EMP Theoretical Notes

Note 11

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Diffusion Approximation Evaluation of the EMP Fields from a Near-Surface Burst

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Abstract

This note presents a derivation of the diffusion approximation expressions for the magnetic field at and above the surface of an earth of arbitrary conductivity with arbitrary electrical current and conductivity time histories, and with the inclusion of an electron turning correction factor.

It can be shown that when the air conductivity near the earth's surface satisfies the following conditions

$$\sigma \gg \epsilon/2\tau \tag{1}$$

and

$$\sigma \gg \frac{2\tau}{\mu r}\left[\frac{1}{2\lambda} + \frac{r}{(2\lambda)^2} + \frac{1}{2c\tau} + \frac{r}{(2c\tau)^2}\right] \tag{2}$$

where

ϵ = $8.854(10^{-12})$

τ = the retarded time

λ = the electrical current and conductivity attenuation length

c = $3(10^{8})$

r = the distance from the axis of a cylindrical coordinate system whose origin is at the projection of the detonation on the earth's surface, and Z is the height above the earth's surface,

then, the magnetic field near an infinitely conducting plane obeys the following equation:

$$\frac{\partial^2 B_\varphi}{\partial Z^2} - \mu\sigma(t)\frac{\partial B_\varphi}{\partial t} = -2\mu J_r(t)\,\delta(Z)\ ^{*} \tag{3}$$

where

$\sigma(t)$ = the time dependent air conductivity

$J_r(t)$ = the radial electrical current.

This equation lends itself to relatively simple solution for $B_\varphi(Z,t)$ by means of Green's functions if it can be expressed in the following standard form with constant coefficients:**

$$\nabla^2\psi - a^2\partial\psi/\partial t = -4\pi\rho(r,t) \quad . \tag{4}$$

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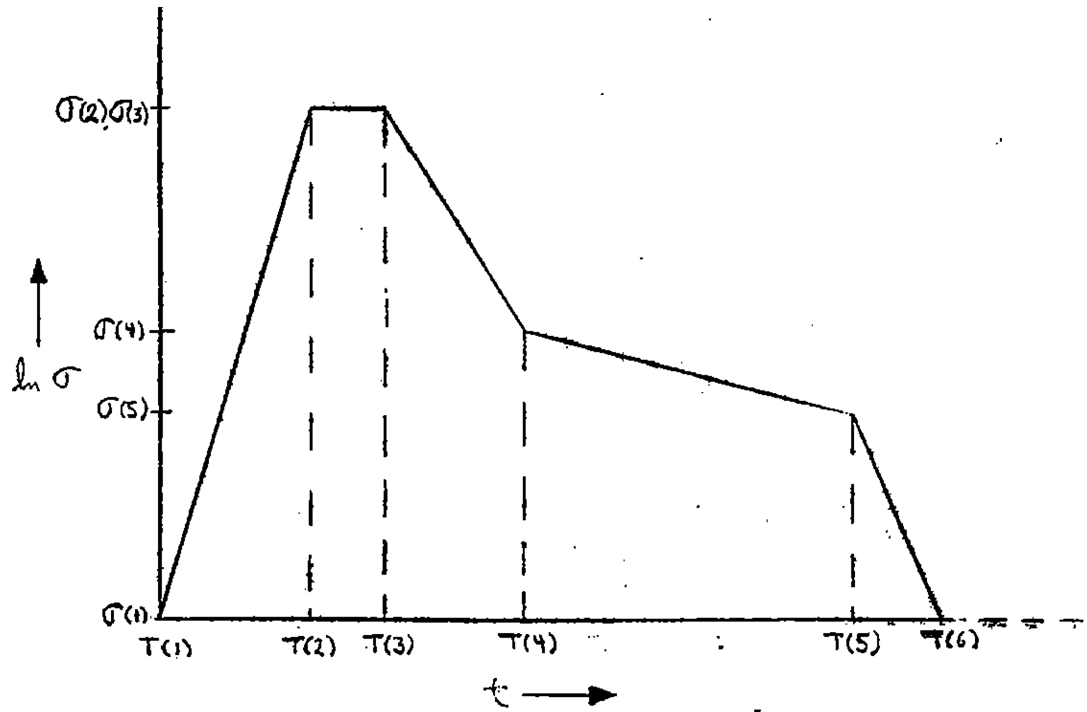
* We have assumed current reversal at Z = 0 representing the image currents.

** This standard form is treated by Morse and Feshbach in Ch. 7, Section 7.4 of Part I of their Methods of Theoretical Physics. The analysis in this presentation utilizes the formalism presented by Morse and Feshbach.

Equation (3) can be easily transformed to the standard form if $\sigma(t)$ is simply defined. For our purposes we will assume that $\sigma(t)$ can be expressed in terms of a piece-wise exponential function as suggested in the following graph.



