Sensor and Simulation Notes

Note 90

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Impedance and Fields of Two Parallel Plates of Unequal Breadths

by

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Abstract

The method of conformal transformation is used to determine the impedance and fields of two parallel plates of unequal breadths. Parallel plates of this type make up the transmission lines that are used to transport electromagnetic energy along certain portions of some nuclear EMP simulators. Pertinent data are displayed graphically.

Acknowledgment

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I. Introduction

In simulating the phenomenon of a nuclear electromagnetic pulse (EMP) by artificial means, a parallel-plate transmission-line has proved to be a practical way of transporting the electromagnetic energy along certain portions of the simulator. In a previous note[1] the properties of two parallel plates of equal breadths were considered. This model also represents the electromagnetically equivalent configuration of a single plate parallel to an infinite ground plate --- a configuration that approximates an actual physical set-up of a plate parallel to a large, but finite, ground plate.

In this note the actual geometry of two parallel plates of unequal breadths, as shown in figure 1, is considered, and the impedance of such a structure is computed. Curves are presented to show how the impedance depends on the size of the ground plate, which varies from a breadth equal to that of the upper plate $(a_1 = a)$ to a breadth of infinite extent $(a_1 = \infty)$.

In addition curves are presented to show the values of a/h and a_1/h corresponding to standard impedance values of 50, 75, 100 and 150 ohms where a and a_1 are the half-breadths of the plates and h is the total spacing between the plates. The latter set of curves is particularly useful because it can be used to design a set of parallel plates of different breadths --- one plate perhaps much larger than the other --- whose impedance is exactly a prescribed value, without resorting to the infinite plate approximation.

Finally the magnitude of the electric field beneath the finite ground plate is compared with the magnitude of the electric field between the parallel plates. This consideration is important if electronic equipment is placed beneath the ground plate.

II. Exact Conformal Transformation

A. E. H. Love gives the conformal transformation for two parallel plates of unequal breadths.[2] For the geometry depicted in figure 1 the transformation is

$$- \frac{iz}{h} = \frac{K(m)}{\pi}\left[Z(w|m_1) + \frac{\pi w}{2K(m)K(m_1)} + \frac{cn(w|m_1)dn(w|m_1)}{\nu + sn(w|m_1)} + \frac{\pi}{2K(m)}\right] . \tag{1}$$

Associated with this transformation are the equations

$$c_1 c_2 = \frac{K(m) - \nu^2 E(m)}{m_1 \nu^2 K(m) - E(m)} \tag{2}$$

$$(\nu + c_1)(\nu + c_2) = \frac{(\nu^2 - 1)(m_1\nu^2 - 1)K(m)}{m_1\nu^2 K(m) - E(m)} . \tag{3}$$

In these equations $Z(w|m_1)$ is Jacobi's Zeta-function; $K(m)$ is the complete elliptic integral; $sn(w|m_1)$, $cn(w|m_1)$, and $dn(w|m_1)$ are Jacobian elliptic functions. The notation used here is consistent with that used in reference 3. The parameter ν relates to the difference in the breadths of the two plates. As $\nu \to \infty$, (1) reduces to the transformation for the case of two plates of equal breadth; as $\nu \to 1/\sqrt{m_1}$, the breadth of the larger plate becomes infinite while the smaller plate remains finite. The range of ν , therefore, is $\nu \geq 1/\sqrt{m_1}$ and throughout this range $a_1 \geq a$.

Evaluating the real part of (1) at $(- a,h)$ and $(- a_1,0)$ gives

$$\frac{a}{h} = \frac{K(m)}{\pi}\left\{v_1 \frac{E(m)}{K(m)} - E(v_1|m) + \frac{\nu}{\frac{\nu}{c_1} + 1}\sqrt{\left(1 - \frac{1}{c_1^2}\right)\left[m_1 - \left(1 - \frac{1}{c_1^2}\right)\right]}\right\} \tag{4}$$

$$\frac{a_1}{h} = \frac{K(m)}{\pi} \left\{ v_2 \frac{E(m)}{K(m)} - E(v_2|m) - \frac{\nu}{\frac{\nu}{c_2} + 1} \sqrt{\left(1 - \frac{1}{c_2^2}\right)\left[m_1 - \left(1 - \frac{1}{c_2^2}\right)\right]} \right\} \tag{5}$$

where v_1 and v_2 satisfy

$$dn(v_1|m) - \frac{1}{c_1} = 0 \quad , \quad 0 \le v_1 \le K(m) \tag{6}$$

and

$$dn(v_2|m) + \frac{1}{c_2} = 0 \quad , \quad 0 \le v_2 \le K(m) \quad . \tag{7}$$

