Theoretical Notes
Note 32

MEMORANDUM
RM-3671-PR
MAY 1963

AIR CONDUCTIVITY PRODUCED BY NUCLEAR EXPLOSIONS

W. J. Karzas and R. Latter

This research is sponsored by the United States Air Force under Project RAND—contract No. AF 49(638)-700 monitored by the Directorate of Development Planning, Deputy Chief of Staff, Research and Development, Hq USAF. Views or conclusions contained in this Memorandum should not be interpreted as representing the official opinion or policy of the United States Air Force.

The RAND Corporation
1700 MAIN ST • SANTA MONICA • CALIFORNIA

PREFACE AND SUMMARY

In view of the paucity of experimental data on the nature of the electromagnetic signals from nuclear explosions and their effects on military systems, it has been necessary to place considerable emphasis on theoretical prediction. Such theoretical prediction on the nature of the signals requires knowledge of the explosion-induced air conductivity. In the past, there has been substantial uncertainty about how to calculate this conductivity, arising from the uncertainty in the spectrum of secondary electrons as they slow down. In the present report, this spectrum is estimated, and the electronic conductivity per electron is shown to be equal to that for a mean electron energy of less than about 0.3 ev. In the presence of an electric field greater than about 2 esu, the mean energy is determined by the electric field.

AIR CONDUCTIVITY PRODUCED BY NUCLEAR EXPLOSIONS

In order to determine the electromagnetic field generated by nuclear explosions in the atmosphere, it is essential to specify the explosion-induced conductivity of the air in the neighborhood of the burst point. It is known that this conductivity depends upon the spectrum of the ionized electrons through the dependence of the conductivity on the energy-dependent electron-atom collision frequency. In the present report, the spectrum of the ionized electrons is determined in terms of their source function. With this spectrum, the electron collision frequency, and hence the electronic conductivity per ionized electron, is estimated.

ELECTRON CONDUCTIVITY

The total electronic contribution to the explosion-induced air conductivity is

$$\sigma_e(t) = \int dE\, \sigma_e(E,t) , \qquad (1)$$

where

$$\sigma_e(E,t) = \frac{e^2}{3m} \frac{n(E,t)}{\nu(E)} \qquad (2)$$

and the factor 3 results from the linear dependence of $\nu(E)$ on E

$$\nu(E) \approx 3 \times 10^9\, p\, E \sec^{-1} . \qquad (3)$$

$\nu(E)$ is the collision frequency of the slow secondary electrons with air molecules[1]. p is the air pressure in mm of mercury and E is

[1] D. R. Bates, ed. Atomic and Molecular Processes, New York, 1962. Chap. 10.

the electron energy in e.v. $n(E,t)\,dE$ is the density of electrons with energies between E and $E + dE$.

An accurate evaluation of $n(E,t)$ can be made using Age Theory—namely, for each collision of an electron with air, the electron is assumed to undergo its mean energy loss. If $\beta(E)$ is the electron attachment rate at energy E, $\gamma(E)$ the recombination rate, and $\lambda(E)$ the fractional energy loss per electron collision with air, then Age Theory gives

$$\frac{\partial n}{\partial t} = -\beta\, n - \gamma N_i\, n + \frac{\partial}{\partial E}(\lambda\, E n\, \nu) + S(E,t), \qquad (4)$$

where $S(E,t)\,dE$ is the production rate of electrons with energy E to $E + dE$ per unit volume and time and N_i is the density of positive ions.

N_i is related to the electron density through the equation

$$\frac{\partial N_i}{\partial t} = -\alpha_o N_i\left[N_i - \int dE\, n(E,t)\right] - N_i \int dE\, \gamma(E)\, n(E,t) + \int dE\, S(E,t) \qquad (5)$$

where α_o is the neutralization rate for positive and negative ions. Charge neutrality implies that the negative ion density is $N_i - \int dE\, n(E,t)$.

