Sensor and Simulation Notes

Note 93

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Minimization of Current Distortions on A Cylindrical Post Piercing A Parallel-Plate Waveguide

by

R. W. Latham, K. S. H. Lee, and R. W. Sassman
Northrop Corporate Laboratories
Pasadena, California

Abstract

The time behavior is obtained of the total current induced on a cylindrical post by a step-function plane wave traveling between a parallel-plate waveguide and impinging on the post which protrudes through one of the guide's walls. This geometry may be considered the idealized shape of some antenna-like structure in a parallel-plate transmission line or surface transmission line.

The resonance frequency and the decay time constant of the fundamental mode (i.e., the mode with the longest wavelength) of the post are found from the response of the post to a step-function incident plane wave and are plotted against the size of the hole in the wall through which the post protrudes. It is found that as the size of the hole is increased, the current amplitude of the fundamental mode decreases away from the corresponding value when the post of twice the length is in free space, whereas the resonance frequency and the decay time constant increase toward the free-space values.

Acknowledgment

We thank Dr. Carl Baum for his helpful suggestions and discussions throughout the course of this work and Mrs. G. Peralta for typing the manuscript and drawing the curves.

I. Introduction

The physics of the surface transmission-line simulator is now fairly well understood, especially when it is used to simulate the low-frequency content of a nuclear EMP. In some instances, there are site structures that rise higher than or as high as the top plate of a surface transmission line. In such cases, schemes are called for to build the line around such structures for minimum electromagnetic interactions between the line and the structures. If the structures are not in electrical contact with the line (that is, they do not physically touch each other), one can be sure that the interactions will be minimal in the low-frequency regime. Thus, the logical way to start is to make holes in the top plate of the line through which the structures protrude, and then to analyze the high-frequency interactions of such structures with the line.

The structures considered in this note will be idealized to be cylindrical in shape and the holes will be taken to be circular. The ground will be assumed to be perfectly conducting and the line will be taken to be two perfectly conducting parallel plates of infinite extent. Throughout this note the cylindrical structures are considered taller than the top plate of the line; the opposite case where the top plate is taller than the structures may be considered in a future note.

In Section II the problem is formulated in terms of a coupled set of integral equations, the unknowns being the aperture electric field in the hole and the total axial current on the part of the post sticking out of the wave-guide (see Fig. 1a). With a knowledge of this aperture electric field the current on the part of the post between the plates can be determined by integration. The Green's functions used in Section II are derived in Section III. Numerical results are presented in graphical form in Section IV. An appendix is devoted to the derivation of a Green's theorem applicable to bodies of revolution and axially symmetric fields.

II. Mathematical Formulation

Consider the situation depicted in figures 1a and 1b where a time-harmonic plane wave travels between two perfectly conducting plates of infinite extent and impinges on the perfectly conducting, cylindrical post that protrudes through a circular hole in the top plate. We wish to calculate the total axial current induced on the post by this plane wave, which will later be generalized to be a step-function pulse. Let us recall the well-known fact that, if only the induced total axial current on an axi-symmetric body is considered, one can restrict his attention only to the axi-symmetric mode of the fields, i.e., the mode whose only non-vanishing field components are H_ϕ, E_z and E_ρ, independent of the azimuthal variation ϕ. H_ϕ is easily seen from Maxwell's equations to satisfy (A·1) (see Appendix).

An application of (A·7) to region (2) (see Fig. 1b) gives

$$H_\phi^{(2)}(\rho,z) = H_\phi^{inc}(\rho) + H_\phi^{ref}(\rho) + i\omega\varepsilon \int_a^b G_2(\rho,z;\rho',s)E_\rho(\rho')\rho' d\rho' \tag{1}$$

where G_2 satisfies (A·6) with boundary conditions

$$\begin{aligned} \frac{\partial}{\partial\rho}(\rho G_2) &= 0 \quad , \quad \text{when} \quad \rho = a \\ \frac{\partial}{\partial z} G_2 &= 0 \quad , \quad \text{when} \quad z = 0 \, , \, s \end{aligned} \tag{2}$$

and H_ϕ^{ref} denotes the field reflected from an infinitely long, perfectly conducting cylinder and will be determined shortly.

The magnetic field vector of the incident plane wave is given by

$$\underline{H}^{inc} = - \underline{e}_y H_o e^{ikx}$$

the axi-symmetric mode of which is

$$H_\phi^{inc}(\rho) = \frac{1}{2\pi}\int_o^{2\pi} \underline{e}_\phi \cdot \underline{H}^{inc} d\phi = - iH_o J_1(k\rho) \tag{3}$$

To find $H_\phi^{ref}(\rho)$ in (1) we take (3) as the wave incident on an infinite, perfectly conducting cylinder. One can easily find that

$$H_\phi^{ref}(\rho) = iH_o \frac{[uJ_1(u)]'}{[uH_1^{(1)}(u)]'} H_1^{(1)}(k\rho) \tag{4}$$

where the prime denotes differentiation with respect to u(=ka).

Substitution of (3) and (4) into (1) gives

$$h_\phi^{(2)}(\rho,z) = - iJ_1(k\rho) + i \frac{[uJ_1(u)]'}{[uH_1^{(1)}(u)]'} H_1^{(1)}(k\rho) + ik\int_a^b G_2 e_\rho \rho' d\rho' \tag{5}$$

where $u = ka$, $h_\phi^{(2)} = H_\phi^{(2)}/H_o$ and $e_\rho = E_\rho/(Z_o H_o)$.

We now apply (A·7) to region (1) (see Fig. 1b) and obtain

$$h_\phi^{(1)}(\rho,z) = - ik\int_a^b G_1(\rho,z;\rho',s) e_\rho(\rho')\rho' d\rho'$$
$$+ \int_s^h h_\phi^{(1)}(a,z')\left\{\frac{\partial}{\partial\rho'}(\rho' G_1)\right\}_{\rho'=a} dz' + (\cdots) , \text{ for } s \le z \le h \tag{6}$$

where $(\cdots)$ denotes the integral over the end cap of the post and has been written out explicitly in a previous note.[1] The Green's function G_1 satisfies (A·6) with boundary condition

$$\frac{\partial}{\partial z} G_1 = 0 \quad , \quad \text{when } z = s \tag{7}$$

and the radiation condition at infinity.

In the hole ($z = s$, $a < \rho \le b$) we have $h_\phi^{(1)} = h_\phi^{(2)}$. Subtracting (5) from (6) in the hole we obtain the first relationship between $h_\phi^{(1)}$ and e_ρ, namely,

$$\int_s^h K(\rho,s;a,z')h_\phi^{(1)}(a,z')dz' + ik \int_a^b \{G_1(\rho,s;\rho',s) + G_2(\rho,s;\rho',s)\}e_\rho(\rho')\rho'd\rho'$$

$$= iJ_1(k\rho) - i\,\frac{[uJ_1(u)]'}{[uH_1^{(1)}(u)]'}\,H_1^{(1)}(k\rho) + (\cdots) \quad , \quad \text{for } a < \rho \le b \tag{8}$$

where

$$K(\rho,z;a,z') = -\left[\frac{\partial}{\partial\rho'}(\rho'G_1)\right]_{\rho'=a} \quad .$$

The second relationship between $h_\phi^{(1)}$ and e_ρ is given by (6) when $\rho = a$. Thus

$$\frac{1}{2}h_\phi^{(1)}(a,z) + \int_s^h K(a,z;a,z')h_\phi^{(1)}(a,z')dz' + ik\int_a^b G_1(a,z;\rho's)e_\rho(\rho')\rho'd\rho'$$

$$= (\cdots) \ , \ \text{for} \quad s \le z \le h \tag{9}$$

Equations (8) and (9) constitute a coupled set of integral equations between $h_\phi^{(1)}(a,z)$ and $e_\rho(\rho)$. Once e_ρ is obtained by numerically solving (8) and (9),

$h_\phi^{(2)}$ can be found from (5).

In the next section we shall find the explicit forms of the Green's functions G_1 and G_2.

III. Green's Functions

The Green's function G_1 that satisfies the equation (A·6) and the boundary condition (7) can be written down by inspection. Thus,

$$G_1(\rho,z;\rho',z') = \int_0^{2\pi} d\phi \cos\phi\left[\frac{e^{ikR}}{4\pi R} + \frac{e^{ikR_s}}{4\pi R_s}\right] \tag{10}$$

where

$$R^2 = (z - z')^2 + \rho^2 + \rho'^2 - 2\rho\rho' \cos\phi$$

$$R_s^2 = (2s - z - z')^2 + \rho^2 + \rho'^2 - 2\rho\rho' \cos\phi$$

The explicit form of the kernel K in (8) and (9) is then given by

$$K(\rho,z;a,z') = -\left[\frac{\partial}{\partial\rho'}(\rho' G_1)\right]_{\rho'=a}$$

$$= \int_0^{2\pi} (\rho - \cos\phi)\left\{\frac{(ikR - 1)e^{ikR}}{4\pi R^3} + \frac{(ikR_s - 1)e^{ikR_s}}{4\pi R_s^3}\right\}_{\rho'=a} d\phi \tag{11}$$

To find G_2 that satisfies

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{1}{\rho^2} + \frac{\partial^2}{\partial z^2} + k^2\right)G_2 = -\delta(z - z')\,\frac{\delta(\rho - \rho')}{\rho'} \tag{12}$$

and

$$\frac{\partial}{\partial z} G_2 = 0 \quad \text{when} \quad z = 0 \ , \ s$$

$$\frac{\partial}{\partial \rho} (\rho G_2) = 0 \quad \text{when} \quad \rho = a$$

we proceed in the following usual way. Let

$$G_2^I = \sum_{n=o}^{\infty} A_n H_1^{(1)}(\lambda_n \rho) \cos(n\pi z/s) \quad , \qquad \text{for } \rho > \rho' \tag{13a}$$

$$G_2^{II} = \sum_{n=o}^{\infty} \alpha_n A_n [P(\lambda_n a) J_1(\lambda_n \rho) - Q(\lambda_n a) H_1^{(1)}(\lambda_n \rho)] \cos(n\pi z/s) \ , \quad \text{for } \rho < \rho' \tag{13b}$$

where $P(x) = [xH_1^{(1)}(x)]'$, $Q(x) = [xJ_1(x)]'$,

$$\alpha_n = \frac{H_1^{(1)}(\lambda_n \rho')}{P(\lambda_n a) J_1(\lambda_n \rho') - Q(\lambda_n a) H_1^{(1)}(\lambda_n \rho')} \quad ,$$

and $\lambda_n^2 = k^2 - (n\pi/s)^2$. Note that (13a) and (13b) satisfy all the boundary conditions (12). To determine the constants A_n , multiply the differential equation (12) by $\cos(n\pi z/s)$ and perform

$$\lim_{\varepsilon \to o} \int_o^s \int_{\rho'-\varepsilon}^{\rho'+\varepsilon} (\cdots) \rho d\rho dz \quad .$$

After some straightforward manipulations we obtain

$$\alpha_n \lambda_n A_n = \frac{\pi i}{s \varepsilon_n} \frac{H_1^{(1)}(\lambda_n \rho')}{H_o^{(1)}(\lambda_n a)} \cos(n\pi z'/s)$$

where $\varepsilon_n = 1$ if $n > 0$, and $\varepsilon_o = 2$. Substituting this into (13a) and (13b) we then have

$$G_2(\rho,z;\rho',z') = \frac{\pi i}{s} \sum_{n=o}^{\infty} \frac{1}{\varepsilon_n} [J_1(\lambda_n\rho_<)H_o^{(1)}(\lambda_n a)$$

$$- J_o(\lambda_n a)H_1^{(1)}(\lambda_n\rho_<)] \frac{H_1^{(1)}(\lambda_n\rho_>)}{H_o^{(1)}(\lambda_n a)} \cos(n\pi z/s)\cos(n\pi z'/s) \qquad (14)$$

where $\rho_<$ ($\rho_>$) denotes the smaller (larger) of ρ and ρ'.

IV. Numerical Results

Equations (8) and (9) were solved on a CDC 6600 computer for two values of a/h, four values of s/h, five values of b/h, and a wide range of kh. Equation (5) was then used to compute $h_\phi^{(2)}$ from the numerical results of e_ρ.

Figures 3a-4d show the variations of the current amplitude along the post with the radius of the hole as a parameter when the frequency of the incident wave is around the resonance frequency of the fundamental mode of the post, i.e., the mode with the longest wavelength. The dimensionless quantity $|I|/I_o$ is related to the field h_ϕ by $|I|/I_o = 2\pi a|h_\phi|/h$.