The constants c_1 and c_2 are obtained from (2) and (3) such that

$$c_{1,2} = \beta \pm \sqrt{\beta^2 - \alpha} \tag{8}$$

where c_1 is the positive value of (8) and c_2 is the negative value. The quantities α and β are given by

$$\alpha = \frac{\nu^2 E(m) - K(m)}{E(m) - m_1 \nu^2 K(m)} \tag{9}$$

$$\beta = \nu \frac{(1 - m/2)K(m) - E(m)}{E(m) - m_1 \nu^2 K(m)} \quad . \tag{10}$$

The impedance of a parallel-plate transmission-line immersed in free-space is[1]

$$Z_L = \frac{\varepsilon_o Z_o}{C} \tag{11}$$

where in this particular case C , the capacitance per unit length, is given by

$$C = \frac{\varepsilon_o K(m)}{K(m_1)} \quad . \tag{12}$$

It is convenient to define a normalized impedance

$$Z_L' = \frac{Z_L}{\frac{h}{2a} Z_o} \tag{13}$$

which, if expressed explicitly, is

$$Z_L' = \frac{2a}{h} \frac{K(m_1)}{K(m)} \quad . \tag{14}$$

In figures 2 through 4 the normalized impedance Z_L' is plotted versus a/h with $d/h = (a_1 - a)/h$ as a parameter. The data in figures 2 and 3 were obtained by choosing a set of values for m_1 and ν and by using these values to compute the corresponding values of a/h , a_1/h and Z_L' . To compute these three quantities, (9) and (10) are used to obtain α and β which are substituted into (8) to compute c_1 and c_2 . The values of c_1

and c_2 are substituted into (6) and (7) and a root-finding technique is used to determine v_1 and v_2. With all of these constants known, (4) and (5) are evaluated to find a/h and a_1/h. The impedance Z_L' is computed from (14).

A different method was used to obtain the data in figure 4. The method, as well as the reason for using it, is discussed in the next section.

The data presented in figures 5 through 7 were obtained in the same way as the data in figures 2 and 3 were obtained except that a root-finding technique was used first to determine from (14) what values of m_1 correspond to specific values of Z_L. In these figures $A = a/a_\Omega$ is plotted versus $A_1 = a_1/a_\Omega$ with Z_L = 50, 75, 100 and 150 ohms as a parameter. The quantity a_Ω is measured in meters and is assigned the subscript Ω to emphasize that it takes on a different value for each ohmic value of Z_L. Below is listed a table of the values of $2a_\Omega/h$ that correspond to each of the four values of Z_L.

Table 1. Values of a_Ω and Z_L

Z_L (ohms)	$\frac{2a_\Omega}{h}$
50	5.998
75	3.618
100	2.457
150	1.337

Physically a_Ω is the half-breadth of two parallel plates of equal breadth that are one meter apart and have an impedance Z_L. Multiplying the values of A and A_1 that correspond to a particular value of Z_L by the appropriate value of $2a_\Omega/h$ listed in Table 1 gives $2a/h$ and $2a_1/h$, respectively, which can be used to design an actual parallel plate structure.

III. Approximate Transformation for Plates Close Together

For the case of two parallel plates that are very close together $(a/h >> 1)$, the parameter m_1 is very small. For example when $2a/h \approx 10$, $m_1 \approx 10^{-14}$; when $2a/h \approx 20$, $m_1 \approx 10^{-24}$. Numerical computation becomes difficult for such small values of m_1; hence, it is convenient to use an approximate conformal transformation for $2a/h \geq 10$. The approximate transformation is[4]

$$\frac{-\pi\tau(z-a)}{h} = \frac{w^2}{2} + (1-\tau)w - \tau \ln w + \frac{1}{2} - \tau \tag{15}$$

where the parameter τ satisfies

$$\frac{\pi d}{h} = \ln \tau + \frac{\tau}{2} - \frac{1}{2\tau} \quad , \quad \tau \geq 1 \quad . \tag{16}$$

The real and imaginary parts of (15) are

$$\frac{-\pi\tau(x-a)}{h} = \frac{r^2}{2}\cos 2\theta + (1-\tau)r\cos\theta - \tau \ln r + \frac{1}{2} - \tau \tag{17}$$

$$\frac{-\pi\tau y}{h} = \frac{r^2}{2}\sin 2\theta + (1-\tau)r\sin\theta - \tau\theta \tag{18}$$

Setting $x = 0$ and $\theta = \pi$ in (17) gives

$$\frac{\pi\tau a}{h} = \frac{r^2}{2} - (1-\tau)r - \tau \ln r + \frac{1}{2} - \tau \tag{19}$$

For a set of values of a/h and τ, two values of r satisfy (19); these radii are denoted r_A and r_B such that $r_A \leq 1$ and $r_B \geq 1$. Values of τ are obtained from (16), which depends on d/h.

In terms of r_A and r_B the transmission line impedance Z_L is[4]

$$Z_L = \frac{\pi}{2} \frac{Z_o}{\ln(r_B/r_A)} \quad . \tag{20}$$

The curves in figure 4 were obtained by using this approximate transformation. In figure 4 the normalized impedance Z_L' defined by (13) is plotted versus a/h with d/h as a parameter. