Theoretical Notes
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EXTRAPOLATING ELECTROMAGNETIC FIELDS FROM VALUES IN A SPHERICAL REGION

by

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ABSTRACT

Methods are developed for extrapolating electromagnetic fields from values of the spherical harmonic expansion coefficients of the spherical components on a sphere surrounding a source region. The region outside the sphere is considered to be homogeneous, isotropic, source free, and nonconducting, except that application of the method of images allows the introduction of an infinitely conducting ground plane. Solutions of the scalar wave equation are made applicable by expressing the spherical harmonic expansion coefficients of the rectangular components in terms of the coefficients of the spherical components. Two methods are developed for obtaining time dependent solutions for fields at points outside the sphere without invoking Fourier analysis. One method results from inserting the expansions of the rectangular components into the Kirchhoff integral solution of the wave equation. The other method involves eliminating the explicit frequency dependence of the eigenfunction solution of the wave equation by introducing a differential operator to replace the Hankel function. These methods are useful when the source has a very wide-band frequency spectrum, so that it is best described in the time domain. This is the case with electromagnetic pulses from nuclear detonations.

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EXTRAPOLATING ELECTROMAGNETIC FIELDS FROM VALUES IN A SPHERICAL REGION

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I. INTRODUCTION

A. Motivation

A companion report by Lt. W. R. Graham[1] of the Air Force Weapons Laboratory describes a method of calculation of electromagnetic fields within the source having the degree of complexity associated with a surface nuclear detonation. The transient conductivities and currents associated with such a source make the solution of Maxwell's equations a matter for numerical evaluation using difference equations and a high-speed digital computer. However, in the region outside the source region, where detectors or targets might be located, the fields obey the wave equation. It is possible, in principle, to continue the solution of Maxwell's equations to any distance of interest. However, the limitations of present day computers make it desirable to explore the possibility of analytical solutions for extrapolating the solutions to points far beyond the source region. This report describes two

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[1] Graham, W. R., The Electromagnetic Fields Produced by a General Current Distribution in a Conductive Environment under Certain Symmetry Conditions, Air Force Weapons Laboratory, Air Force Systems Command Report WL-TR-64-153 (Dec., 1964), U.

methods that have been found to be practical solutions to the extrapolation problem. These methods were designed to utilize field values in the form generated by Lt. Graham's computer solutions, but are thought to be of more general interest.

B. Problem Postulation

It is assumed that electromagnetic fields are known within and on the surface of a sphere whose center lies in an infinitely conducting ground plane. There may be current sources, charges, and conductivity in the space within the sphere and above the ground plane, but these must be negligible outside the sphere. The problem is to find the fields as a function of position and time in the upper half space outside the sphere. Using the method of images the problem may be replaced by one having only a spherical boundary in which the fields in the lower half space are the images of those in the upper half space. The material outside the sphere is assumed to be homogeneous and isotropic. Attention shall be restricted to those problems where the field possesses azimuthal symmetry about a vertical axis and is zero before the initiation of a disturbance inside the sphere at a time designated by $t = 0$.

C. Review of Separation of Wave Equation

For such a problem the fields, $\vec{E}$ and $\vec{H}$ satisfy the vector wave equation without conductivity or sources which has the form

$$\nabla\nabla\cdot\vec{E} - \nabla\times\nabla\times\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0 \quad .$$

A further simplification can be made if only the rectangular components of the fields are considered. These scalars, E_x, E_y, E_z, H_x, H_y, and H_z, (as well as the corresponding components of the electromagnetic potential), satisfy the scalar wave equation

$$\nabla^2 \psi - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = 0 \quad , \tag{1}$$

where ψ is any of the components mentioned, and c is the velocity of propagation in the medium.

It is still possible to describe the rectangular components in terms of the spherical coordinates, r, θ, and ϕ, if the correct form for the Laplacian operator is used:

$$\nabla^2 \psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2} \quad .$$

The usual method of solving this equation is to reduce it to the Helmholz equation by Fourier analysis of the time dependence of the variable ψ, such that

$$\psi (\vec{x}, t) = \int_{-\infty}^{\infty} \psi (\vec{x}, \omega) e^{-i \omega t} d\omega$$

and

$$(\nabla^2 + k^2) \psi (\vec{x}, \omega) = 0 \quad ,$$

with $k^2 = \omega^2 / c^2$. The separation of the Helmholz equation into three equations by use of the substitution

$$\psi(x, \omega) = f(r)\, \Theta(\theta)\, \Phi(\phi)$$

yields the differential equations

$$\frac{d^2\Phi}{d\phi^2} + m^2 \Phi = 0$$

$$\frac{1}{\sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{d\Theta}{d\theta}\right) + \left[n(n+1) - \frac{m^2}{\sin^2\theta} \right] \Theta = 0$$

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{df}{dr}\right) + \left[k^2 - \frac{n(n+1)}{r^2} \right] f = 0 \quad ,$$

where m and n are integers. The equation for f is identified with Bessel's equation, and the Θ equation can be reduced to Legendre's equation. The Φ equation has normalized solutions of the form

$$\Phi = \frac{1}{\sqrt{2\pi}} e^{im\phi} \quad .$$

The general solution of interest for Θ is the normalized associated Legendre function of the first kind:

$$\Theta = \overline{P_n^m}(\cos\theta) = \sqrt{\frac{2n+1}{2} \frac{(n-m)!}{(n+m)!}}\; P_n^m(\cos\theta) \quad .$$

Since the present problem is confined to outgoing waves, the radial function of interest is the spherical Hankel function of the first kind:*

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* The fact that an outgoing wave is associated with a Hankel function of the first kind rather than of the second kind is associated with the choice of convention for defining $\psi(\vec{x}, \omega)$ in terms of $\psi(\vec{x}, t)$ using a negative exponential.

$$f(r) = h_n^{(1)}(kr)$$

$$= \left(\frac{\pi}{2kr}\right)^{1/2} \left[J_{n+\frac{1}{2}}(kr) + i N_{n+\frac{1}{2}}(kr) \right] ,$$

where $J_{n+\frac{1}{2}}$ and $N_{n+\frac{1}{2}}$ are the half-odd-integer order Bessel and Neumann functions that arise in problems of cylindrical symmetry. Thus the general solution of Eq. (1) may be written in the form

$$\psi(\vec{x}, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_{nm}(\omega) \, h_n^{(1)}(kr) \, \overline{P_n^m}(\cos\theta) \, e^{i(m\phi-\omega t)} \, d\omega \quad , \quad (2)$$

where the A_{nm} are arbitrary complex functions of ω.

D. Expansion of the Rectangular Components

The general eigenfunction method of solving the postulated problem would then proceed from a knowledge of the function ψ only on the surface of the sphere for which the fields are specified. Assume that the fields are given as spherical components expanded in a Legendre series:

$$E_r(r, \theta, t) = \sum_{n=1}^{\infty} E_{nr}(r, t) \, \overline{P}_n(\cos\theta) \quad (3)$$

$$E_\theta(r, \theta, t) = \sum_{n=1}^{\infty} E_{n\theta}(r, t) \, \overline{P_n^1}(\cos\theta)$$

$$H_{\phi}(r, \theta, t) = \sum_{n=1}^{\infty} H_{n\phi}(r, t)\, \overline{P_n^1}(\cos\theta) \quad .$$

Note that E_r is expanded in the normal Legendre functions, whereas E_θ and H_ϕ are expanded in the associated Legendre functions with index m equal to 1. This assumption results from the formulation of the near field problem. It turns out to be a quite fortunate choice for present purposes. All the other spherical components are zero since azimuthal symmetry is assumed. It is desired to find the fields at another position (r', θ', ϕ') outside the sphere. The assumption of azimuthal symmetry permits ϕ' to be chosen to be 0 without loss of generality. Some of the geometric relations are illustrated by Fig. 1. The fact that it may be desirable to obtain

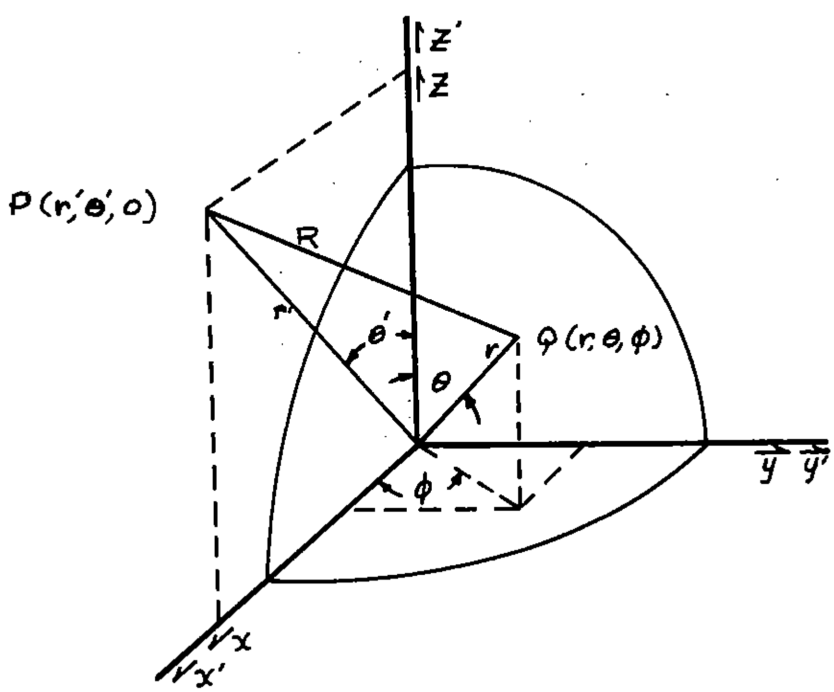

Fig. 1 - Problem Geometry

as results $E_{r'}$, $E_{\theta'}$, and $H_{\phi'}$, rather than the rectangular components causes very little difficulty, since one has the relations

$$E_{r'} = E_{x'} \sin\theta' + E_{z'} \cos\theta' \qquad (4)$$

$$E_{\theta'} = E_{x'} \cos\theta' - E_{z'} \sin\theta'$$

$$H_{\phi'} = H_{y'}$$

and the only rectangular components one must solve for are $E_{x'}$, $E_{z'}$, and $H_{y'}$; i.e., these are the functions ψ in Eq. (2).