For a step-function incident wave, $\underline{H}^{inc}(x,t) = -\underline{e}_y H_o U(t - x/c)$, the time behavior of the current at z = 0 is given in figures 5a-6d with the radius of the hole as a parameter. For comparison purposes, the response of the post without the top plate is also given in those figures.

Figures 7a and 7b summarize the results derived from figures 3a-4d. The resonance frequency as well as the current amplitude at z = 0 of the fundamental mode are plotted against the radius of the hole with the plate spacing as a parameter.

The decay time constant, α^{-1}, of the fundamental mode given in figures 8a and 8b was obtained by fitting the envelopes of the curves in figures 5a-6d to an exponential curve of the form $\exp(-\alpha ct/h)$. They show that, except for the expected result that α^{-1} increases with the hole's size for a given plate spacing, α^{-1} decreases as the plate spacing is increased for a fixed hole size.

In figures 3a-4d and in figures 7a and 7b, k_o is the wave number corresponding to the resonance frequency.

Appendix

Let $u(\rho,z)$ satisfy the equation

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{1}{\rho^2} + \frac{\partial^2}{\partial z^2} + k^2\right)u = 0 \tag{A·1}$$

in the space exterior to V, the volume of a body of revolution (see Fig. 2), and let u satisfy the radiation condition at infinity. We wish to express u in terms of its value and its normal derivative on S. Since $u(\rho,z)\cos\phi$ satisfies the three-dimensional Helmholtz equation

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2} + k^2\right)(u\cos\phi) = 0 \quad ,$$

an application of the ordinary Green's theorem gives

$$u(\rho,z)\cos\phi = \iint\limits_S \left\{u\cos\phi'\,\frac{\partial G}{\partial n'} - G\,\frac{\partial}{\partial n'}(u\cos\phi')\right\}dS' \tag{A·2}$$

where G is the free space Green's function given by

$$G(\rho,z,\phi;\rho',z',\phi') = \frac{e^{ik\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\phi-\phi') + (z-z')^2}}}{4\pi\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\phi-\phi') + (z-z')^2}} \quad .$$

Multiplying (A·2) by $\cos\phi$ and integrating the resulting equation with respect to ϕ from 0 to 2π we get

$$u(\rho,z) = \frac{1}{\pi}\int_0^{2\pi} \cos\phi \, d\phi \iint_S \{u \cos\phi' \frac{\partial G}{\partial n'} - G \cos\phi' \frac{\partial}{\partial n'} u\} dS' \quad . \qquad (A\cdot 3)$$

For a function $F(\phi)$ having the properties $F(\phi) = F(-\phi)$ and $F(\phi) = F(2\pi + \phi)$, it can be easily shown that

$$\int_0^{2\pi} \cos\phi \, d\phi \int_0^{2\pi} F(\phi - \phi')\cos\phi' \, d\phi' = \pi \int_0^{2\pi} F(\psi)\cos\psi \, d\psi \quad .$$

Since G and $(\partial/\partial n')G$ have the same properties as F, (A·3) becomes

$$u(\rho,z) = \int (u \frac{\partial G^{(2)}}{\partial n'} - G^{(2)} \frac{\partial u}{\partial n'})\rho' d\ell' \qquad (A\cdot 4)$$

where

$$G^{(2)}(\rho,z;\rho',z') = \int_0^{2\pi} \cos\psi \frac{e^{ik\sqrt{(z-z')^2 + \rho^2 + \rho'^2 - 2\rho\rho' \cos\psi}}}{4\pi\sqrt{(z-z')^2 + \rho^2 + \rho'^2 - 2\rho\rho' \cos\psi}} d\psi \qquad (A\cdot 5)$$

which can be shown to satisfy the equation

$$(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{1}{\rho^2} + \frac{\partial^2}{\partial z^2} + k^2)G^{(2)} = -\delta(z-z') \frac{\delta(\rho-\rho')}{\rho'} \quad . \qquad (A\cdot 6)$$

Let us consider the case where the only non-vanishing magnetic field component is H_ϕ and has the same properties as u, and where the volume V is a right circular cylinder. Remembering that $i\omega\varepsilon E_\rho = (\partial/\partial z)H_\phi$ and $i\omega\varepsilon E_z = -(\partial/\partial\rho + 1/\rho)H_\phi$, we have from (A·4)

$$H_\phi(\rho,z) = \int\limits_{z'=\text{constant}} (\underline{n}' \cdot \underline{e}_{z'})\left\{H_\phi \frac{\partial}{\partial z'} \underline{G^{(2)}} - i\omega\varepsilon E_\rho G^{(2)}\right\}\rho' d\rho'$$

$$+ \int\limits_{\rho'=\text{constant}} \left\{H_\phi \frac{\partial}{\partial \rho'} (\rho' G^{(2)}) + i\omega\varepsilon\rho' E_z G^{(2)}\right\}dz' \quad . \tag{A·7}$$

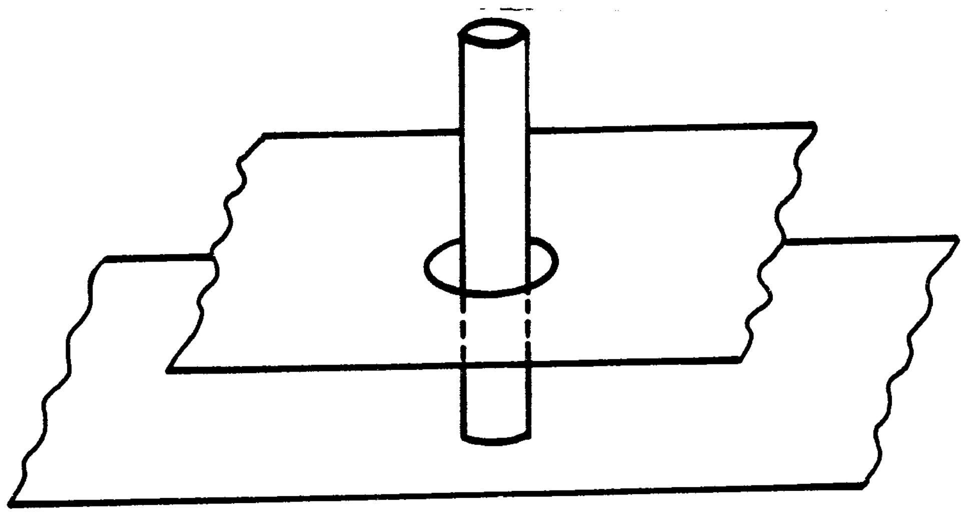

Figure 1a. The geometry of the problem.

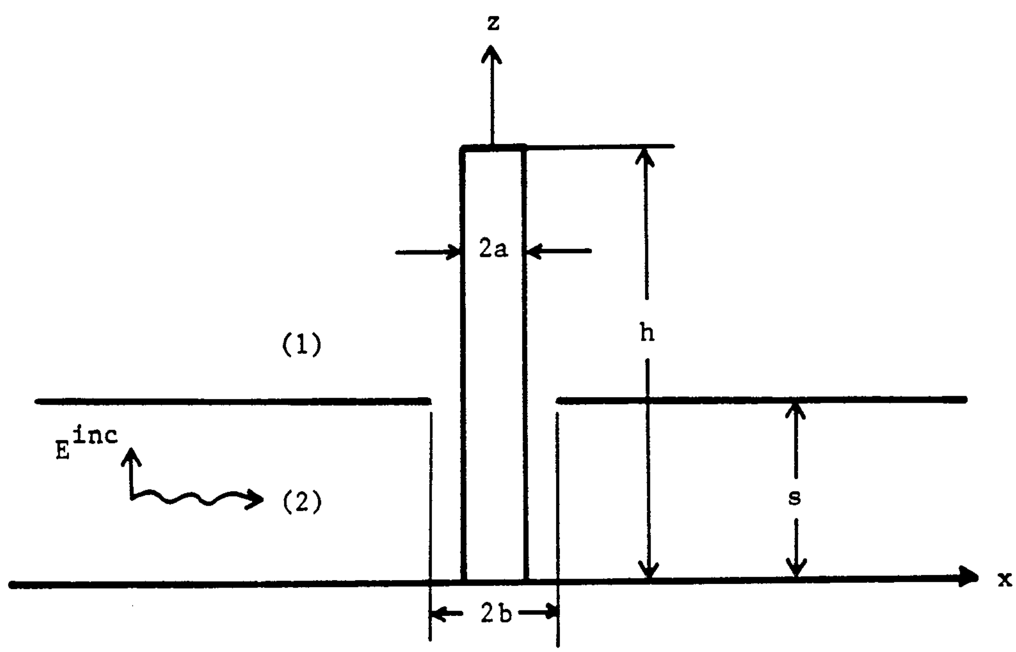

Figure 1b. Side view of the geometry of the problem.

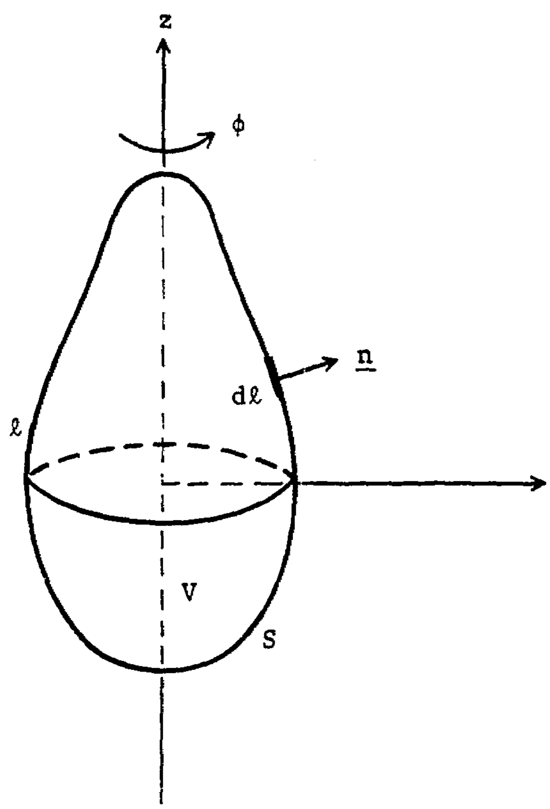

Figure 2. Body of revolution.

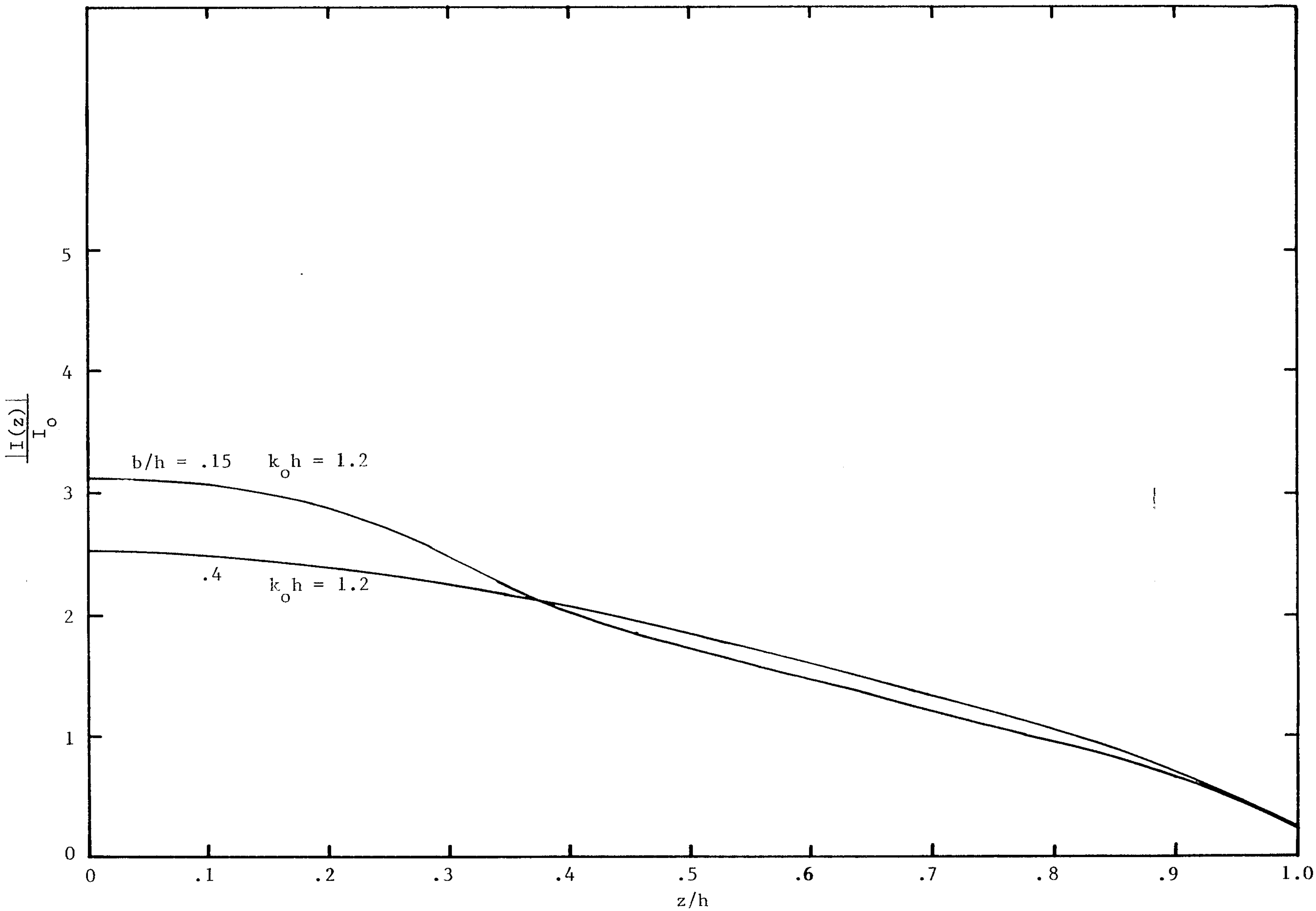

FIGURE 3a. Current vs. position around resonance frequency.
$s/h = .3$, $a/h = .1$,