A straightforward solution of Eq. (4) gives the electron density in terms of the source strength and the density of positive ions:

$$n(E, t) = \frac{1}{\lambda E \nu} \int_E^\infty dE' \; S\left(E', t - \int_E^{E'} \frac{dE''}{E''\lambda''\nu''}\right) \tag{6}$$

$$\cdot \exp\left[-\int_E^{E'} dE'' \, \frac{\beta(E'') + \gamma(E'')\, N_i\left(t - \int_E^{E''} \frac{dE'''}{E'''\lambda'''\nu'''}\right)}{E''\lambda''\nu''}\right].$$

Accordingly, the electron conductivity is

$$\sigma_e(t) = \frac{e^2}{3m} \int_0^\infty \frac{dE}{E\lambda\, \nu^2} \int_E^\infty dE' \, S\left(E', t - \int_E^{E'} \frac{dE''}{E''\lambda''\nu''}\right) \tag{7}$$

$$\cdot \exp\left[-\int_E^{E'} dE'' \, \frac{\beta(E'') + \gamma(E'')\, N_i\left(t - \int_E^{E''} \frac{dE'''}{E'''\lambda'''\nu'''}\right)}{E''\lambda''\nu''}\right]$$

and the total electron density is

$$n_e(t) = \int_0^\infty dE \; n(E, t). \tag{8}$$

Hence,

$$\sigma_e(t) = \frac{e^2}{m} \, n_e(t) \left\langle \frac{1}{\nu} \right\rangle , \tag{9}$$

where

$$\left\langle \frac{1}{\nu} \right\rangle = \frac{1}{3} \left\{ \int_0^\infty \frac{dE}{E\lambda\nu} \, \frac{1}{\nu} \int_E^\infty dE' \, S\left(E', t - \int_E^{E'} \frac{dE''}{E''\lambda''\nu''}\right) \cdot \right.$$

$$\left. \cdot \exp\left[-\int_E^{E'} dE'' \, \frac{\beta(E'') + \gamma(E'')\, N_i\left(t - \int_E^{E''} \frac{dE'''}{E'''\lambda'''\nu'''}\right)}{E''\lambda''\nu''}\right]\right\}.$$

$$\left\{\int_0^\infty \frac{dE}{E\lambda\nu}\int_E^\infty dE'\, S\left(E', t-\int_E^{E'}\frac{dE''}{E''\lambda''\nu''}\right)\cdot \right.$$
$$\left. \cdot \exp\left[-\int_E^{E'} dE''\,\frac{\beta(E'')+\gamma(E'')\,N_i\left(t-\int_E^{E''}\frac{dE'''}{E'''\lambda'''\nu'''}\right)}{E''\lambda''\nu''}\right]\right\}^{-1} . \tag{10}$$

Since, in general, N_i depends on n_e (by Eq. (5)), this is only a formal expression for $\left\langle\frac{1}{\nu}\right\rangle$. In many important cases, however, the recombination rate γN_i can be neglected compared to the attachment rate β, so that $\left\langle\frac{1}{\nu}\right\rangle$ depends only on the source function and can therefore be evaluated explicitly.

APPROXIMATIONS TO $\sigma_e(t)/n_e(t)$

Evaluation of $\sigma_e(t)$ requires specifying $S(E,t)$. Since, however, $S(E,t)$ depends upon the design of the particular explosion device, we cannot evaluate $\sigma_e(t)$ in complete generality. We can, however, obtain an accurate approximation to $\sigma_e(t)/n_e(t)$, the electronic conductivity per electron, which is valid for any specified $S(E,t)$, provided electron recombination can be neglected compared to attachment.[2]

For this purpose, we shall assume an approximate form for $S(E),t)$, neglect recombination, and evaluate $\sigma_e(t)/n_e(t)$ within these approximations. Then we will indicate the conditions under which recombination may be neglected and show that our result for $\sigma_e(t)/n_e(t)$

[2]In the Appendix the effect of retaining electron recombination is discussed.

is independent of the assumed form for S(E, t).

We shall assume that the secondary electrons are all produced at a single energy E_1; then

$$S(E, t) = \dot{n}(t)\ \delta(E - E_1), \tag{11}$$

where $\dot{n}(t)$ is the total number of electrons produced per unit volume and time.