At point Q on the sphere the situation is more complicated. First, one needs field values for all ϕ to describe the boundary conditions. The rotation matrix yields

$$E_x = (E_r \sin\theta + E_\theta \cos\theta) \cos\phi \qquad (5)$$

$$E_z = E_r \cos\theta - E_\theta \sin\theta$$

$$H_y = H_\phi \cos\phi \quad .$$

Next, the method calls for the boundary conditions to be expressed in terms of the spherical harmonics:

$$Y_{nm}(\theta, \phi) = \Phi_m(\phi)\, \Theta_{nm}(\theta)$$

and simply inserting Eqs. (3) into Eqs. (4) does not yield answers in this form. For the E components one obtains θ functions of the form $\sin\theta$ $\overline{P}_n(\cos\theta)$, $\cos\theta\, \overline{P}_n(\cos\theta)$, $\sin\theta\, \overline{P_n^1}(\cos\theta)$, and $\cos\theta\, \overline{P_n^1}(\cos\theta)$. These can be put in the form of the associated Legendre functions using the following relations, given by Condon and Shortley[2] on page 53.

$$\sin\theta\, \overline{P_n^m}(\cos\theta) = -\sqrt{\frac{(m+n+1)(m+n+2)}{(2n+1)(2n+3)}}\, \overline{P_{n+1}^{m+1}}(\cos\theta) + \sqrt{\frac{(n-m)(n-m-1)}{(2n-1)(2n+1)}}\, \overline{P_{n-1}^{m+1}}(\cos\theta)$$

or

$$\sin\theta\, \overline{P_n^m}(\cos\theta) = \sqrt{\frac{(n-m+1)(n-m+2)}{(2n+1)(2n+3)}}\, \overline{P_{n+1}^{m-1}}(\cos\theta) - \sqrt{\frac{(n+m)(n+m-1)}{(2n-1)(2n+1)}}\, \overline{P_{n-1}^{m-1}}(\cos\theta) \quad (6)$$

and

$$\cos\theta\, \overline{P_n^m}(\cos\theta) = \sqrt{\frac{(n+1-m)(n+1+m)}{(2n+1)(2n+3)}}\, \overline{P_{n+1}^{m}}(\cos\theta) + \sqrt{\frac{(n+m)(n-m)}{(2n-1)(2n+1)}}\, \overline{P_{n-1}^{m}}(\cos\theta)$$

Combining Eqs. (3), (5), and (6) one can obtain the desired forms

$$E_x = \sum_{n=2}^{\infty} \frac{E_{xn2}}{2} \left[Y_{n,1}(\theta,\phi) + Y_{n,1}^{*}(\theta,\phi) \right]$$

$$E_z = \sum_{n=0}^{\infty} E_{zn2}\, Y_{n,0}(\theta,\phi) \quad (7)$$

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[2] Condon, E. U., and Shortley, G. H., The Theory of Atomic Spectra, Cambridge University Press, London (1953).

$$H_y = \sum_{n=1}^{\infty} \frac{H_{yn2}}{2} \left[Y_{n,1}(\theta, \phi) + Y^*_{n,1}(\theta, \phi) \right] ,$$

where the coefficients are given by:

$$\sqrt{2\pi}\, E_{xn2} = -\sqrt{\frac{(n+1)\, n}{(2n-1)(2n+1)}}\, E_{n-1,r} + \sqrt{\frac{(n-1)(n+1)}{(2n-1)(2n+1)}}\, E_{n-1,\theta}$$

$$+ \sqrt{\frac{(n+1)\, n}{(2n+1)(2n+3)}}\, E_{n+1,r} + \sqrt{\frac{(n+2)\, n}{(2n-1)(2n+3)}}\, E_{n+1,\theta} \qquad (8)$$

$$\sqrt{2\pi}\, E_{zn2} = \sqrt{\frac{n^2}{(2n-1)(2n+1)}}\, E_{n-1,r} - \sqrt{\frac{(n-1)\, n}{(2n-1)(2n+1)}}\, E_{n-1,\theta}$$

$$+ \sqrt{\frac{(n+1)^2}{(2n+1)(2n+3)}}\, E_{n+1,r} + \sqrt{\frac{(n+2)(n+1)}{(2n+1)(2n+3)}}\, E_{n+1,\theta}$$

$$\sqrt{2\pi}\, H_{yn2} = H_{\phi n} .$$

E. The Eigenfunction Method of Solution

Comparing Eqs. (8) with Eq. (2) and using the relation

$$Y_{n,m}(\theta, \phi) = (-1)^m Y^*_{n,-m}(\theta, \phi) ,$$

it is found that for E_x and H_y :

$$A_{n,m,x} = A_{n,m,y} = 0 \text{ for } |m| \neq 1$$

$$A_{n,1,x} = -A_{n,-1,x} , \quad A_{n,1,y} = -A_{n,-1,y}$$

and for E_z

$$A_{nmz} = 0 \text{ for } m \neq 0 \quad .$$

The relations between the coefficients are

$$E_{xn2}(r,t) = 2 \int_{-\infty}^{\infty} A_{n,1,x} h_n^{(1)}(kr) e^{-i\omega t} d\omega$$

$$E_{zn2}(r,t) = \int_{-\infty}^{\infty} A_{n,0,z} h_n^{(1)}(kr) e^{-i\omega t} d\omega$$

$$H_{yn2}(r,t) = 2 \int_{-\infty}^{\infty} A_{n,1,y} h_n^{(1)}(kr) e^{-i\omega t} d\omega \quad .$$

Thus the coefficients are given by

$$A_{n,1,x}(\omega) = \frac{1}{4\pi h_n^{(1)}(kr)} \int_0^{\infty} E_{xn2}(r,t) e^{i\omega t} dt$$

$$A_{n,0,z}(\omega) = \frac{1}{2\pi h_n^{(1)}(kr)} \int_0^{\infty} E_{zn2}(r,t) e^{i\omega t} dt \qquad (9)$$

$$A_{n,1,y}(\omega) = \frac{1}{4\pi h_n^{(1)}(kr)} \int_0^{\infty} H_{yn2}(r,t) e^{i\omega t} dt \quad .$$

Thus, given the fields at a particular r, one can obtain all the coefficients in Eq. (2) for each of the three variables. The rectangular components of the fields at Q (r', θ', ϕ') are then given by

$$E_{x'} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} (A_{n,1,x} - A_{n,-1,x})\, h_n^{(1)}(kr')\, \overline{P}_n^1(\cos\theta') \cos\phi'\, e^{-i\omega t}\, d\omega$$

$$E_{z'} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} A_{n,0,z}\, h_n^{(1)}(kr')\, \overline{P}_n(\cos\theta')\, e^{-i\omega t}\, d\omega \qquad (10)$$

$$H_{y'} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} (A_{n,1,y} - A_{n,-1,y})\, h_n^{(1)}(kr)\, \overline{P}_n^1(\cos\theta') \cos\phi'\, e^{-i\omega t}\, d\omega \, .$$

Application of Eqs. (4) completes the problem.

If the frequency spectrum of the signal at P (r', θ', ϕ') is the desired answer, then the integration of Eqs. (10) need not be performed, and the eigenfunction method is probably the best method of obtaining answers. However, if the desired answers are time histories of the field components,there are two alternative methods that will be developed in the next two sections that look more attractive. These methods solve the problem in the time domain, while keeping part, but not all, of the eigenfunction formalism. In that they use a spherical harmonic expansion and the scalar wave equation, they require the expansion of the rectangular components introduced above.

Basically, they are thought to be better for generating time histories because they require only one numerical integration (over retarded time) in addition to the summation over the index n. Equations (9) and (10) really represent a double numerical integration and a summation. Also, the fact that each of these integrals is highly periodic indicates that there may be serious numerical analysis difficulties connected with their evaluation.

One important consequence of the above eigenfunction theory is that it demonstrates, by the uniqueness of the coefficients $A_{n,m}$, that the values of the fields on the surface of the sphere are necessary and sufficient to determine the fields outside the sphere. That is, they constitute a proper set of boundary conditions for the wave equation.

II. THE HANKEL OPERATOR METHOD

A. General Theory

In this section the starting point will be Eq. (2) of the preceding section. The goal is to obtain a form of the general solution of the scaler wave equation for outgoing waves which is equivalent to Eq. (2) but does not contain an integral over ω. That is, a solution which does not explicitly contain the frequency spectrum of the waves is sought. This will be accomplished by expressing the spherical Hankel function $h_n^{(1)}(kr)$ in a particular form which allows one to remove its r-dependent part from under the integral sign in Eq. (2). When this has been accomplished, one finds that the retarded time, $t^* = t-r/c$, has replaced t in the integrand of Eq. (2) and the integral becomes simply the Fourier transform of the arbitrary functions $A_{nm}(\omega)$ (times $-i/k^{n+1}$) from the frequency domain to the retarded time domain. Since the functions $A_{nm}(\omega)$ are arbitrary, the functions of retarded time resulting from the Fourier transform are also arbitrary. Hence, the desired form of the general solution in the time domain is found. The arbitrary functions of retarded time can be interpreted physically as representing multipole sources at the origin.

The general method used here was brought to the attention of the authors by a report by John S. Wicklund[3]. He used it to develop a technique to extrapolate the spherical field components using a dipole approximation.

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[3] Wicklund, John S., Extrapolation of the Electromagnetic Field, TR-1058, Diamond Ordnance Fuze Laboratories, 2 July 1962.

