Theoretical Notes
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ELECTROMAGNETIC PROPAGATION ALONG STRATIFIED MEDIA

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PREFACE

The work reported in this memorandum is a contribution to RAND's continuing study of electromagnetic propagation through various environments. In particular, this work on propagation along stratified media provides results germane to global VLF communications, electromagnetic detection of atmospheric nuclear bursts, and transmission line analysis and design.

Digital computer programs have been written to exploit the analysis presented in the following pages. Copies of these programs are available from the authors.

SUMMARY

In this paper an iterative technique is developed for calculating the propagation coefficient for electromagnetic waves traveling along a stratified medium. The strata are required only to be "parallel" and have scalar electrical parameters, ε, σ, and μ; no restrictions are placed upon the distribution of these parameters. The boundary conditions at the top and bottom strata may be of a variety of forms; for example, a surface impedance may be specified or outgoing waves only may be required.

The technique is explicitly presented for rectangular, cylindrical, and spherical geometries. In cylindrical geometry, the "parallel" strata are concentric cylinders, and in spherical geometry, they are concentric spheres.

The technique is designed to be used on a digital computer; it has been programmed and used to solve several practical problems.

CONTENTS

I. INTRODUCTION

Stratified media are those media whose electrical properties are independent of one of the coordinates used to describe the configuration. For example, an infinite parallel plate transmission line containing any material in layers parallel to the plates is an example of a stratified medium, and parallel plate transmission lines of finite widths are often good approximations to stratified media for the purpose of understanding their electrical propagation characteristics. In another geometry, a space made up of concentric spheres of material, each of uniform electrical properties, is a stratified medium.

Configurations which closely approximate stratified media occur in a number of electromagnetic propagation problems. In particular, the propagation of radio waves in the region between the earth and the ionosphere is a problem of this type. In this earth-ionospheric waveguide, one way to describe the propagation is in terms of normal modes, and it is this mode description that will be used in the following.*

For previous work which treats the earth-ionospheric waveguide propagation problem analytically and numerically, see Wait[(1)] and Budden.[(2)]

At frequencies sufficiently high that the wavelength is small compared with the waveguide height, a large number of modes are required to describe the distant signal from any reasonable source, and so the normal mode description is of limited value. However, when the wavelength becomes comparable to the waveguide height, then only one or at most a few normal modes are required to describe distant propagation phenomena, and the normal mode description becomes quite useful.

The substance of this paper begins in Section II, with a statement of the problem to be solved, the form assumed by Maxwell's equations in plane geometry and a derivation of the iterative technique for finding the propagation constant for arbitrary plane parallel strata.

Part III then extends the analysis to spherical geometry, and Part IV completes the discussion for two cylindrical geometries. In Part V, the mode orthogonality relation is derived so that given field distributions may be resolved into their modal components.

*The more general earth-ionospheric propagation problem considers ε, σ, and μ to be tensors of the second rank, but these parameters will be considered only as scalars in the present paper.

II. ELECTROMAGNETIC PROPAGATION ALONG STRATIFIED MEDIA IN PLANE GEOMETRY

In this section the electromagnetic propagation along media of the type shown in Fig. 1 will be discussed.

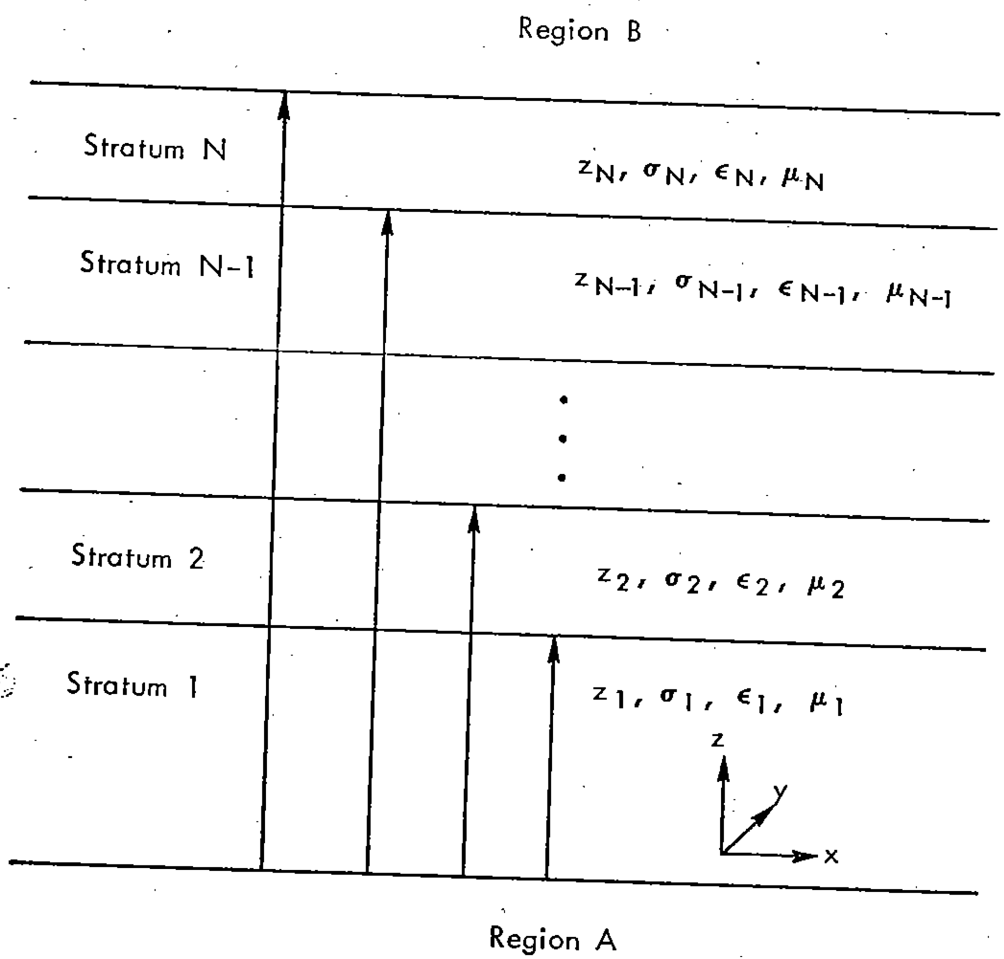

Fig. 1—The medium

Each layer is assumed to be uniform and homogeneous with given height z_i, and electrical parameters σ_i, μ_i, and ε_i, which extend to infinity in the x and y direction.

The electromagnetic waves considered are sinusoidal in time, independent of one of the coordinates parallel to the strata (y), and moving in the positive x direction:

$$F = F_o(z)e^{i(\omega t - kx)} \quad .$$

The propagation is characterized by the wave number $k(\omega)$, and the central problem is the determination of $k(\omega)$.