If we perform the following transformation from the t- or time-domain to the x-domain, we will obtain an equation in the standard form.

$$x(t) = \frac{-e^{-\alpha[t-T(N)]}}{\mu\sigma(N)\alpha} \qquad \text{if } \alpha \neq 0 \qquad (5)$$

or

$$x(t) = [t - T(N)]/\mu\sigma(N) \qquad \text{if } \alpha = 0 \qquad (6)$$

where

α, the exponential rate of change of the conductivity σ, is given by

$$\alpha = \log[\sigma(N+1)/\sigma(N)] \Big/ [T(N+1) - T(N)] \quad . \qquad (7)$$

The desired standard form equation for B_φ in the x-domain is

$$\frac{\partial^2 B_\varphi}{\partial Z^2} - \frac{\partial B_\varphi}{\partial x} = -2\mu J_r[x(t)]\,\delta(Z) \quad . \qquad (8)$$

The appropriate one-dimensional infinite domain Green's function, g, is

$$g(x, Z/x_0, Z_0) = \frac{2\sqrt{\pi}}{\sqrt{x - x_0}} \exp[-(Z - Z_0)^2/4(x - x_0)] \qquad (9)$$

for $x > x_0$, otherwise $g = 0$.

This function yields the field at x, Z, given a point source at x_0, Z_0 .

The magnetic field B_φ is then explicitly given by

$$B_\varphi[Z, t(x)] = \int_{x_1}^{x} \left[\frac{2\mu J_r[x(t)]}{4\pi}\right] g(x, Z/x_0, 0)\, dx_0$$

$$+ \int_{-\infty}^{\infty} \frac{B(x_1, Z_0)}{4\pi} g(x, Z/x_1, Z_0)\, dZ_0 \qquad (10)$$

where

$$x_1 = x[T(N)] .$$

The first integral represents the inhomogeneous or driven contribution to B_φ . The second integral represents the diffusion of the initial conditions. Equation (10) can be used to step through the successive time or x-domain intervals (or phases). The total field at the end of any phase becomes the initial condition, $B(x_1, Z_0)$ for the next phase.

The driven portion of the magnetic field, $B_i(0,t)$ at the earth's surface can be expressed very simply if J(t) is proportional to $\sigma(t)$.

$$B_i(Z = 0,t) = \frac{\sqrt{\mu}\, J_r(N)}{\sqrt{\sigma(N)\,\alpha}} \sqrt{\pi}\, [e^{\alpha(\tau/2)} - 1] \quad \text{if } \alpha > 0 \qquad (11)^*$$

$$B_i(Z = 0,t) = \frac{2\sqrt{\mu}\, J_r(N)}{\sqrt{\pi\sigma(N)}} \sqrt{\tau} \qquad \text{if } \alpha = 0 \qquad (12)^*$$

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*Presented in RM-4905, Close-In EMP, by W. R. Graham and D. Babb.

$$B_i(Z = 0,t) = \frac{-J_r(N)\sqrt{\mu}}{\sqrt{-\pi\alpha\sigma(N)}} \; e^{\alpha(\tau/2)} \log \left| \frac{\sqrt{1 - e^{\alpha\tau}} - 1}{\sqrt{1 - e^{\alpha\tau}} + 1} \right| \qquad (13)$$

$$\text{if } \alpha < 0$$

where

$$\tau = t - T(N) \; .$$

Expressions (10), (11) and (12) are helpful in estimating the magnetic field near the earth's surface when they consist mainly of the driven component. In any case, numerical solution of equation (10) can provide more complete and accurate fields estimates.

In many real applications the assumption that the earth is an infinite conductor is not valid. In fact the earth's conductivity can cause orders of magnitude differences in the field estimates. Another strong influence on the fields is the turning of the electrons due to the driven magnetic fields. This turning of the electrons which compose the driving current is another order of magnitude correction in many situations. Therefore the use of equation (10) in evaluating the EMP fields may result in a gross over-estimate.

These two corrections can be made by modifying the integrand of the inhomogeneous (driver) term in equation (10), and by subdividing each x-domain phase into many steps and feeding the magnetic field from the previous steps into the integrand for the next step.