The spherical Hankel function of the first kind can be written

$$h_n^{(1)}(\rho) = \frac{(-i)^{n+1}\, e^{i\rho}}{\rho} \left[P_{n+\frac{1}{2}}(\rho) + i\, Q_{n+\frac{1}{2}}(\rho) \right] \tag{11}$$

$$P_{n+\frac{1}{2}}(\rho) = 1 + \sum_{j=1}^{\substack{n/2,\ n \text{ even} \\ (n-1)/2,\ n \text{ odd}}} (-1)^j \frac{n(n^2-1)(n^2-4)\cdots\left[n^2-(2j-1)^2\right](n+2j)}{2^{2j}\,(2j)!\,\rho^{2j}}$$

$$Q_{n+\frac{1}{2}}(\rho) = \frac{n(n+1)}{2\rho} + \sum_{j=1}^{\substack{(n-2)/2,\ n \text{ even} \\ (n-1)/2,\ n \text{ odd}}} (-1)^j \frac{n(n^2-1)(n^2-4)\cdots\left[n^2-(2j)^2\right](n+2j+1)}{2^{2j+1}\,(2j+1)!\,\rho^{2j+1}} .$$

By combining the expansions for $P_{n+\frac{1}{2}}$ and $i\,Q_{n+\frac{1}{2}}$, and bringing the factor $(-i)^n$ into the brackets, $h_n^{(1)}$ can be written

$$h_n^{(1)}(\rho) = \frac{(-i)\, e^{i\rho}}{\rho} \Bigg\{ (-i)^n + (-i)^{n-1} \frac{n(n+1)}{2 \cdot 1!\, \rho} + (-i)^{n-2} \frac{n(n^2-1)(n+2)}{2^2 \cdot 2!\, \rho^2} + \cdots \tag{12}$$

$$+ (-i)^{n-j} \frac{n(n^2-1)(n^2-4)\cdots\left[n^2-(j-1)^2\right](n+j)}{2^j \cdot j!\, \rho^j}$$

$$+ \frac{n(n^2-1)(n^2-4)\cdots\left[n^2-(n-1)^2\right](2n)}{2^n \cdot n!\, \rho^n} \Bigg\} .$$

Substituting (kr) for ρ in Eq. (12) and introducing t as a dummy variable, one can write $h_n^{(1)}(kr)$ as

$$h_n^{(1)}(kr) = \exp(i\omega t) \left\{ \frac{1}{c^n r} \frac{d^n}{dt^{*n}} + \frac{n(n+1)}{2 \cdot 1!\, c^{n-1} r^2} \frac{d^{n-1}}{dt^{*n-1}} + \frac{n(n^2-1)(n+2)}{2^2 \cdot 2!\, c^{n-2} r^3} \frac{d^{n-2}}{dt^{*n-2}} + \cdots \right.$$

$$+\frac{n(n^2-1)(n^2-4)\cdots[n^2-(j-1)^2]\ (n+j)}{2^j\ j!\ c^{n-j}\ r^{j+1}}\ \frac{d^{n-j}}{d t^{*n-j}} + \cdots \tag{13}$$

$$+\left.\frac{n(n^2-1)(n^2-4)\cdots[n^2-(n-1)^2]\ (2n)}{2^n\ n!\ r^{n+1}}\right\}\ \frac{\exp[-i(\omega t^* + \pi/2)]}{k^{n+1}}\ ,$$

where $t^* = t - r/c$.

For brevity, the symbol $\Xi_n(r)$ will henceforth be used to represent the operator in the braces of Eq. (13). Using this shorthand notation, $h_n^{(1)}(kr)$ can be written simply as

$$h_n^{(1)}(kr) = \exp(i\omega t)\,\Xi_n(r)\left[e^{-i(\omega t^* + \pi/2)}/k^{n+1}\right] \quad . \tag{14}$$

Equation (2) can now be written

$$\psi(\vec{x}, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} A_{nm}(\omega)\,\Xi_n(r)\left[\frac{e^{-i(\omega t^* + \pi/2)}}{k^{n+1}}\right]$$

$$\times\ \overline{P}_n^m(\cos\theta)\, e^{im\phi}\ d\omega \quad . \tag{15}$$

Since the operator $\Xi_n(r)$ does not involve ω it can be removed from under the integral sign (assuming, of course, that $A_{nm}(\omega)$ is sufficiently well-behaved). The functions $\overline{P}_n^m(\cos\theta)$ and $e^{im\phi}$ can likewise be taken from under the integral sign. Thus one is allowed to write Eq. (15) in the form

$$\psi(\vec{x}, t) = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \Xi_n(r) \int_{-\infty}^{+\infty} \frac{A_{nm}(\omega) e^{-i(\omega t^* + \pi/2)}}{k^{n+1}}\ d\omega$$

$$\times\ \overline{P}_n^m(\cos\theta)\, e^{im\phi} \quad . \tag{16}$$

Equation (16) can be rewritten with $(\cos m\phi + i \sin m\phi)$ substituted for $e^{im\phi}$ as follows

$$\psi(\vec{x}, t) = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \Xi_n(r) \left\{ A'_{n0}(t^*) \overline{P}_n(\cos\theta) + \sum_{m=1}^{n} \left[A'_{nm}(t^*) \cos m\phi + B'_{nm}(t^*) \sin m\phi \right] \overline{P}_n^m \right\} , \tag{17}$$

where $$A'_{n0}(t^*) = \int_{-\infty}^{+\infty} \frac{A_{n0}(\omega)\, e^{-i(\omega t^* + \pi/2)}}{k^{n+1}}\, d\omega$$

$$A'_{nm}(t^*) = \int_{-\infty}^{+\infty} \frac{[A_{n,m}(\omega) + A_{n,-m}(\omega)]\, e^{-i(\omega t^* + \pi/2)}}{k^{n+1}}\, d\omega$$

$$B'_{nm}(t^*) = \int_{-\infty}^{+\infty} \frac{[A_{n,m}(\omega) - A_{n,-m}(\omega)]\, e^{-i(\omega t^* + \pi/2)}}{k^{n+1}}\, d\omega .$$

Since the functions $A_{nm}(\omega)$ are arbitrary complex functions of ω, they can be chosen such as to make $A'_{nm}(t^*)$ and $B'_{nm}(t^*)$ any functions of t^* desired. Thus Eq. (17) is the general solution of the scaler wave equation for outgoing waves containing the arbitrary functions A'_{nm} and B'_{nm}. The operator $\Xi_n(r)$ will be referred to as the Hankel operator.

B. The Boundary Value Problem - Method of Solution

Now that the general solution has been obtained in the time domain, the problem of determining the arbitrary functions A'_{nm} and B'_{nm}

for given field values on the surface of a sphere will be attacked. Suppose that on the surface of a sphere of radius r_o the field ψ has the spherical harmonic expansion

$$\psi(\theta, \phi, t) = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \left\{ \alpha'_{n0}(t^*) \overline{P}_n(\cos\theta) + \sum_{m=1}^{n} \left[\alpha'_{nm}(t^*) \cos m\phi + \beta'_{nm}(t^*) \sin m\phi \right] \overline{P}_n^m(\cos\theta) \right\}, \quad (18)$$

where $t^* = t - r_o/c$.

Then by equating coefficients in Eq. (17) and Eq. (18) one has

$$\Xi_n(r_o) A'_{nm}(t^*) = \alpha'_{nm}(t^*)$$

and $$\Xi_n(r_o) B'_{nm}(t^*) = \beta'_{nm}(t^*), \; n = 0, 1, \ldots, \infty; \; m = 0, \ldots, n. \quad (19)$$

The problem becomes that of solving the ordinary differential equations of Eq. (19).

In solving Eqs. (19) it is convenient to define a dimensionless retarded time by the formula $\tau = t^* c/r$. It must be remembered that τ only has meaning at radius r. At another radius, $\tau' = t^* c/r' = \tau r/r'$. In terms of the dimensionless retarded time, the Hankel operator can be written

$$\Xi_n(r) = \frac{1}{r^{n+1}} \left\{ \frac{d^n}{d\tau^n} + \frac{n(n+1)}{2 \cdot 1!} \frac{d^{n-1}}{d\tau^{n-1}} + \frac{n(n^2-1)(n+2)}{2^2 \cdot 2!} \frac{d^{n-2}}{d\tau^{n-2}} + \cdots \right.$$

$$+ \frac{n(n^2-1)(n^2-4)(n^2-9)\cdots[n^2-(j-1)^2]\,(n+j)}{2^j\, j!} \frac{d^{n-j}}{d\tau^{n-j}} + \cdots \tag{20}$$

$$\left. + \frac{n(n^2-1)(n^2-4)(n^2-9)\cdots[n^2-(n-1)^2]\,(2n)}{2^n\, n!} \right\} .$$

The equations to be solved can now be written

$$\Xi_n(r_o) A_{nm}(\tau_o) = \alpha_{nm}(\tau_o)$$

and

$$\Xi_n(r_o) B_{nm}(\tau_o) = \beta_{nm}(\tau_o) \quad , \tag{21}$$

where $\tau_o = t^* c/r_o$, $A_{nm}(\tau_o) = A'_{nm}(t^*) = A'_{nm}(\tau_o r_o / c)$,

$B_{nm}(\tau_o) = B'_{nm}(\tau_o r_o / c)$, $\alpha_{nm}(\tau_o) = \alpha'_{nm}(t_o r_o / c)$, and

$\beta_{nm}(\tau_o) = \beta'_{nm}(\tau_o r_o / c)$.

To solve Eqs. (21), Green's functions will be found that satisfy the equations

$$\Xi_n(r_o) G_n(\tau_o, \tau'_o) = \delta(\tau_o - \tau'_o) , \quad n = 0, \ldots, \infty \quad , \tag{22}$$

with the initial conditions, $G_n(0,\tau'_o) = G_n^{(1)}(0, \tau'_o) = \ldots = G_n^{(n-1)}(0,\tau'_o) = 0$, where $G_n^{(j)}(0, \tau'_o) = d^j G_n(\tau_o, \tau'_o) / d\tau_o^j \Big|_{\tau_o = 0}$. The source functions A_{nm} and B_{nm} will then be given by

$$A_{nm}(\tau_o) = \int_0^{\tau_o} G_n(\tau_o, \tau'_o)\, \alpha_{nm}(\tau'_o)\, d\tau'_o \tag{23}$$

and

$$B_{nm}(\tau_o) = \int_0^{\tau_o} G_n(\tau_o, \tau'_o)\, \beta_{nm}(\tau'_o)\, d\tau'_o \quad .$$

To solve Eqs. (22) the homogeneous equations must first be solved. Since the equations are linear with constant coefficients, this is merely a matter of finding the roots of the auxiliary equation which, for Eq. (22), is

$$\begin{aligned} z^n &+ \frac{n(n+1)}{2 \cdot 1!} z^{n-1} + \frac{n(n^2-1)(n+2)}{2^2 \cdot 2!} z^{n-2} + \dots \\ &+ \frac{n(n^2-1)(n^2-4)(n^2-9) \dots [n^2-(j-1)^2]\,(n+j)}{2^j\, j!} z^{n-j} + \dots \\ &+ \frac{n(n^2-1)(n^2-4)(n^2-9) \dots [n^2-(n-1)^2]\,(2n)}{2^n \cdot n!} = F_n(z) = 0 \quad . \end{aligned} \tag{24}$$

The roots of $F_n(z)$ are the roots of $H^{(1)}_{n+\frac{1}{2}}(iz)$, where $H^{(1)}_{n+\frac{1}{2}}(iz)$ is the half-odd-integer order Hankel function of the first kind. This can be seen by identifying the $P_{n+\frac{1}{2}}(\rho)$ and the $Q_{n+\frac{1}{2}}(\rho)$ following Eq. (11) with the similar functions in Jahnke and Emde[4] using the relations on page 358

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[4] Jahnke, E., and Emde, F., Tables of Functions (Dover Publications, 1945), pp. 136, 137.