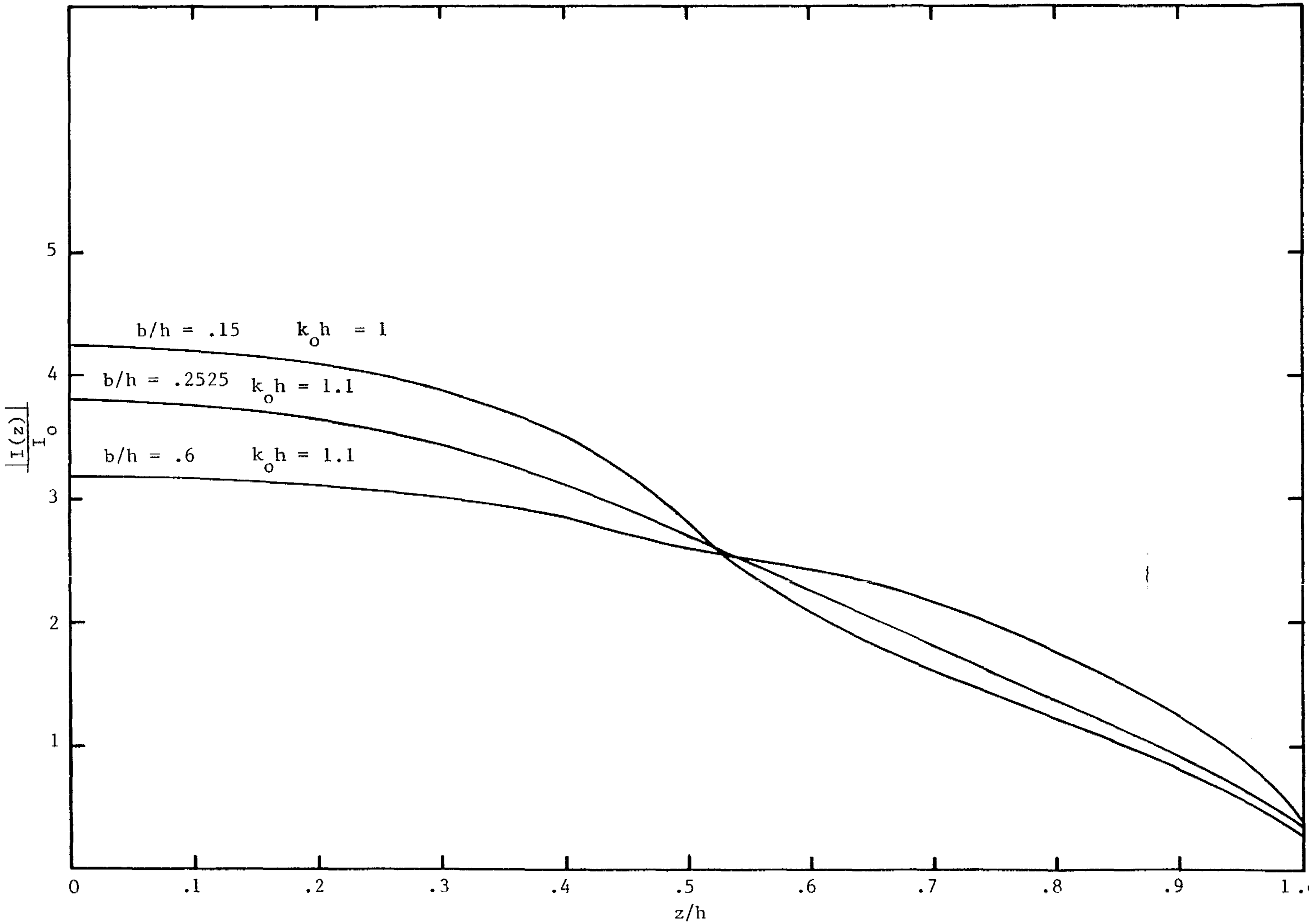

FIGURE 3b. Current vs. position around resonance frequency.
s/h = .5, a/h = .1.

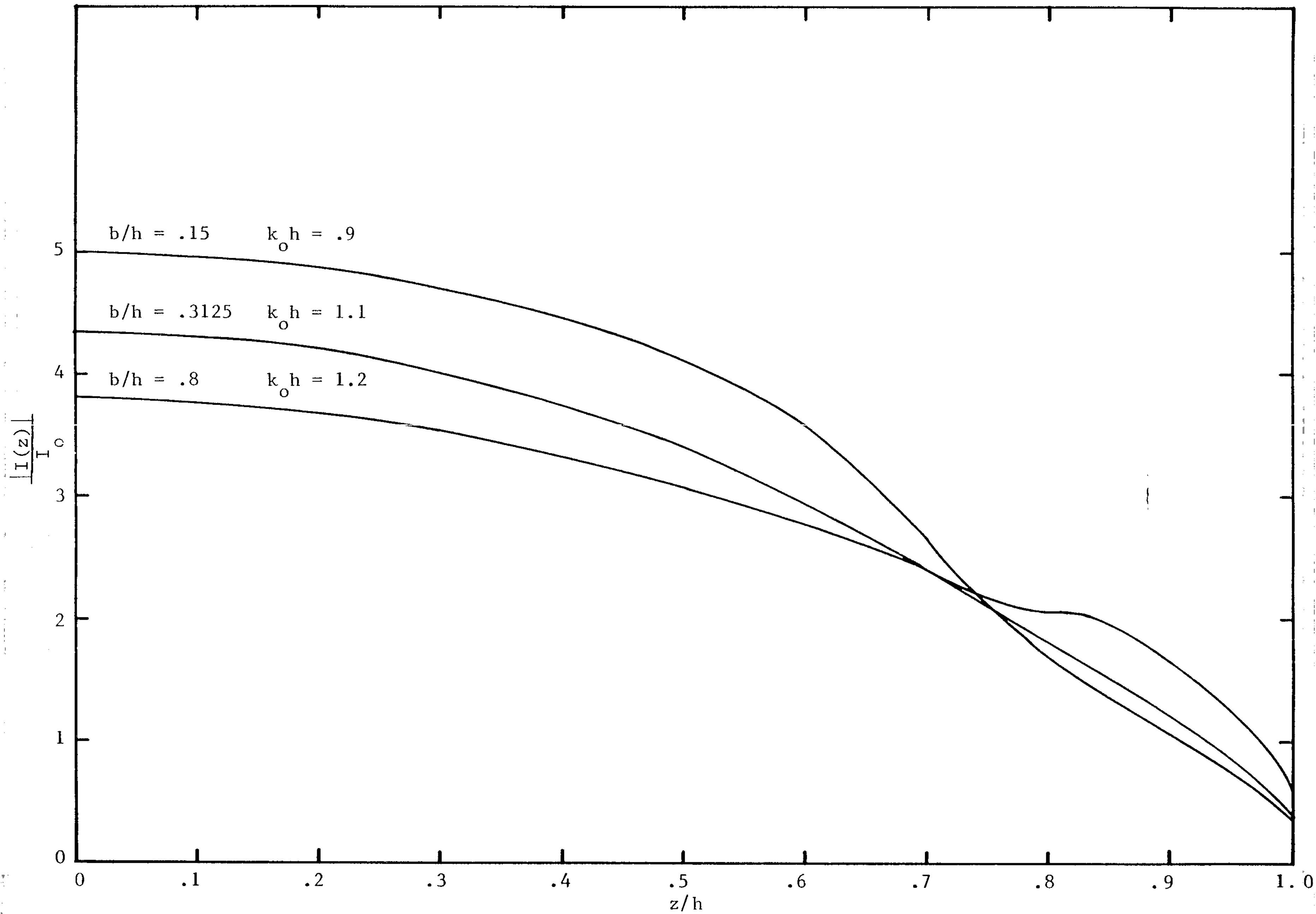

FIGURE 3c. Current vs. position around resonance frequency. $s/h = .7$, $a/h = .1$.

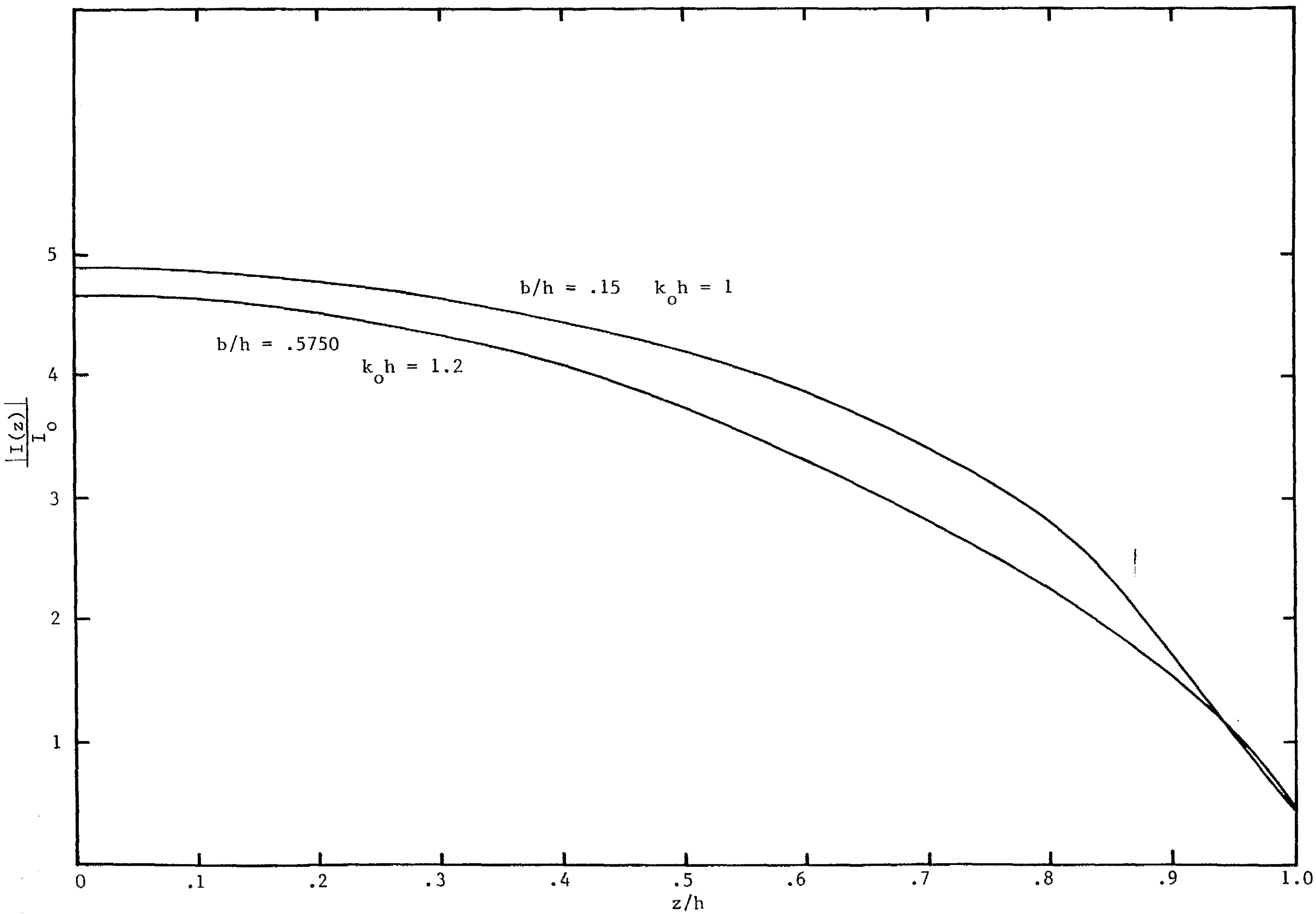

FIGURE 3d. Current vs. position around resonance frequency.
$s/h = .9$, $a/h = .1$.

