2}} - \theta_L)\Bigg[4\pi J^{n}_{r_{k,L+\frac{1}{2}}} \\
&\quad - \frac{4\pi\sigma_a e^{-Y_a}}{1 - e^{-Y_a}}\left\{E^{n-1}_{k,L-1+\frac{1}{2}}(1-\Delta_a) + \Delta_a E^{n-1}_{k,L+\frac{1}{2}}\right\}\Bigg] \\
&\quad + \frac{4\pi\sigma_0(Z_{L+1} - Z_{L+\frac{1}{2}})e^{-Y_0}}{1 - e^{-Y_0}}\left\{E^{n-1}_{k,L+1+\frac{1}{2}}(1-\Delta_g) + \Delta_g E^{n-1}_{k,L+\frac{1}{2}}\right\}
\end{aligned}\right\} . \quad 2.27$$

The sixth step in a cycle is that of solving for F and E_r implicitly for all ℓ at each k. To this end, we first write the F and E_r equations in the forms (deleting n + 1 and k from E_r and F),

$$F_\ell + B_\ell E_{\ell+\frac{1}{2}} + C_\ell E_{\ell-1+\frac{1}{2}} = D_\ell \,, \qquad 2.28$$

for ℓ = 2, 3, ···, L and ℓ = L + 1, L + 2, ···, L + n_{Zs} + n_{Zf}, and

$$E_{\ell+\frac{1}{2}} + \left(\frac{1-\Delta_\theta}{\Delta_\theta}\right) E_{\ell-1+\frac{1}{2}} + \overline{B}_\ell F_{\ell+1} + \overline{C}_\ell F_\ell = \overline{D}_\ell \qquad 2.29$$

for ℓ = 1 to L - 1, and

$$E_{\ell+\frac{1}{2}} + \frac{\Delta_Z}{1-\Delta_Z} E_{\ell+1+\frac{1}{2}} + \overline{B}_\ell F_{\ell+1} + \overline{C}_\ell F_\ell = \overline{D}_\ell \qquad 2.30$$

for ℓ = L + 1 to L + n_{Zs} + n_{Zf} - 1, where we define (from Eqs. 2.12, 2.13, and 2.16 to 2-19)

$$\left.\begin{aligned}
B_\ell &= -\frac{\left[1-\frac{1}{X}(1-e^{-X})\right]}{2\pi\sigma^{n+1}_{k,\ell}(\theta_{\ell+\frac{1}{2}}-\theta_{\ell-\frac{1}{2}})} \\
C_\ell &= -B_\ell \\
D_\ell &= e^{-X}F^{n+1}_{k-1,\ell} + \left[-e^{-X}+\frac{1}{X}(1-e^{-X})\right]\alpha^{n+1}_{k-1,\ell} \\
&\quad - \left[1-\frac{1}{X}(1-e^{-X})\right]\left[\frac{2r_k J^{n+1}_{\theta k,\ell}}{\sigma^{n+1}_{k,\ell}} + G^{n+1}_{k,\ell}\right]
\end{aligned}\right\}, \quad 2.31$$

for ℓ = 2 to L, and

$$\left.\begin{aligned}
B_\ell &= -\frac{(1-e^{-X_0})\epsilon_0 r_k}{4\pi\sigma_0(Z_{\ell+\frac{1}{2}}-Z_{\ell-\frac{1}{2}})} \\
C_\ell &= -B_\ell \\
D_\ell &= F^{n-1}_{k,\ell}e^{-X_0} + (1-e^{-X_0})\left\{\left(\frac{-\sqrt{\epsilon_0}}{2\pi\sigma_0}\frac{\partial F}{\partial r}\right)^n_{k,\ell}\right. \\
&\quad \left. + \frac{\epsilon_0 r_k\left[E^{n-1}_{k,\ell+\frac{1}{2}} - E^{n-1}_{k,\ell-1+\frac{1}{2}}\right]}{4\pi\sigma_0(Z_{\ell+\frac{1}{2}}-Z_{\ell-\frac{1}{2}})} - G^n_{k,\ell}\right\}
\end{aligned}\right\}, \quad 2.32$$

for ℓ = L + 1 to L + n_{Zs} + n_{Zf} and ($\sigma = \sigma^n_{k,c}$), and

$$\left.\begin{aligned}
\overline{B}_\ell &= -\frac{(1-e^{-Y})\sin\theta_{\ell+1}}{16\pi\sigma\Delta_\theta r_k^2 \sin\theta_c(\theta_{\ell+1}-\theta_\ell)} \\
\overline{C}_\ell &= \frac{(1-e^{-Y})\sin\theta_\ell}{16\pi\sigma\Delta_\theta r_k^2 \sin\theta_c(\theta_{\ell+1}-\theta_\ell)} \\
\overline{D}_\ell &= e^{-Y}\left[E^{n-1}_{k,\ell+\frac{1}{2}} + \frac{(1-\Delta_\theta)}{\Delta_\theta}E^{n-1}_{k,\ell-1+\frac{1}{2}}\right] \\
&\quad - \frac{(1-e^{-Y})J_r}{\sigma\Delta_\theta} - \overline{B}_\ell\left[F^{n-1}_{k,\ell+1} - 2G^n_{k,\ell+1}\right] \\
&\quad - \overline{C}_\ell\left[F^{n-1}_{k,\ell} - 2G^n_{k,\ell}\right]
\end{aligned}\right\}, \quad 2.33$$

for ℓ = 1, L - 1, and

$$\left.\begin{aligned}
\overline{B}_\ell &= -\frac{(1-e^{-Y_0})}{16\pi\sigma_0 r_k(Z_{\ell+1}-Z_\ell)(1-\Delta_Z)} \\
\overline{C}_\ell &= -\overline{B}_\ell \\
\overline{D}_\ell &= e^{-Y_0}\left[E^{n-1}_{k,\ell-\frac{1}{2}} + \frac{\Delta_Z}{(1-\Delta_Z)}E^{n-1}_{k,\ell+1+\frac{1}{2}}\right] \\
&\quad + \overline{C}_\ell\left[F^{n-1}_{k,\ell+1} - 2G^n_{k,\ell+1} - F^{n-1}_{k,\ell} + 2G^n_{k,\ell}\right]
\end{aligned}\right\}, \quad 2.34$$

for ℓ = L + 1 to L + n_{Zs} + n_{Zf} - 1. Thus Eq. 2.28 is Eq. 2.12 when Eqs. 2.31 are inserted, and is Eq. 2.13 when Eqs. 2.32 are inserted. Equations 2.29 and 2.33 give Eq. 2.18, and Eqs. 2.30 and 2.34 give Eq. 2.19. We now use Eq. 2.28 to eliminate $F^{n+1}_{k,\ell}$ and $F^{n+1}_{k,\ell+1}$ from Eqs. 2.26, 2.29, and 2.30 and find

$$E_{\ell+\frac{1}{2}}\left[1-\overline{B}_\ell C_{\ell+1}-\overline{C}_\ell B_\ell\right]+E_{\ell+1+\frac{1}{2}}\left[-\overline{B}_\ell B_{\ell+1}\right]+E_{\ell-1+\frac{1}{2}}\left[-\overline{C}_\ell C_\ell+\frac{1-\Delta_\theta}{\Delta_\theta}\right]=\left[\overline{D}_\ell-\overline{B}_\ell D_{\ell+1}-\overline{C}_\ell D_\ell\right]$$