Maxwell's equations are

$$\nabla \times \vec{H} = (\sigma + i\omega\varepsilon)\vec{E}$$

$$\nabla \times \vec{E} = - i\omega\mu \vec{H}$$

$$\nabla \cdot [(\sigma + i\omega\varepsilon)\vec{E}] = 0$$

$$\nabla \cdot (\mu\vec{H}) = 0 \quad .$$

We can eliminate E from the equations for H, and vice versa, to obtain

$$\nabla^2\vec{H} = - [\nabla \log(\sigma + i\omega\varepsilon)] \times (\nabla \times H) + i\omega\mu (\sigma + i\omega\varepsilon)\vec{H}$$

$$\nabla^2\vec{E} = - [\nabla \log(i\omega\mu)] \times (\nabla \times E) + i\omega\mu (\sigma + i\omega\varepsilon)\vec{E} \quad .$$

Within each stratum, the electrical parameters are constant,* so that in the n^{th} stratum we have

$$\nabla^2\vec{H} = i\omega\mu_n(\sigma_n + i\omega\varepsilon_n)\vec{H}$$

$$\nabla^2\vec{E} = i\omega\mu_n(\sigma_n + i\omega\varepsilon_n)\vec{E} \quad .$$

The solutions for H and E must have the same general form, since they satisfy the same equation. The general solution for waves propagating in the positive x direction is

*If the electrical parameters of a problem in fact vary continuously, then the constant strata may be chosen to approximate, as closely as one wishes, the continuous parameters.

$$\vec{H}_n = (\vec{H}_n^{+} e^{b_n z} + \vec{H}_n^{-} e^{-b_n z})\, e^{i(\omega t - kx)}$$

where

$$b_n = [k^2 + i\omega\mu_n(\sigma_n + i\omega\varepsilon_n)]^{1/2}$$

and the complex constant vector coefficients $\vec{H}_n^{+}$ and $\vec{H}_n^{-}$ are arbitrary. Notice that b_n depends upon the value of the unknown propagation constant $k(\omega)$.

Since the vector coefficients H_n^{+} and H_n^{-} are arbitrary, a conceptual and calculational simplification is achieved if the fields are divided into those which have only a $\vec{y}$ component of the magnetic field (transverse magnetic, or TM, waves) and those which do not have a $\vec{y}$ component of the magnetic field. The second group may also be characterized as having only a $\vec{y}$ component of the electric field (TE waves).

In order to be specific, only the propagation of the TM waves will be discussed in detail, but the calculation of the TE wave propagation follows in a completely analogous manner.

First, the boundary condition equations that apply to the interface between two strata will be found. These equations relate the fields in adjacent strata. The various boundary condition equations for the fields at the top and bottom of the strata, at the region A and region B interface (see Fig. 1) will be discussed.

Taken together, all the boundary condition equations may be written as a set of n linear homogeneous equations in n unknowns. Only for particular values of k will this set of equations have a nonzero solution. The various values of k which permit solutions characterize the modes of propagation along the strata.

THE BOUNDARY CONDITIONS AT THE INTERFACE BETWEEN TWO STRATA

The complete set of boundary conditions at the interface between two media, neither of which is infinitely conducting, is for the $\vec{z}$ (perpendicular to the boundary) components of the field

$$\mu_1 H_{z1} = \mu_2 H_{z2}$$

$$(\sigma_1 + i\omega\varepsilon_1)E_{z1} = (\sigma_2 + i\omega\varepsilon_2)E_{z2}$$

and for the $\vec{x}$ and $\vec{y}$ (tangent to the boundary) components

$$H_{x1} = H_{x2} \qquad H_{y1} = H_{y2}$$

$$E_{x1} = E_{x2} \qquad E_{x1} = E_{x2} \ .$$

For the TM modes, these boundary conditions reduce to

$$(\sigma_1 + i\omega\varepsilon_1)E_{z1} = (\sigma_2 + i\omega\varepsilon_2)E_{z2}$$

$$E_{x1} = E_{x2}$$

$$H_{y1} = H_{y2} \ .$$

These conditions are not independent. The z component of the ∇xH equation is in strata 1 and 2

$$\frac{\partial H_{y1}}{\partial x} = (\sigma_1 + i\omega\varepsilon_1)E_{z1}$$

$$\frac{\partial H_{y2}}{\partial x} = (\sigma_2 + i\omega\varepsilon_2)E_{z2}$$

But if H_y is continuous across the boundary, then

$$\frac{\partial H_{y1}}{\partial x} = \frac{\partial H_{y2}}{\partial x} \quad \text{at the boundary,}$$

and the boundary conditions on H_y and E_z are dependent; one may be derived from the other. The condition on H_y will be used in what follows, since it results in somewhat less algebra.

The magnetic field continuity boundary condition at the interface between layers n and n + 1 is then

$$H_n^+ e^{b_n z_n} + H_n^- e^{-b_n z_n} - H_{n+1}^+ e^{b_{n+1} z_n} - H_{n+1}^- e^{b_{n+1} z_n} = 0$$

which separates into homogeneous 2 equations, one real and one imaginary, each in ten unknowns: the real and imaginary parts of H_n^+, H_n^-, H_{n+1}^+, H_{n+1}^- and k.

A similar equation results from the continuity of the tangential component of the electric field, E_x.

THE INDUCTIVE-ITERATIVE METHOD OF SOLUTION

Since the details of the inductive-iterative method of solution are sufficiently intricate that they may obscure the physics involved, we first present a summary of the method, with the calculational details removed. The summary will also be used to define many of the variables used later in the more detailed analysis.

Starting as before with a source-free stratified medium and a wave propagating in the positive x-direction, Maxwell's equations may be written in the form

$$\frac{d\psi_\mu(n,z)}{dz} = \sum_{\nu=1}^{4} Q_{\mu\nu}(n)\, \psi_\nu(n,z)$$

or, in the implicit sum tensor contraction notation that will be used in the following,

$$\frac{d\psi_\mu(n,z)}{dz} = Q_{\mu\nu}(n)\ \psi_\nu(n,z)$$

where $\psi_\mu(n,z)$ is treated as a vector with the four components

$$\psi_\mu(n,z) = \begin{bmatrix} Re[E_x] \\ Im[E_x] \\ Re[H] \\ Im[H] \end{bmatrix}$$

and $Q_{\mu\nu}$ is a matrix which does not depend upon the fields, but depends only on the properties of the n^{th} layer of the strata. The solution for ψ in the n^{th} layer is

$$\psi_\mu(n,z) = (e^{Qz})_{\mu\nu}\ C_\nu \quad .$$

Since both H and E_x are continuous across each stratum interface, the constant C_ν must be chosen to connect the fields in the n^{th} layer with those in the n-1th layer:

$$\psi_\mu(n,z) = [e^{Q(n)(z-z_n)}]_{\mu\nu}\ \psi_\nu(n-1,z_n).$$

Given the fields at the region A interface, we can then find the fields anywhere in the guide by successively calculating the fields from one region to the next, finally obtaining

$$\psi_\mu(n,z) = [e^{Q(n)(z-z_{n-1})}]_{\mu\alpha}[e^{Q(n-1)(z_{n-1}-z_{n-2})}]_{\alpha\beta} \ldots [e^{Q(1)z_1}]_{\gamma\delta}\psi_\delta(1,0)\ .$$