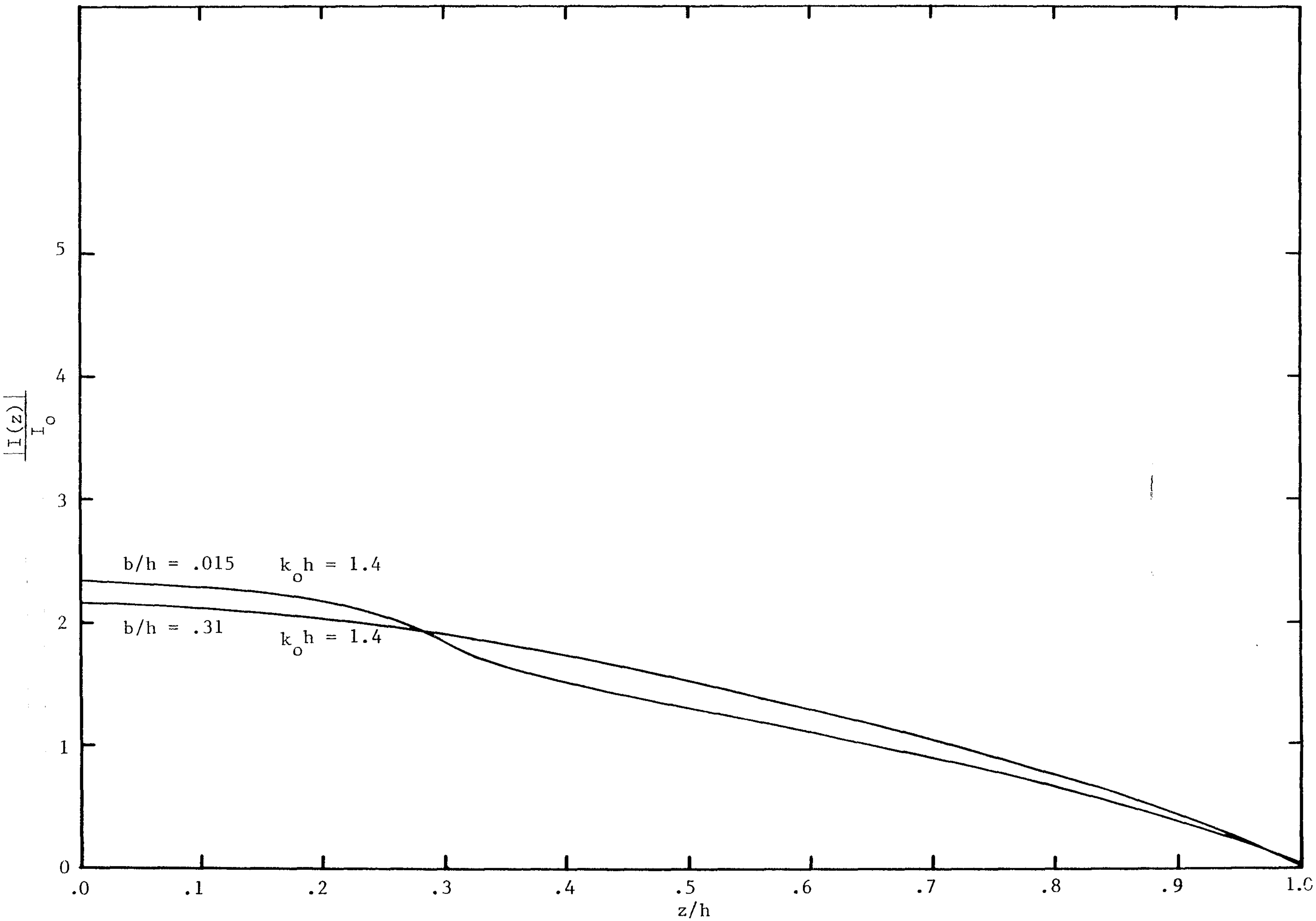

FIGURE 4a. Current vs. position around resonance frequency.
a/h = .01, s/h = .3.

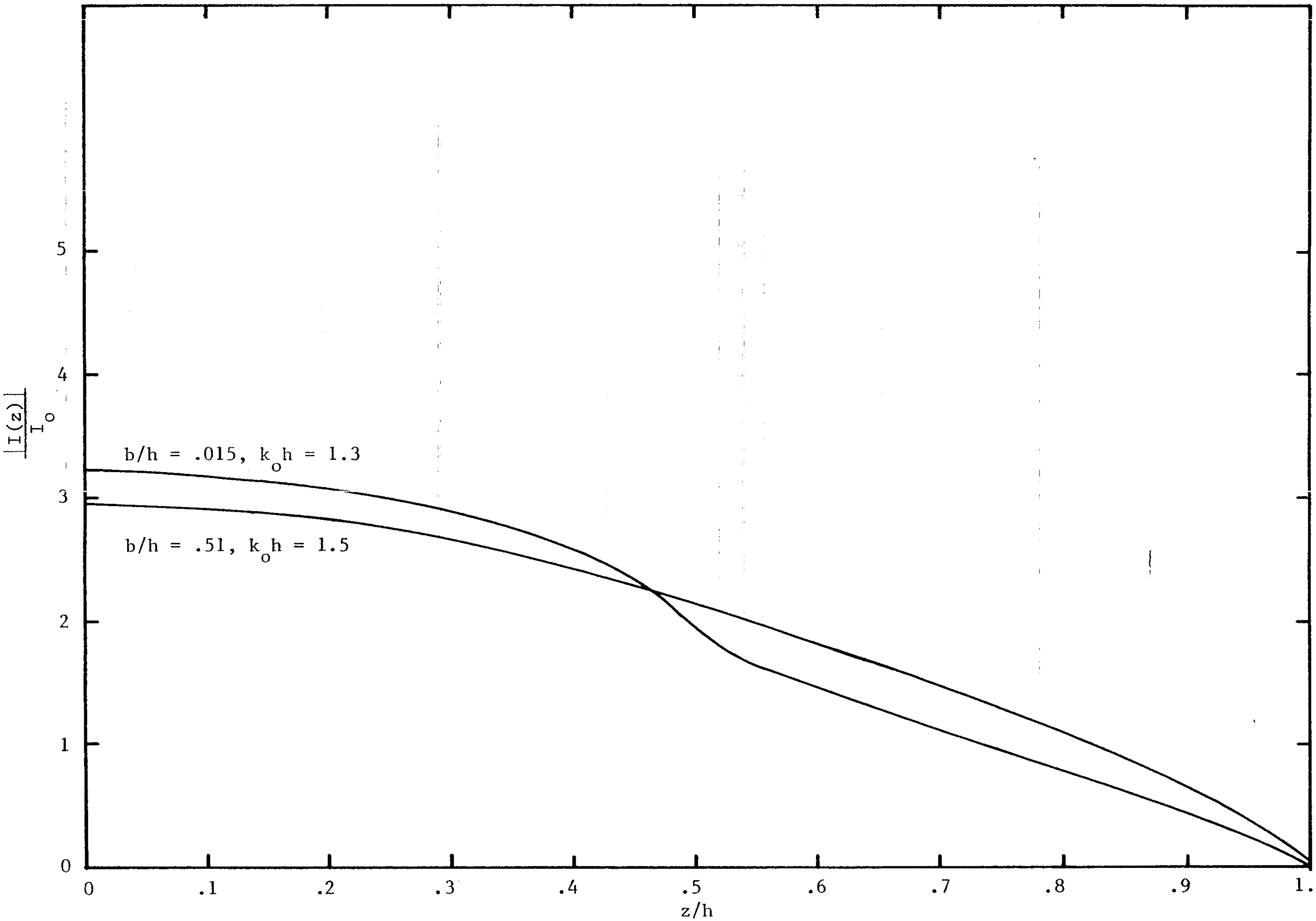

FIGURE 4b. Current vs. position around resonance frequency. $a/h = .01$, $s/h = .5$.

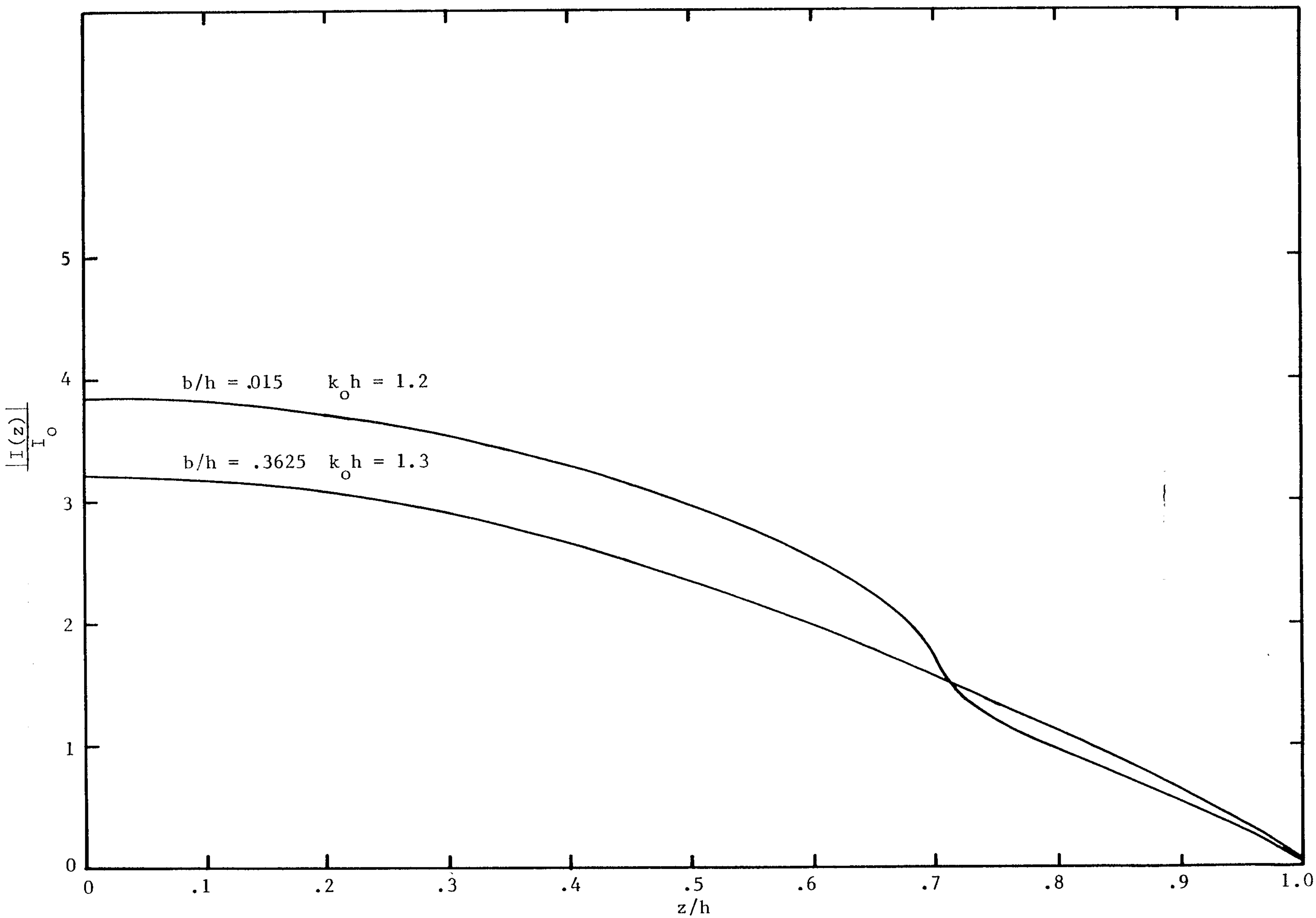

FIGURE 4c. Current vs. position around resonance frequency.
$a/h = .01$, $s/h = .7$.