for $\ell = 1$ to $L - 1$, and

$$E_{\ell+\frac{1}{2}}\left[-H_L B_\ell+H_{L+\frac{1}{2}}-H_{L+1}C_{\ell+1}\right]+E_{\ell+1+\frac{1}{2}}\left[-H_{L+1}B_{\ell+1}+H_{L+\frac{3}{2}}\right]+E_{\ell-1+\frac{1}{2}}\left[H_{L-\frac{1}{2}}-H_L C_\ell\right]$$

$$=\left[\overline{H}-H_L D_\ell-H_{L+1}D_{\ell+1}\right] \quad . \quad 2.35$$

for $\ell = L$, and

$$E_{\ell+\frac{1}{2}}\left[1-\overline{B}_\ell C_{\ell+1}-\overline{C}_\ell B_\ell\right]+E_{\ell+1+\frac{1}{2}}\left[-\overline{B}_\ell B_{\ell+1}+\frac{\Delta_Z}{1-\Delta_Z}\right]+E_{\ell-1+\frac{1}{2}}\left[-\overline{C}_\ell C_\ell\right]=\left[\overline{D}_\ell-\overline{B}_\ell D_{\ell+1}-\overline{C}_\ell D_\ell\right]$$

for $\ell = L + 1$ to $L + n_{Zs} + n_{Zf} - 1$

We write

$$E_{\ell+\frac{1}{2}}A_\ell + E_{\ell+1+\frac{1}{2}}A_{2_\ell} + E_{\ell-1+\frac{1}{2}}A_{3_\ell} = A_{4_\ell} , \quad 2.36$$

for $\ell = 1$ to $L + n_{Zs} + n_{Zf} - 1$ where these A's are defined by Eqs. 2.35. Next define e_ℓ and f_ℓ by

$$E_{\ell+\frac{1}{2}} = e_\ell E_{\ell+1+\frac{1}{2}} + f_\ell , \quad 2.37$$

for $\ell = 1$ to $L + n_{Zs} + n_{Zf} - 1$. Then,

$$E_{\ell-1+\frac{1}{2}} = e_{\ell-1}E_{\ell+\frac{1}{2}} + f_{\ell-1} ,$$

substituted into Eq. 2.36 gives

$$E_{\ell+\frac{1}{2}} = \left(\frac{-A_{2\ell}}{A_\ell + A_{3\ell}e_{\ell-1}}\right)E_{\ell+1+\frac{1}{2}} + \left(\frac{A_{4\ell} - A_{3\ell}f_{\ell-1}}{A_\ell + A_{3\ell}e_{\ell-1}}\right) . \quad 2.38$$

Comparing Eqs. 2.37 and 2.38 gives the recursion relation for e_ℓ and f_ℓ; i.e.,

$$\left.\begin{aligned} e_\ell &= \frac{-A_{3\ell}}{A_\ell + A_{3\ell}C_{\ell-1}} \\ f_\ell &= \frac{A_{4\ell} - A_{3\ell}f_{\ell-1}}{A_\ell + A_{3\ell}e_{\ell-1}} \end{aligned}\right\} . \quad 2.39$$

Our boundary condition at $\ell = 1$ gives $F_1 = 0$; i.e., for an equation like Eq. 2.28 we have $\overline{B}_1 = C_1 = D_1 = 0$. Substituting this condition, and remembering that $\Delta_\theta = 1$ where the mesh is uniform, into the first of Eqs. 2.35 we have

$$E_{1+\frac{1}{2}} = \frac{\overline{B}_1 B_2}{1 - \overline{B}_1 C_2} E_{2+\frac{1}{2}} + \frac{\overline{D}_1 - \overline{B}_1 D_2}{1 - \overline{B}_1 C_2} , \quad 2.40$$

or

$$\left.\begin{aligned} e_1 &= \frac{\overline{B}_1 B_2}{1 - \overline{B}_1 C_2} \\ f_1 &= \frac{\overline{D}_1 - \overline{B}_1 D_2}{1 - \overline{B}_1 C_2} \end{aligned}\right\} . \quad 2.41$$

With Eqs. 2.41, one can iterate Eqs. 2.39 for e_ℓ and f_ℓ from $\ell = 2$ to $L + n_{Zs} + n_{Zf} - 1$. Our boundary condition in the ground is that $E_{L+n_{Zs}+n_{Zf}} = 0$ for all k and n; i.e., to run a problem we select n_{Zs}, n_{Zf}, and Z_0 so that the mesh goes to such a depth that the fields will not diffuse to this depth by the latest time desired for this problem. With this boundary condition and the values of e_ℓ and f_ℓ, we iterate (backward) by Eq. 2.37 to find $E^{n+1}_{k,\ell+\frac{1}{2}}$ for $\ell = L + n_{Zs} + n_{Zf} - 1$ to 1. Then, using these values of E_r and B_ℓ, C_ℓ, and D_ℓ as calculated before, one solves Eq. 2.28 for values of $F^{n+1}_{k,\ell}$ from $\ell = 2$ to $L + n_{Zs} + n_{Zf}$ to complete the implicit solution of E_r and F for all ℓ at each r_k.

2.6 Choice of Time Step

Step 2, the calculation of $\delta\tau$ for the next cycle, is done by selecting the smallest of three different time increments. The first of the time increments, $\delta\tau_1$, is determined during the air-ion equations calculation. Here, for each mesh point a number, $\xi^{n+1}_{k,\ell}$, is formed as

$$\xi^{n+1}_{k,\ell} = 0.3 - 20.0\left(\sigma^{n+1}_{k,\ell}\delta r_k\right)^2 + 2.25\left(\frac{\delta r_k}{r_{k+\frac{1}{2}}\delta\theta_\ell}\right)^2 , \qquad 2.42$$

where $\delta r_k = r_{k+1} - r_k$, $r_{k+\frac{1}{2}} = \frac{1}{2}(r_{k+1} + r_k)$, and $\delta\theta_\ell = \theta_{\ell+1} - \theta_\ell$. If $\xi^{n+1}_{k,\ell} \leq 0$, nothing is done. If $\xi^{n+1}_{k,\ell} > 0$, then a time increment equal to

$$f_s \frac{\delta r_k}{\sqrt{\xi^{n+1}_{k,\ell}}}$$

is formed. $\delta\tau_1$ is the smallest such time increment formed when k and ℓ vary over the entire (air) mesh. Here $f_s \leq 1$ is called the fraction of stability and is an input number. This method of determining a $\delta\tau$ is the result of an empirical study as given in the Richtmyer paper referenced above. For the ground we set

$$\delta\tau_c = f_s \delta Z_{min} \sqrt{\epsilon_0} \qquad \text{(Courant)} \qquad 2.43$$

and

$$\delta\tau_d = f_3 2\pi\sigma_0 \sqrt{\epsilon_0}\, \delta Z^2_{min} \qquad \text{(diffusion)} , \qquad 2.44$$

where $\delta Z_{min} = Z_0/2$. If $2\pi\delta Z_{min}\sigma_0 \leq 1$, then $\delta\tau_2 = \delta\tau_c$. Otherwise, $\delta\tau_2 = \delta\tau_d$ for our second time increment. Last, if a source function is e-folding in a time $1/\beta$, we form

$$\delta\tau_3 = \frac{f_{acc}}{\beta} , \qquad 2.45$$

where $f_{acc} \leq 1$ is called the fraction of accuracy and is an input number. The $\delta\tau$ for the next cycle is the smallest of $\delta\tau_1$, $\delta\tau_2$, and $\delta\tau_3$. This smallest value is also restricted to vary by no more than ±2% of $\delta\tau^n$ each cycle.