The actual calculations of the exponents raised to matrix powers (which is defined as the series $e^M = I + M + \frac{1}{2!} M \cdot M + \frac{1}{3!} M \cdot M \cdot M + \ldots$) is considerably expedited by noting that Q has the property that it can always be expressed as eight coefficients times eight linearly independent 4 x 4 matrices, and that these eight 4 x 4 matrices form a group.* The group property is particularly important, since it means that the product of any two of the matrices is a third matrix of the group. Therefore, to calculate the lengthy matrix products in the solution, we just use the group multiplication table and keep track of the coefficient of each of the eight matrices.

The boundary conditions which are imposed at the top of the guide (the region B interface) put additional constraints upon the fields at that boundary which will be satisfied only when certain values of the propagation constant k are used in the Q matrices. An iterative numerical scheme which adjusts the fields to satisfy the boundary conditions at the upper boundary is then used to converge upon the correct value of k.

DETAILS OF THE INDUCTIVE METHOD CALCULATION

As before, we shall treat only the TM case. Using the notation of the previous sections, let the 4 component vector $\vec{\psi}$ be defined as

$$\psi_\mu(z) = \begin{pmatrix} \mathrm{Re}\ E_x(z) \\ \mathrm{Im}\ E_x(z) \\ \mathrm{Re}\ H_y(z) \\ \mathrm{Im}\ H_y(z) \end{pmatrix}$$

Then, in the nth stratum,

$$\frac{d\psi_\mu}{dz} = Q_{\mu\nu}(n)\psi_\nu \ ,$$

*For those interested in esoterica, the algebra of this group is that of a subgroup of the Clifford, or hypercomplex, algebra.

where the matrix $Q_{\mu\nu}(n)$ takes the form

$$Q_{\mu\nu}(n) = \begin{pmatrix} 0 & 0 & A_n & B_n \\ 0 & 0 & -B_n & A_n \\ -\sigma_n & \omega\varepsilon_n & 0 & 0 \\ -\omega\varepsilon_n & -\sigma_n & 0 & 0 \end{pmatrix}$$

with

$$A_n = -\frac{k^2\sigma_n}{(\sigma_n)^2 + (\omega\varepsilon_n)^2}$$

$$B_n = \omega\mu_n\left(1 + \frac{\varepsilon_n}{\mu_n}\frac{A_n}{\sigma_n}\right) .$$

Define the eight base matrices as follows:*

$$S_x(+) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$iS_y(+) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$S_z(+) - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

*The notation is clearly related to that of spinors in quantum mechanics. This just reflects the authors' background.

$$iS_x(-) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$S_y(-) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$iS_z(-) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$N = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The algebra of matrix multiplication is:

$$S_x(p) \; S_x(q) = S_y(p) \; S_y(q) = S_z(p) \; S_z(q)$$

$$= I \text{ if } pq = 1;$$

$$= - iN \text{ if } pq = - 1.$$

$$N^2 = -I$$

$$I^2 = I$$

$$S_x(p)\ S_y(q) = -S_y(q)\ S_x(p) = iS_z(pq)$$

$$S_y(p)\ S_z(q) = -S_x(q)\ S_z(p) = iS_x(pq)$$

$$S_z(p)\ S_x(q) = -S_x(q)\ S_z(p) = iS_y(pq)$$

$$S_{\substack{x\\y\\z}}(p)N = NS_{\substack{x\\y\\z}}(p) = iS_{\substack{x\\y\\z}}(-p)$$

In terms of the base matrices, Q can be written as

$$Q_n = \frac{1}{2}\Big\{(A_n-\sigma_n)\ S_x(+) + (A_n+\sigma_n)\ iS_y(+)$$

$$+ (\omega\varepsilon_n+B_n)\ iS_x(-) + (\omega\varepsilon_n-B_n)\ S_y(-)\Big\}\ .$$

Using the multiplication rules, we can now calculate

$$e^{Qz} = I + Qz + \frac{Q^2z^2}{2!} + \frac{Q^3z^3}{3!} + \ldots$$

We find that

$$Q^2 = (k^2 - \omega^2\varepsilon\mu)\ I - \omega\mu\sigma\cdot N,$$

where we have suppressed the subscript "n" denoting the n^{th} stratum.

We note that, in the notation of the first section,

$$Q^2 = \text{Re}\,[b^2] \cdot I - \text{Im}\,[b^2].$$

For convenience, define

$$f_n = \text{Re}\,[b_n^2] = k^2 - \omega^2 \varepsilon_n \mu_n$$

$$g_n = \text{Im}\,[b_n^2] = \omega \mu_n \sigma_n,$$

so that

$$A^2 = f \cdot I - g \cdot N.$$

Then

$$Q^{2m} = (f \cdot I - g \cdot N)^m$$

$$= f^m \cdot I - m f^{m-1}\, gN + \frac{m(m-1)}{2}\, f^{m-2} g^2 N^2$$

$$= (f^m - \frac{m(m-1)}{2}\, f^{m-2}\, g^2 + \ldots)\; I$$

$$- (m f^{m-1} g - \frac{m(m-1)(m-2)}{3!}\, f^{m-3}\, g^3 + \ldots) N,$$

since $N^2 = -1$.

After a good deal of algebra and rearrangement, this can be recognized as:

$$Q^{2m} = \left\{ (f^2+g^2)^{m/2}\, \text{Re}[e^{i \cdot m \cdot \text{Tan}^{-1}(g/f)}] \cdot I - \text{Im}[e^{i \cdot m \cdot \text{Tan}^{-1}(g/f)}] \cdot N \right\}$$

and

$$Q^{qm+1} = Q^{2m} \cdot Q$$

defining

$$M = N \cdot Q \equiv Q \cdot N,$$

$$Q^{2m+1} = (f^2+g^2)^{m/2} \left\{ \mathrm{Re}[e^{i \cdot m \cdot \mathrm{Tan}^{-1}(g/f)}] \cdot Q - \mathrm{Im}[e^{i \cdot m\ \mathrm{Tan}^{-1}(g/f)}] \cdot M \right\}$$

finally,

$$e^{Qz} = \sum_{m=0}^{\infty} \left[\frac{z^{2m} Q^{2m}}{(2m)!} + \frac{z^{2m+1} Q^{2m+1}}{(2m+1)!} \right]$$