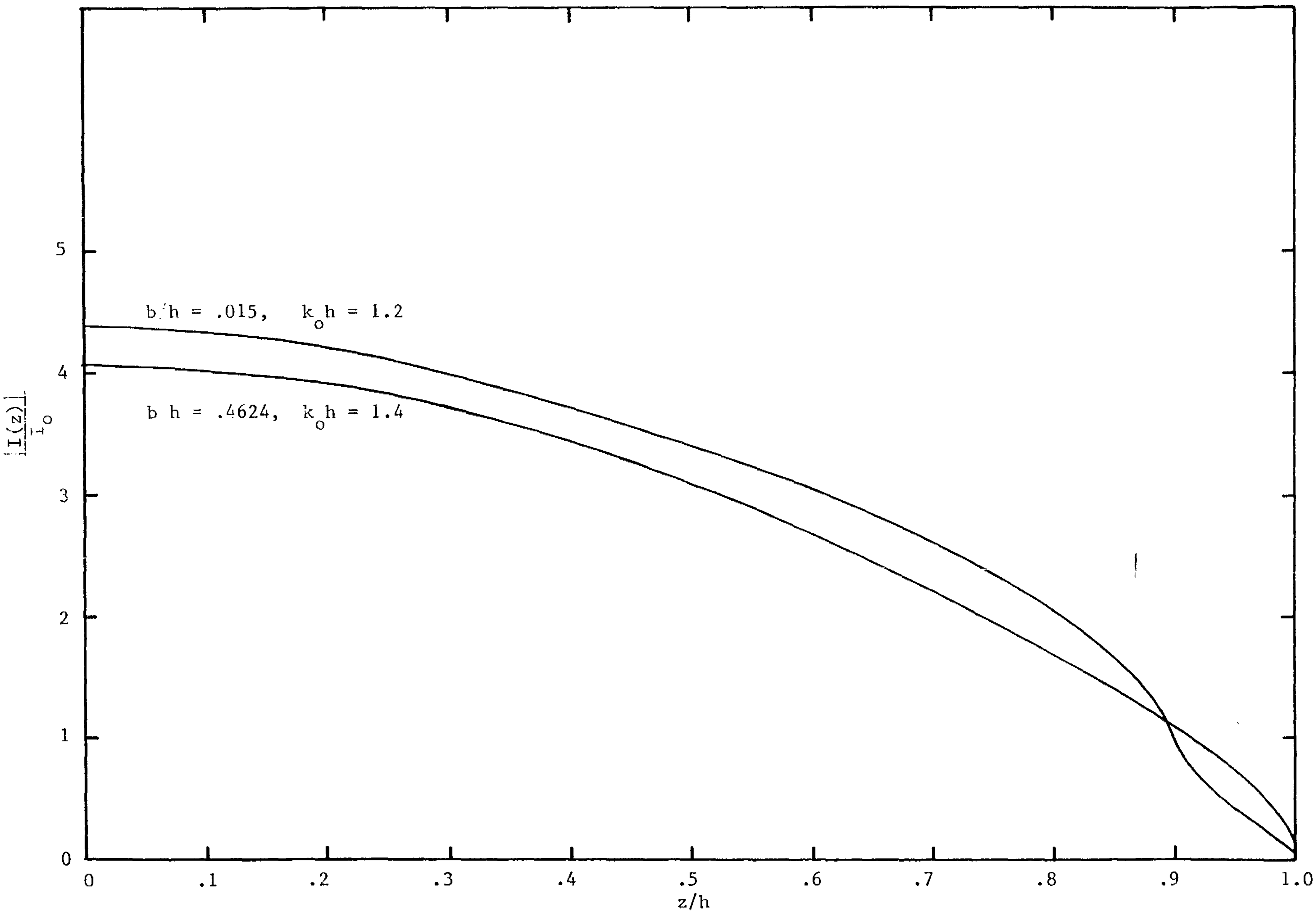

FIGURE 4d. Current vs. position around resonance frequency. a/h = .01, s/h = .9.