2.7 The Inner Boundary Condition

Step 4, the calculations of the mesh values at k = 1, is done in several steps. First, the G equations (2.10 and 2.11) are solved for $G^{n+1}_{1,\ell}$ for $\ell = 2$ to $L + n_{Zs} + n_{Zf}$. In this operation we approximate (off-center)

$$\left.\frac{\partial G}{\partial r}\right)^n_{1,\ell}$$

by

$$\left.\frac{\partial G}{\partial r}\right)^n_{1,\ell} = \frac{G^n_{2,\ell} - G^n_{1,\ell}}{r_2 - r_1} . \qquad 2.46$$

Because of our boundary condition that F = -G ($E_\theta = 0$) at $r = r_0 = r_1$, we now know $F^{n+1}_{1,\ell}$ for $\ell = 2$ to $L + n_{Zs} + n_{Zf}$. There is thus no need to solve the F and E_r equations implicitly as was the case in step 6. The new values of E_r are determined in three steps. First $E^{n+1}_{1,\ell+\frac{1}{2}}$ for $\ell = 1$ to $L - 1$ may be solved (since $\Delta_\theta = 1$ at $\ell = 1$ in a uniform mesh) by Eqs. 2.16 and 2.18. Second, $E^{n+1}_{1,\ell+\frac{1}{2}}$ for $\ell = L + n_{Zs} + n_{Zf} - 1$ to $L + 1$ ($\Delta_Z = 0$ in uniform mesh) may be solved by Eqs. 2.17 and 2.19 in the backward direction indicated. Third, $E_{L+\frac{1}{2}}$ is found by solving Eq. 2.36. One should note that since $F^{n-1}_{1,\ell}$ is needed in the calculation of E_r, F should not be "back stored" until after the new values of E_r have been calculated.

2.8 The Outer Boundary Condition

The theory of the outer boundary condition has been discussed in Chapter 1.5. We now formulate this theory in difference equations for use by the code in providing an outer boundary condition for G, and in calculating output values of the fields at radii beyond the boundary.

The standard difference equations for F and G, Eqs. 2.10 to 2.13, are used for k = 2 to $n_r - 1$. In the ground, to advance in time F and G at $k = n_r - 1$, we need values of F and G at $k = n_r$; these are set equal to zero, in accordance with the assumption that the ground conductivity is infinite beyond the boundary. In the air, we do not need a value for F at $k = n_r$, but we do need a value for G at $k = n_r$.

We first pick out the ℓ = 1, 3, and 5 parts of G. For brevity let

$$G(\theta_m) = G(r_{n_r-1}, \theta_m, \tau^n) , \qquad 2.47$$

where θ_m are the angles at which G is carried. Then the ℓ = 1, 3, 5 parts of G are found from

$$\left.\begin{aligned} g_1 &= \sum_{m=2}^{L} q(\ell,m)G(\theta_m) \\ g_3 &= \sum_{m=2}^{L} q(3,m)G(\theta_m) \\ g_5 &= \sum_{m=2}^{L} q(5,m)G(\theta_m) \end{aligned}\right\} . \qquad 2.48$$

Here the $q(\ell,m)$ are the adjoint functions, over our θ-mesh, of the appropriate spherical harmonics. We take these spherical harmonics to be $-\partial/\partial\theta\ P_\ell(\cos\theta)$, and have

$$\left.\begin{aligned} Y_1(\theta) &= \sin\theta \\ Y_3(\theta) &= \frac{1}{2}\sin\theta[12 - 15\sin^2\theta] \\ Y_5(\theta) &= \frac{1}{8}\sin\theta[120 - 420\sin^2\theta + 315\sin^4\theta] \end{aligned}\right\} . \qquad 2.49$$

The adjoint functions are calculated during the setup part of the code, and are changed during a mesh change. They are calculated on the assumption that $G(\theta)$ is linearly interpolated between θ_m and θ_{m+1}. On this assumption, each point θ_m is allowed to contribute a triangular part to $G(\theta)$, as indicated in Fig. 4. The $q(\ell,m)$ are then found by integrating these triangular functions against the appriately normalized spherical harmonics. The results are as follows: define

$$L = n_{\theta f} + n_{\theta s} . \qquad 2.50$$

Then for m = 2, 3, 4, $\cdots$, L - 2

$$q(\ell,m) = \frac{2\ell+1}{\ell(\ell+1)}\left[\frac{1}{\theta_m - \theta_{m-1}}\left\{\varphi_\ell(\theta_m) - \varphi_\ell(\theta_{m-1}) - \theta_{m-1}\Big(\psi_\ell(\theta_m) - \psi_\ell(\theta_{m-1})\Big)\right\} + \frac{1}{\theta_{m+1} - \theta_m}\left\{\theta_{m+1}\Big(\psi_\ell(\theta_{m+1}) - \psi_\ell(\theta_m)\Big) - \varphi_\ell(\theta_{m+1}) + \varphi_\ell(\theta_m)\right\}\right] .$$

For m = L - 1 only

$$q(\ell,m) = \frac{2\ell+1}{\ell(\ell+1)}\left[\frac{1}{\theta_m - \theta_{m-1}}\left\{\varphi_\ell(\theta_m) - \varphi_\ell(\theta_{m-1}) - \theta_{m-1}\Big(\psi_\ell(\theta_m) - \psi_\ell(\theta_{m-1})\Big)\right\} + \frac{1}{\theta_{m+1} - \theta_m}\left\{\theta_{m+1}\left(\psi_\ell\left(\frac{\pi}{2}\right) - \psi_\ell(\theta_m)\right) - \varphi_\ell\left(\frac{\pi}{2}\right) + \varphi_\ell(\theta_m)\right\}\right] .$$

For m = L only

$$q(\ell,m) = \frac{2\ell+1}{\ell(\ell+1)}\left[\frac{1}{\theta_m - \theta_{m-1}}\left\{\varphi_\ell\left(\frac{\pi}{2}\right) - \varphi_\ell(\theta_{m-1}) - \theta_{m-1}\left(\psi_\ell\left(\frac{\pi}{2}\right) - \psi_\ell(\theta_{m-1})\right)\right\}\right] .$$

(2.51)