$$= \mathrm{Re} \left\{ \mathrm{Cosh}\ [z(f^2+g^2)^{1/4}\ e^{i\ 1/2\ \mathrm{Tan}^{-1} g/f}] \right\} \cdot I$$

$$- \mathrm{Im} \left\{ \mathrm{Cosh}\ [z(f^2+g^2)^{1/4}\ e^{i\ 1/2\ \mathrm{Tan}^{-1} g/f}] \right\} \cdot N$$

$$+ \mathrm{Re} \left\{ \frac{\mathrm{Sinh}\ [z(f^2+g^2)^{1/4}\ e^{i\ 1/2\ \mathrm{Tan}^{-1} g/f}]}{(f^2+g^2)^{1/4}\ e^{i\ 1/2\ \mathrm{Tan}^{-1} g/f}} \right\} \cdot Q$$

$$- \mathrm{Im} \left\{ \frac{\mathrm{Sinh}\ [z(f^2+g^2)^{1/4}\ e^{i\ 1/2\ \mathrm{Tan}^{-1} g/f}]}{(f^2+g^2)^{1/4}\ e^{i\ 1/2\ \mathrm{Tan}^{-1}\ g/f}} \right\} \cdot M$$

where the matrices I, N, Q have already been defined and

$$M(n) = N\cdot Q = \frac{1}{2}\left\{ -(\omega\varepsilon_n + B_n)\ S_x(+) + (\omega\varepsilon_n - B_n)\ iS_y(+) + (A_n - \sigma_n)\ iS_x(-) - (A_n + \sigma_n)\ S_y(-) \right\} .$$

We have thus exhibited the matrix e^{Qz} in terms of its components on the eight basis vectors; the multiplication of a series of these exponentials for succeeding strata can then be accomplished using the multiplication laws. (We note that the basis set is independent of the properties of the stratum, although, of course, the coefficients are not.) This scheme has been programmed and provides a very efficient way of integrating Maxwell's equations to find the fields.

DETERMINING THE INCOMING WAVE PORTION OF THE FIELDS

Finally, it remains for us to exhibit explicitly how we pick out the incoming wave portion of the fields (which we may then make zero as our extremum boundary conditions which are needed to completely define the problem) and we shall also give the iterative correction procedure which we have used.

Using our previous notation, we note that we can write, in the n^{th} stratum.

$$E_x(z) = E_+ e^{b_n(z-z_{n-1})} + E_- e^{-b_n(z-z_{n-1})}$$

so that

$$\frac{dE_x(z)}{dz} = b_n[\tilde{E}_+(z) - \tilde{E}_-(z)],$$

or

$$\tilde{E}_+(z) = \frac{1}{2}\left[E_x(z) + \frac{1}{b_n}\frac{dE_x(z)}{dz}\right] .$$

But from Maxwell's equations

$$\frac{dE_x(z)}{dz} = -\left[\frac{k^2}{\sigma_n + i\omega\varepsilon_n} + i\omega\mu_n\right] H_y(z)$$

$$= -\left[\frac{k^2 - \omega^2\varepsilon_n\mu_n + i\omega\mu_n\sigma_n}{\sigma_n + i\omega\varepsilon_n}\right] H_y(z)$$

$$= -\frac{(b_n)^2}{\sigma_n + i\omega\varepsilon_n} H_y(z) ,$$

thus

$$\tilde{E}_+(z) = \frac{1}{2}\left[E_x(z) - \frac{b_n}{\sigma_n + i\omega\varepsilon_n} H_y(z)\right] .$$

That is, the incoming wave portion of $E_x(z)$ can be expressed in terms of the fields at z.

To find the modes of propagation, we adjust k to make $\tilde{E}_+ = 0$.

This can be done very simply, since we have two quantities to make zero ($Re[\tilde{E}_+]$ and $Im[\tilde{E}_+]$) and two parameters to adjust (Re[k] and Im[k]). Setting $R \equiv Re[E_+]$ and $I \equiv Im[E_+]$, $k_1 \equiv Re[k]$, and $k_2 \equiv Im[k]$, we begin a Taylor expansion of R and I about the present values of k_1 and k_2. Since we want to find new values of k_1 and k_2 which make R = 0 and I = 0, we write

$$0 = R + \Delta k_1 \frac{\partial R}{\partial k_1} + \Delta k_2 \frac{\partial R}{\partial k_2}$$

and

$$0 = I + \Delta k_1 \frac{\partial I}{\partial k_1} + \Delta k_2 \frac{\partial I}{\partial k_2}$$

we now have two equations in two unknowns, Δk_1 and Δk_2, which are the corrections in k_1 and k_2 used to approach $R = 0$ and $I = 0$. We make these corrections iteratively until R and I are sufficiently small to provide the desired accuracy.*

*This is simply Newton's method in two dimensions.

III. ELECTROMAGNETIC PROPAGATION ALONG STRATIFIED MEDIA IN SPHERICAL GEOMETRY

Now consider the geometry shown in Fig. 2, which might, for example, represent a spherically symmetric ionosphere surrounding the earth.

In each uniform stratum, Maxwell's equations still have the form

$$\nabla^2\vec{H} = i\omega\mu(\sigma + i\omega\varepsilon)\vec{H} \ .$$

Putting the origin of the spherical coordinate system at the center of the spheres, we will look for modes having only a ϕ component of the magnetic field.

Using the conventional separation of variables technique, we finally obtain the equations

$$\frac{r^2}{R}\frac{d^2R}{dr^2} - i\omega\mu(\sigma + i\omega\varepsilon)r^2 = n(n+1)$$

$$-\frac{1}{\Theta}\frac{d}{d\theta}\left[\frac{1}{\sin\theta}\frac{d}{d\theta}(\Theta\sin\theta)\right] = n(n+1)$$

where $rH_\phi = R(r)\cdot\Theta(\theta)$ and n is the separation constant, and is to be determined.

Although the solutions to the r equation are known exactly, the correspondence between the solutions in plane and spherical geometries is clearer if we use the following approximation to the r equation in each stratum

$$\frac{r^2}{R}\frac{d^2R}{dr^2} - i\omega\mu(\sigma + i\omega\varepsilon)r^2 \approx n(n+1)\frac{r^2}{r_o^2}$$