Equation (10) then becomes:

$$\left\langle \frac{1}{\nu} \right\rangle = \frac{1}{3} \left\{ \int_o^{E_1} \frac{dE}{E\lambda\nu} \frac{1}{\nu}\ \dot{n}\left(t - \int_E^{E_1} \frac{dE'}{E'\lambda'\nu'}\right) \cdot \exp\left[- \int_E^{E_1} dE' \frac{\beta(E') + \gamma(E')N_i\left(t - \int_E^{E'} \frac{dE''}{E''\lambda''\nu''}\right)}{E'\lambda'\nu'} \right] \right\} \cdot$$

$$\left\{ \int_o^{E_1} \frac{dE}{E\lambda\nu}\ \dot{n}\left(t - \int_E^{E_1} \frac{dE'}{E'\lambda'\nu'}\right) \cdot \exp\left[- \int_E^{E_1} dE' \frac{\beta(E') + \gamma(E')N_i\left(t - \int_E^{E'} \frac{dE''}{E''\lambda''\nu''}\right)}{E'\lambda'\nu'} \right] \right\}^{-1} . \tag{12}$$

Neglecting γN_i compared to β, this simplifies to

$$\left\langle\frac{1}{v}\right\rangle = \frac{1}{3}\,\frac{\int_0^{E_1}\frac{dE}{E\lambda\nu}\;\frac{1}{\nu}\;\dot{n}\left(t-\int_E^{E_1}\frac{dE'}{E'\lambda'\nu'}\right)\;\exp\left[-\int_E^{E_1}dE'\,\frac{\beta(E')}{E'\lambda'\nu'}\right]}{\int_0^{E_1}\frac{dE}{E\lambda\nu}\;\dot{n}\left(t-\int_E^{E_1}\frac{dE'}{E'\lambda'\nu'}\right)\;\exp\left[-\int_E^{E_1}dE'\,\frac{\beta(E')}{E'\lambda'\nu'}\right]}\;. \tag{13}$$

Replacing E by

$$x = \int_E^{E_1} dE'\,\frac{\beta(E')}{E'\lambda'\nu'} \tag{14}$$

as the integration variable,

$$\left\langle\frac{1}{v}\right\rangle = \frac{1}{3}\,\frac{\int_0^{\infty} dx\,\frac{1}{\beta}\;\frac{1}{\nu}\;\dot{n}\left(t-\int_0^{x}\frac{dx'}{\beta}\right)\,e^{-x}}{\int_0^{\infty} dx\,\frac{1}{\beta}\,\dot{n}\left(t-\int_0^{x}\frac{dx'}{\beta}\right)e^{-x}}\;. \tag{15}$$

If the electron production rate is slow compared to the attachment rate β, then for the important values of x in the integration,

$$\dot{n}\left(t-\int_0^{x}\frac{dx'}{\beta}\right) \approx \dot{n}(t), \tag{16}$$

and

$$\left\langle\frac{1}{v}\right\rangle \approx \frac{1}{3}\,\frac{\int_0^{\infty} dx\;e^{-x}\,\frac{1}{\beta}\;\frac{1}{\nu}}{\int_0^{\infty} dx\;e^{-x}\,\frac{1}{\beta}} = \frac{1}{3}\,\frac{\int_0^{E_1}\frac{dE}{E\lambda\nu}\;\frac{1}{\nu}\;e^{-\int_E^{E_1}dE'\,\frac{\beta(E')}{E'\lambda'\nu'}}}{\int_0^{E_1}\frac{dE}{E\lambda\nu}\;e^{-\int_E^{E_1}dE'\,\frac{\beta(E')}{E'\lambda'\nu'}}}\;. \tag{17}$$

Thus for slow production rates, $\left\langle \frac{1}{\nu} \right\rangle$ is independent of the source rate and depends only on the initial electron energy.

If, on the other hand, the electron production rate is fast, or comparable to the attachment rate, the approximation of Eq. (16) cannot be made. For this case, it is convenient to assume that the rate is exponentially increasing; that is,

$$\dot{n}(t) \sim e^{\alpha t}; \qquad \alpha \gtrsim \beta \quad . \tag{18}$$

Then

$$\left\langle \frac{1}{\nu} \right\rangle = \frac{1}{S} \frac{\int_0^\infty dx \, \frac{1}{\beta} \, \frac{1}{\nu} \, e^{-x} \exp\left\{ -\alpha \int_0^x \frac{dx'}{\beta} \right\}}{\int_0^\infty dx \, \frac{1}{\beta} \, e^{-x} \exp\left\{ -\alpha \int_0^x \frac{dx'}{\beta} \right\}} \quad . \tag{19}$$

<u>Basic Physical Data</u>

The initial energy of the secondary electrons is about 10 e.v., since on the average 33 e.v. are expended per ion pair, and about one–half of this energy goes into ionization. The electrons are slowed down by collisions with air molecules, losing some energy by elastic collisions (on the average 0.37×10^{-4} of their energy per collision), but most by inelastic collisions which excite vibrations (at the higher energies) and rotations of the air molecules. After enough collisions the energy will be reduced to thermal energies $\sim 1/25$ e.v. (at $kT \sim \frac{1}{40}$ e.v.). At thermal energies, the collision frequency is $\sim 10^{11}$ sec^{-1}.

