EMP Theoretical Notes
Note 26

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Calculation of the EMP from High Altitude Nuclear Detonations

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Abstract

A numerical technique for solving the high frequency approximation EMP equations is presented, as appropriate to high-altitude source geometries.

1. Source Terms and Field Calculations

A set of differential equations describing both the radiating and non-radiating electric and magnetic fields, assuming the high frequency approximation, has been developed by Karzas and Latter, who also derived equations for the source terms (current density and conductivity). The three electric field equations are[1]

$$\frac{\partial E_r}{\partial \tau} + \frac{j_r}{\varepsilon_o} = 0 \tag{1}$$

$$\frac{\partial r E_\theta}{\partial r} + \frac{\mu_o c}{2} j_\theta = 0 \tag{2}$$

$$\frac{\partial r E_\phi}{\partial r} + \frac{\mu_o c}{2} j_\phi = 0 \tag{3}$$

(In rationalized MKSA Units)

- τ: retarded time
- j: current density
- μ_o: permeability of vacuum ($4\pi(10^{-7})\ \frac{\text{henry}}{\text{m}}$)
- c: velocity of light
- ε_o: permittivity of vacuum ($8.854\ (10^{-12})$ farad/m)

A local right handed coordinate system is constructed at R, θ, ϕ, with unit vector $\hat{r}$, a radial from the burst point,

$\hat{\phi} = \frac{-\hat{r} \times \vec{B}_e}{|B_e|}$ where B_e is the earth's magnetic field, and

$\hat{\theta} = \hat{\phi} \times \hat{r}$. (See figure 1)

The source terms are conveniently expressed in retarded time by adding up the contributions from individual electrons that are created along the line of sight and arrive at the point R at the same time. The procedure involves calculation of the total γ-ray flux, conversion into the number of primary electrons by the Compton process, and determination of the rate of energy loss and components of velocity for the electrons when they are at R.

The total number of γ rays at a point in space R meters away from the burst point is given by

$$n_\gamma(R) = \frac{N_\gamma e^{-\sigma_c \int_o^R N_a d_r}}{4\pi R^2} \tag{4}$$

1. W. J. Karzas and Richard Latter, Detection of the Electromagnetic Radiation from Nuclear Explosions in Space, RM-4306, Oct 1964, EMP Theoretical Note 40.