of Stratton[5]. In Jahnke and Emde's notation, $F_n(z) = z^n S_{n+\frac{1}{2}}(2z) = z^n \sqrt{\frac{1}{2}\pi z}$ $\exp(z)(i)^{n+3/2}\ H^{(1)}_{n+\frac{1}{2}}(iz)$. Note that $H^{(1)}_{n+\frac{1}{2}}(iz)$ has a singular point at $z = 0$ which annihilates the zero and branch point in its coefficient; thus $F_n(z)$ is analytic and nonzero at $z = 0$. The general behavior of the roots of $H^{(1)}_{n+\frac{1}{2}}(iz)$ can be deduced from the graph on page 243 of Jahnke and Emde (Ref. 4). It is found that for n odd, $F_n(z)$ has one real negative root and (n-1) complex roots which appear in complex conjugate pairs and have negative real parts. For n even, all of the n roots of $F_n(z)$ are complex (appearing, of course, in complex conjugate pairs) and have negative real parts. It is significant to note that all the roots are distinct. Thus, the solution of the homogeneous equation can now be written explicitly. Let $D_n(\tau_o)$ satisfy the equation $\Xi_n(r_o)\, D_n(\tau_o) = 0$, then

$$D_n(\tau_o) = \sum_{j=1}^{\substack{n/2,\ n \text{ even} \\ (n-1)/2,\ n \text{ odd}}} \exp(p_{nj}\tau_o) \left(c_j' \sin q_{nj}\,\tau_o + d_j' \cos q_{nj}\,\tau_o \right) \tag{25}$$

$$+ f' \exp\left(p_{n\frac{n+1}{2}}\,\tau_o\right) \quad ;$$

where $f' = 0$ if n is even; c_j', d_j', and f' are arbitrary constants; the complex roots of $F_n(z)$ are given by $p_{nj} \pm i\, q_{nj}$; the real root of $F_n(z)$ (if n is odd) is $p_{n\frac{n+1}{2}}$. To find the Green's functions, the arbitrary

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[5] Stratton, J. A., Electromagnetic Theory, McGraw Hill, New York (1941).

constants in Eq. (25) must be evaluated such that the resulting expression satisfies Eqs. (22) with their initial conditions. To facilitate this operation it is convenient to define the constants slightly differently. Let the Green's function be given by

$$G_n(\tau_o, \tau_o') = r_o^{n+1} \left\{ \sum_{j=1}^{\substack{n/2,\, n \text{ even} \\ (n-1)/2,\, n \text{ odd}}} \exp\left[p_{nj}(\tau_o - \tau_o')\right] \left[c_{nj} \sin q_{nj}(\tau_o - \tau_o') + d_{nj} \cos q_{nj}(\tau_o - \tau_o')\right] + f_n \exp\left[p_{n\frac{n+1}{2}}(\tau_o - \tau_o')\right] \right\}, \quad (26)$$

where, again, $f_n = 0$ if n is even. The derivatives of this expression for $G_n(\tau_o, \tau_o')$ can be written

$$\frac{d^k G_n(\tau_o, \tau_o')}{d\tau_o^k} = r_o^{n+1} \left\{ \sum_{j=1}^{\substack{n/2,\, n \text{ even} \\ (n-1)/2,\, n \text{ odd}}} r_{nj}^k \exp\left[p_{nj}(\tau_o - \tau_o')\right] \left[(d_{nj} \cos k\theta_{nj} + c_{nj} \sin k\theta_{nj}) \cos q_{nj}(\tau_o - \tau_o') + (c_{nj} \cos k\theta_{nj} - d_{nj} \sin k\theta_{nj}) \sin q_{nj}(\tau_o - \tau_o')\right] + f_n p_{n\frac{n+1}{2}}^k \exp\left[p_{n\frac{n+1}{2}}(\tau_o - \tau_o')\right] \right\}, \quad (27)$$

where $r_{nj} \exp(i\,\theta_{nj}) = p_{nj} + i\,q_{nj}$. The initial conditions below Eq. (22) state that, if two sets of constants c_{nj}, d_{nj}, and f_n are defined, the first applying for $0 \leq \tau_o < \tau'_o$ and another for $0 < \tau'_o < \tau_o$, then the first set is given by

$$\sum_{j=1}^{\substack{n/2,\, n \text{ even} \\ (n-1)/2,\, n \text{ odd}}} r_{nj}^k \exp(-p_{nj}\tau'_o) \Big[(d_{nj} \cos k\,\theta_{nj} + c_{nj} \sin k\,\theta_{nj}) \cos(-q_{nj}\tau'_o)$$

$$+ (c_{nj} \cos k\,\theta_{nj} - d_{nj} \sin k\,\theta_{nj}) \sin(-q_{nj}\tau'_o)\Big] \tag{28}$$

$$+ f_n p_{n\frac{n+1}{2}}^k \exp(-p_{n\frac{n+1}{2}}\tau'_o) = 0, \; k=0, 1, \ldots, n-1$$

for all τ'_o (>0). Thus, for $\tau_o < \tau'_o$, all the constants are zero and the Green's function is zero. Integration of both sides of Eq. (22) from $\tau'_o - \epsilon$ to $\tau'_o + \epsilon$ and taking the limit as $\epsilon \longrightarrow 0$ reveals that $d^{n-1} G_n(\tau_o, \tau'_o)/d\tau_o^{n-1}$ must have a positive discontinuous jump of magnitude r_o^{n+1} at the point $\tau_o = \tau'_o$. Thus

$$\lim_{\epsilon \to 0} d^{n-1} G_n(\tau_o, \tau'_o)/d\tau_o^{n-1} \Big|_{\tau_o = \tau'_o + \epsilon} = r_o^{n+1}$$

Since the lower order derivatives must be continuous at $\tau_o = \tau'_o$ to satisfy Eq. (22), the equations determining the constants in the Green's function for $\tau_o > \tau'_o$ are

$$\sum_{j=1}^{\substack{n/2,\, n \text{ even} \\ (n-1)/2,\, n \text{ odd}}} r_{nj}^k (d_{nj} \cos k\,\theta_{nj} + c_{nj} \sin k\,\theta_{nj}) + f_n p_{n\frac{n+1}{2}}^k = \delta_{k,\, n-1} ,$$

$$k = 0, 1, \ldots, n-1 \quad , \tag{29}$$

where δ_{ij} is the Kronecker delta.

Now that the Green's functions are determined, the next step will be to find explicit expressions for $\Xi_n(r)\,A_{nm}(\tau_o)$ and $\Xi_n(r)\,B_{nm}(\tau_o)$. Terms of the form

$$\frac{1}{r^{n+1}}\;\frac{d^k A_{nm}(\tau_o)}{d\tau^k}\;,\quad k=1,\,2,\,\ldots.,\,n$$

must be evaluated. By succeeding differentiations of Eqs. (23) one obtains

$$\frac{d^k A_{nm}(\tau_o)}{d\tau_o^k} = \int_0^{\tau_o} G_n^{(k)}(\tau_o,\tau_o')\,\alpha_{nm}(\tau_o')\,d\tau_o'\;,\quad k=1,\,\ldots.,\,n-1 \tag{30}$$

$$\frac{d^n A_{nm}(\tau_o)}{d\tau_o^n} = \int_0^{\tau_o} G_n^{(n)}(\tau_o,\tau_o')\,\alpha_{nm}(\tau_o')\,d\tau_o' + r_o^{n+1}\,\alpha_{nm}(\tau_o)\quad ,$$

where $G_n^{(j)}(\tau_o,\tau_o') = d^j G_n(\tau_o,\tau_o')/d\tau_o^j$. Noting that $\tau_o = \tau\, r/r_o$ and hence that $\frac{d^k}{d\tau^k} = \left(\frac{r}{r_o}\right)^k \frac{d^k}{d\tau_o^k}$, one can write

$$\frac{1}{r^{n+1}}\;\frac{d^k A_{nm}}{d\tau^k} = \frac{1}{r^{n-k+1}\,r_o^k}\left\{\int_0^{\tau r/r_o} G_n^{(k)}(\tau r/r_o,\tau_o')\,\alpha_{nm}(\tau_o')\,d\tau_o' + \delta_{kn}\,r_o^{n+1}\,\alpha_{nm}(\tau_o)\right\}\;,\quad k=1,\,2,\,\ldots.,\,n\quad . \tag{31}$$

To obtain the final expression for the terms of $\Xi_n(r) A_{nm}$, the expression for $G_n^{(k)}$ given by Eq. (27) is substituted in Eq. (31) which yields

$$\frac{1}{r^{n+1}} \frac{d^k A_{nm}}{d\tau^k} = \left(\frac{r_o}{r}\right)^{n-k+1} \left\{ \int_0^{\frac{t^* c}{r_o}} \left[\sum_{j=1}^{\substack{n/2,\, n \text{ even} \\ (n-1)/2,\, n \text{ odd}}} r_{nj}^k \right.\right.$$

$$\times \exp\left[p_{nj}\left(\frac{t^* c}{r_o} - \tau'_o\right)\right] \left[\left(d_{nj} \cos k\,\theta_{nj} + c_{nj} \sin k\,\theta_{nj}\right) \cos q_{nj}\right.$$

$$\times \left(\frac{t^* c}{r_o} - \tau'_o\right) + \left(c_{nj} \cos k\,\theta_{nj} - d_{nj} \sin k\,\theta_{nj}\right) \sin q_{nj} \left.\left(\frac{t^* c}{r_o} - \tau'_o\right)\right] \tag{32}$$

$$\left. + f_n p_{n\,\frac{n+1}{2}}^k \exp\left[p_{n\,\frac{n+1}{2}} \left(\frac{t^* c}{r_o} - \tau'_o\right)\right] \right] \alpha_{nm}(\tau'_o)\, d\tau'_o$$

$$\left. + \delta_{kn}\, \alpha_{nm}\left(\frac{t^* c}{r_o}\right) \right\} .$$

The identical steps [Eqs. (30), (31), and (32)] can be performed with B_{nm} in place of A_{nm} and β_{nm} in place of α_{nm}. Note that the integral in Eq. (32) is independent of r , that is, independent of the radius of observation of the field. Hence, for a given source, the integration need only be performed once to give field values everywhere outside the sphere.