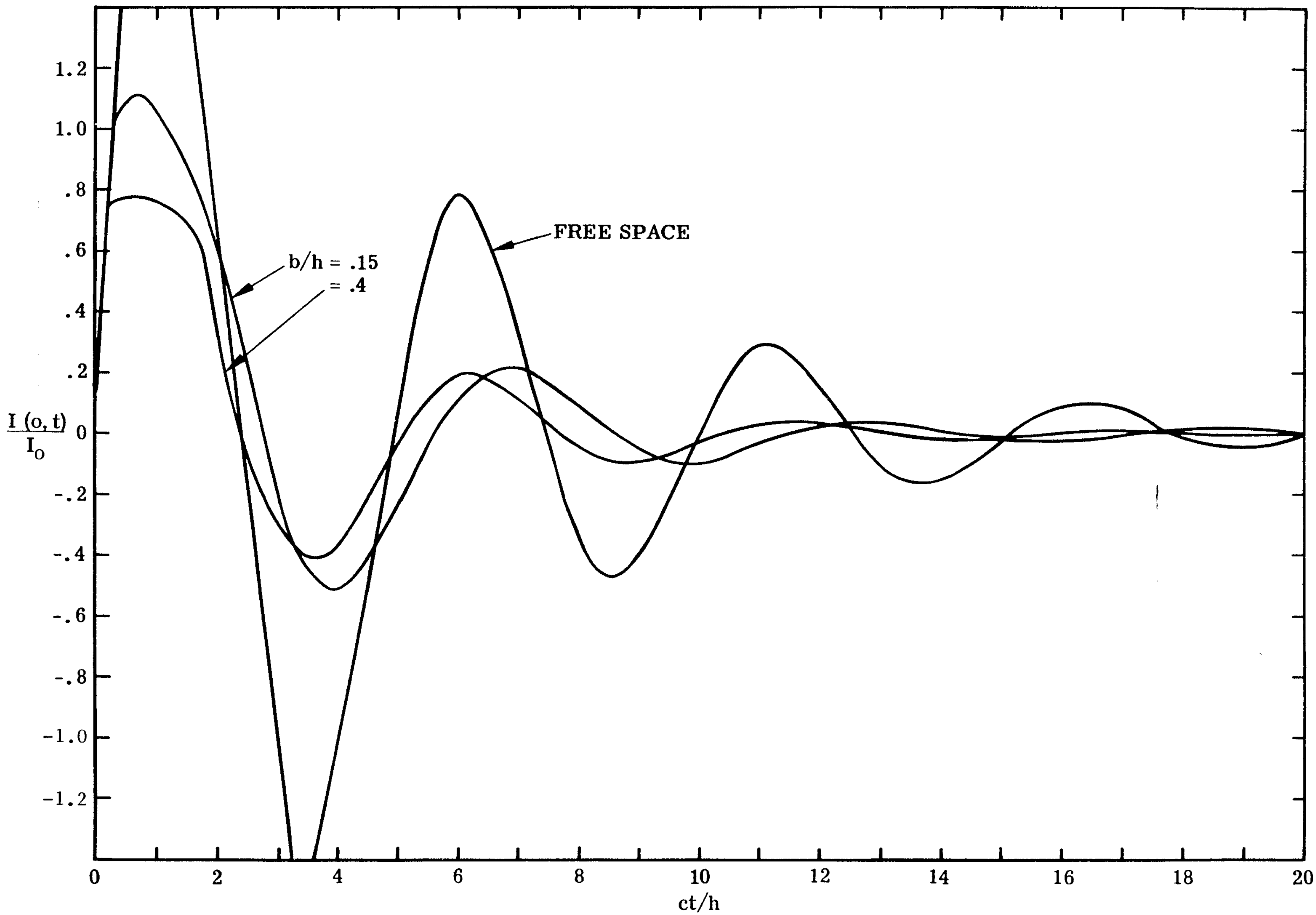

FIGURE 5a. CURRENT VS TIME AT $z = 0$ FOR $a/h = .1$, $s/h = .3$

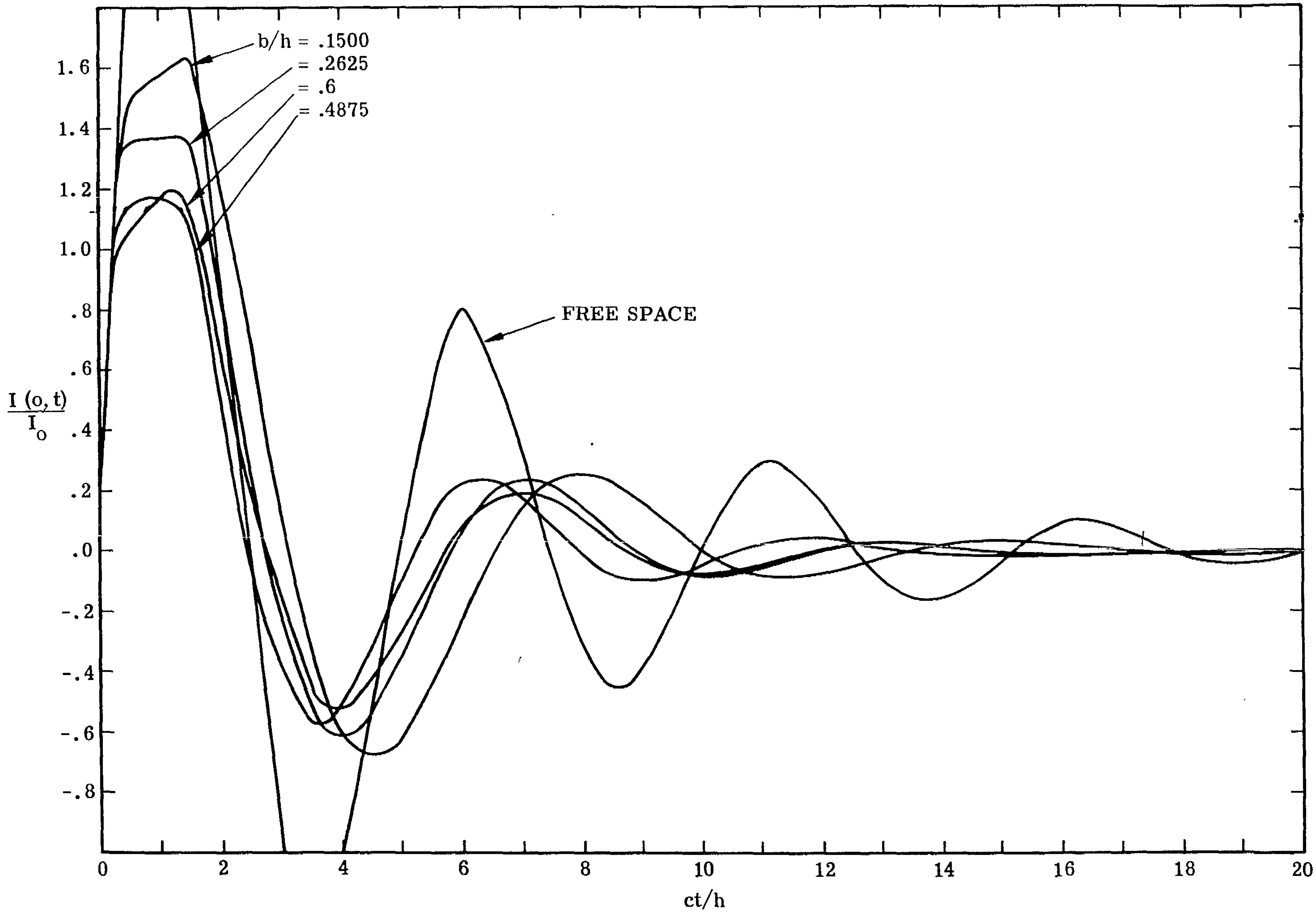

FIGURE 5b. POST CURRENT VS TIME AT z = 0 FOR a/h = .1, s/h = .5

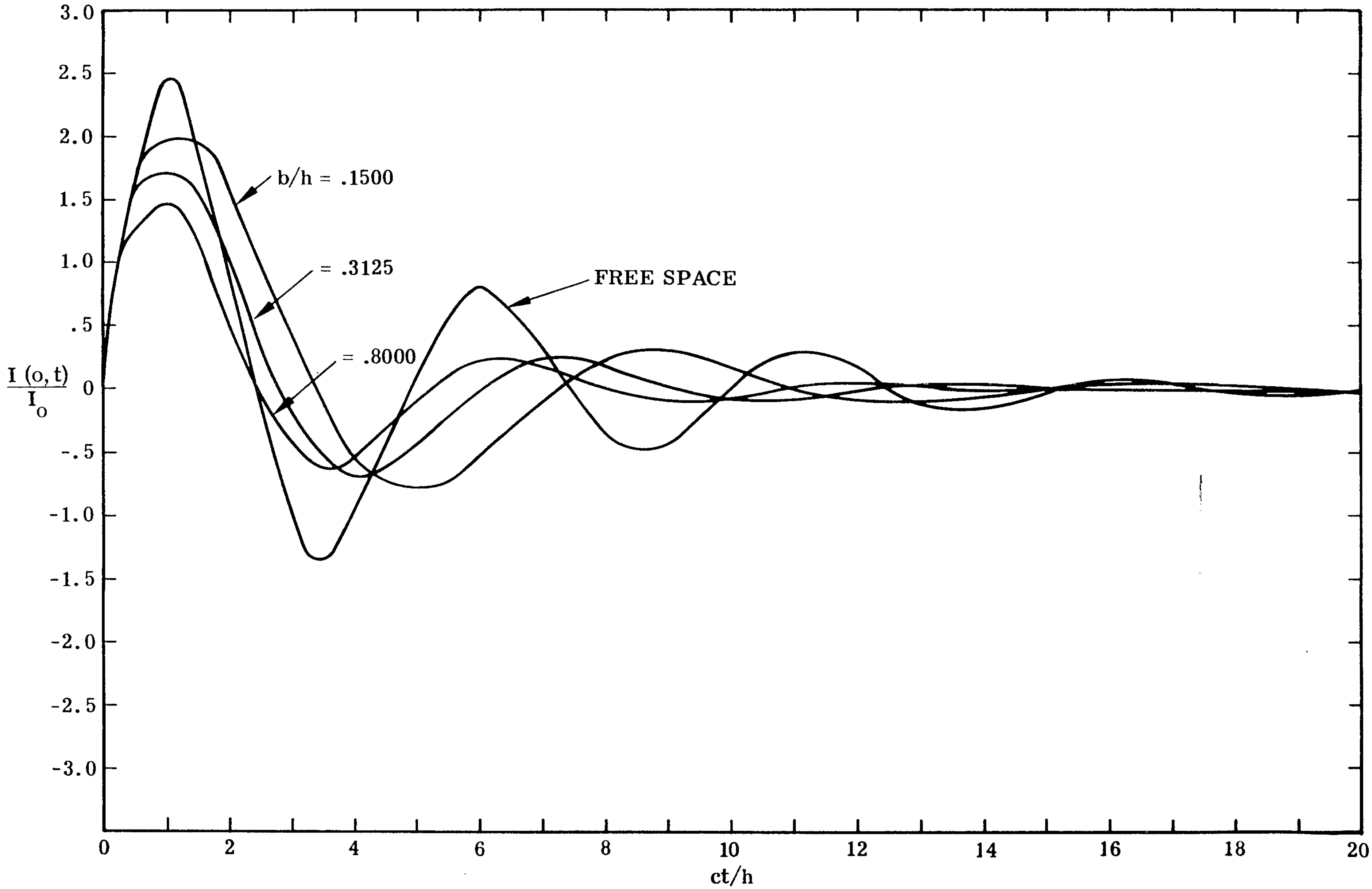

FIGURE 5c. **POST CURRENT VS TIME AT z = 0 FOR a/h = .1, s/h = .7**

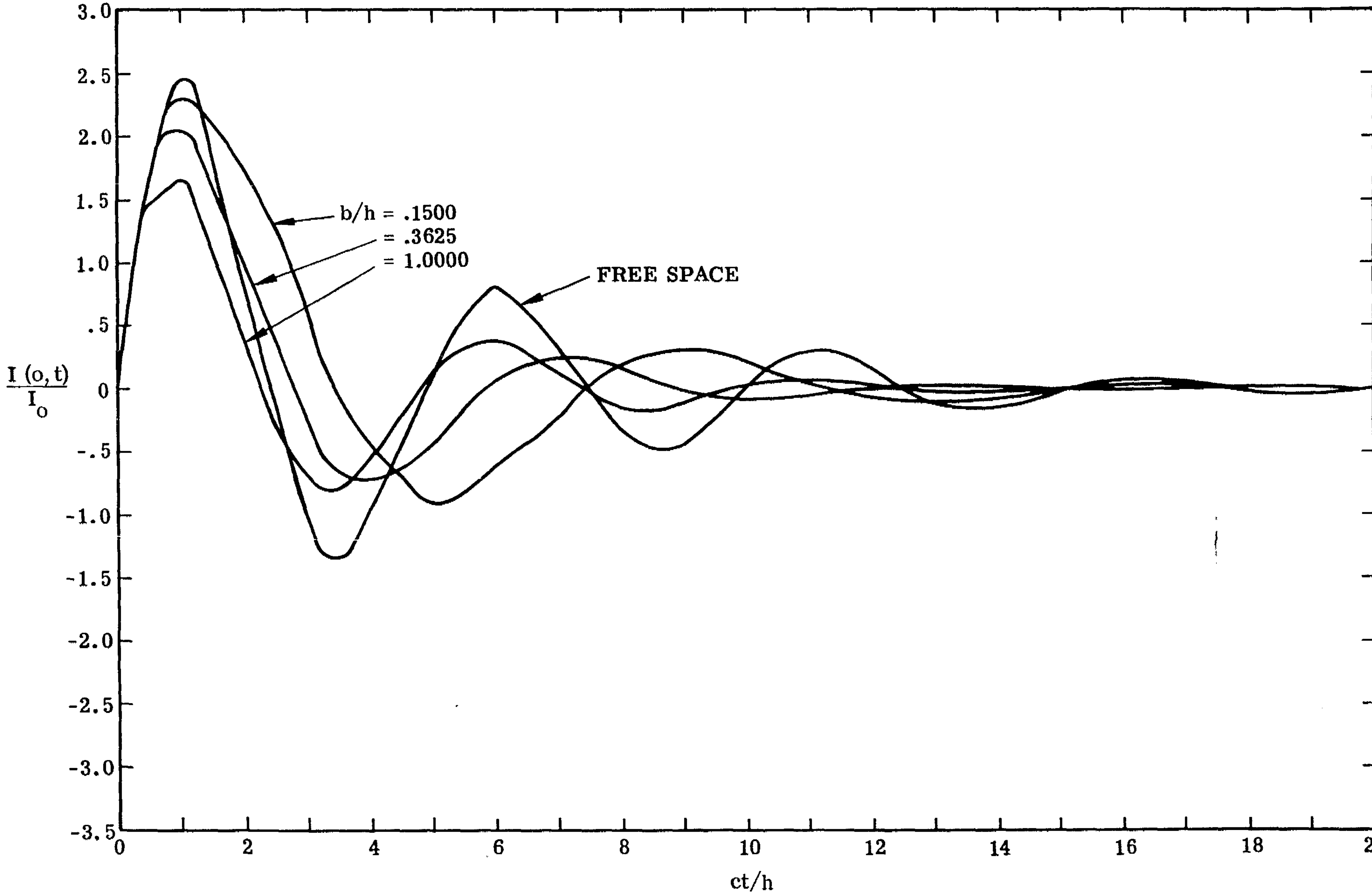

FIGURE 5d. POST CURRENT VS TIME AT $z = 0$ FOR $a/h = .1$, $s/h = .9$

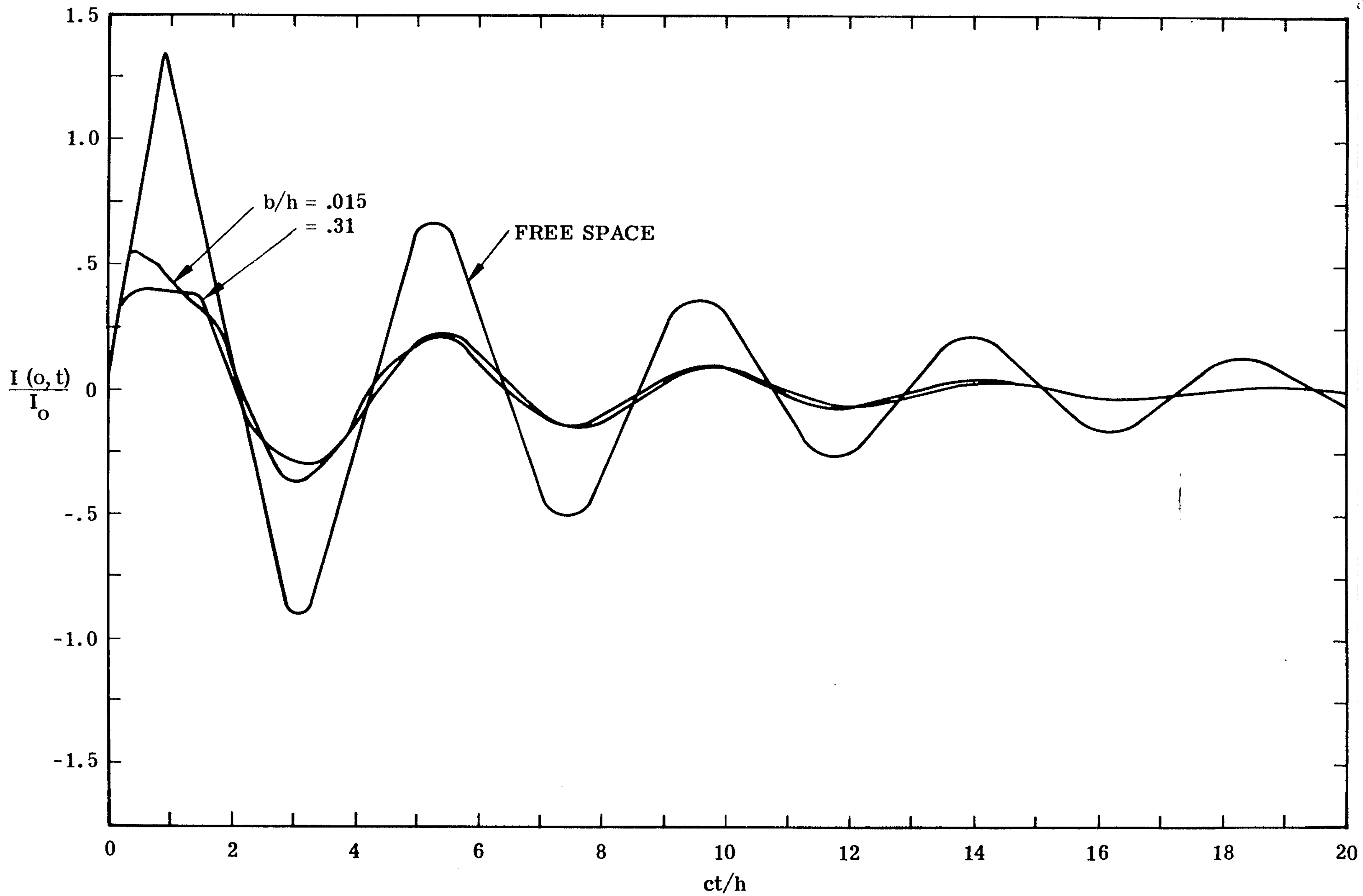

FIGURE 6a. **POST CURRENT VS TIME AT z = 0 FOR a/h = .01, s/h = .3**