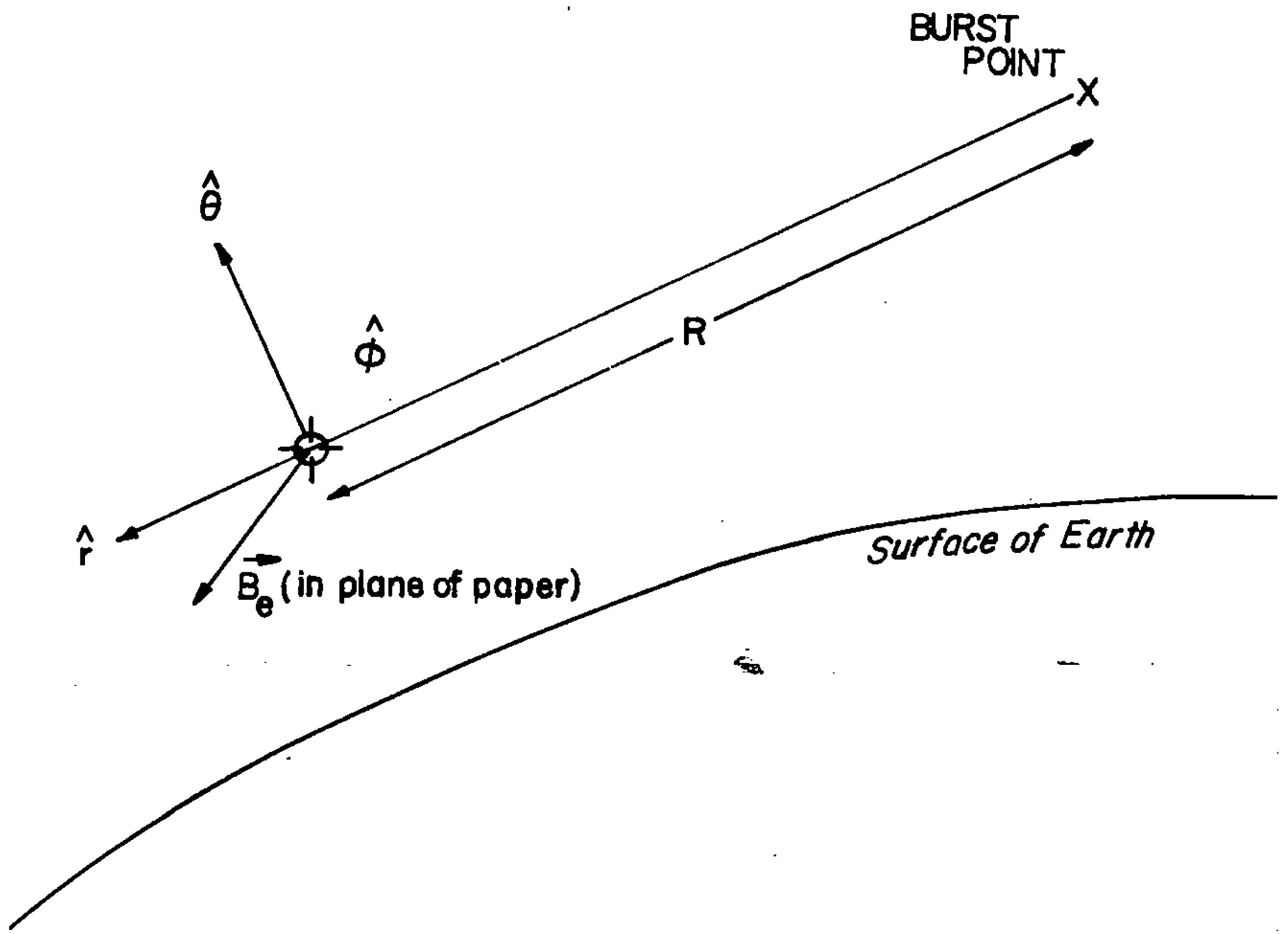


Fig.1 - COORDINATE SYSTEM FOR HIGH ALTITUDE EMP CALCULATIONS

N_a is the number density of air molecules at r; σ_c is the collision cross section for the Compton effect and is a function of the γ-ray energy; N_γ is the total number of γ rays emitted at the device. Only the first collision of a gamma ray is considered since the times involved in scattering back into the radial are longer than times of interest. Then the number of primary Compton electrons created at R is

$$n_{pri}(R) = \frac{N_\gamma e^{-\sigma_c \int_0^R N_a d_r}}{4\pi R^2} \sigma_c N_a(R) \qquad (5)$$

and the production rate (number of primaries per m^3 - sec) is

$$\dot{n}_p(R,\tau) = n_{pri}(R) f(\tau). \qquad (6)$$

$f(\tau)$ describes the rate of γ-ray emission at the burst point normalized so that

$$\int_{-\infty}^{\infty} f(\tau)\, d\tau = 1 \qquad (7)$$

The average energy of the electrons created at R can be found by using the Klein-Nishina formulas for the average scattering and collision cross sections. By using this average energy, the primary current* and ionization produced by a single electron are calculated. While losing energy to ionizing collisions with the air, the electron is being turned by the $\vec{v} \times \vec{B}$ force of the earth's magnetic field. By substitution into the equations of motion of the electron and neglecting the effects of self-generated E and B fields, we obtain the electron's turning frequency as a function of velocity, strength of the earth's magnetic field, and θ, the angle of intersection between $\hat{r}$ and $\vec{B}_e$.

$$\omega = q \frac{B_e (1-\beta^2)^{1/2}}{m_o} \qquad (8)$$

m_o: rest mass of the electron

β: v/c

q: electronic charge

The normalized velocity components (β_r, β_θ, β_ϕ) and rate of energy loss (dE/dt) are found as the electron slows down and spirals about the field lines. If t_1 is absolute time since the birth of the electron, and the calculations are carried on for the total time of flight (t_f), the following equations describe the electron's motion:

*The primary currents are produced by the Compton electrons directly, as opposed to the conduction currents.

$$E(t_1) = E_o - \int_o^{t_1} \left(\frac{dE}{dt}\right) dt \tag{9}$$

$\frac{dE}{dt}$ can be obtained from stopping power formulas for high energy electrons in air.