C. The Boundary Value Problem - Solution Applied to Present Problem

The coefficients E_{xn2}, E_{zn2}, and H_{yn2} of Eqs. (7) are the functions represented by α_{nm} in Eq. (32). The integrals, appearing in Eq. (32), needed to extrapolate the field are

$$E_{xsnj} = \int_0^{\frac{t^* c}{r_o}} \sin q_{nj} \left(\frac{t^* c}{r_o} - \tau'_o\right) \exp\left[p_{nj}\left(\frac{t^* c}{r_o} - \tau'_o\right)\right] E_{xn2}(\tau'_o)\, d\tau'_o \quad , \qquad (33)$$

$$E_{xcnj} = \int_0^{\frac{t^* c}{r_o}} \cos q_{nj} \left(\frac{t^* c}{r_o} - \tau'_o\right) \exp\left[p_{nj}\left(\frac{t^* c}{r_o} - \tau'_o\right)\right] E_{xn2}(\tau'_o)\, d\tau'_o \quad , \qquad (34)$$

$$E_{zsnj} = \int_0^{\frac{t^* c}{r_o}} \sin q_{nj} \left(\frac{t^* c}{r_o} - \tau'_o\right) \exp\left[p_{nj}\left(\frac{t^* c}{r_o} - \tau'_o\right)\right] E_{zn2}(\tau'_o)\, d\tau'_o \quad , \qquad (35)$$

$$E_{zcnj} = \int_0^{\frac{t^* c}{r_o}} \cos q_{nj} \left(\frac{t^* c}{r_o} - \tau'_o\right) \exp\left[p_{nj}\left(\frac{t^* c}{r_o} - \tau'_o\right)\right] E_{zn2}(\tau'_o)\, d\tau'_o \quad , \qquad (36)$$

$$H_{ysnj} = \int_0^{\frac{t^* c}{r_o}} \sin q_{nj} \left(\frac{t^* c}{r_o} - \tau'_o\right) \exp\left[p_{nj}\left(\frac{t^* c}{r_o} - \tau'_o\right)\right] H_{yn2}(\tau'_o)\, d\tau'_o \quad , \qquad (37)$$

$$H_{ycnj} = \int_0^{\frac{t^* c}{r_o}} \cos q_{nj} \left(\frac{t^* c}{r_o} - \tau'_o\right) \exp\left[p_{nj}\left(\frac{t^* c}{r_o} - \tau'_o\right)\right] H_{yn2}(\tau'_o)\, d\tau'_o \quad , \qquad (38)$$

where $j = 1, 2, \ldots, n/2$ if n is even; if n is odd $j = 1, 2, \ldots, (n+1)/2$ and $q_{n\,\frac{n+1}{2}}$ is taken to be zero ($p_{n\,\frac{n+1}{2}}$ is, of course, the real root of $F_n(z) = 0$ (Eq. (24)).

Let μ_{nm} be the coefficients of the derivatives in Eq. (20) so that the Hankel operator can be written

$$\Xi_n(r) = \frac{1}{r^{n+1}} \sum_{m=0}^{n} \mu_{nm} \frac{d^{n-m}}{d\tau^{n-m}} \qquad (39)$$

The general expressions for the coefficients of the expansion of the fields at a new radius r, expanded in the form of Eq. (7), Section I, are

$$E_{xn2}(r) = \sum_{m=0}^{n} \mu_{nm} \left(\frac{r_o}{r}\right)^{m+1} \left\{ \sum_{j=1}^{\substack{n/2,\, n \text{ even} \\ (n-1)/2,\, n \text{ odd}}} r_{nj}^{(n-m)} \left[(d_{nj} \cos(n-m)\theta_{nj} + c_{nj} \sin(n-m)\theta_{nj}) E_{xcnj} + (c_{nj} \cos(n-m)\theta_{nj} - d_{nj} \sin(n-m)\theta_{nj}) E_{xsnj} \right] + f_n p_{n\frac{n+1}{2}}^{(n-m)} E_{xcn\frac{n+1}{2}} \right\} + \frac{r_o}{r} E_{xn2}(r_o) \qquad (40)$$

Equations for the E_z and H_y components are identical to Eq. (40) except for the substitution of the quantities pertaining to those components. The fields at radius r can then be written

$$E_x(r) = \frac{1}{\sqrt{2\pi}} \sum_{\substack{n=2 \\ \text{(even values)}}}^{\infty} E_{xn2}(r)\, \overline{P}_n^1(\cos\theta) \cos\phi \qquad (41)$$

$$E_z(r) = \frac{1}{\sqrt{2\pi}} \sum_{\substack{m=0 \\ \text{(even values)}}}^{\infty} E_{zn2}(r)\ \overline{P}_n(\cos\theta) \tag{42}$$

$$H_y(r) = \frac{1}{\sqrt{2\pi}} \sum_{\substack{n=1 \\ \text{(odd values)}}}^{\infty} H_{yn2}(r)\ \overline{P}_n^1(\cos\theta)\cos\phi \quad . \tag{43}$$

Only even order terms of the E components and odd order terms of H_y are non-zero due to the image symmetry of the problem.

III. THE KIRCHHOFF INTEGRAL METHOD

A. General Considerations

The method starts with the Kirchhoff integral representation in terms of the values and the derivatives of a function ψ that satisfies the scalar wave equation given on a boundary surface S. The integral is expressed in terms of the distance, R, shown in Fig. 1, between the point where field values are desired (a point inside the volume V) and an arbitrary point on the surface S, the surface enclosing V. It also involves the derivatives with respect to the outward normal denoted by $\frac{\partial}{\partial n}$, relative to the volume V. If there are no sources within V the function ψ at point $P(r', \theta', \phi')$ of Fig. 1 can be expressed in terms of the values of ψ on the surface at a retarded time, $T = t - \frac{R}{c}$. The expression as given by Stratton[6], p. 427, is

$$\psi (r', \theta', \phi', t) = \frac{1}{4\pi} \int_S \left\{ \left[\frac{1}{R} \frac{\partial}{\partial n} \psi (r, \theta, \phi, T)\right] - \left[\frac{\partial}{\partial n} \left(\frac{1}{R}\right)\right] \psi (r, \theta, \phi, T) + \frac{1}{cR} \frac{\partial R}{\partial n} \frac{\partial}{\partial T} \left[\psi (r, \theta, \phi, T)\right] \right\} da \quad . \tag{44}$$

For the present problem, the volume V is the space outside the sphere surrounding the source, the surface S is the spherical surface, and the outward normal derivative $\frac{\partial}{\partial n}$ is $-\frac{\partial}{\partial r}$.

The logical way to set up the surface integral is to integrate first over the variable that is orthogonal to the time retardation, represented by the distance R as shown in Fig. 2. This implies setting up a new spherical coordinate system with the z axis along r' for defining the positions on the sphere. This will be the (r, γ, β) system referred to the x'', y'', z'' set of axes as shown in Fig. 3.

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[6] Ibid.