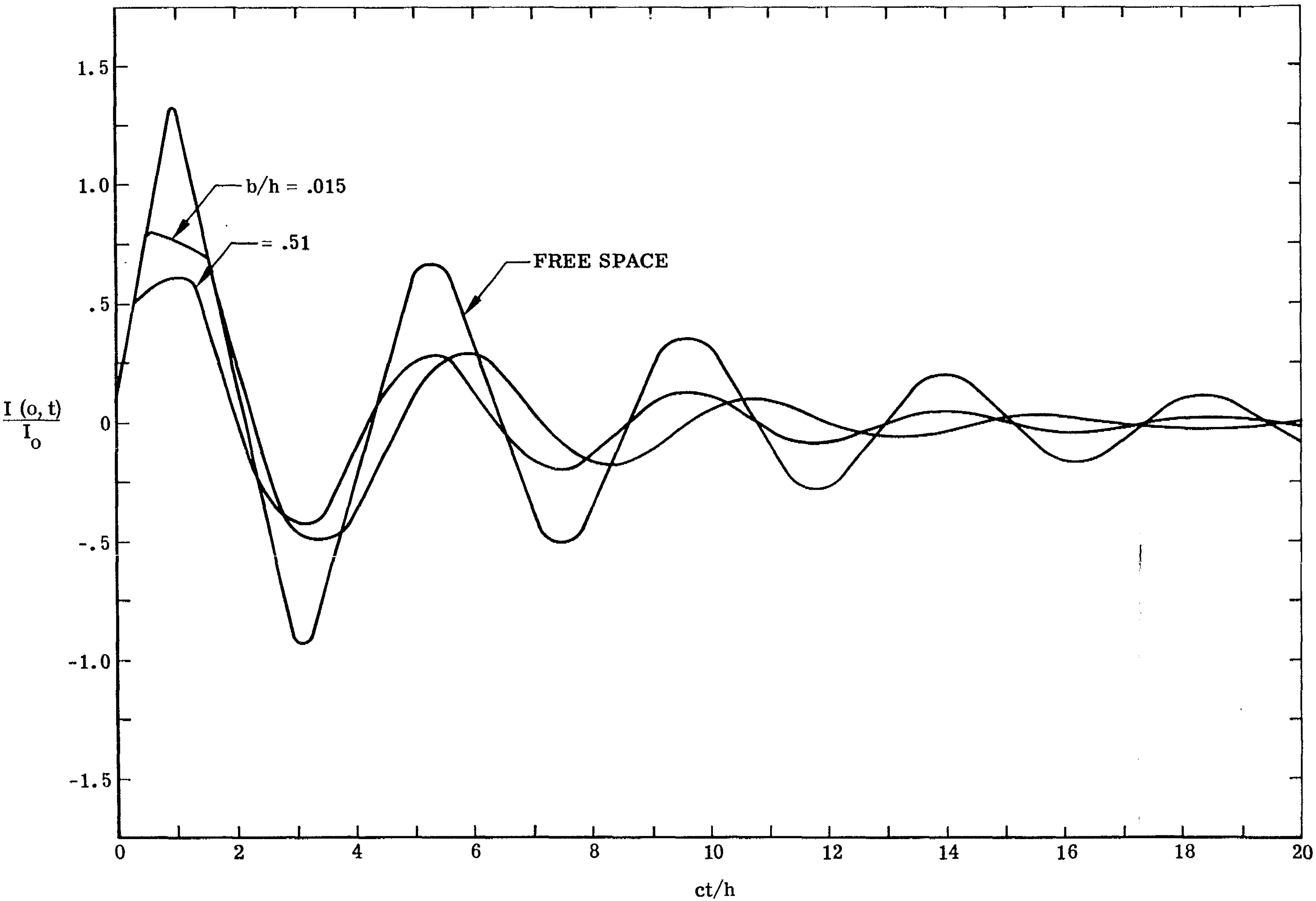

FIGURE 6b. POST CURRENT VS TIME AT $z = 0$ FOR $a/h = .01$, $s/h = .5$

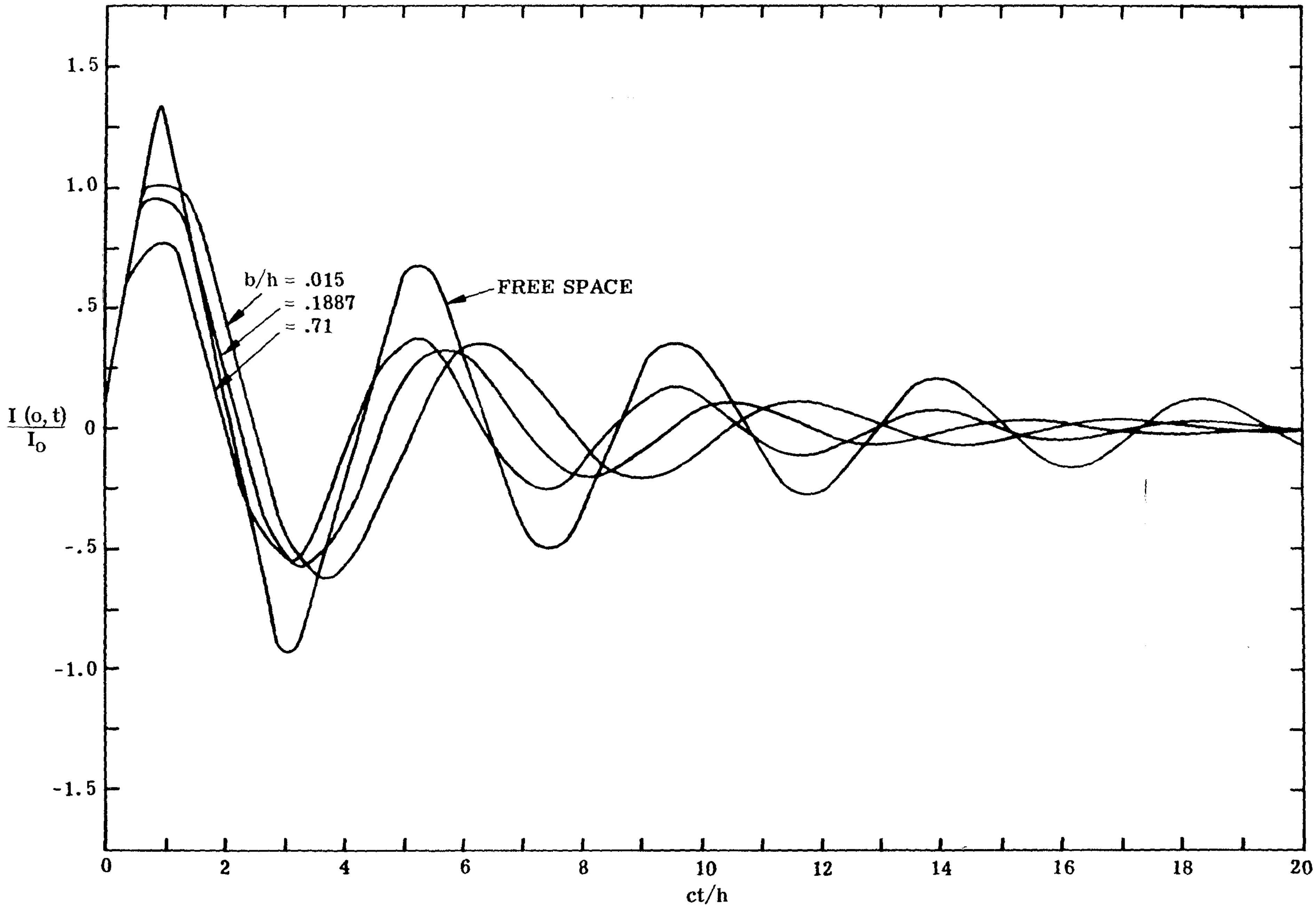

FIGURE 6c. POST CURRENT VS TIME AT z = 0 FOR a/h = .01, s/h = .7

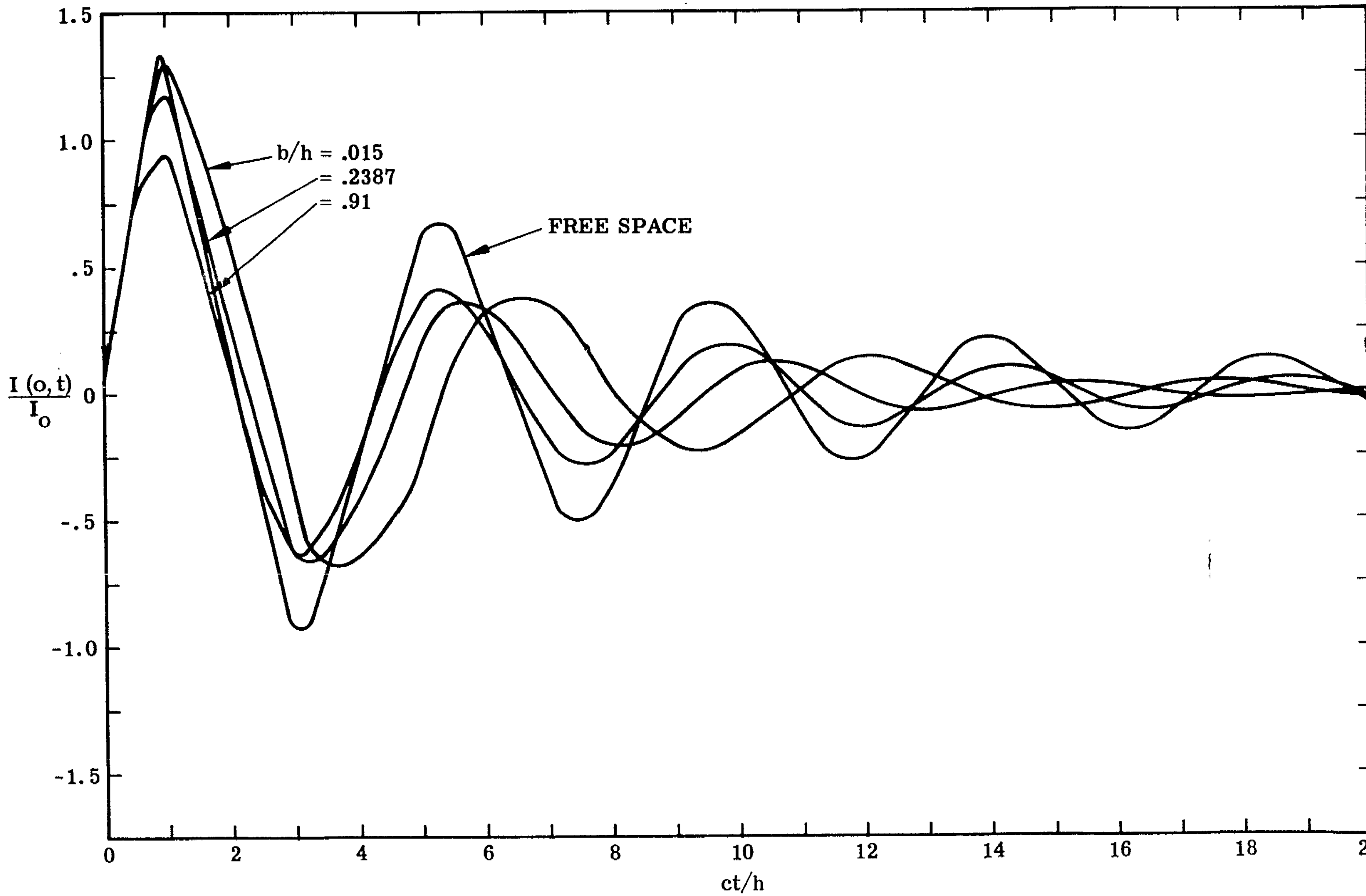

FIGURE 6d. POST CURRENT VS TIME AT z = 0 FOR a/h = .01, s/h = .9

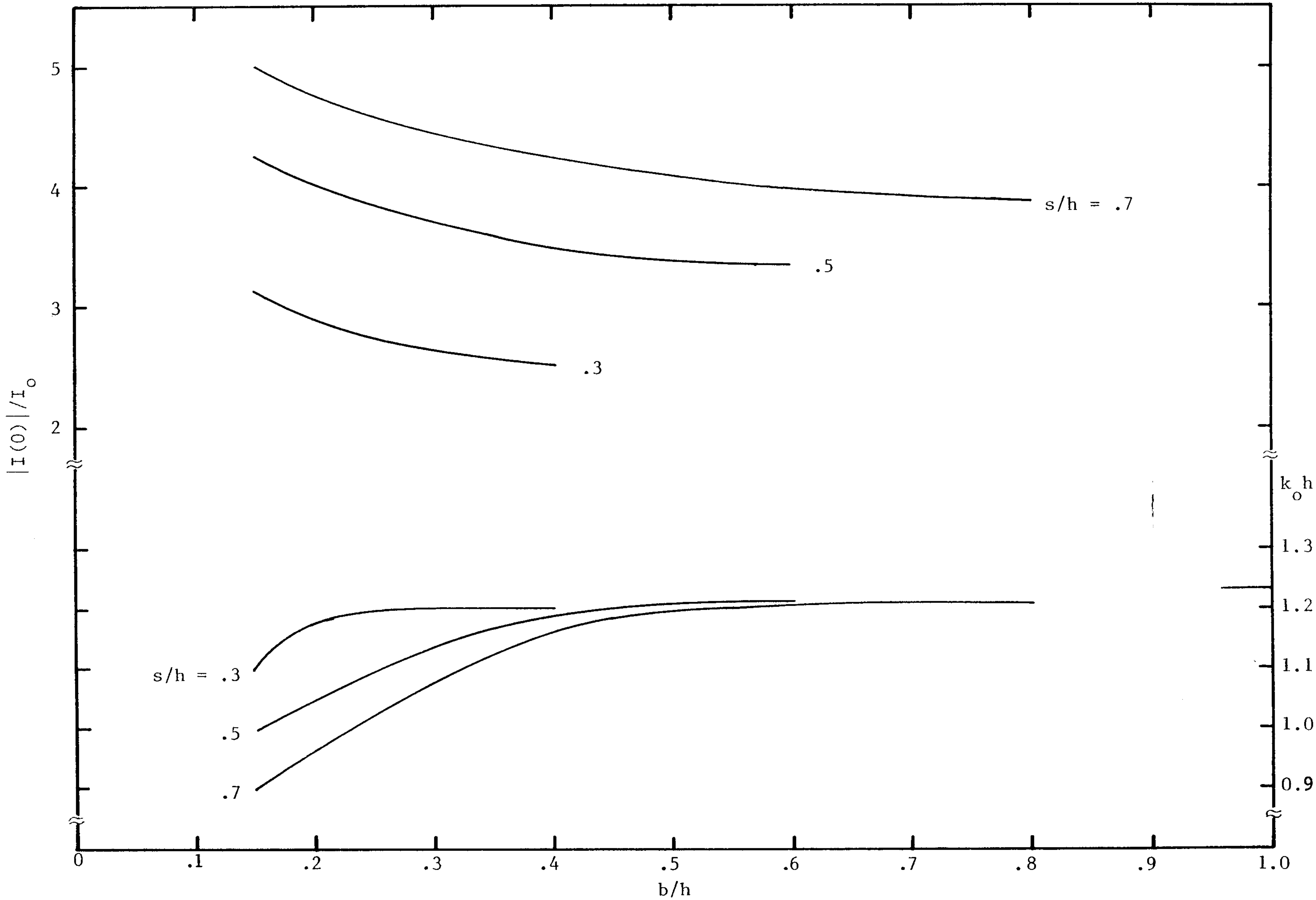

FIGURE 7a. Resonance frequency and current at resonance frequency vs. hole's size for a/h = .1. (Post in free space: $|I(0)|/I_o = 8.4$, $k_oh = 1.23$.)