The finite ground conductivity correction is based on the results of Carl E. Baum's analysis in EMP Theoretical Note XIX. Carl obtained as a correction factor, F_σ, for the step function driver case ($\alpha = 0$)

$$F_\sigma = \frac{1}{1 + \sqrt{\sigma(x)/\sigma_g}} \qquad (14)$$

where

σ_g = the ground conductivity, and

$$\sigma(x) = \sigma(N) \; e^{\alpha[T(x)-T(N)]} \; . \qquad (15)$$

The applicability of this finite conductivity correction to phases with any value of α is based on two facts. First, any real driving current and conductivity can be represented by a succession of step functions, and second, the diffusion characteristics of initial values are not influenced by the ground conductivity. This second fact was observed in Baum's analysis. (See Appendix A)

An approximate magnetic turning correction factor, F_m, was derived by W. R. Graham.* This correction factor considered the loss of momentum and the loss of velocity of the electron as it traversed its range. This process causes the electrons to tighten their turning radius and, thus, displace slightly in the forward direction even for turning radii much smaller than the electron range. The magnetic turning correction factor, F_m, is

$$F_m = \frac{1}{1 + \left(0.6(10^3)\ B[Z = 0,\ t(x_0)]\right)^2} \tag{16}$$

with B in webers/meter^2.

Thus, the desired expression for $B_\varphi(t,Z)$ is

$$B_\varphi(Z,\ T_i) = \int_{x(T_{i-1})}^{x(T_i)} g[x(T_i),\ Z/x_0,\ 0] \left(\frac{\mu 2J_r[t(x_0)]}{4\pi}\right) \left[\frac{1}{1 + \sqrt{\frac{\sigma(x)}{\sigma_g}}}\right] dx_0 \left(\frac{1}{1 + [0.6(10^3)\ B_\varphi(0,\ T_{i-1})]^2}\right) + \int_{-\infty}^{\infty} g[x(T_i),\ Z/x(T_{i-1}),\ Z_0]\ \frac{B[Z_0,\ x(T_{i-1})]}{4\pi}\ dZ_0 \tag{17}$$

where the $(T_i - T_{i-1})$ are chosen small enough such that B(0,t) doesn't change appreciably in the interval. This equation has been programmed in FORTRAN and is presently being evaluated at General Atomic.

*The formula used here has been evaluated from a more general form by myself, using estimates of electron energy and range.

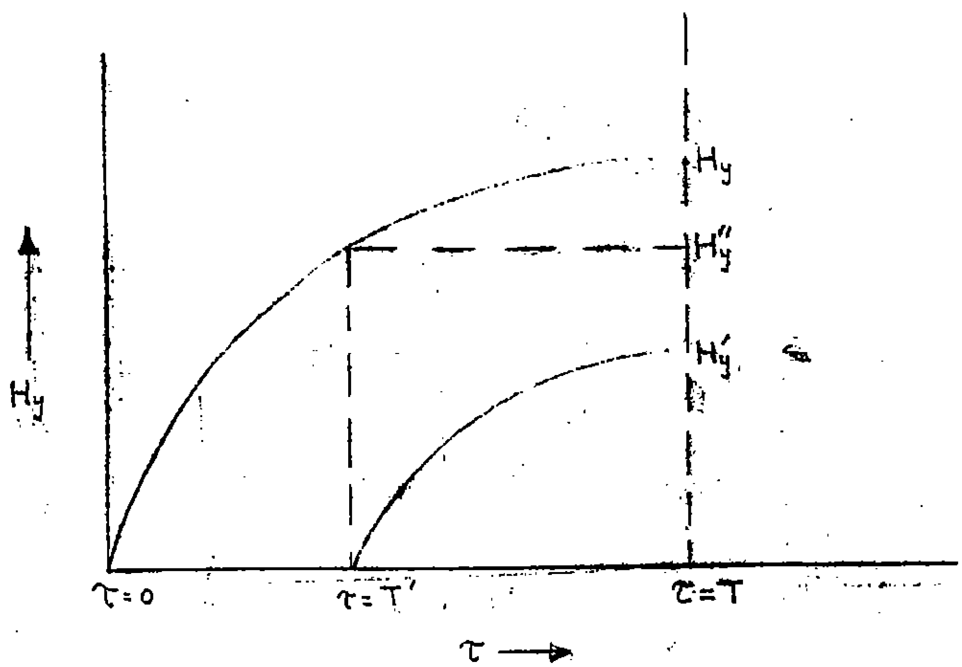
Appendix A

Justification of the Application of the Finite Ground Conductivity Correction to Time Varying Air Conductivity Problems

Consider the expression (114) for magnetic field with arbitrary ground conductivity for a step function driver contained in EMP Theoretical Note XIX.* This expression is

$$H_{y_0} = \left[1 + \sqrt{\frac{\mu_1 \sigma_0}{\mu_0 \sigma_1}}\right]^{-1} J_0 c t_{r_0} \left\{ 2 \sqrt{\frac{\tau}{\pi t_{r_0}}} e^{-(t_{z_0}/\tau)} - 2 \sqrt{\frac{t_{z_0}}{t_{r_0}}} \operatorname{erfc}\left[\sqrt{\frac{t_{z_0}}{\tau}}\right] \right\} \qquad (114)$$