or

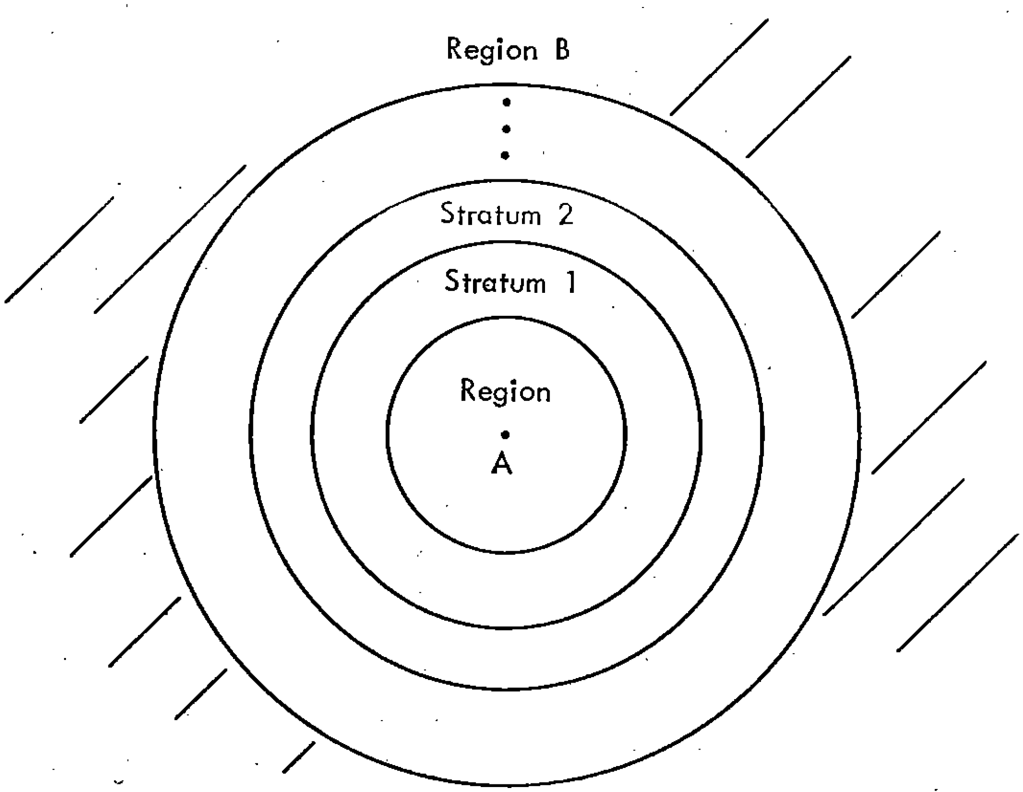

Fig.2—The medium in spherical geometry

$$\frac{d^2R}{dr^2} - i\omega\mu(\sigma + i\omega\varepsilon)R - \frac{n(n+1)}{r_o^2} R = 0.$$

This approximation should be valid in computing the propagation constant when

$$\left|\left(\frac{r}{r_o}\right)^2 - 1\right| << \left|\frac{i\omega\mu(\sigma + i\omega\varepsilon)r^2}{n(n+1)} + 1\right| \; *$$

for any r in the stratum, with r_o the mean radius of the stratum.

This equation may be identified with the equation for the vertical (z) dependence of H in place geometry by associating

$$k^2 \text{ with } \frac{n(n+1)}{r_o^2} \quad .$$

However, it is now n(n+1), not k^2, which is constant throughout the strata. Thus, if we set

$$k_1^2 = \frac{n(n+1)}{r_1^2}$$

then

$$k_2^2 = \frac{n(n+1)}{r_2^2} = k_1^2 \times \left(\frac{r_1}{r_2}\right)^2$$

$$k_3^2 = k_1^2 \left(\frac{r_1}{r_3}\right)^2$$

or, if we set $k_1 = k$, then

$$k_i^2 = k^2 \left(\frac{r_1}{r_i}\right)^2$$

k is now the constant, but k_i must be used in each spherical stratum in place of the constant k that was used in the plane geometry case.

*Wait,[1] p. 157, has expressed this inequality analytically in closed form in terms of the physical parameters of the medium for certain tractable cases.

The determination of k will provide the propagation constant in the strata in the following way:

$$n(n+1) = r_1^2 k^2$$

or

$$n = -\frac{1}{2} + \sqrt{\frac{1}{4} + r_1^2 k^2}.$$

The solution to the equation for Ⓗ are P_n^1 (cos θ) and Q_n^1 (cos θ), the Legendre function. Since k_1 is on the order of the inverse of the free space wave length, then for strata of radius much greater than the wavelength, n will be much greater than 1. For large n,

$$n \approx -\frac{1}{2} + r_1 k,$$

$$P_n^1(\cos\theta) = \frac{\Gamma(n+2)}{\Gamma(n+\frac{3}{2})} \left(\frac{1}{2\pi \sin\theta}\right)^{1/2} \cos\left[(n+\frac{1}{2})\theta + \frac{\pi}{4}\right] + 0(\frac{1}{n}),$$

and

$$Q_n^1(\cos\theta) = \frac{\Gamma(n+2)}{\Gamma(n+\frac{3}{2})} \left(\frac{\pi}{2\pi \sin\theta}\right)^{1/2} \sin\left[(n+\frac{1}{2})\theta + \frac{\pi}{4}\right] + 0(\frac{1}{n}).$$

Thus, for large n , we can find a linear form which represents waves propagating in the θ direction from the pole

$$Ⓗ(\theta) \propto P_n^1(\cos\theta) - \frac{1}{\pi} Q_n^1(\cos\theta) = C \frac{1}{\sqrt{\sin\theta}} e^{-i(n+\frac{1}{2})\theta} + 0(\frac{1}{n})$$

or

$$\text{Ⓗ}(\theta) \propto = \frac{C}{\sqrt{\sin\theta}} e^{-ikr\theta} .$$

In terms of the path length S along the surface of the sphere from the pole,

$$\text{Ⓗ}(S) \propto \frac{C}{\sqrt{\sin\theta}} e^{-ikS}.$$

This form parallels the x dependence obtained in the plane wave case.

Since the solution for H_ϕ is being expressed in terms of $rH_\phi = R$Ⓗ, the expressions for the boundary conditions must be reconsidered.

For the equation of continuity of H, it is clear that if rH is continuous for a fixed value of r, then H is also continuous, so that the continuity condition may be applied to either H or rH.

In the equation for the continuity of E_θ (which corresponds to Ex in the plane geometry), we use the relation

$$\nabla \times H = (\sigma + i\omega\varepsilon)E$$

or

$$-\frac{1}{r}\frac{\partial(rH_\phi)}{\partial r} = (\sigma + i\omega\varepsilon)E_\theta$$

$$-\frac{1}{\sigma + i\omega\varepsilon}\frac{\partial(rH_\phi)}{\partial r} = (rE_\theta).$$

Thus, it is not H but rH = RⒽ which we must differentiate to obtain the value of E_o (or rE_θ) to use in the continuity equation.

In summary, for large radius compared with the wavelengths involved, by changing k to $k(\frac{r_o}{r_i})^2$ in the i^{th} layer and using rH_ϕ and rE_θ in place of H_y and E_x, all of the formulas derived for the plane geometry case may be used for the spherical geometry case as well. The connection of results in the two geometries as $r \to \infty$ is manifest.