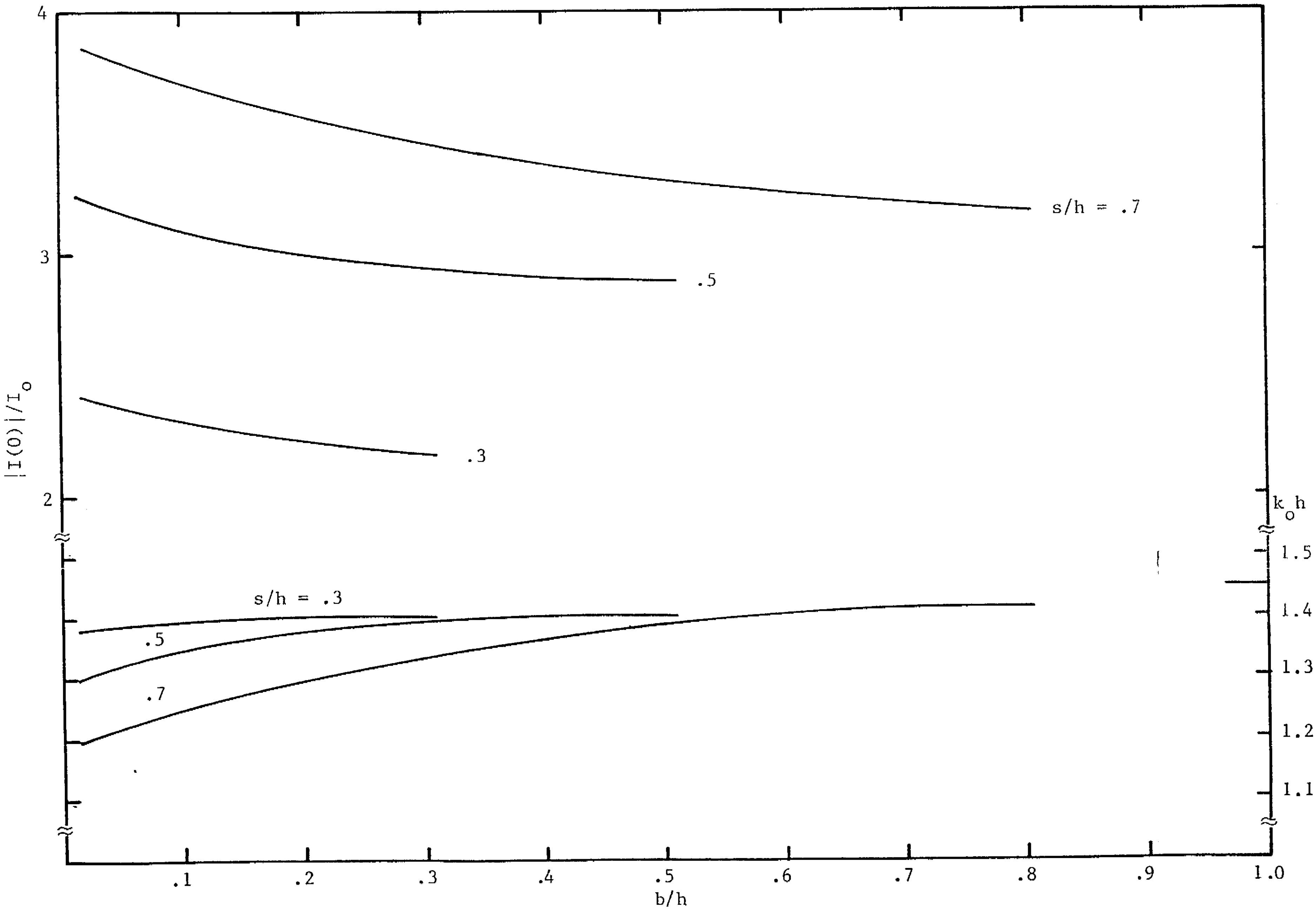

FIGURE 7b. Resonance frequency and current at resonance frequency vs. hole's size for a/h = .01. (Post in free space: $|I(0)|/I_o = 7.2$, $k_oh = 1.45$.)

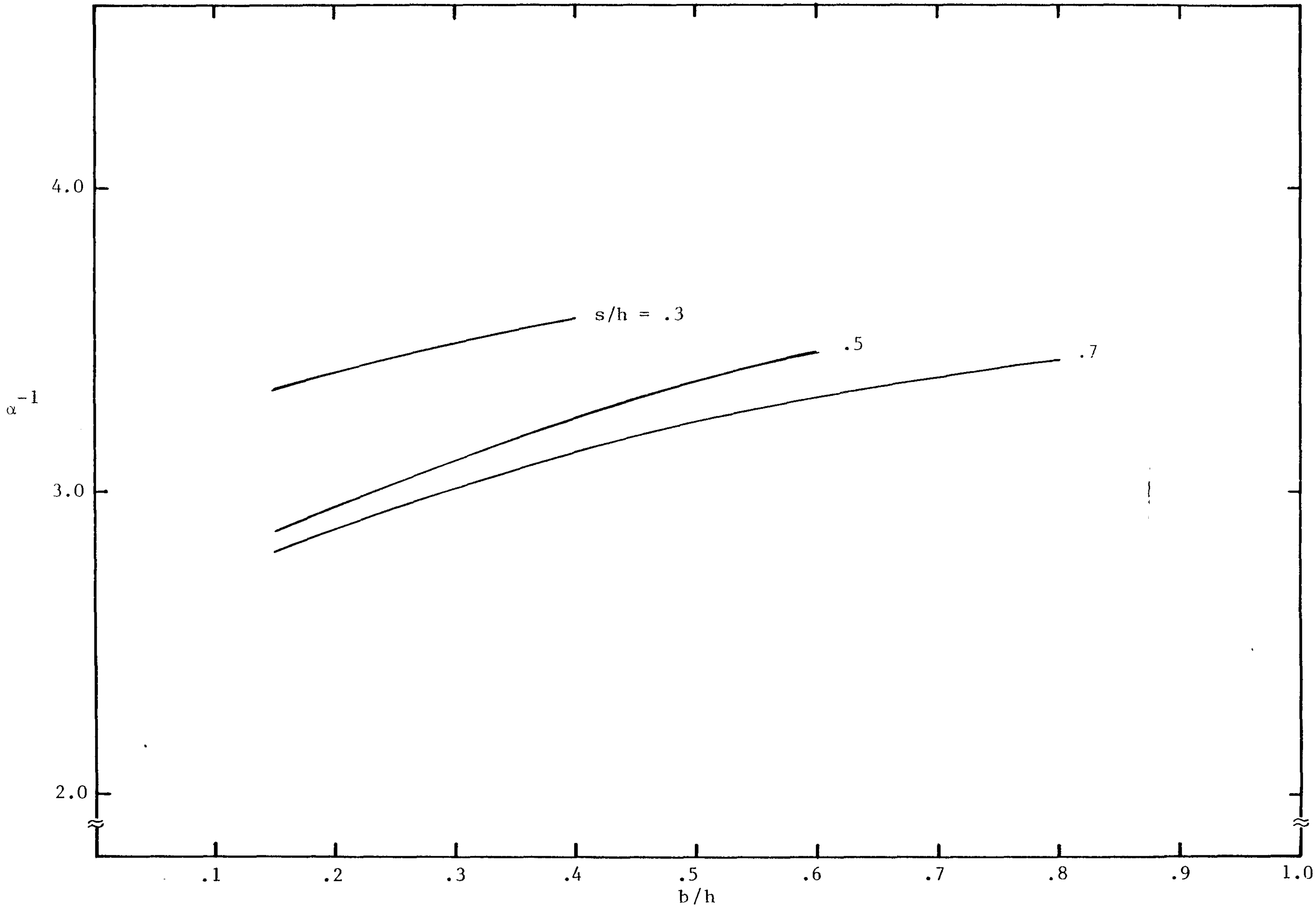

FIGURE 8a. Decay time constant vs. hole's size for a/h = .1.
(Free space: α^{-1} = 4.6.)

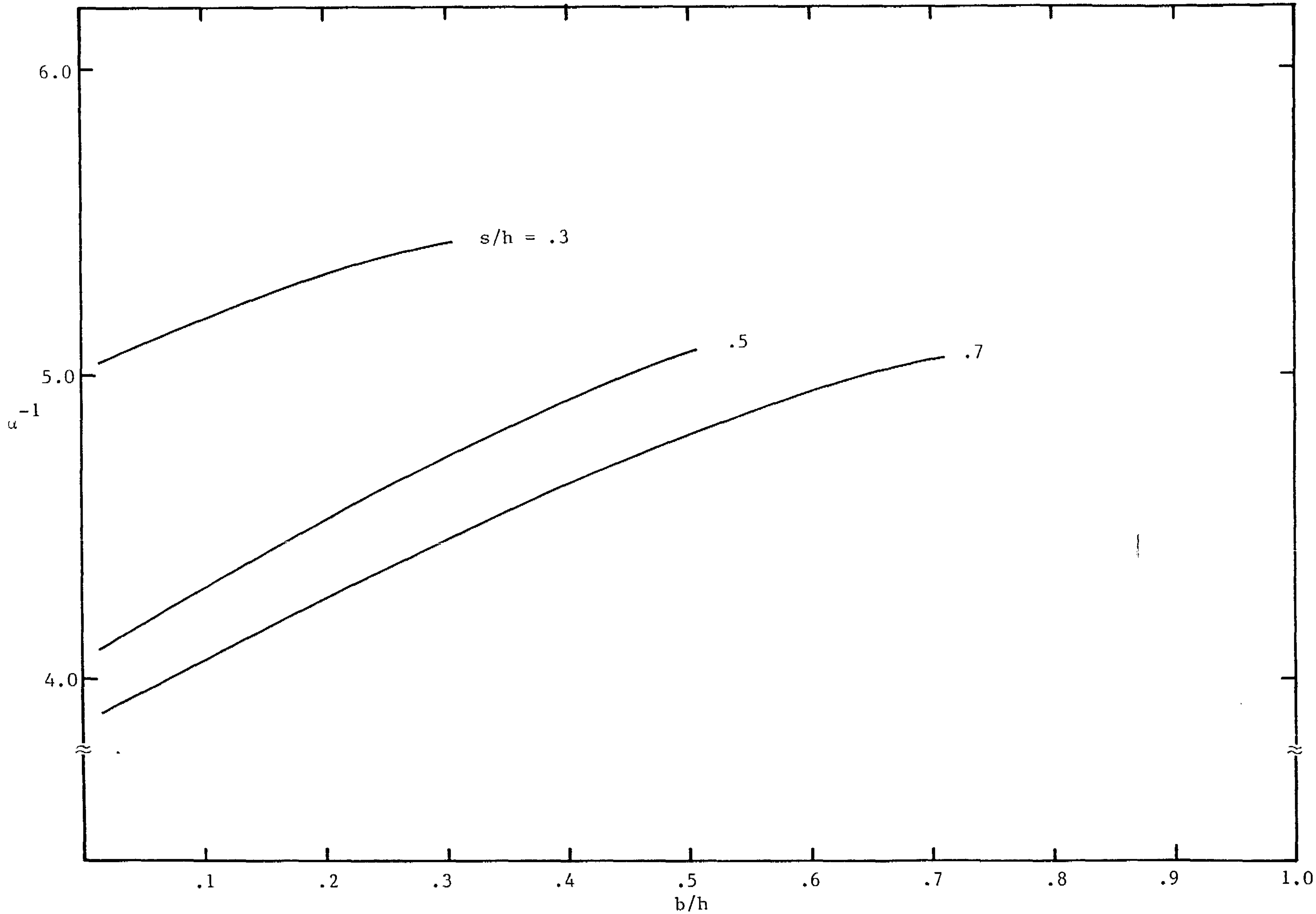

FIGURE 8b. Decay time constant vs. hole's size for a/h = .01.
(Free space: $\alpha^{-1} = 7.8$.)

References

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