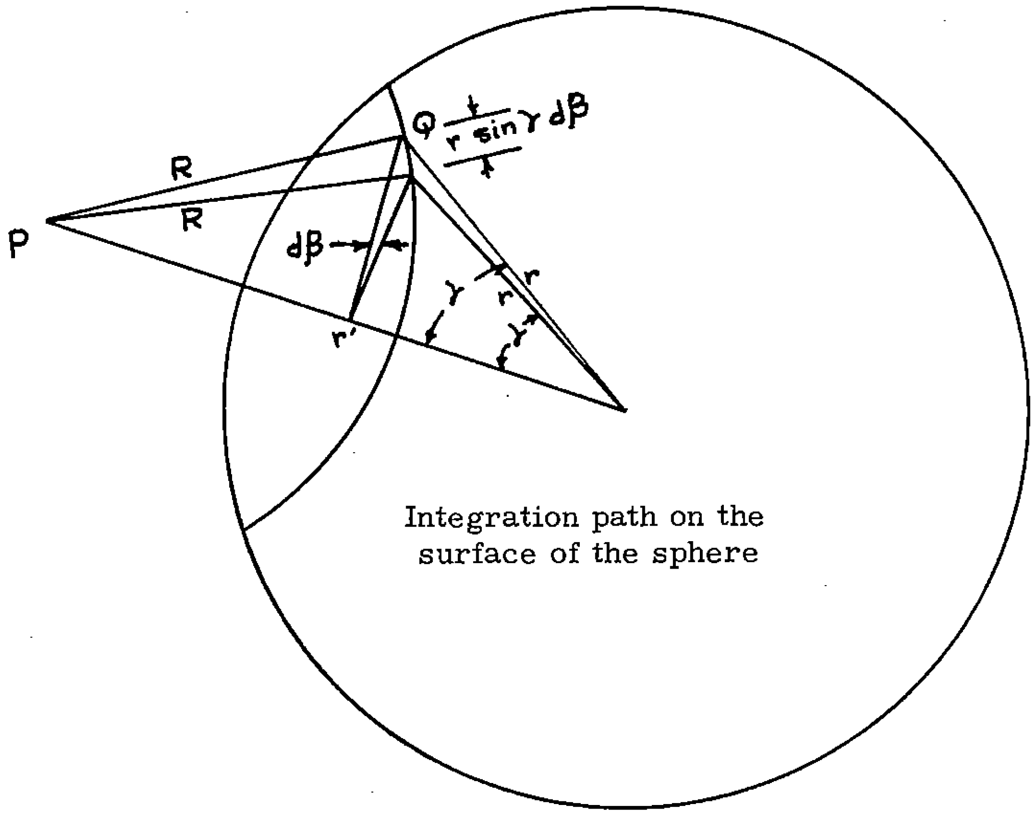

Fig. 2 - The Integration Path of Constant Retarded Time

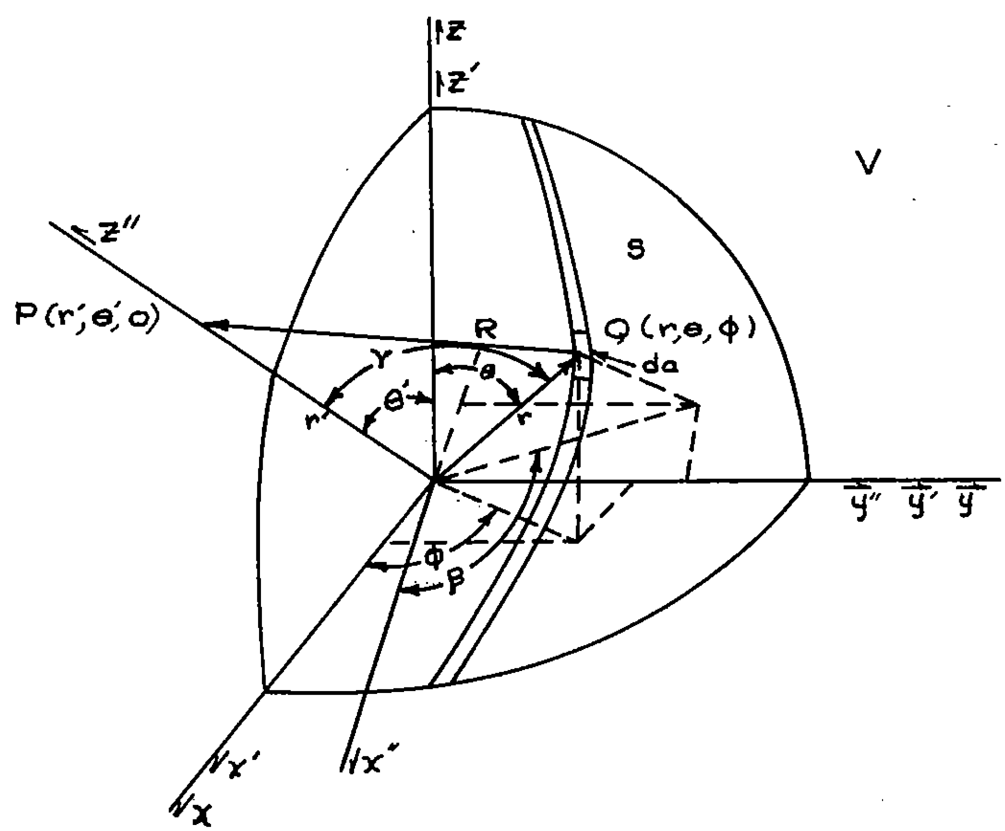

Fig. 3 - The Rotated Coordinate System

The element of area is

$$da = r^2 \sin\gamma \, d\beta \, d\gamma$$

but

$$R^2 = r'^2 + r^2 - 2 r r' \cos\gamma$$

so

$$2R \frac{dR}{d\gamma} = 2r r' \sin\gamma$$

or

$$d\gamma = \frac{R}{r r'} \frac{dR}{\sin\gamma}$$

$$da = \frac{rR}{r'} \, dR \, d\beta \quad .$$

This allows one to write Eq. (44) in the form

$$\psi (r', \theta', 0, t) = \frac{r}{4\pi r'} \int_{r'-r}^{r'+r} I(r, R, T) \, dR \quad ,$$

where

$$I(r, R, T) = R \int_0^{2\pi} I_\beta (r, R, \beta, T) \, d\beta \qquad (45)$$

and I_β is the integrand of Eq. (44). If I_β can be put in the form of a spherical harmonic expansion of the angles γ and β, then each term will have a β dependence of the form $e^{im\beta}$ and the β integration can be performed analytically, and all the terms for $m \neq 0$ will disappear. This is the approach that will be followed.

It should be noted that, provided the radial derivative and the values of the field are specified as a function of time on the sphere as input data, what has been presented thus far constitutes a method of solution, regardless of the form of the input data. What follows will merely be a method of simplifying the numerical work when the input data is supplied in the form of a Legendre expansion.

The first step is to separate the three terms of the integrand by the set of definitions

$$I_\beta = I_{\beta 1} + I_{\beta 2} + I_{\beta 3} \qquad ,$$

where

$$I_{\beta 1} = -\frac{1}{R}\frac{\partial}{\partial r}\psi(r, R, \beta, T)\Big|_{\theta, \phi}$$

$$I_{\beta 2} = \frac{\partial}{\partial r}\frac{1}{R}\quad \psi(r, R, \beta, T) \tag{46}$$

$$I_{\beta 3} = -\frac{1}{cR}\frac{\partial R}{\partial r}\frac{\partial}{\partial T}\psi(r, R, \beta, T) \qquad ,$$

and to confine attention to one term, say $I_{\beta 2}$. The factor

$$\frac{\partial}{\partial r}\frac{1}{R} = \frac{r' \cos\gamma - r}{R^3}$$

is independent of β. Thus it is sufficient for purposes of the β integration to expand ψ in spherical harmonics. Since ψ is E_x, E_z, or H_y, Eqs. (7) and (8) give the spherical harmonic expansion of the ψ functions in the angles θ and ϕ. The problem is to find the expansion in a coordinate system rotated through an angle θ'.

B. Rotation of the Expansion

The terms of the (θ, ϕ) expansion of ψ have either the form $\overline{P}_n(\cos\theta)$ or $Y_{n,1}(\theta,\phi)$. The standard form of the addition theorem for spherical harmonics suffices to convert $P_n(\cos\theta)$ to a series of spherical harmonics of the angles γ and β. (cf. Jackson[7], p. 68)

$$P_n(\cos\theta) = \frac{4\pi}{2n+1} \sum_{m=-n}^{n} Y^*_{nm}(\theta', \pi-\phi')\, Y_{nm}(\gamma,\beta) \quad . \tag{47}$$

An extension of the addition theorem is necessary to convert $Y_{n,1}(\theta,\phi)$. This expansion will now be derived.

The components of the angular momentum operator $\vec{L}$ can conveniently be written in the form (cf. Jackson, p. 542)

$$L_+ = L_x + i\,L_y$$
$$L_- = L_x - i\,L_y$$
$$L''_+ = L''_x + i\,L''_y$$
$$L''_- = L''_x - i\,L''_y$$

for the two coordinate systems under consideration. The L_+ operator applied to the left side of Eq. (47) will yield

$$L_+\,\overline{P}_n(\cos\theta) = \sqrt{n(n+1)\,2\pi}\quad Y_{n,1}(\theta,\phi) \quad . \tag{48}$$

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[7] Jackson, J. D., Classical Electrodynamics, Wiley, New York (1962).

Before applying it to the right side of the equation it is desirable to express it in terms of the operators L''_+, L''_-, and L''_z. The components tranform as the components of an ordinary vector; thus, one obtains

$$L_x = \cos\theta' \, L''_x + \sin\theta' \, L''_z$$

$$= \frac{1}{2} \cos\theta' \, (L''_+ + L''_-) + \sin\theta' \, L''_z$$

$$L_y = L''_y = \frac{1}{2i} (L''_+ - L''_-) \tag{49}$$

$$L_+ = \frac{1}{2} \left[(1 + \cos\theta') \, L''_+ - (1 - \cos\theta') \, L''_- \right] + \sin\theta' \, L''_z \quad .$$

The operators L''_+, L''_-, and L''_z have the following effect on the terms of the right side of Eq. (47):

$$L''_+ Y_{nm}(\gamma,\beta) = \sqrt{(n-m)(n+m+1)} \; Y_{n,\, m+1}(\gamma,\beta)$$

$$L''_- Y_{nm}(\gamma,\beta) = \sqrt{(n+m)(n-m+1)} \; Y_{n,\, m-1}(\gamma,\beta) \tag{50}$$

$$L''_z Y_{nm}(\gamma,\beta) = m \, Y_{n,\, m}(\gamma,\beta) \quad .$$

Combining Eqs. (49) and (50)

$$L_+ Y_{nm}(\gamma,\beta) = \tfrac{1}{2} \Big[(1 + \cos\theta') \sqrt{(n-m)(n+m+1)} \; Y_{n,\, m+1}(\gamma,\beta)$$

$$- (1 - \cos\theta') \sqrt{(n+m)(n-m+1)} \; Y_{n,\, m-1}(\gamma,\beta) \Big] \tag{51}$$

$$+ \sin\theta' \, m \, Y_{nm}(\gamma,\beta) \quad .$$

Using the factor for normalizing the Legendre polynomial of Eq. (47), one obtains

$$Y_{n,0}(\theta, \phi) = \sqrt{\frac{4\pi}{2n+1}} \sum_{m=-n}^{n} Y^*_{n,m}(\theta', \pi - \phi') \, Y_{n,m}(\gamma, \beta) \quad . \qquad (52)$$

Applying Eq. (48) to Eq. (52), the desired expansion is

$$Y_{n,1}(\theta, \phi) = \sqrt{\frac{4\pi}{(2n+1)\, n(n+1)}} \sum_{m=-n}^{n} Y^*_{n,m}(\theta', \pi - \phi') \left[L_+ Y_{n,m}(\gamma, \beta)\right] \quad . \qquad (53)$$

Equations (51), (52), and (53) define the necessary transformations.

By using the definitions

$$Y_{n,m,1} = \sqrt{\frac{4\pi}{(2n+1)\, n(n+1)}} \left\{ \frac{Y^*_{n,m}(\theta', \pi - \phi') \left[L_+ Y_{n,m}(\gamma, \beta)\right]}{2} + \frac{Y(\theta', \pi - \phi') \left[L_+ Y_{nm}(\gamma, \beta)\right]^*}{2} \right\} \qquad (54)$$

$$Y_{n,m,0} = \sqrt{\frac{4\pi}{2n+1}} \; Y^*_{n,m}(\theta', \pi - \phi') \, Y_{n,m}(\gamma, \beta) \quad ,$$

equations (7) can be written in the form

$$E_x = \sum_{\substack{n=0 \\ \text{(even values)}}}^{\infty} \; \sum_{m=-n}^{n} E_{xn2} Y_{n,m,1}$$

$$E_z = \sum_{\substack{n=0 \\ \text{(even values)}}}^{\infty} \sum_{m=-n}^{n} E_{zn2} Y_{n,m,0} \tag{55}$$

$$H_y = \sum_{\substack{n=1 \\ \text{(odd values)}}}^{\infty} \sum_{m=-n}^{n} H_{yn2} Y_{n,m,1} \quad .$$

C. Evaluation of the β Integral

Since the coefficients are independent of θ, the problem of evaluating the β integral, I(r, R, T) of Eq. (44), reduces to the evaluation of the integrals defined as follows:

$$I_{y0} = \int_0^{2\pi} Y_{n,m,0}\, d\beta$$

$$I_{y1} = \int_0^{2\pi} Y_{n,m,1}\, d\beta \quad . \tag{56}$$

Taking them in order, one obtains

$$I_{y0} = \sqrt{\frac{4\pi}{2n+1}}\; Y^*_{n,0}(\theta', \pi - \phi') \left[\int_0^{2\pi} e^{im\beta}\, d\beta \right] \frac{1}{\sqrt{2\pi}}\; \overline{P}_n^m(\cos\gamma)$$

$$= \begin{cases} \dfrac{\sqrt{4\pi}\ \sqrt{2\pi}}{\sqrt{2n+1}}\; Y^*_{n,0}(\theta', \pi - \phi')\, \overline{P}_n(\cos\gamma) \text{ if } m = 0 \\ 0 \text{ if } m \neq 0 \end{cases}$$

$$= \sqrt{\frac{4\pi}{2n+1}} \quad \overline{P}_n(\cos\theta')\,\overline{P}_n(\cos\gamma)\,\delta_{m,0} \quad , \tag{57}$$

where $\delta_{m,0}$ is the Kronecker delta function.