IV. ELECTROMAGNETIC PROPAGATION ALONG STRATIFIED MEDIA IN CYLINDRICAL GEOMETRY

Propagation of TM waves in the r direction and in the ϕ direction will be discussed. Since propagation in the r direction is simpler, it will be discussed first.

Consider the geometry shown in Fig. 3.

We shall look for modes which have only an H_ϕ independent of ϕ, and propagate in the r direction.

In each stratum Maxwell's equations reduce to

$$\vec{\nabla}^2 \vec{H} = i\omega\mu(\sigma + i\omega\varepsilon)\vec{H}$$

or

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial H_\phi}{\partial r}\right) + \frac{d^2 H_\phi}{dz^2} - \frac{1}{r^2} H_\phi = i\omega\mu(\sigma + i\omega\varepsilon)H_\phi \ .$$

Using separation of variables, this is written as two equations with $H_\phi = Rz$

$$\frac{1}{Rr}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) - i\omega\mu(\sigma + i\omega\varepsilon) + \frac{1}{r^2})\frac{1}{R} = - b^2$$

and

$$\frac{1}{z}\frac{d^2 z}{dz} = b^2 \ .$$

The separation constant b is to be determined. Multiplying the first of these equations by R, and we obtain Bessel's equations, which has outgoing wave solutions

$$R = H_1^{(2)}(kr)$$

where

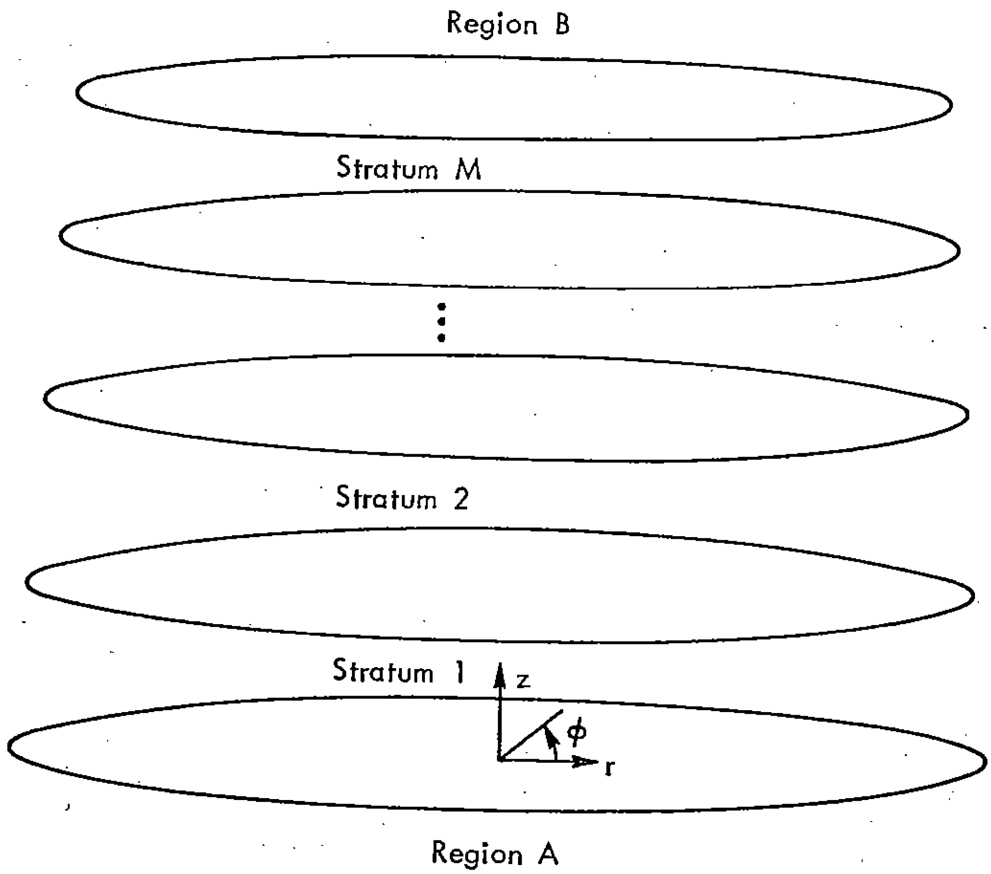

Fig.3—The medium in cylindrical geometry : R, Φ symmetry

$$k = [b^2 - i\omega\mu(\sigma + i\omega\varepsilon)]^{1/2} .$$

Since

$$b = [k^2 + i\omega\mu(\sigma + i\omega\varepsilon)]^{1/2},$$

and the solution for the z variation is

$$z = c_+ e^{bz} + c_- e^{-bz},$$

just as in the plane geometry case, k should correspond to the propagation constant k used in plane geometry. That the correspondence is complete can be seen by noting that for large kr,

$$H_1^{(2)} (\nu) \rightarrow (\frac{i-1}{\sqrt{2}})\sqrt{\frac{2}{\pi kr}}\, e^{-ikr}$$

$$kr \rightarrow \infty$$

and so, aside from constants and the geometrical factor $\frac{1}{\sqrt{r}}$, k gives the wave phase shift and attenuation with distance.

The equations for the continuity of H across the strata interfaces are the same as in plane geometry. The continuity of E_r becomes an equation involving only H through the relation

$$\nabla \times H = (\sigma + i\omega\varepsilon)E$$

or

$$\frac{\partial H_\phi}{\partial z} = (\sigma + i\omega\varepsilon)E_r$$

which is the same equation that was found in plane geometry. It follows that the E_r continuity equations are the same in this cylindrical geometry and in plane geometry. Therefore, the propagation constant k calculated in plane geometry applies, with no change in the calculations, equally to this cylindrical geometry. Only the geometrical factor $\frac{1}{\sqrt{r}}$ and the large distance from the origin compared with the wavelength must be kept in mind when going from one geometry to the other.

CYLINDRICAL GEOMETRY-PROPAGATION IN THE Ø DIRECTION

Now consider the geometry shown in Fig. 4. We shall discuss propagation in the Ø direction. In this orientation, the TM mode has only a z component of H, and that component is independent of z.