Data on the average energy loss per collision of slow electrons with air is obtained mainly by swarm measurements and has uncertainties of interpretation. Massey and Burhop[3] present two sets of data for λ as a function of $\bar{E}$, the mean energy of the electron swarm, which disagree considerably due to differences in experimental determination of the relation of $\bar{E}$ and F/p. (F is the electric field causing the drift of the swarm.) However, for mean electron energies above 0.2 e.v., both sets of data agree that $\lambda \geq 1.3 \times 10^{-3}$. Moreover, recent, more reliable data quoted by Huxley and Crompton for very slow electrons, $\bar{E} \lesssim 0.1$ e.v., gives $\lambda \gtrsim 4 \times 10^{-4}$. For our numerical estimates, we shall assume $\lambda = 1.3 \times 10^{-3}$ for $E > 0.2$ e.v., and $\lambda = 4 \times 10^{-4}$ for $E < 0.2$ e.v. Since in both cases we take the minimum values for λ, the effect is definitely to underestimate the slowing down; that is, to overestimate the effective $\left\langle \frac{1}{\nu} \right\rangle^{-1}$.

Electron attachment to neutral oxygen forming O_2^- involves a three-body process for which at sea level the rate is about 10^8 sec^{-1}.[4] At thermal energy (1/25 e.v.)

---

[3]Massey, H. S. W. and E. H. S. Burhop, Electronic and Ionic Impact Phenomena, Oxford, 1952, p. 279.

[4]D. R. Bates, ed. Atomic and Molecular Processes, New York, 1962, Chap. 6.

the rate is 0.75×10^8; it rises to $\sim 1.5 \times 10^8$ at 0.1 e.v., then falls slowly to $\sim 0.4 \times 10^8$ at 1 e.v. For numerical estimates, we shall take β = constant $\approx 10^8$ sec^{-1}, at sea level; or $\beta \approx 10^8 (\rho/\rho_0)^2$ sec^{-1} at other densities.

The recombination rate of electrons and positive ions, γ, is about 10^{-6} cm^3 sec^{-1} for the dissociative process

$$e + O_2^+ \rightarrow O + O.$$

In addition, at high pressures, there is a three-body process involved. At sea level, this has a rate of the order of 10^{-6} cm^3 sec^{-1}, and, of course, becomes less important at high altitudes. Thus, the term γN_i can be neglected compared to β if [5]

$$\gamma N_i << \beta$$

or

$$10^{-6} N_i << 10^8 (\rho/\rho_0)^2; \quad N_i << 10^{14} (\rho/\rho_0)^2.$$

The positive ion density for a typical nuclear explosion can be estimated by computing the γ-ray energy deposition, assuming 33 e.v. per ion pair produced. If E_γ is the total prompt γ-ray energy, and Y is the total yield in kilotons, the ion density at distance r in meters is

---

[5] See Appendix.

$$N_i \approx .6 \times 10^{17} \frac{Y}{r^2} \frac{\rho}{\rho_o} e^{-r/300\, \rho_o/\rho},$$

where it is assumed that $E_\gamma = 3 \times 10^{-4}$ Y. For $N_i << 10^{14} (\rho/\rho_o)^2$,

$$\therefore \frac{Y}{r^2} e^{-r/300\, \rho_o/\rho} << 1.6 \times 10^{-3} \rho/\rho_o .$$

For example, at sea level, for a 1 KT explosion, the inequality is satisfied at distances $r \gtrsim 25$ meters. For a 1 megaton explosion, the distance is about 400 meters. At higher altitudes, for kiloton yields the distance increases as $(\rho_o/\rho)^{1/2}$; for megaton yields at very high altitudes $\left(\rho/\rho_o \lesssim 10^{-3}\right)$, $r > 800[Y_{MT}\, \rho_o/\rho]^{1/2}$ meters. Thus, if we consider distances[6] greater than this value of r, we can neglect νN_i compared to β, and use the approximate formulae for $\left\langle \frac{1}{\nu} \right\rangle$.