The integral I_{y1} is defined in terms of a simpler integral, I_{y2}

$$I_{y1} = \sqrt{\frac{4\pi}{(2n+1)n(n+1)}} \left\{ \frac{Y^*_{n,m}(\theta', \pi - \phi')I_{y2} + Y_{n,m}(\theta', \pi - \phi')I^*_{y2}}{2} \right\} ,$$

where

$$I_{y2} = \int_0^{2\pi} \left[L_+ Y_{n,m}(\gamma, \beta) \right] d\beta \quad . \tag{58}$$

Inserting Eq. (51) into Eq. (58) one obtains

$$\begin{aligned} I_{y2} = \frac{1}{2} \Big[&(1 + \cos\theta')\sqrt{(n-m)(n+m+1)}\; I_{y,m+1} \\ &- (1 - \cos\theta')\sqrt{(n+m)(n-m+1)}\; I_{y,m-1} \Big] \\ &+ \sin\theta' \; m \; I_{y,m} \quad , \end{aligned} \tag{59}$$

where

$$\begin{aligned} I_{y,m+1} &= \int_0^{2\pi} Y_{n,m+1}\, d\beta = \sqrt{2\pi}\;\overline{P}_n(\cos\gamma)\,\delta_{m,-1} \\ I_{y,m-1} &= \int_0^{2\pi} Y_{n,m-1}\, d\beta = \sqrt{2\pi}\;\overline{P}_n(\cos\gamma)\,\delta_{m,1} \\ I_{y,m} &= \int_0^{2\pi} Y_{n,m}\, d\beta = \sqrt{2\pi}\;\overline{P}_n(\cos\gamma)\,\delta_{m,0} \quad . \end{aligned} \tag{60}$$

Equations (56) through (60) define a method for evaluating the β integral of the expansion defined by Eqs. (54) and (55). This enables evaluation of $\int_0^{2\pi} I_{\beta 2}\, d\beta$ that occurs if Eq. (46) is inserted in Eq. (45). One must now show that the terms resulting from $I_{\beta 1}$ and $I_{\beta 3}$ can be handled in a similar fashion. To evaluate $I_{\beta 1}$, one must have the radial derivatives of the Legendre expansion coefficients of the fields on the sphere. One can then write the Legendre expansion of the radial derivatives of the field in the following fashion

$$\frac{\partial E_\theta}{\partial r} = \sum_{n=1}^{\infty} \frac{\partial E_{\theta n}}{\partial r} \overline{P}_n^1 (\cos\theta)$$

$$\frac{\partial E_r}{\partial r} = \sum_{n=1}^{\infty} \frac{\partial E_{rn}}{\partial r} \overline{P}_n (\cos\theta) \tag{61}$$

$$\frac{\partial H_\phi}{\partial r} = \sum_{n=1}^{\infty} \frac{\partial H_{\phi n}}{\partial r} \overline{P}_n^1 (\cos\theta) \quad .$$

The analysis for the spherical harmonic expansion of the rectangular components of the radial derivatives in terms of γ and β will then proceed in the same way as for the components themselves, as described in Section I and in Eqs. (54) and (55), above. The same is true for the T derivatives that occur in considering $I_{\beta 3}$. The r and T derivatives of $E_{\theta, n}$, $E_{r, n}$, and $H_{\phi, n}$ can be obtained from finite difference approximations to the input data.

Thus, one knows how to find the coefficients in the following six equations, analogous to Eqs. (55):

$$\frac{\partial E_x}{\partial r} = \sum_{\substack{n=0 \\ \text{(even values)}}}^{\infty} \sum_{m=-n}^{n} E_{xn3} \, Y_{n,m,1}$$

$$\frac{\partial E_z}{\partial r} = \sum_{\substack{n=0 \\ \text{(even values)}}}^{\infty} \sum_{m=-n}^{n} E_{zn3} \, Y_{n,m,0}$$

$$\frac{\partial H_y}{\partial r} = \sum_{\substack{n=1 \\ \text{(odd values)}}}^{\infty} \sum_{m=-n}^{n} H_{yn3} \, Y_{n,m,1} \qquad (62)$$

$$\frac{\partial E_x}{\partial T} = \sum_{\substack{n=0 \\ \text{(even values)}}}^{\infty} \sum_{m=-n}^{n} E_{xn4} \, Y_{n,m,1}$$

$$\frac{\partial E_z}{\partial T} = \sum_{\substack{n=0 \\ \text{(even values)}}}^{\infty} \sum_{m=-n}^{n} E_{zn4} \, Y_{n,m,0}$$

$$\frac{\partial H_y}{\partial T} = \sum_{\substack{n=1 \\ \text{(odd values)}}}^{\infty} \sum_{m=-n}^{n} H_{yn4} Y_{n,m,1} \quad ,$$

The additional factor in $I_{\beta 3}$ is

$$\frac{\partial R}{\partial r} = \frac{r - r' \cos\gamma}{R}$$

which again is independent of β. Thus the β integration will follow the same pattern. One now has all the formalism necessary to solve the problem. The results will now be pulled together into the simplest possible expression for the field components.

D. Summary

Assume that E_{xnj}, E_{znj}, and H_{ynj} for j=2 to 4 have been determined. For the sake of definiteness, let $\psi = E_{x''}$. Applying Eqs. (57) through (60) to the evaluation of the β integral of $I_{\beta 1}$ yields

$$\int_0^{2\pi} I_{\beta 1}\, d\beta = \frac{-1}{R} \sum_{\substack{n=0 \\ \text{(even values)}}}^{\infty} \Bigg\{ E_{xn3} \sqrt{\frac{8\pi^2}{(2n+1)\,n(n+1)}}\ \overline{P}_n(\gamma)$$

$$\frac{1}{4}\Bigg[(1+\cos\theta')\ \sqrt{n(n+1)}\ \left\{Y^*_{n,-1}(\theta', \pi-\phi')+Y_{n,-1}(\theta', \pi-\phi')\right\}$$

$$-(1-\cos\theta')\sqrt{n(n+1)}\ \left\{Y^*_{n,1}(\theta', \pi-\phi')+Y_{n,1}(\theta', \pi-\phi')\right\}\Bigg]\Bigg\} \quad ,$$

where the $\overline{P}_n(\gamma)$ is written as shorthand for $\overline{P}_n(\cos\gamma)$, but

$$Y^{*}_{n,-1} = -Y_{n,1} = -\frac{1}{\sqrt{2\pi}}\,\overline{P}^{1}_{n}(\theta')\,e^{i(\pi-\phi')} \quad .$$

This yields

$$Y^{*}_{n,1}(\theta,\pi-\phi') + Y_{n,1}(\theta',\pi-\phi') = \frac{2}{\sqrt{2\pi}}\,\overline{P}^{1}_{n}(\theta')\cos(\pi-\phi')$$

$$= -Y^{*}_{n,-1}(\theta',\pi-\phi') - Y_{n,-1}(\theta',\pi-\phi') \quad .$$

Thus the integral can be written

$$\int_0^{2\pi} I_{\beta 1}\, d\beta = -\frac{1}{R} \sum_{\substack{n=0\\ \text{(even values)}}}^{\infty} \left\{ -\sqrt{\frac{4\pi}{2n+1}}\; E_{xn3}\,\overline{P}^{1}_{n}(\theta')\,\overline{P}_{n}(\gamma)\cos(\pi-\phi') \right\} \quad .$$

But $\cos(\pi-\phi') = -1$ due to the choice of coordinate systems. The result is

$$\int_0^{2\pi} I_{\beta 1}\, d\beta = -\frac{1}{R} \sum_{\substack{n=0\\ \text{(even values)}}}^{\infty} \sqrt{\frac{4\pi}{2n+1}} \left[E_{xn3}\,\overline{P}^{1}_{n}(\theta')\right] \overline{P}_{n}(\gamma) \quad . \qquad (63)$$

Likewise

$$\int_0^{2\pi} I_{\beta 2}\, d\beta = \frac{r'\cos\gamma - r}{R^3} \sum_{\substack{n=0\\ \text{(even values)}}}^{\infty} \sqrt{\frac{4\pi}{2n+1}} \left[E_{xn2}\,\overline{P}^{1}_{n}(\theta')\right] \overline{P}_{n}(\gamma)$$

$$\int_0^{2\pi} I_{\beta 3}\, d\beta = \frac{r - r'\cos\gamma}{c R^2} \sum_{\substack{n=0\\ \text{(even values)}}}^{\infty} \sqrt{\frac{4\pi}{2n+1}} \left[E_{xn4}\,\overline{P}^{1}_{n}(\theta')\right] \overline{P}\,(\gamma) \quad .$$

One now considers the R integration of Eq. (45). θ is a function of R, but θ' is not. The coefficients $E_{n,j}$ are functions of θ' and T, where $T = t - \frac{R}{c}$, and are therefore functions of R as far as the integration on R is concerned. t, r, r', θ', and c are regarded as constants. Thus after writing

$$E_x(r', \theta', 0, t) = \frac{r}{4\pi r'} \int_{r'-r}^{r'+r} R \sum_{\substack{n=0 \\ \text{(even values)}}}^{\infty} \sqrt{\frac{4\pi}{2n+1}} \left\{ \frac{r' \cos\gamma - r}{R^3} \left[E_{xn2} \overline{P}_n^1(\theta') \right] - \frac{1}{R} \left[E_{xn3} \overline{P}_n^1(\theta') \right] - \frac{1}{cR} \frac{r - r' \cos\gamma}{R} \left[E_{xn4} \overline{P}_n^1(\theta') \right] \right\} \overline{P}_n(\gamma) \, dR \quad ,$$

one defines the R integral as follows:

$$\rho_{xn}(r', \theta', t) = \int_{r'-r}^{r'+r} R \left\{ \frac{r' \cos\gamma - r}{R^3} E_{xn2} - \frac{E_{xn3}}{R} - \frac{1}{cR} \frac{r - r' \cos\gamma}{R} E_{xn4} \right\} \overline{P}_n(\gamma) \, dR \quad .$$

Then the field component is given by

$$E_x(r', \theta', 0, t) = \frac{r}{4\pi r'} \sum_{\substack{n=2 \\ \text{(even values)}}}^{\infty} \sqrt{\frac{4\pi}{2n+1}} \left[\rho_{xn}(r', \theta', t) \overline{P}_n^1(\theta') \right] \quad .$$