In a particular stratum, Maxwell's equations reduce to

$$\vec{\nabla}^2 \vec{H} = i\omega\mu(\sigma + i\omega\varepsilon)\vec{H}$$

or

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial H_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 H_z}{\partial \phi^2} - i\omega\mu(\sigma + i\omega\varepsilon)H_z = 0 \ .$$

Using standard separation of variables, with the substitution $H_z = R(r)\Phi(\phi)$, the equation can be reduced to

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} + [-\, i\omega\mu(\sigma + i\omega\varepsilon)r^2 - \lambda^2]\, R = 0$$

and

$$\frac{d^2\Phi}{d\phi^2} = -\,\lambda^2\Phi \ .$$

The r equation may be put in more familiar form with the substitution

$$\rho = \sqrt{r}R$$

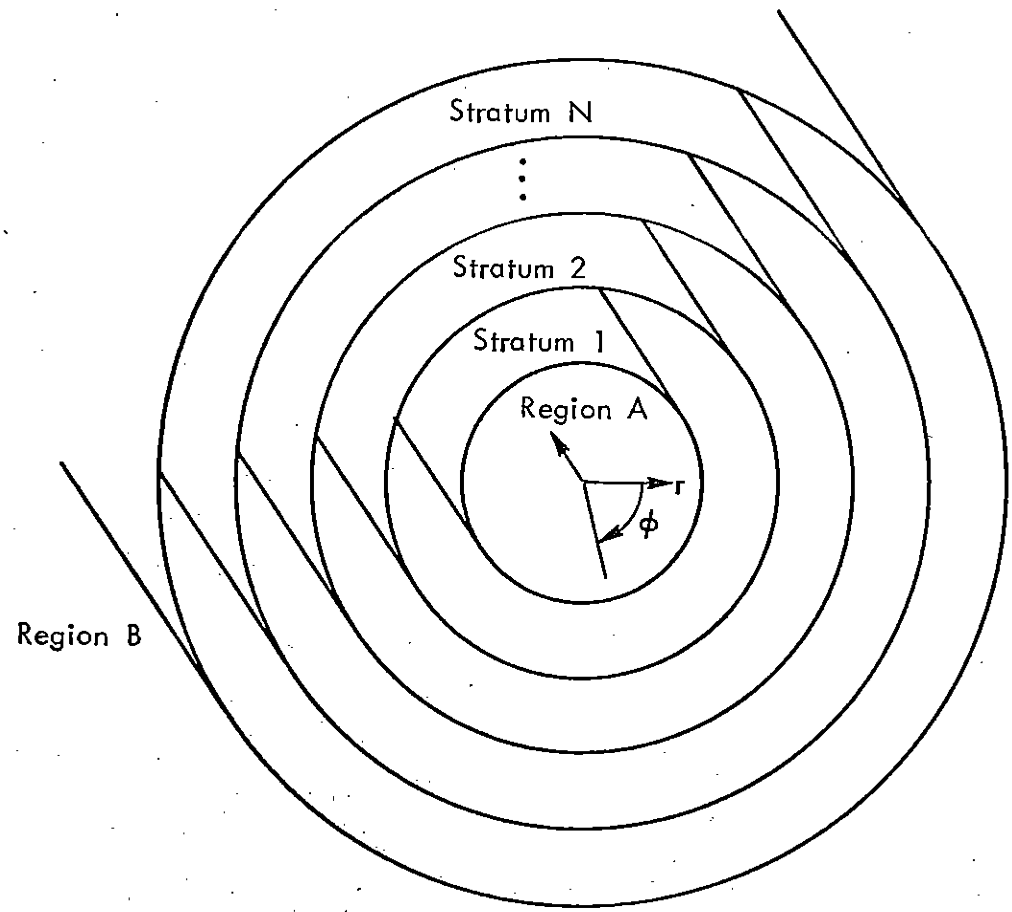

Fig.4—The medium in cylindrical geometry : Φ, Z symmetry

in terms of which the r equation is

$$\frac{d^2\rho}{dr^2} - [i\omega\mu(\sigma + i\omega\varepsilon) + \frac{\lambda^2 - \frac{1}{4}}{r^2}]\rho = 0.$$

When

$$| (\frac{r}{\bar{r}})^2 - 1| << | \frac{i\omega\mu(\sigma + i\omega\varepsilon)r^2}{\lambda^2 - \frac{1}{4}} + 1 |,$$

then the equation may be approximated by

$$\frac{d^2\rho}{dr^2} - [i\omega\mu(\sigma + i\omega\varepsilon) + K^2]\ \rho = 0$$

where

$$K^2 = \frac{\lambda^2 - \frac{1}{4}}{\bar{r}^2}$$

$\bar{r}$ = the mean value of r in the stratum.

With this approximation, the radial equation takes on the same form as the plane geometry z equation.

However, when moving to the next stratum, one must keep in mind that it is the quantity λ^2, not K^2, that is constant throughout the strata. Therefore, in the i^{th} layer, the equation is

$$\frac{d^2\rho}{dr^2} - [i\omega\mu(\sigma + i\omega\varepsilon) + K^2\ \frac{\bar{r}^2}{\bar{r}_i^2}\ ]\ \rho = 0$$

just as in the spherical geometry case, and has solutions

$$\rho = A_+ e^{\left(r\sqrt{i\omega\mu(\sigma+i\omega\varepsilon) + K^2\ \frac{\bar{r}^2}{\bar{r}_i^2}}\right)} + A_- e^{-\left(r\sqrt{i\omega\mu(\sigma+i\omega\varepsilon) + K^2\ \frac{\bar{r}^2}{\bar{r}_i^2}}\right)}.$$

The $\emptyset$ equation has the solution

$$\Phi = C_+ e^{i\lambda\phi} + C_- e^{-i\lambda\phi} \quad .$$

This can also be written

$$\Phi = C_+ e^{i\sqrt{K^2\bar{r}^2 - \frac{1}{4}}\,\phi} + C_- e^{-i\sqrt{K^2\bar{r}^2 - \frac{1}{4}}\,\phi} \quad .$$

If $\bar{r}$ is much greater than the wavelengths of concern, $K^2\bar{r}^2 >> 1$ and the 1/4 may be ignored. Then in terms of the path length S at the reference radius $\bar{r}$ in the ϕ direction,

$$\Phi = C_+ e^{iKS} + C_- e^{-iKS} \quad .$$

For waves moving in the + ϕ direction, $C_+ = 0$.

The continuity equations also must be reviewed. If $\rho\Phi$ is continuous across one r = constant boundary, then $rH_z = \rho\Phi$ is also continuous, and so the H_z continuity equation may equally well be expressed as a $\rho\Phi$ continuity equation. For the E_ϕ continuity, E_ϕ is expressed in terms of H_z by the relation

$$\frac{-\partial H_z}{\partial r} = (\sigma + i\omega\varepsilon)E_\phi$$

or, in terms of $\rho\Phi$,

$$-\ \Phi\left(\frac{1}{\sqrt{r}}\frac{d\rho}{dr} - \frac{1}{2r^{3/2}}\rho\right) = (\sigma + i\omega\varepsilon)E_\phi$$

which may be written

$$\frac{-\Phi}{\sigma+i\omega\varepsilon}\; A_+\left[\frac{\sqrt{i\omega\mu(\sigma+i\omega\varepsilon)+K^2\,\frac{\bar{r}^2}{\bar{r}_i^2}}}{\sqrt{r}} - A_-\;\frac{\sqrt{(i\omega\mu(\sigma+i\omega\varepsilon)+K^2\,\frac{\bar{r}^2}{\bar{r}_i^2}}}{\sqrt{r}} - \frac{\rho}{2r^{3/2}}\right] = E_\emptyset$$

or

$$\frac{-\Phi}{\sigma+i\omega\varepsilon}\left[A_+\left(\frac{\sqrt{i\omega\mu(\sigma+i\omega\varepsilon)+K^2\,\frac{\bar{r}^2}{\bar{r}_i^2}}}{\sqrt{r}} - \frac{1}{2r^{3/2}}\right)\right.$$

$$\left. - A_-\left(\frac{\sqrt{i\omega\mu(\sigma+i\omega\varepsilon)+K^2\,\frac{\bar{r}^2}{\bar{r}_i^2}}}{\sqrt{r}} + \frac{1}{2r^{3/2}}\right)\right] = E_\theta\ ,$$

and so the A_+ or A_- coefficients are somewhat different from those found in the plane and spherical cases.