NUMERICAL ESTIMATE OF $\sigma_e(t)/n_e(t)$

It is a straightforward matter to substitute the energy dependent functions λ, β, and ν into the approximations, Eqs. (17) and (19), and evaluate $\sigma_e(t)$ numerically. However, it is sufficiently accurate to approximate these functions by

$$\beta(E) = \beta = 10^8 \text{ sec}^{-1}$$

$$\nu(E) = aE, \quad a = 2?3 \times 10^{12} \text{ sec}^{-1} \text{ e.v.}^{-1}$$

$$\lambda(E) = \lambda_1 \approx 1.3 \times 10^{-3}, \quad E > E_b$$

$$= \lambda_2 \approx 4 \times 10^{-4}, \quad E < E_b$$

[6] If $E_\gamma > 3 \times 10^{-4}$ Y, the critical values of r are increased. For example, with $E_\gamma = 3 \times 10^{-2}$ Y, which is a reasonable upper bound of E_γ, $r \sim 200$ meters for Y = 1 kiloton and $r \sim 1100$ meters for Y = 1 megaton at sea level.

where $E_b \approx 0.2$ e.v. These approximations enable Eqs. (17) and (19) to be evaluated analytically.

For the initial energy we take $E_1 = 10$ e.v. Then

$$x = \int_E^{E_1} dE \frac{\beta}{E\lambda\nu} = \frac{\beta}{a\lambda_1}\left(\frac{1}{E} - \frac{1}{E_1}\right), \quad E > E_b$$

$$= x_b + \frac{\beta}{a\lambda_2}\left(\frac{1}{E} - \frac{1}{E_b}\right), \quad E < E_b, \qquad (20)$$

where

$$x_b = \frac{\beta}{a\lambda_1}\left(\frac{1}{E_b} - \frac{1}{E_1}\right). \qquad (21)$$

Then

$$\frac{1}{\nu} = \frac{1}{aE}$$

$$= \frac{1}{aE_1} + \frac{\lambda_1}{\beta} x \qquad x < x_b$$

$$= \frac{1}{aE_b} + (x - x_b)\frac{\lambda_2}{\beta} \qquad x > x_b. \qquad (22)$$

Thus Eq. (19) becomes

$$\left\langle\frac{1}{\nu}\right\rangle = \frac{\alpha+\beta}{3\beta}\left[\int_0^{x_b} dx\, e^{-x(1+\frac{\alpha}{\beta})}\left(\frac{1}{aE_1} + \frac{\lambda_1}{\beta}x\right)\right.$$

$$\left. + \int_{x_b}^{\infty} dx\, e^{-x(1+\alpha/\beta)}\left(\frac{1}{aE_b} + (x - x_b)\frac{\lambda_2}{\beta}\right)\right] \qquad (23)$$

$$= \frac{\lambda_1}{3(\alpha+\beta)} + \frac{1}{3aE_1} - \frac{\lambda_1-\lambda_2}{3(\alpha+\beta)}\, e^{-\frac{\alpha+\beta}{\lambda_1 a}\left(\frac{1}{E_b}-\frac{1}{E_1}\right)} .$$

Substituting numerical values for the parameters, and setting $\alpha \approx 10^8\ \mathrm{sec}^{-1}$, we find from Eq. (19)

$$\left\langle\frac{1}{\nu}\right\rangle \approx 0.12 \times 10^{-11}\ \mathrm{sec},$$

or $\left\langle\frac{1}{\nu}\right\rangle^{-1} \approx 8 \times 10^{11}\ \mathrm{sec}^{-1}$, which corresponds to an average electron energy of about 0.3 e.v. This result is essentially independent of the initial energy E_1.