Here the functions ψ_ℓ and θ_ℓ are defined by

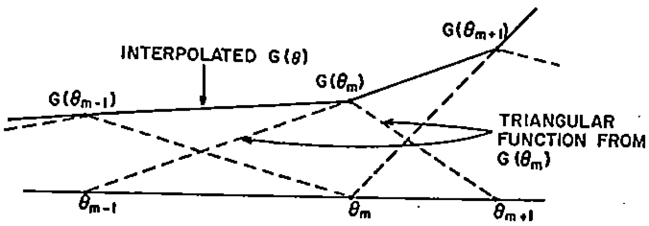

Fig. 4. Analysis of interpolated $G(\theta)$ as a sum of triangular functions.

$$\left.\begin{aligned} B\ell 1 &\equiv \frac{1}{r_1^3} I_3(b_\ell) = \frac{1}{r^1}\int I_2(b_\ell(\tau))d\tau \\ B\ell 2 &\equiv \frac{1}{r_1^4} I_4(b_\ell) = \frac{1}{r^1}\int B\ell 1(\tau)d\tau \\ &\text{etc.} \end{aligned}\right\}, \qquad 2.53$$

$$\left.\begin{aligned} \psi_1(\theta) &= \frac{1}{4}[2\theta - \sin(2\theta)] \\ \varphi_1(\theta) &= \frac{1}{8}[2\theta^2 - 2\theta\sin(2\theta) - \cos(2\theta)] \\ \psi_3(\theta) &= \frac{3}{64}[4\theta + 8\sin(2\theta) - 5\sin(4\theta)] \\ \varphi_3(\theta) &= \frac{3}{256}[8\theta^2 + 32\theta\sin(2\theta) + 16\cos(2\theta) - 20\,\theta\sin(4\theta) - 5\cos(4\theta)] \\ \psi_5(\theta) &= \frac{15}{512}[4\theta + 5\sin(2\theta) + 7\sin(4\theta) - 7\sin(6\theta)] \\ \varphi_5(\theta) &= \frac{5}{2048}[24\theta^2 + 60\theta\sin(2\theta) + 30\cos(2\theta) + 84\sin(4\theta) + 21\cos(4\theta) - 84\theta\sin(6\theta) - 14\cos(6\theta)] \end{aligned}\right\}. \qquad 2.52$$

This completes the formulae needed to determine the adjoint functions $q(\ell,m)$.

Return now to the g_ℓ found from Eq. 2.48. The next step is to calculate the $b_\ell(\tau)$ of Section 1.5. For convenience in the code, we modify the notation of that section, starting from Eq. 1.47 First define

$$r_1 \equiv r_{n_r-1} ,$$

the maximum radius at which the difference equations for G are solved. Then define

$$B\ell \equiv \frac{1}{r_1^2} I_2(b_\ell)$$

and the integrals

and also the derivatives

$$B\ell A \equiv \frac{1}{r_1} I_1(b_\ell) = r_1 \frac{\partial}{\partial\tau} I_2(b_\ell) ,$$

and

$$B\ell B \equiv b_\ell = r_1 \frac{\partial}{\partial\tau}(B\ell A) .$$

We have to determine the $B\ell$ quantities from the g_ℓ. For $\ell = 1$ the problem is simple; from Eq. 1.40,

$$B1N = g_1 , \qquad 2.54$$

where we use an additional letter N to indicate the new value of the quantity whose old value is B1. Next

$$\left.\begin{aligned} B1AN &= r_1(B1N - B1)\delta\tau^n \\ B1B &= 2r_1(B1AN - B1A)/(\delta\tau^n + \delta\tau^{n-1}) \end{aligned}\right\}. \qquad 2.55$$

At this point the new quantities are stored in place of the old.

For $\ell = 3$ we have to solve Eq. 1.46, or 1.43. First we calculate preliminary values (indicated by final letter P) of the first and second integrals of B3:

$$\left.\begin{aligned} B31P &= B31 + B3\delta\tau^n/r_1 \\ B32P &= B32 + \frac{1}{2}(B31 + B31P)\delta\tau^n/r_1 \end{aligned}\right\} . \quad 2.56$$

The preliminary value of B3 is calculated, from Eq. 1.46,

$$B3P = \frac{1}{6}\, g_3 - 5B31P - \frac{15}{2}\, B32P . \quad 2.57$$

Using the preliminary values, we calculate final new values,

$$\left.\begin{aligned} B31N &= B31 + \frac{1}{2}(B3 + B3P)\delta\tau^n/r_1 \\ B32N &= B32 + \frac{1}{2}(B31 + B31N)\delta\tau^n/r_1 \\ B3N &= \frac{1}{6}\, g_3 - 5B31N - \frac{15}{2}\, B32N \end{aligned}\right\} . \quad 2.58$$

This two-step iteration gives second-order accuracy. Next, one calculates the derivatives

$$\left.\begin{aligned} B3AN &= r_1(B3N - B3)/\delta\tau^n \\ B3B &= 2r_1(B3AN - B3A)/(\delta\tau^n + \delta\tau^{n-1}) \end{aligned}\right\} , \quad 2.59$$

and, finally, stores the new quantities in place of the old.

For $\ell = 5$ the procedure is similar to that for $\ell = 3$, except there are more terms. The preliminary values are:

$$\left.\begin{aligned} B51P &= B51 + B5\delta\tau^n/r_1 \\ B52P &= B52 + \frac{1}{2}(B51 + B51P)\delta\tau^n/r_1 \\ B53P &= B53 + \frac{1}{2}(B52 + B52P)\delta\tau^n/r_1 \\ B54P &= B54 + \frac{1}{2}(B53 + B53P)\delta\tau^n/r_1 \\ B5P &= \frac{1}{15}\, g_5 - 14B51P - 84B52P - 252B53P \\ &\quad - 315B54P \end{aligned}\right\} . \quad 2.60$$

The new values are:

$$\left.\begin{aligned} B51N &= B51 + \frac{1}{2}(B5 + B5P)\delta\tau^n/r_1 \\ B52N &= B52 + \frac{1}{2}(B51 + B51N)\delta\tau^n/r_1 \\ B53N &= B53 + \frac{1}{2}(B52 + B52N)\delta\tau^n/r_1 \\ B54N &= B54 + \frac{1}{2}(B53 + B53N)\delta\tau^n/r_1 \\ B5N &= \frac{1}{15}\, g_5 - 14B51N - 84B52N - 252B53N \\ &\quad - 315B54N \\ B5AN &= r_1(B5N - B5)/\delta\tau^n \\ B5B &= 2r_1(B5AN - B5A)/(\delta\tau^n + \delta\tau^{n-1}) \end{aligned}\right\} . \quad 2.61$$

Finally, the new values are stored in place of the old.

It will be noticed that the derivatives $B\ell A$ and $B\ell B$ are not centered in time at the same time as the other quantities. These derivatives are not used in the boundary condition for G, but only in the extrapolation of the fields to large distances, which does not affect the main part of the calculation.