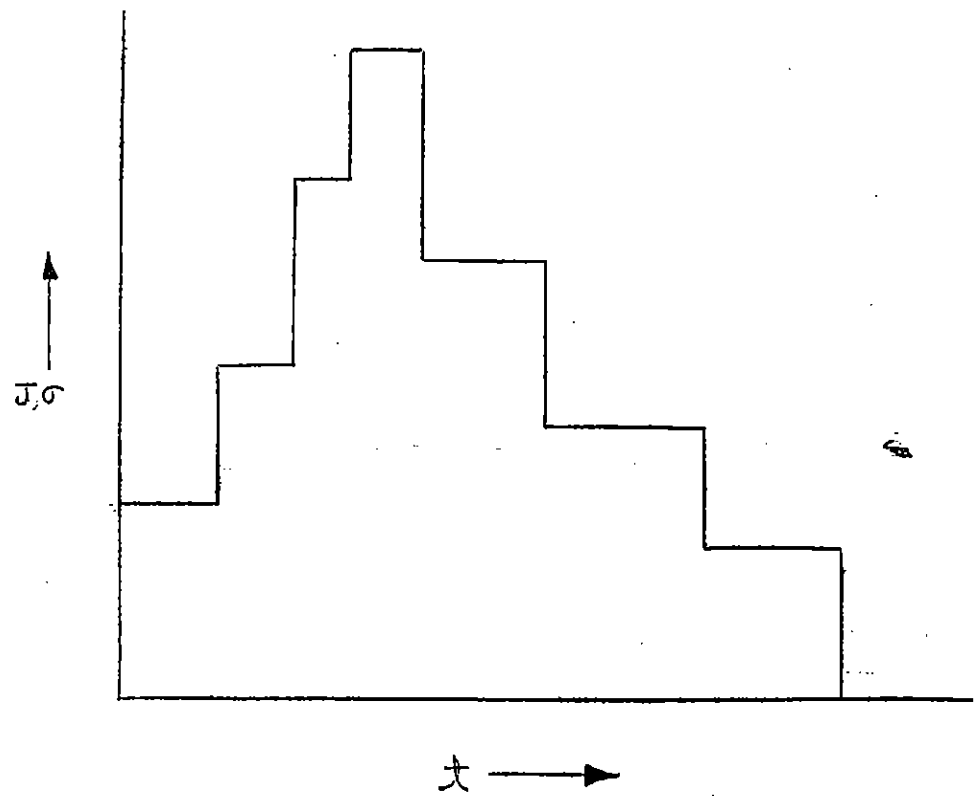
Consider also a typical plot of field versus time from $\tau = 0$ to $\tau = T$.



*A Technique for the Approximate Solution of EMP Fields from a Surface Burst in the Vicinity of an Air-Ground or an Air-Water Interface, by Lt. Carl E. Baum (September 11, 1965).

If we had stopped evaluating H_{y_0} at $\tau = T'$ and restarted the calculation, we would expect the y_0 driven fields in the interval $\tau = T' \rightarrow T$ to be a duplicate of those driven from $\tau = 0 \rightarrow (T - T')$. We would also have an initial value at $\tau = T'$ of $H_{y_0} = H_y''$ which would have to diffuse to a value $H_y - H_y'$ in order to obtain the value H_y at $\tau = T$. Thus, the fractional change in the initial value fields in the interval $\tau = T'$ to $\tau = T$ is $(H_y - H_y')/H_y''$. This diffusion of the initial values of fields is dependent on the air conductivity, but independent of ground conductivity. Furthermore, since we can choose T' arbitrarily, the fact that the diffusion of initial values of fields is independent of ground conductivity is itself independent of the field distribution in the air because this distribution is a function of T'.

Thus, since we can represent any realistic driving current and conductivity time histories by sums of step functions (see graph below) we should be able to calculate the electromagnetic fields generated by the realistic time history and include the appropriate finite ground conductivity correction.



We can do this because we can evaluate the fields driven in any step interval using equation (114), and add an additional contribution to the field at the end of the step which represents the decay of the initial values (the total field at the end of the previous step). We have demonstrated above that we know how these initial values diffuse with finite ground conductivity.

We will now note that the fields obtained by marching through infinitessimal steps using equation (114) and addition of the diffusion of initial values are the same as the fields obtained by marching through infinitessimal steps using our expression (17). This can be seen by removing J(x) and the correction factor from the integrand and looking at the integral of the Green's function.

But the validity of expression (17) is independent of the way the time-(x-) domain is divided. Therefore, expression (17) and its treatment of the ground conductivity correction over the piece-wise exponential time intervals yields the correct fields.