V. MODE ORTHOGONALITY

With the propagational aspects of individual modes well in hand, the question arises of the propagation of a given electromagnetic field having a specified spatial and temporal distribution. To understand how a given field would propagate, it must first be decomposed into its fourier components at each point over a suitable boundary. Then the spatial distribution of each fourier component must be resolved into the modes discussed earlier. The modes are then propagated individually to the observation point, where the fields are summed over all modes and frequencies to produce the final propagated waveform.* To effect the mode decomposition, one must know the orthogonality relation for the modes.

The relation will be derived for the TM modes in rectangular coordinates; the relation for the TE modes and for cylindrical and spherical coordinates follows in exactly the same manner.

THE ORTHOGONALITY CONDITION IN RECTANGULAR GEOMETRY

When the field variation in the direction of propagation, $e^{i(\omega t-k_\ell x)}$, is factored from the fields, Maxwell's equations may be written

$$-\frac{dH_\ell}{dz} = (\sigma + i\omega\varepsilon)E_{x_\ell}$$

*This assumes a completeness theorem for the modal decomposition. For the complex variables considered here, we have not been able to prove, or to find, such a theorem. In fact, there is reason to believe that there may be other terms (related to branch cuts, as distinguished from poles) required for the decomposition of an arbitrary spatial distribution. These added terms probably do not influence long distance propagation; in any case, we are not alone in ignoring them.

$$- ik_\ell H_\ell = (\sigma + i\omega\varepsilon)E_{z_\ell}$$

$$\frac{dE_{x_\ell}}{dz} + ik_\ell E_{z_\ell} = - i\omega\varepsilon H_\ell$$

where the superscript ℓ denotes the fields corresponding to mode ℓ, having propagation constant k_ℓ. With the elimination of E_z, the second and third equations become

$$\frac{dE_{x_\ell}}{dz} = - (\frac{k_\ell^2}{\sigma+i\omega\varepsilon} + i\omega\mu)H_\ell .$$

If this equation is multiplied by H_m and the resultant equation is subtracted from the similar equation with ℓ and m interchanged, one obtains

$$H_m \frac{dE_{x_\ell}}{dz} - H_\ell \frac{dE_{x_m}}{dz} = \frac{k_m^2 - k_\ell^2}{\sigma+i\omega\varepsilon} H_\ell H_m .$$

Performing similar operations with the equation for dH_ℓ/dz leads to

$$E_{x_\ell} \frac{dH_m}{dz} - E_{x_m} \frac{dH_\ell}{dz} = 0 .$$

The addition of these two bilinear equations produces

$$\frac{d}{dz} (E_{x_\ell} H_m - E_{x_m} H_\ell) = \frac{k_m^2 - k_\ell^2}{\sigma + i\omega\varepsilon} H_\ell H_m .$$

Noting that the fields must go to zero for $z = \pm \infty$, the integration of this equation provides the desired orthogonality condition:

$$(k_m^2 - k_\ell^2) \int_{\infty}^{\infty} \frac{dz}{\sigma + i\omega\varepsilon} H_\ell H_m = 0 \, .$$

In practice, this form of the orthogonality condition is not the most convenient. The fields are usually calculated explicitly over a finite region such as that between planes A and B in Fig. 5 and some boundary condition is imposed at A and B, such as that corresponding to a perfectly conducting sheet, or corresponding to outgoing waves propagating into a uniform medium. Therefore, it is convenient to break the orthogonality integral into three parts: $-\infty$ to A, A to B, and B to ∞.

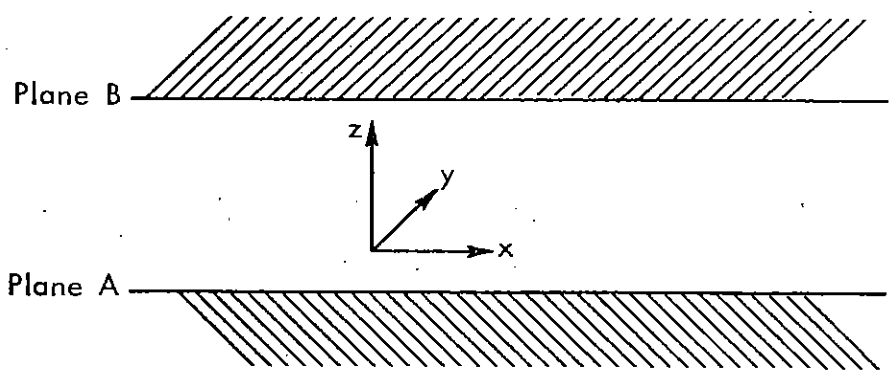

Fig.5—Boundary regions

Since from the bilinear differential equation, over any interval L, U,

$$(k_m^2-k_\ell^2) \int_L^U \frac{dz}{\sigma+i\omega\varepsilon} H_\ell H_m = (E_{x_\ell} H_m - E_{x_m} H_\ell) \Big|_L^U \quad ;$$

it follows that

$$(k_m^2-k_\ell^2) \int_{-\infty}^{\infty} \frac{dz}{\sigma+i\omega\varepsilon} H_\ell H_m = (E_{x_\ell} H_m - E_{x_m} H_\ell) \Big|_B^A + (k_m^2-k_\ell^2) \int_A^B \frac{dz}{\sigma+i\omega\varepsilon} H_\ell H_m = 0$$

and since the fields are usually calculated explicitly in the interval A to B, the most useful form of the orthogonality integral is

$$(k_m^2-k_\ell^2) \int_A^B \frac{dz}{\sigma+i\omega\varepsilon} H_\ell H_m - (E_{x_\ell} H_m - E_{x_m} H_\ell) \Big|_A^B = 0 \ .$$

The normalization constants may be obtained by direct integration.

The orthogonality relation for the fields in cylindrical and spherical coordinate systems follows from Maxwell's equations in a way completely analogous to the above.

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2. Budden, K. G., The Waveguide Mode Theory of Wave Propagation, Logos Press, London, 1961.