For the boundary condition on G, define

$$\eta \equiv r_{n_r-1}/r_{n_r} . \quad 2.62$$

Then calculate

$$\left.\begin{aligned} D1 &\equiv 30(1 - \eta)B31 + 45(1 - \eta^2)B32 \\ D2 &\equiv 210(1 - \eta)B51 + 1260(1 - \eta^2)B52 \\ &\quad + 3780(1 - \eta^3)B53 + 4725(1 - \eta^4)B54 \end{aligned}\right\} . \quad 2.63$$

Then the extrapolated value of G at r_{n_r} is

$$G(r_{n_r}, \theta_m) = \eta^2[G(r_1, \theta_m) - Y_3(\theta_m)D1 - Y_5(\theta_m)D2] . \quad 2.64$$

The extrapolation of the fields to large distances is not done every cycle in LEMP, but only at those times when output is stored on the output tape. For this extrapolation we define some additional D's. If the extrapolated radius is r, let

$$\eta \equiv r_{n_r-1}/r \ . \qquad 2.65$$

Then the extrapolated $G(r,\theta_m)$ is given by Eqs. 2.64 and 2.63 where the present η is to be used. For F and E_r, define

$$\left.\begin{aligned}
D3 &\equiv 2(1-\eta)B1A + (1-\eta^2)B1 \\
D4 &\equiv 12(1-\eta)B3A + 36(1-\eta^2)B3 + 60(1-\eta^3)B31 + 45(1-\eta^4)B32 \\
D5 &\equiv 30(1-\eta)B5A + 225(1-\eta^2)B5 + 1050(1-\eta^3)B51 + 3150(1-\eta^4)B52 + 5670(1-\eta^5)B53 + 4725(1-\eta^6)B54 \\
D6 &\equiv 2(1-\eta)B1 \\
D7 &\equiv 72(1-\eta)B3 + 180(1-\eta^2)B31 + 180(1-\eta^3)B32 \\
D8 &\equiv 450(1-\eta)B5 + 3150(1-\eta^2)B51 + 12600(1-\eta^3)B52 + 28350(1-\eta^4)B53 + 28350(1-\eta^5)B54
\end{aligned}\right\} . \qquad 2.66$$

Then the extrapolated F and E_r are

$$F(r,\theta_m) = F(r_1,\theta_m) - Y_1(\theta_m)D3 - Y_3(\theta_m)D4 - Y_5(\theta_m)D5 \qquad 2.67$$

$$E_r(r,\theta_m) = \eta^2[E_r(r_1,\theta_m) - P_1(\theta_m)D6 - P_3(\theta_m)D7 - P_5(\theta_m)D8] \ . \qquad 2.68$$

For easy reference, the $P_\ell(\theta)$ are

$$\left.\begin{aligned}
P_1(\theta) &= \cos\theta \\
P_3(\theta) &= \frac{1}{2}\cos\theta(5\cos^2\theta - 3) \\
P_5(\theta) &= \frac{1}{8}\cos\theta(63\cos^4\theta - 70\cos^2\theta + 15)
\end{aligned}\right\} . \qquad 2.69$$

Note that the BℓB quantities (second derivatives of Bℓ) are not used in these extrapolations. In LEMP they are used in certain print-outs which are used to examine the accuracy of the extrapolation.

III. TEST PROBLEMS

3.1 Introduction

To test the accuracy of the code, it is desirable to find problems which are similar to the real problems, but for which the solutions can be found analytically or by independent means. In this Chapter we present two such problems. The first is called the wave test problem, and tests the accuracy at early times (in the real problem) when the air conductivity is still negligible. The second is called the diffusion test problem, and tests the accuracy in the diffusion phase when the conduction current dominates the displacement current, and the magnetic field is diffusing into the air from the ground-air interface. In both cases the ground conductivity is assumed infinite.

3.2 The Wave Test Problem

Our procedure will be to look for a simple solution, and see what kind of Compton source current is needed to yield this solution. We assume that the air conductivity is zero.

Since Maxwell's equations have solutions which are separable into radial and angular parts with the angular parts being spherical harmonics, we shall look for such solutions. In the real problem, the magnetic field is small except near the ground surface. We can match this property by taking spherical harmonics of an imaginary argument, as will be seen below. In addition, we look for solutions growing exponentially with τ, another desirable feature.

We start with the Eqs. 1.15 to 1.17 for $\sigma = J_\theta = 0$; i.e.,

$$\frac{\partial E_r}{\partial \tau} = -4\pi J_r + \frac{1}{2r^2 \sin\theta} \frac{\partial\{\sin\theta(F - G)\}}{\partial\theta} , \qquad 3.1$$

$$\frac{\partial F}{\partial \tau} = \frac{\partial E_r}{\partial \theta} , \qquad 3.2$$

$$2 \frac{\partial G}{\partial \tau} = \frac{\partial G}{\partial r} - \frac{\partial E_r}{\partial \theta} . \qquad 3.3$$

We let J_r, E_r, F, and G be proportional to $e^{\alpha\tau}$ (only τ dependence) so that,

$$\alpha E_r = -4\pi J_r + \frac{1}{2r^2 \sin\theta} \frac{\partial\{\sin\theta(F - G)\}}{\partial\theta} , \qquad 3.4$$

$$\frac{\partial F}{\partial r} = \frac{\partial E_r}{\partial \theta} , \qquad 3.5$$

$$2\alpha G = \frac{\partial G}{\partial r} - \frac{\partial E_r}{\partial \theta} , \qquad 3.6$$

where we have used the same symbols for J_r, E_r, F, and G to represent only their space parts. We eliminate J_r by defining E_1 as

$$E_r = -\frac{4\pi J_r}{\alpha} + E_1 . \qquad 3.7$$

The boundary condition on E_r is that it be zero on the ground ($\theta = \pi/2$), and, therefore,

$$E_1(r,\theta)\Big|_{\theta=\frac{\pi}{2}} = \frac{4\pi J_r}{\alpha} . \qquad 3.8$$

J_r is chosen as a function of r only and will be determined by Eq. 3.8. Next, we let

$$E_1 = E_1(r)\psi(\theta) , \qquad 3.9$$

$$F = F(r)X(\theta) , \qquad 3.10$$

$$G = G(r)X(\theta) ; \qquad 3.11$$

and, as can be seen from Eqs. 3.4 to 3.6,

$$X(\theta) = \frac{\partial\psi(\theta)}{\partial\theta} = \psi' . \qquad 3.12$$

We now assume that ψ is a spherical harmonic of imaginary argument by assuming

$$\frac{1}{\sin\theta} \frac{\partial(\sin\theta\psi')}{\partial\theta} = \beta^2\psi , \qquad 3.13$$

where β is a real constant, to be chosen later. Then Eqs. 3.4 to 3.6 give

$$\alpha E_1(r) = \frac{\beta^2}{2r^2} \{F(r) - G(r)\} , \qquad 3.14$$

$$F'(r) = E_1(r) , \qquad 3.15$$

$$G'(r) = E_1(r) + 2\alpha G(r) , \qquad 3.16$$

where $F'(r) \equiv \partial F/\partial r$, etc. These equations are most easily solved by expressing all quantities in terms of

$$H(r) \equiv F(r) - G(r) \qquad 3.17$$

and its derivatives. Subtracting Eq. 3.15 from Eq. 3.16, one finds

$$G(r) = -\frac{1}{2\alpha}\frac{\partial H}{\partial r} = -\frac{1}{2} H'(x) , \qquad 3.18$$

where we have defined a new independent variable

$$x \equiv \alpha r . \qquad 3.19$$

Then from Eq. 3.15,

$$E_1(r) = F'(r) = \alpha F'(x) = \alpha[H'(x) + G'(x)]$$
$$= \alpha[H'(x) - \frac{1}{2} H''(x)].$$