Equation (17) leads to the same result, with α set equal to zero. Neglecting unimportant terms, we find

$$\left\langle\frac{1}{\nu}\right\rangle \approx \frac{\lambda_1}{3\beta} - \frac{\lambda_1-\lambda_2}{3\beta}\, e^{-\beta/\lambda_1 aE_b}, \qquad (24)$$

or expanding,

$$\left\langle\frac{1}{\nu}\right\rangle \approx \frac{\lambda_2}{3\beta} + \left(1-\frac{\lambda_2}{\lambda_1}\right)\frac{1}{3aE_b} \approx 1.8 \times 10^{-12}\ \mathrm{sec},$$
$$\left\langle\frac{1}{\nu}\right\rangle^{-1} \approx 5.4 \times 10^{11}\ \mathrm{sec}^{-1}, \qquad (25)$$

which again corresponds to an energy of $\sim$ 0.2 e.v.

The calculations above have assumed that the electrons in colliding with the air tend to lose all their energy. Actually, of course, they only tend to slow down to $\sim$ 0.04 e.v., the ambient thermal energy. Since the effective average energy is several times as large as this, no significant correction is necessary.

If there is an electric field present, however, the electrons at equilibrium would have a drift velocity which will increase their energy above thermal. If this new equilibrium energy is greater than the 0.3 e.v., which we found above, then it will determine the air conductivity.

APPENDIX

It is obvious from Eq. (7) that if recombination to positive ions is not neglected, then $\sigma_e(t)$ is less than we have estimated. This lessening results from two effects. First, the electron density decays more rapidly due to the additional loss mechanism. Second, and perhaps more important, the lessening is greatest at the low (near thermal) energies which tends to raise the mean electron energy and lower $\left\langle \frac{1}{\nu} \right\rangle$.

Since the dependence of $\gamma(E)$ on E is unknown, it is not possible quantitatively to determine the change in $\sigma_e(t)$ or $\left\langle \frac{1}{\nu} \right\rangle$ resulting from recombination effects. Nonetheless, it is possible to indicate more precisely than done in the text the conditions under which the neglect of recombination is unimportant.

If recombination is to be unimportant, then, as before,

$$\gamma N_i << \beta .$$

In the text we assumed that this inequality must hold for N_i equal to the total number of positive ions which are formed. Actually N_i, which obeys Eq. (5), attains this value only if $S(t)$ is a δ – function, and then only initially. A more reasonable estimate of N_i can be obtained from Eq. (5) if it is assumed that $\alpha_o = \gamma(E)$, a constant. Experimentally, $\alpha_o \approx \gamma(E \sim 0)$ and, at higher energies, $\gamma(E)$

decreases somewhat. Under the approximate assumption, $\alpha_o \approx \gamma(E)$, we find

$$\frac{\partial N_i}{\partial t} = -\alpha_o N_i^2 + S(t) .$$

This equation can be solved in two limits. First, if $S(t) = S_o\,\delta(t)$,

$$N_i = \frac{S_o}{1+\alpha_o S_o t} ,$$

where S_o is the total number of positive ions produced, and, second, if $\frac{\partial N_i}{\partial t} \approx 0$,

$$N_i \approx \sqrt{\frac{S(t)}{\alpha_o}} .$$

For the impulsive source,

$$\gamma N_i \approx \alpha_o N_i << \beta$$

implies that ($\alpha_o \approx \gamma$)

$$\frac{\gamma S_o}{1+\gamma S_o t} << \beta .$$

This inequality is satisfied for all times in the region indicated in the text and in all regions about the burst point for times $t >> \beta^{-1}$.

However, since the production of gammas in a nuclear explosion extends over times T substantially longer than $1/\beta$ (β evaluated at sea level), it is a better approximation at sea level to assume that

$$N_i = \sqrt{\frac{S_o}{T\gamma}} \; .$$

Since $T \sim$ several x 10^{-8} and $\nu \sim 10^{-6}$ (at sea level),

$$N_i \sim \sqrt{\frac{10^2 S_o}{(\text{several})}} \; .$$

Or

$$\gamma N_i << \beta$$

gives

$$S_o << \text{several} \times 10^{14}$$

at sea level. This condition is somewhat less restrictive than in the text and decreases the radius at sea level outside of which $\gamma N_i << \beta$ by a factor of 2 or so.

Furthermore, since N_i decays as

$$N_i \sim \frac{\sqrt{S_o / T\nu}}{1 + \gamma\sqrt{S_o / T\gamma}\,(t - T)}$$

after the gamma–ray source cuts off ($t > T$), $\gamma N_i << \beta$ is satisfied everwhere after times

$$t - T >> \beta^{-1} \; .$$