Likewise, for the other electric field components

$$E_z(r', \theta', 0, t) = \frac{r}{4\pi r'} \sum_{\substack{n=1 \\ \text{(even values)}}}^{\infty} \sqrt{\frac{4\pi}{2n+1}} \left[\rho_{zn}(r', \theta', t)\, \overline{P}_n(\theta')\right] ,$$

where

$$\rho_{zn}(r', \theta', t) = \int_{r'-r}^{r'+r} R \left\{ \frac{r' \cos\gamma - r}{R^3} E_{zn2} - \frac{E_{zn3}}{R} - \frac{1}{cR} \frac{r - r' \cos\gamma}{R} E_{zn4} \right\} \overline{P}_n(\gamma)\, dR .$$

For H_y, one obtains a form

$$H_y(r', \theta', 0, t) = \frac{r}{4\pi r'} \sum_{\substack{n=1 \\ \text{(odd values)}}}^{\infty} \sqrt{\frac{4\pi}{2n+1}}\, \rho_{yn}(r', \theta', t)\, \overline{P}_n^1(\theta') ,$$

where

$$\rho_{yn}(r', \theta', t) = \int_{r'-r}^{r'+r} R \left\{ \frac{r' \cos\gamma - r}{R^3} H_{yn2} - \frac{H_{yn3}}{R} - \frac{1}{cR} \frac{r - r' \cos\gamma}{R} H_{yn4} \right\} \overline{P}_n(\gamma)\, dR .$$

Since no generalizations can be made about the coefficients E_{xnj}, E_{znj} and H_{ynj}, other than the method of calculating them described above, one knows that the ρ_n's cannot be analytically evaluated. All that can be done now is to show how to obtain $\overline{P}_n(\theta)$, $\overline{P}_n(\theta')$, and $P_n^1(\theta')$ for use in the above sums and integrals. Note that by $\overline{P}_n(\gamma)$ one means $\overline{P}_n(\cos\gamma)$. $\overline{P}_n(\cos\gamma)$ and $\overline{P}_n(\cos\theta')$ can be obtained from the recurrence formula,

$$P_{n+1}(x) = \frac{(2n+1)\,x\,P_n(x) - n\,P_{n-1}(x)}{n+1} \quad ,$$

starting with $P_o(x) = 1$ and $P_1(x) = x$ and normalizing using

$$\overline{P}_n(x) = \sqrt{\frac{2n+1}{2}}\,P_n(x) \quad .$$

For $P_n(\cos\theta')$ the argument is an input constant. For $P_n(\cos\gamma)$, $\cos\gamma$ is given by

$$\cos\gamma = \frac{r^2 + (r')^2 - R^2}{2\,r\,r'} \quad .$$

The $\overline{P}_n^1(\cos\theta')$ can be obtained from the following formulae

$$P_n^1(x) = -\sqrt{1-x^2}\,\frac{d\,P_n(x)}{d\,x} \quad ,$$

where the derivatives can be obtained from the recurrence formula

$$\frac{d\,P_n(x)}{d\,x} = \frac{n\,P_{n-1}(x) - n\,x\,P_n(x)}{(1-x^2)} \quad .$$

The normalization is accomplished using

$$\overline{P}_n^1(x) = \sqrt{\frac{2n+1}{2n(n+1)}}\,P_n^1(x) \quad .$$

IV. CONCLUSIONS AND PLANS

In this report two mathematical tools have been developed which can be used to extrapolate to arbitrary radii outgoing waves which are known on the surface of a sphere. An infinitely-conducting infinite ground plane that bisects the sphere has been assumed. FORTRAN programs have been written to evaluate extrapolated fields by both the Kirchhoff integral method and the Hankel operator method. Computer results using each of the methods have been compared to each other and with results obtained from Lt. Graham's computer solution of Maxwell's equations. Output from the program utilizing the Hankel operator method has been compared to the result of a test problem which was solved analytically. All of these cross-checks indicate that the methods are proper and their associated computer programs are working correctly. The results of the test program are given in the appendix. A later classified report, to be written in collaboration with AFWL personnel, will discuss the numerical results of efforts to interpret weapon test data using these methods along with the solutions of Maxwell's equations.

Each method has certain advantages and limitations relative to the other method. The Kirchhoff integral method is faster if it is desired to extrapolate to only one distant point, but it must treat each receiver point as an essentially separate problem. On the other hand, the time-consuming part of the Hankel operator method is the calculation of the expansion coeff-cients. Once this is done, answers may be obtained very rapidly for several receiver points. It is harder to obtain answers for very high orders of Legendre polynomials using the Hankel operator method because of difficulty in evaluating the roots of the Hankel function for high orders. A double-precision program for evaluating these roots with an IBM 7044 computer was only capable of obtaining usefully accurate roots to the sixteenth order due to a very high degree of cancellation between the terms of the polynomial that defines the Hankel function.

APPENDIX

A sample problem was devised to serve two purposes. It was to serve as a check on the correctness of the Hankel operator method computer code and to supply an illustrative example that could be used in an unclassified report. In order to serve as a computer-code check a problem was chosen that could be solved analytically. The expressions representing the analytic solution are fairly complicated and a separate computer code was written to evaluate them. The same problem was then supplied numerically to the more general code and the results were compared and found to agree.

The field used in the test problem corresponds to B_ϕ with the symmetry assumed in the first section of this report. For the sake of generality, the dimensionless retarded time $\tau = ct^* / r_o$ and dimensionless radius $\mu = r/r_o$ will be used in presenting the test problem. On the sphere $\mu = 1$, $B_{\phi\ \mathrm{test}}$ was assumed to be

$$B_{\phi\ \mathrm{test}} = \frac{f(\tau)}{\sqrt{2\pi}} \left[\overline{P_1^1} (\cos\theta) + \overline{P_3^1} (\cos\theta) \right] ,$$

where

$$f(\tau) = \begin{cases} 0, & 0 \geq \tau \quad \text{and} \quad .1923 \leq \tau \\ \sin(\pi\tau / .1923), & 0 \leq \tau \leq .1923 \end{cases} .$$

The appropriate convolution integrals of $f(\tau)$ were performed analytically and algebraic expressions were obtained for B_ϕ for arbitrary μ. Four curves are plotted in Fig. 4; two curves are the radiation $(1/r)$ terms at $\mu = 1$ and 2, $\theta = \pi/2$ and two are the total field curves at the same locations. As μ increases to larger values, the total field approaches the radiation field and decreases in magnitude inversely with μ. The curves plotted in Fig. 4 were calculated for values of τ to 9.5; no second crossover point was found; the curves asymptotically approach zero.

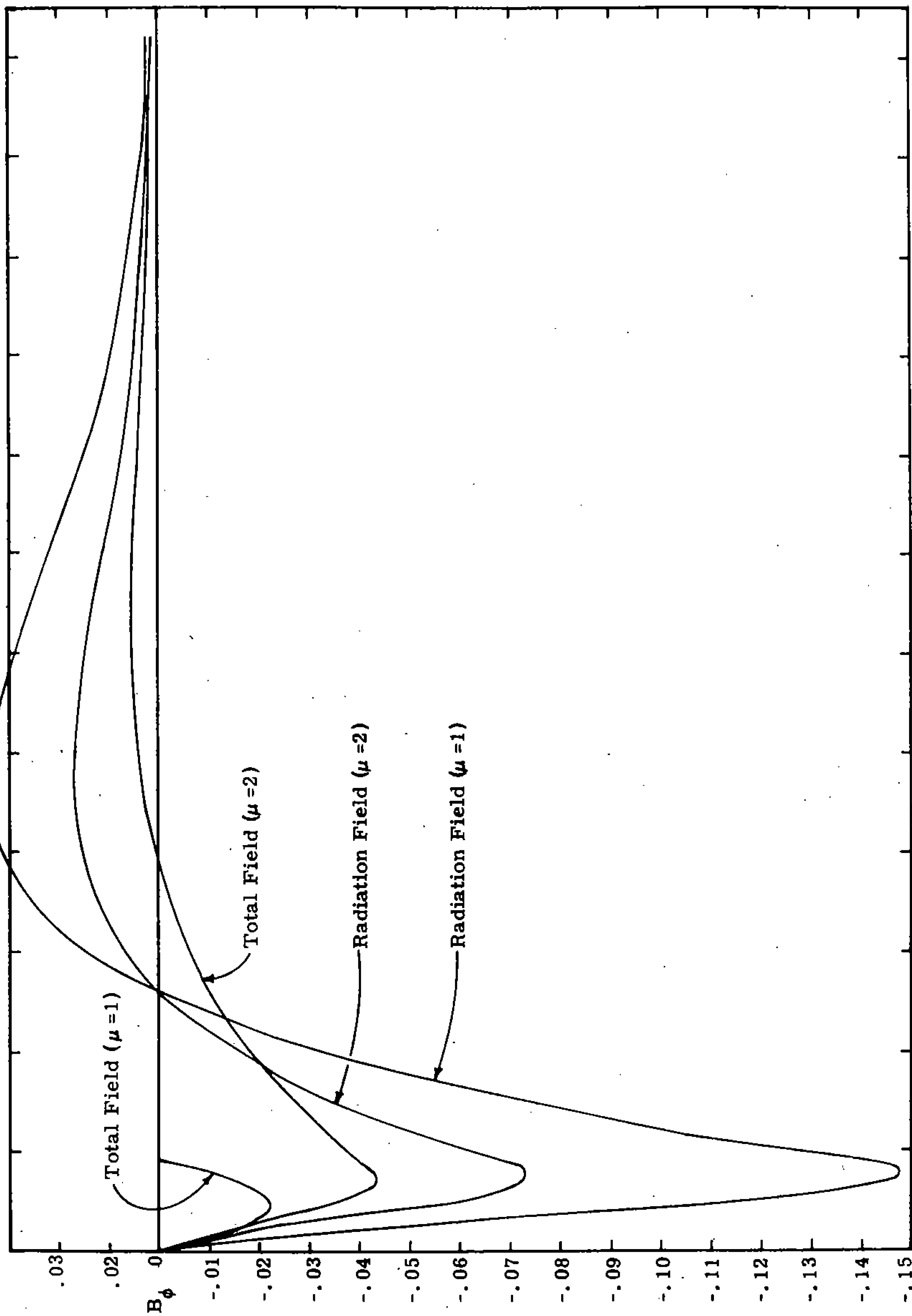

Fig. 4 - Test Problem Results