Using this result to eliminate E_1 from Eq. 3.14, we find the final equation for H,

$$H''(x) - 2H'(x) + \frac{\beta^2}{x^2} H = 0 . \qquad 3.20$$

Once H has been found, G is found from Eq. 3.18, F is found from

$$F = H + G = H(x) - \frac{1}{2}H'(x) , \qquad 3.21$$

and E_1 is found from Eq. 3.14, which can be written as

$$E_1 = \frac{\alpha\beta^2}{2x^2} H(x) . \qquad (3.14')$$

Returning now to Eq. 3.13 for the angular function, we notice that near $\theta = \pi/2$, $\sin\theta \approx 1$, there is one solution of the type

$$\psi(\theta) \approx \exp\left[-\beta\left(\frac{\pi}{2} - \theta\right)\right] .$$

This has the behavior that we want, decreasing roughly exponentially with distance (angle) away from the ground surface. In the test problem we have arbitrarily chosen

$$\beta = 12 , \qquad 3.22$$

so that the e-folding angle is approximately 5°.

While the imaginary spherical harmonics are known, we chose to integrate Eq. 3.13 numerically, using a fine mesh to guarantee accuracy much better than LEMP 1 (which uses a fairly coarse mesh away from the ground). We used the difference equation

$$\psi_{i+1} = \frac{1}{\sin\theta_{i+\frac{1}{2}}}\left\{\psi_i\left[\sin\theta_{i+\frac{1}{2}} + \sin\theta_{i-\frac{1}{2}} + \delta\theta^2\beta^2 \sin\theta_i\right] - \psi_{i-1}\sin\theta_{i-\frac{1}{2}}\right\} . \qquad 3.23$$

$\delta\theta$ is a θ increment which is a small fraction of the smallest increment to be expected in the LEMP 1 mesh. The angular integration starts at $\theta = 0$ and sets $\psi(\theta) = \psi_{i-1} = 1$ and $\psi_i = 1 + \beta^2/4\ \delta\theta^2$. The integration proceeds to $\theta = \pi/2$, with the values of $\psi(\theta)$ and $X(\theta)$ being stored at each θ_l and $\theta_{l+\frac{1}{2}}$ of the LEMP 1 mesh. [$X(0) = 0$]. $\psi(\theta)$ and $X(\theta)$ are then divided by $\psi(\pi/2)$, so that the final $\psi(\pi/2) = 1$.

We now consider the radial equation, 3.20, bearing in mind that β has been fixed. For large x there are two possible asymptotic forms. One is

$$H \sim e^{2x} ,$$

but this is clearly not a desirable form. The other asymptotic form is

$$H(x) = 1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \frac{a_3}{x^3} + \cdots , \qquad 3.24$$

where, by substitution in Eq. 3.20, one finds

$$a_1 = -\frac{\beta^2}{2}, \quad a_2 = \frac{1}{8}\beta^2(\beta^2+2) , \quad a_3 = -\frac{1}{48}\beta^2(\beta^2+2)(\beta^2+6) , \quad \cdots, \quad a_{n+1} = -\frac{[n(n+1)+\beta^2]}{2(n+1)} a_n . \qquad 3.25$$

This is a desirable asymptotic form, since H is proportional to rB_ϕ.

The procedure therefore is to start at large x and integrate Eq. 3.20 inward. The actual integration was done numerically, using a fine mesh and the difference equation

$$H_{i-1} = \frac{-H_{i+1}(1 - \delta x) + H_i(2 - \beta^2\delta x^2/x_i^2)}{1 + \delta x} . \qquad 3.26$$

The first (outer) two values are found from the asymptotic form.

In LEMP 1 we use the inner boundary condition that $E_\theta = 0$ at $r = r_0$. Thus we want

$$0 = F(r_0) + G(r_0) = H(x_0) - H'(x_0) . \qquad 3.27$$

We therefore integrate Eq. 3.26 inward until Eq. 3.27 is satisfied. This happens (for $\beta = 12$) at

$$\alpha r_0 = x_0 = 10.306 . \qquad 3.28$$

Therefore we must choose

$$\alpha = 10.306/r_0 . \qquad 3.29$$

The Compton current needed to produce this solution is then determined by Eqs. 3.8 and 3.14′, or

$$4\pi J_r = \frac{\alpha^2\beta^2}{2x^2} H(x) e^{\alpha\tau} . \qquad 3.30$$

(It is seen that $J_r \sim 1/r^2$ for large r.) This Compton current is then used as the source in LEMP 1 and the resulting fields are compared with those determined above.

In the wave test problem we used $\beta = 12$, $\delta x = 10^{-3}$, $x_{max} = 2 \times 10^3$. For the LEMP 1 problem, $r_0 = 3 \times 10^3$ cm. This, with Eq. 3.29, gives an exponential growth rate in real time (rather than τ = ct) of

$$\alpha c = 1.03062 \times 10^{8} \text{ sec}^{-1} , \qquad 3.31$$

which is a reasonable value. In addition, we chose $r_{max} = 3.55 \times 10^5$ cm, $\delta r_0 = 1 \times 10^3$ cm, $n_r = 65$, $n_{\theta s} = 9$, $n_{\theta f} = 6$, $n_{Zs} = 7$, $n_{Zf} = 4$, and $Z_0 = 0.01$ cm. The air conductivity was set equal to 10^{-10} cm^{-1} (a negligible value) and the ground conductivity was set equal to 10^5 cm^{-1} (a large value). The problem was started with theoretically obtained values for the fields and run for about 7 e-folding periods.

In Table I, theoretical and LEMP 1 values of B_φ and of E_θ are compared, at r = 300 meters on the ground, at various times. The error in B is about 0.7% and the error in E_θ is about 0.3%.

TABLE I

r = 300 METERS, ON GROUND

time, shakes	B_φ, LEMP 1	B_φ, theory	E_θ, LEMP 1	E_θ, theory
0.1116	1.005 (-4)	1.007 (-4)	0.995 (-4)	1.000 (-4)
1.009	2.833 (-4)	2.818 (-4)	2.805 (-4)	2.799 (-4)
1.997	7.845 (-4)	7.795 (-4)	7.766 (-4)	7.743 (-4)
2.998	2.202 (-3)	2.188 (-3)	2.180 (-3)	2.173 (-3)
4.005	6.215 (-3)	6.175 (-3)	6.153 (-3)	6.134 (-3)

In Table II, theoretical and LEMP 1 values of B_φ and of E_r are compared, at retarded time = 6.844 shakes and r = 300 meters, for various angles $\pi/2 - \theta$ above the ground. The error in B_φ is about 0.7% of the maximum value (surface value) of B_φ. The error in E_r is about 0.6% of the maximum value (large $\pi/2 - \theta$) of E_r.