For $T = E/m_o c^2$

$$\beta(t_1) = \sqrt{\frac{T(t_1)\ [2 + T(t_1)]}{[1 + T(t_1)]^2}} \tag{10}$$

$$\phi(t_1) = \int_o^{t_1} \omega(t)\ dt \tag{11}$$

[$\phi(t_1)$ is the total angle through which the electron has turned at t_1]

$$\beta_r(t_1) = [\cos^2\theta + \sin^2\theta \cos\phi\ (t_1)]\ \beta(t_1) \tag{12}$$

$$\beta_\phi(t_1) = \sin\theta \sin\phi\ (t_1)\ \ \beta(t_1) \tag{13}$$

$$\beta_\theta(t_1) = -\sin\theta\cos\theta\ [1 - \cos\phi\ (t_1)]\ \ \beta(t_1) \tag{14}$$

Finally the components of velocity are multiplied by the appropriate production rates ($\dot{n}_p$ from equation 6), and the product is summed for all velocities in the electron's lifetime to obtain the current density.

$$j_r\ (R,\tau) = qc \int_o^{t_f} \beta_r(t_2)\ \dot{n}_p\ (R,\tau_1)\ dt_2 \tag{15}$$

$$j_\theta\ (R,\tau) = qc \int_o^{t_f} \beta_\theta(t_2)\ \dot{n}_p\ (R,\tau_1)\ dt_2 \tag{16}$$

$$j_\phi\ (R,\tau) = qc \int_o^{t_f} \beta_\phi(t_2)\ \dot{n}_p\ (R,\tau_1)\ dt_2 \tag{17}$$

with

$$\tau_1 = \tau - t_2 + \int_o^{t_2} \beta_r(t_1)\ dt_1 \tag{18}$$

A similar operation with the energy loss terms and an additional integration over retarded time gives the total energy lost to the production of secondary electrons. One secondary electron is created for each 34 electron volts deposited by the Compton electrons.

$$n_{sec}\ (R,\tau) = \frac{1}{34}\int_{0}^{\tau} d\tau_2 \int_{0}^{t_f} \left(\frac{dE}{dt_2}\right) \dot{n}_p\ (R,\tau_1)\ dt_2 \qquad (19)$$

(n_{sec} is the number of secondary electrons) with

$$\tau_1 = \tau_2 - t_2 + \int_{0}^{t_2} \beta_r(t_1)\ dt_1 \qquad (20)$$

We have assumed that an electron does not see changes in the atmospheric density during its time of flight, so that the velocity history of one electron applies to all those created at some earlier time along the ray. To account for scattering of electrons off the radial, the number of primary electrons contributing to the current density is reduced by the cosine of the scattering angle.

The total current density j must also include the conduction currents generated by the electric fields. This current is expressed by

$$\vec{j}_{sec} = \sigma\vec{E} \qquad (21)$$

If the ion motion is neglected, the conductivity is given by

$$\sigma = n_{sec}\ q\ \mu_e \qquad (22)$$

where μ_e is the electronic mobility.

The mobility as a function of electric field and atmospheric pressure has been determined experimentally,[2] and the results are presented in figure 2. By using fits to this data the conduction currents can easily be calculated as a function of electric field, atmospheric density, and the density of secondary electrons. There are limitations to this procedure for large fields at high altitudes because the equilibrium temperature of the electrons becomes large, and the drift velocities associated with the calculated mobilities are great enough to cause further ionization in the air.