TABLE II

RETARDED t = 6.844 SHAKES, r = 300 METERS

$\frac{\pi}{2} - \theta$, radians	B_φ, LEMP 1	B_φ, theory	E_r, LEMP 1	E_r, theory
0			-2.453 (-5)	0
2.789 (-4)	1.156 (-1)	1.148 (-1)		
5.578 (-4)			-1.145 (-4)	-0.895 (-4)
8.367 (-4)	1.148 (-1)	1.141 (-1)		
1.116 (-3)			-2.038 (-4)	-1.784 (-4)
1.673 (-3)	1.137 (-1)	1.129 (-1)		
2.231 (-3)			-3.808 (-4)	-3.544 (-4)
3.347 (-3)	1.114 (-1)	1.107 (-1)		
4.462 (-3)			-7.278 (-4)	-6.994 (-4)
6.694 (-3)	1.071 (-1)	1.063 (-1)		
8.925 (-3)			-1.394 (-3)	-1.362 (-3)
1.339 (-2)	9.888 (-2)	9.809 (-2)		
1.785 (-2)			-2.626 (-3)	-2.586 (-3)
2.677 (-2)	8.435 (-2)	8.352 (-2)		
3.570 (-2)			-4.729 (-3)	-4.673 (-3)
5.355 (-2)	6.145 (-2)	6.057 (-2)		
7.140 (-2)			-7.795 (-3)	-7.718 (-3)
1.071 (-2)	3.275 (-2)	3.189 (-2)		
1.428 (-1)			-1.107 (-2)	-1.100 (-2)
2.142 (-1)	9.571 (-3)	8.876 (-3)		
2.856 (-1)			-1.298 (-2)	-1.299 (-2)
4.284 (-1)	1.044 (-3)	6.999 (-4)		
5.712 (-1)			-1.340 (-2)	-1.342 (-2)
7.140 (-1)	8.255 (-5)	2.468 (-5)		

In Table III, theoretical and LEMP 1 values of B_φ and of E_θ are compared, at retarded time = 6.844 shakes and on the ground, for various radii. For radii greater than 130 meters, the error in both B_φ and E_θ is less than 0.7%.

We shall use a Cartesian coordinate system (instead of spherical) and restrict our attention to times at which the skin depth in the air is small compared with r, so this introduces negligible error. We use the radius r from the burst, the height z above the ground, and y in such a direction that r,z,y form a right-handed orthogonal system. Note that z and y are in the $-\theta$ and $-\varphi$ directions. We shall have electric fields E_r and E_z, and a magnetic field B in the y direction. Maxwell's equations are

$$\frac{1}{c}\frac{\partial B}{\partial t} = -\frac{\partial}{\partial r} E_z + \frac{\partial}{\partial z} E_r , \qquad 3.36$$

$$\frac{1}{c}\frac{\partial E_z}{\partial t} + 4\pi\sigma E_z = -\frac{\partial}{\partial r} B , \qquad 3.37$$

$$\frac{1}{c}\frac{\partial E_r}{\partial t} + 4\pi\sigma E_r + 4\pi J_r = \frac{\partial}{\partial z} B . \qquad 3.38$$

We now transform to independent variables $\tau = ct - r$, $r' = r$, and drop the prime on r' as usual so that

$$\frac{1}{c}\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial \tau} , \qquad 3.39$$

$$\frac{\partial}{\partial r} \rightarrow \frac{\partial}{\partial r'} - \frac{\partial}{\partial \tau} . \qquad 3.40$$

In the resulting Maxwell equations, we neglect $\partial/\partial r'$ compared with $\partial/\partial\tau$ and $\partial/\partial z$ and obtain

$$\frac{\partial B}{\partial \tau} = \frac{\partial E_z}{\partial \tau} + \frac{\partial E_r}{\partial z} , \qquad 3.41$$

TABLE III

RETARDED t = 6.844 SHAKES, ON GROUND

r, meters	B_φ, LEMP 1	B_φ, theory	E_θ, LEMP 1	E_θ, theory
30	1.188 (-3)	1.117 (-3)	0	0
40	8.081 (-3)	7.814 (-3)	4.737 (-3)	4.389 (-3)
51.43	2.222 (-2)	2.194 (-2)	1.678 (-2)	1.655 (-2)
130	1.078 (-1)	1.073 (-1)	1.039 (-1)	1.034 (-1)
300	1.160 (-1)	1.152 (-1)	1.152 (-1)	1.144 (-1)
580	8.386 (-2)	8.335 (-2)	8.370 (-2)	8.320 (-2)
900	6.138 (-2)	6.105 (-2)	6.133 (-2)	6.100 (-2)
1750	3.528 (-2)	3.515 (-2)	3.527 (-2)	3.514 (-2)
3550	1.843 (-2)	1.841 (-2)	1.843 (-2)	1.841 (-2)

These accuracies are entirely adequate, especially for the small numbers of mesh cells used in LEMP 1.

3.3 The Diffusion Test Problem

In this test problem, we imagine a conductivity which is constant in time,

$$\sigma = \frac{10^6}{r^2} \text{ cm}^{-1} , \qquad 3.32$$

and a Compton current which is turned on at $\tau = 0$,

$$J_r = 0 , \quad \tau < 0$$
$$= -\frac{10^6}{r^2} , \quad \tau > 0 . \qquad 3.33$$

We shall examine the fields at

$$r = 3 \times 10^4 \text{ cm} = 300\text{m} , \qquad 3.34$$

where the conductivity is

$$\sigma = \frac{1}{900} \text{ cm}^{-1} . \qquad 3.35$$

With this high conductivity there is very little propagation of waves, and we can obtain analytically an almost exact solution of Maxwell's equations. We assume the ground conductivity is infinite.

$$\frac{\partial E_z}{\partial \tau} + 4\pi\sigma E_z = \frac{\partial B}{\partial \tau} , \qquad 3.42$$

$$\frac{\partial E_r}{\partial \tau} + 4\pi\sigma E_r + 4\pi J_r = \frac{\partial}{\partial z} B . \qquad 3.43$$

Comparing the first two of these we see that

$$E_z = \frac{1}{4\pi\sigma} \frac{\partial E_r}{\partial z} . \qquad 3.44$$

Therefore, from Eq. 3.41,

$$\frac{\partial B}{\partial \tau} = \frac{\partial \mathcal{E}}{\partial z} , \qquad 3.45$$

where we have defined

$$\mathcal{E} \equiv E_r + \frac{1}{4\pi\sigma} \frac{\partial E_r}{\partial \tau} . \qquad 3.46$$