2. Lt Carl E. Baum, EMP Theoretical Note 12, Electron Thermalization and Mobility in Air, 16 July 1965.

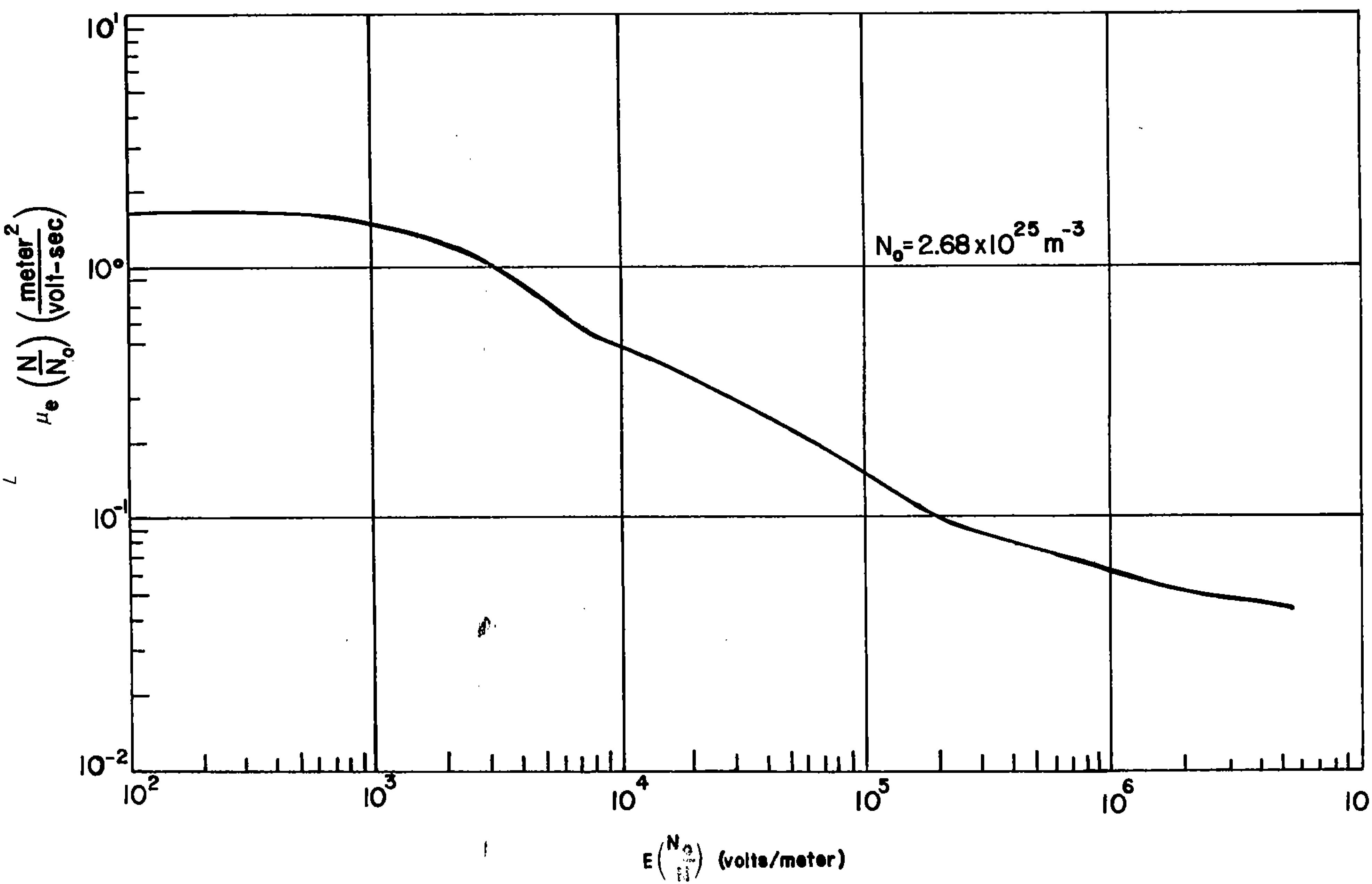


Fig. 2 - ELECTRON MOBILITY IN AIR

The net current density is the sum of the Compton induced currents and the conduction currents.

$$\vec{j} = \vec{j}_c + \sigma\vec{E} \tag{23}$$

2. Numerical Solutions

The three differential equations for the electric field are of the form

$$\frac{dy}{dx} = f(x,y) \tag{24}$$

A numerical technique applicable to equations of this type is the Runge Kutta method.

To solve (24) between x_o and $x_o + \Delta x$, when $y_o = y(x_o)$ is given as an initial condition, set

$$k_1 = \Delta x \; f(x_o, y_o) \tag{25}$$

$$k_2 = \Delta x \; f(x_o + \Delta x/2, \; y_o + k_1/2) \tag{26}$$

$$k_3 = \Delta x \; f(x_o + \Delta x/2, \; y_o + k_2/2) \tag{27}$$

$$k_4 = \Delta x \; f(x_o + \Delta x, \; y_o + k_3) \tag{28}$$

then

$$y_1 = y_o + \frac{1}{6}\,(k_1 + 2(k_2 + k_3) + k_4) \tag{29}$$

The three differential equations to be solved are

$$\frac{\partial E_r}{\partial \tau} = -\,\frac{1}{\varepsilon_o}\,(j_{cr} + \sigma E_r) \tag{30}$$

for constant r, and

$$\frac{\partial E_\theta}{\partial r} = -\,\frac{E_\theta}{r} - \frac{\mu_o c}{2}\,(j_{c\theta} + \sigma E_\theta) \tag{31}$$

$$\frac{\partial E_\phi}{\partial r} = -\,\frac{E_\phi}{r} - \frac{\mu_o c}{2}\,(j_{c\phi} + \sigma E_\phi) \tag{32}$$

for constant τ.

These equations are independent except for the mobility, which is a function of total electric field.

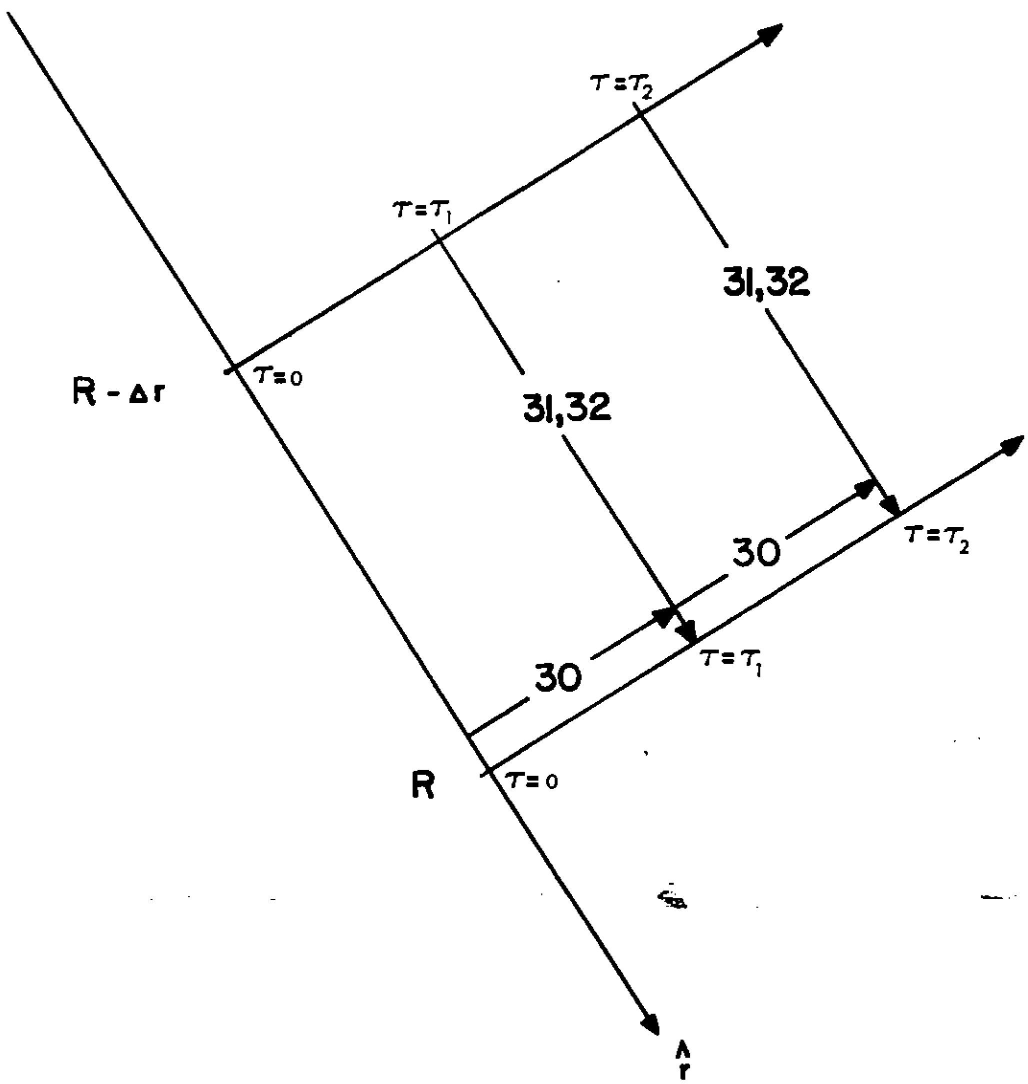


Fig.3-STEPWISE PROCEDURE IN SOLUTION OF EQUATIONS 30-32

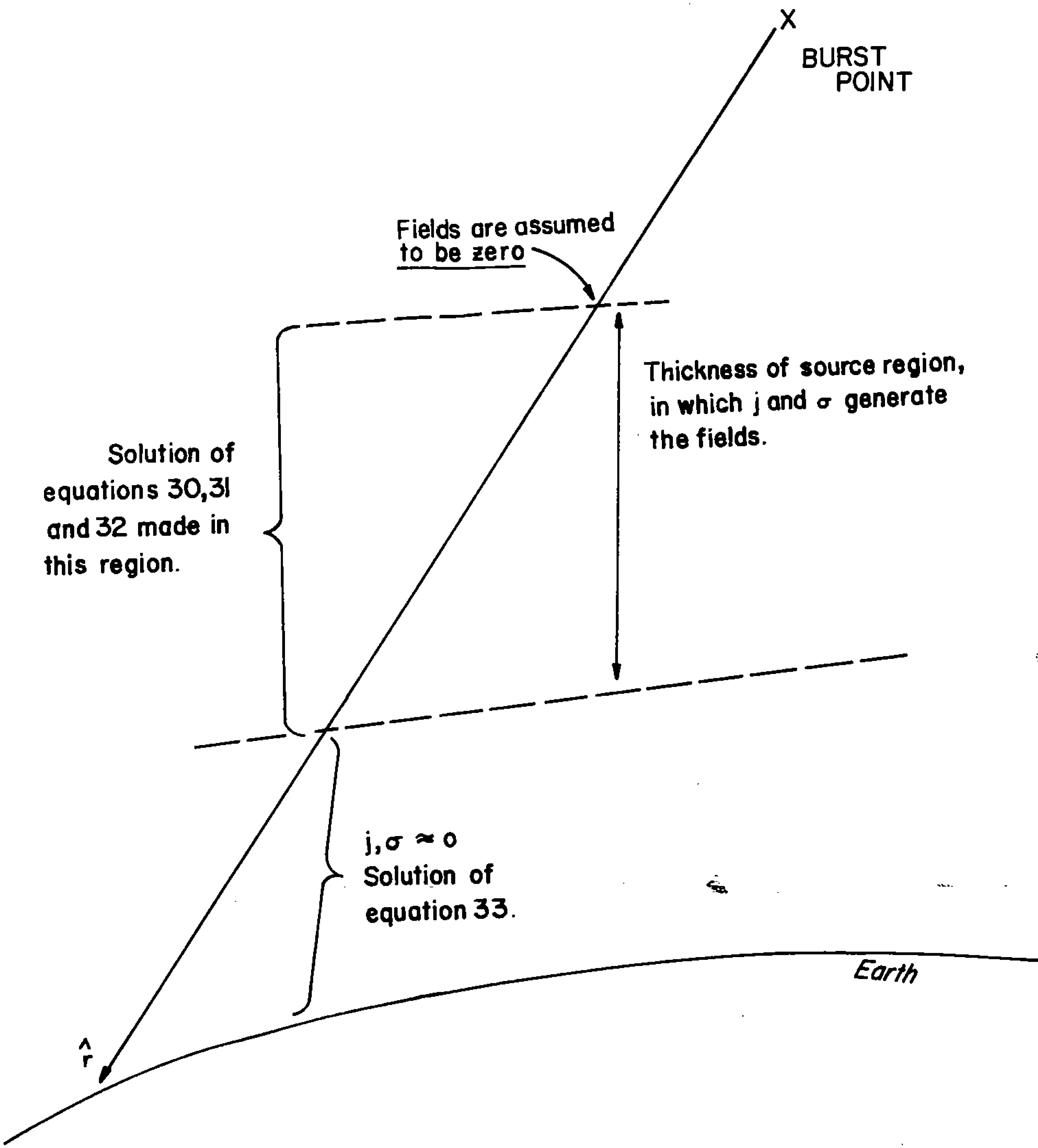


Fig.4 - REGIONS OF EMP CALCULATIONS

A complete solution requires the integrals of (31) and (32) along the entire radial from the burst point to the ground, and the solution of (30) up to times of interest at all values of r along the ray. The fields are initialized by making $E_r = 0$ for $\tau = 0$ at all R and by setting the transverse fields to zero for all time at an initial R_o near the burst point. Δr's and $\Delta\tau$'s are chosen so that changes in the source terms are small over each interval.

Starting at R and $\tau = 0$ the calculations are proceeding in this order (see figure 3):

1. A solution of (30) by performing steps 25-29 from $\tau = 0$ to $\tau = \tau_1$ at R.

2. Solution of (31) and (32) from R- Δr to R at $\tau = \tau_1$ in the same manner.

3. Solution of (30) from $\tau = \tau_1$ to $\tau = \tau_2$ at R.

4. Solution of (31) and (32) from R-Δr to R at $\tau = \tau_2$.

5. Continue the procedure up to $\tau = \tau$ max; then r is incremented and the process begins again at $\tau = 0$.

By following this stepwise procedure the total electric field $(E_r^2+E_\theta^2+E_\phi^2)^{1/2}$ can be found quite accurately for every calculation.

In the radial field calculation, E_θ and E_ϕ are approximated by the neighboring tangential field values. Equations 31 and 32 then know the exact value for E_r.