Then Eq. 3.43 may be written in the form

$$\frac{\partial B}{\partial z} = 4\pi J + 4\pi\sigma\mathcal{E} . \qquad 3.47$$

Equations 3.45 and 3.47 are the usual diffusion equations for the skin effect. Note that on the ground we must have $\mathcal{E} = 0$. For $\tau < 0$, $\mathcal{E} = B = 0$ everywhere. For $\tau > 0$, $\mathcal{E} \rightarrow J/\sigma$ (=1 in our case), and $B \rightarrow 0$ for large z. The solutions of the equations are well known, with the results

$$\mathcal{E} = - \frac{J_r}{\sigma} \frac{2}{\sqrt{\pi}} \int_0^a e^{-x^2} dx \qquad 3.48$$

and

$$B = -4 \frac{J_r}{\sigma} \left[\sqrt{\sigma\tau}\, e^{-a^2} - 2\sqrt{\pi}\, \sigma z \int_a^\infty e^{-x^2} dx \right] , \qquad 3.49$$

where

$$a \equiv z \sqrt{\frac{\pi\sigma}{\tau}} . \qquad 3.50$$

From Eq. 3.46,

$$E_r = e^{-4\pi\sigma\tau} \int_0^\tau 4\pi\sigma\mathcal{E}(\tau') e^{4\pi\sigma\tau'} d\tau' . \qquad 3.51$$

While some of these results are a little complicated, the value of B at z = 0 is quite simple:

$$B(z = 0) = -4 \frac{J_r}{\sigma} \sqrt{\sigma\tau} . \qquad 3.52$$

Also, when $4\pi\sigma\tau >> 1$,

$$E_r \approx \mathcal{E} . \qquad 3.53$$

We shall use these results to test LEMP 1. At 300 meters, we find

$$B = \frac{4}{30} \sqrt{\tau} .$$

The LEMP 1 problem used

$$n_{\theta f} = 6 , \quad n_{\theta s} = 9 ,$$

$$\sigma_{gnd} = 10^5 \text{ cm}^{-1} = 3.33 \times 10^5 \text{ mho/m} ,$$

$$z_0 = 0.01 \text{ cm} ,$$

$$n_{Zs} = 7 , \quad n_{Zf} = 4 .$$

Other parameters were as usual, and the code was allowed to choose its own $\delta\tau$.

Table IV compares B on the ground at 300 meters

TABLE IV

B ON THE GROUND

τ, cm	Cycle No.	B_φ, LEMP 1	B_φ, Theory	% Error
0.581	1	-0.121	-0.102	20
9.87	17	-0.420	-0.419	0.2
20.32	35	-0.604	-0.601	0.5
44.70	77	-0.897	-0.891	0.7
109.14	188	-1.400	-1.393	0.5
202.03	348	-1.906	-1.895	0.6
424.97	732	-2.763	-2.749	0.5
996.24	1716	-4.231	-4.208	0.5
2190.4	3773	-6.272	-6.240	0.5
4264.8	7346	-8.747	-8.707	0.5

from LEMP 1 and from theory. The agreement is excellent. Even on the first cycle the error is only 20%, and after the 17th cycle the error is never larger than 0.7%.

For the last cycle computed, τ = 4264.8 cm, and $4\pi\sigma\tau$ = 59.6. By this time the approximation Eq. 3.53 should be very accurate. In Table V, E_r as a function of z as calculated by LEMP 1 is compared with the theoretical result. Again the agreement is excellent, the largest error being 1.7% of the saturated field, 1 esu.

TABLE V

COMPARISON OF E_r

AT τ = 4264.8 cm; r = 300 m; $\sqrt{\frac{\tau}{\pi\sigma}}$ = 1.10541 x 10^3 cm; $4\pi\sigma\tau$ = 59.6

z cm	$z\sqrt{\frac{\pi\sigma}{\tau}}$	E_r, LEMP 1	E_r, Theory	Error
0	0	0.0018	0	0.0018
16.7	0.0151	0.0192	0.0170	0.0018
33.4	0.0303	0.0365	0.0342	0.0023
66.9	0.0606	0.0711	0.0683	0.0028
133.9	0.1211	0.1398	0.1359	0.0039
267.7	0.2422	0.2738	0.2680	0.0058
535.5	0.4844	0.5165	0.5067	0.0098
1071	0.9689	0.8468	0.8294	0.0174
2142	1.9377	0.9892	0.9938	-0.0046
4284	3.8755	0.9999	1.0000	-0.0001
8568	7.7509	1.0000	1.0000	0.0000

It can be seen from these results that radial derivatives dropped in the theory are no more than about 1% of the z-derivatives.

IV. RELATED TOPICS

There are two Los Alamos reports which serve as companion reports to this one. These reports are numbered LA-4347 and LA-4348.

LA-4348 is titled "Compton Current in Presence of Fields for LEMP 1." If a Compton electron is produced by a gamma ray of energy E_γ in air and with an electromagnetic field (E_r,E_θ,B_φ) present, it will slow to a stop at an average position (DX,DY). DX is the average distance the electron travels in the direction of the gamma ray and DY is the average distance the electron travels perpendicular to the direction of the gamma ray. LA-4348 gives, as a final result, the fitted values of the functions DX(E_γ,E_r,E_θ,B_φ) and DY(E_γ,E_r,E_θ,E_φ). This result is used in LA-4347 in the source calculations.

LA-4347 is titled "Sources, Parameter Study, and the Output Library for LEMP 1." This report is classified Secret-Restricted Data. The first portion of this report, "Sources," gives the formulae for γ-transport and the prescription for obtaining J_r, J_θ, and $\dot{\gamma}$. The second portion gives the effect on peak electromagnetic fields caused by changing one (or more) of the parameters used for input to LEMP 1. The last portion of this report is an attempt to summarize the results of the Confidential-Restricted Data LEMP 1 "Library." This Library consists of 8 rolls of 35-mm film with approximately 20000 exposed frames. Each roll of film is for a specific problem and contains output time plots and tables for the fields and sources at a large number of points in the air and in the ground. Each problem was run for a "typical" yield. Problems 1 to 6 are for yields of 0.1, 1, 10, 100, 1000, and 10000 kiloton devices, respectively. Problems 7 and 8 are for 10 and 1000 kiloton devices, respectively. Problems 1 to 6 were run with "typical" ground parameters of σ_0 = 0.02 mho/m and ϵ/ϵ_0 = 16. Problems 7 and 8 had "ground" parameters of σ_0 = 4.3 mho/m and ϵ/ϵ_0 = 81, as might be appropriate to sea water. This Library was distributed nationally in August 1969, and is available to anyone in need.

The present Library should be regarded only as an initial attempt to satisfy the needs that exist.

We expect that the Library will continue to grow in response to requests for additional or special problems.

APPENDIX. UNITS

Gaussian units were used in the code because of the resulting simplicity of Maxwell's equations. The transformations to $\tau = ct - r$ and to F and G are not cluttered with unnecessary constants, and it is convenient to have $E = B$ for free waves.

To connect Gaussian units with MKS units, the following relations apply.

$$E(\text{volts/meter}) = 3 \times 10^4 \; E(\text{Gaussian} = \text{esu})$$
$$B(\text{webers/m}^2) = 10^{-4} \; B(\text{Gaussian} = \text{emu})$$
$$J(\text{amps/m}^2) = 10^5 \; J(\text{Gaussian} = \text{abamps/cm}^2)$$
$$\sigma(\text{mho/m}) = \frac{10}{3} \; \sigma(\text{Gaussian} = \text{cm}^{-1})$$
$$\epsilon(\text{MKS}) = \epsilon_0 \; \epsilon(\text{Gaussian})$$

Here ϵ_0 is the dielectric constant of free space, which is well known by users of MKS units.