It is necessary to solve equations 30-32 only in the regions along the radial in which j and σ are large. For values of R near the surface of the earth, the sources become very small. Here the radial fields become equal to zero, and an analytic solution of

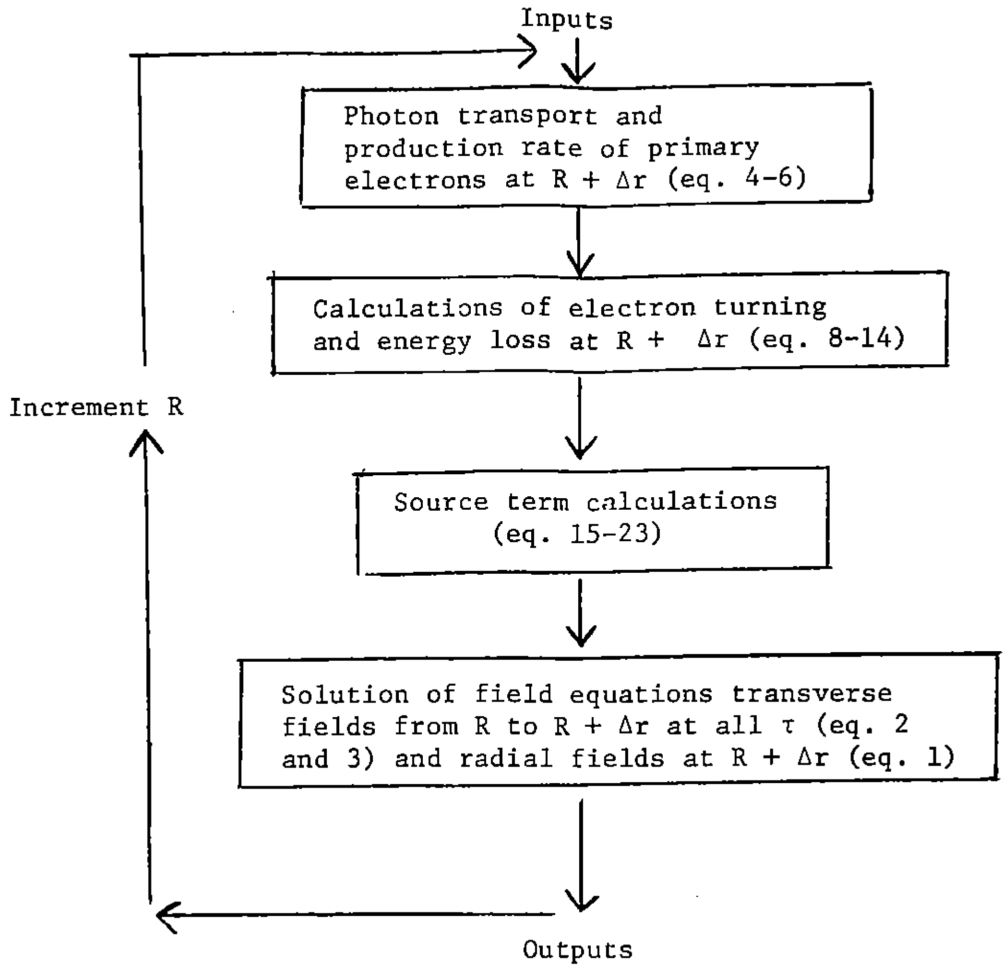
$$\frac{\partial rE}{\partial r} = 0 \qquad (33)$$

can be used for the transverse components. Figure 4 summarizes the geometry of the field solutions.

The remainder of the numerical work is primarily associated with the calculation of source terms and can be accomplished by simple differencing methods. j and n_{sec} are given in equations 15-20. The energy loss terms required in equations 9 and 19 may be obtained from a variety of stopping power formulae or empirical fits for the rate of energy loss of fast electrons in air.

3. High Altitude Production Codes at AFWL.

The numerical methods presented in section two have been programmed on the CDC 6600 for production running at AFWL. A series of production codes has been generated in the following format:



The appendix contains the formulas for the Compton cross sections and a plot of the model atmosphere used in the codes. The size and oritentation of the earth's magnetic field for a known latitude, longitude, and altitude are curve fitted in a routine provided by the Theoretical Branch at AFWL.

APPENDIX

1. Earth's Atmosphere
2. Compton Cross Section

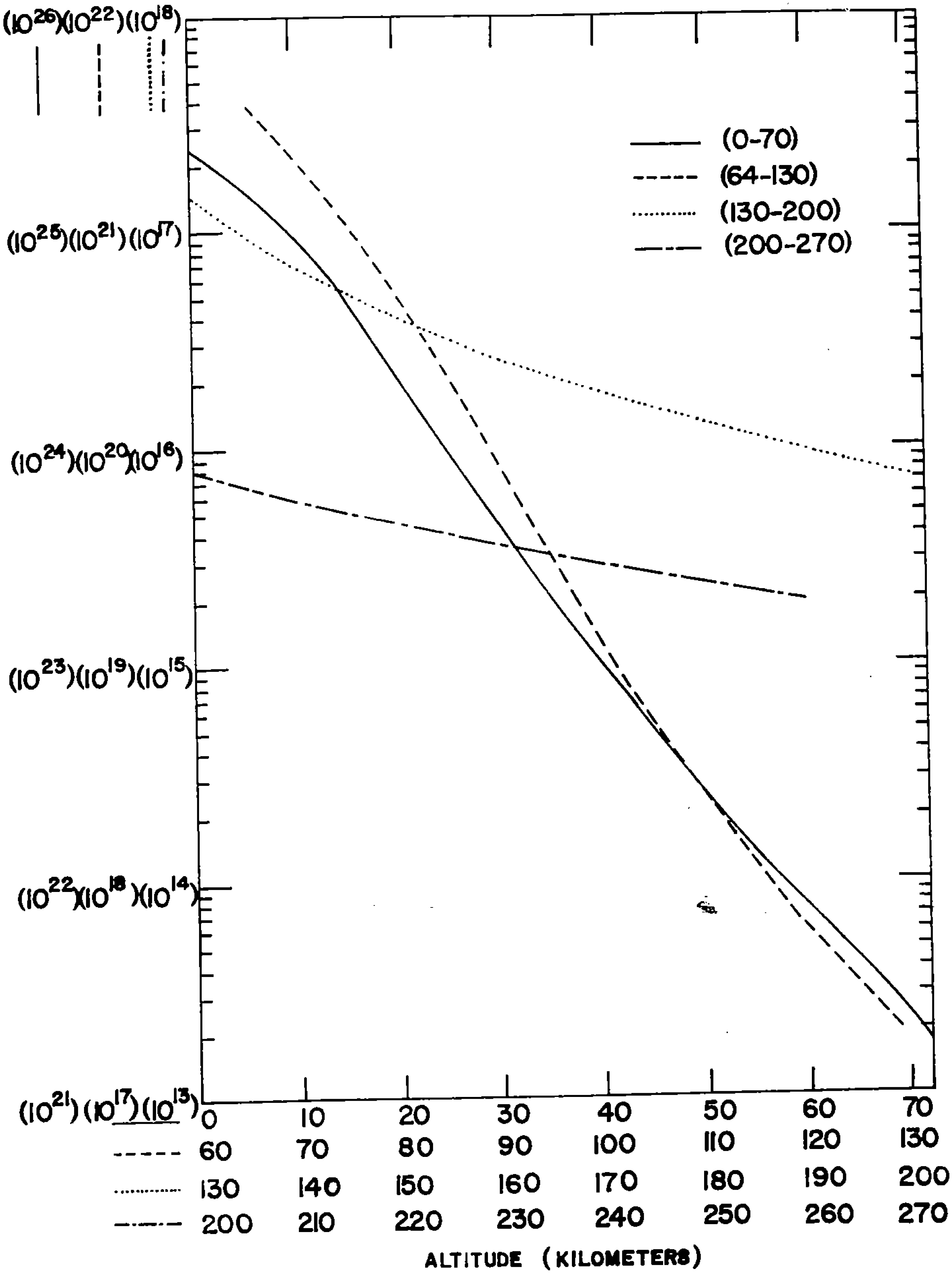


FIGURE 5 - EARTH ATMOSPHERE 0–270 km

Compton Cross Sections[3]

For $\alpha = h\nu_o / m_o c^2$ and $r_o = 2.818\ (10^{-15})$ m.

$$_e\sigma = 2\pi r_o^2 \left\{ \frac{1+\alpha}{\alpha^2} \left[\frac{2(1+\alpha)}{1+2\alpha} - \log \frac{1+2\alpha}{\alpha} \right] + \frac{\log(1+2\alpha)}{2\alpha} - \frac{1+3\alpha}{(1+2\alpha)^2} \right\} \frac{m^2}{\text{electron}}$$

$$_e\sigma_s = \pi r_o^2 \left[\frac{\log(1+2\alpha)}{\alpha^3} + \frac{2(1+\alpha)(2\alpha^2-2\alpha-1)}{\alpha^2(1+2\alpha)^2} + \frac{8\alpha^2}{3(1+2\alpha)^3} \right] \frac{m^2}{\text{electron}}$$

$_e\sigma$: average collision cross section

$_e\sigma_s$: average scattering cross section

The average energy per recoil electron is

$$\frac{(T)_{ave}}{h\nu_o} = 1 - \frac{_e\sigma_s}{_e\sigma}$$

3. Evans, R. D., Handbuch der Physic, V. 34.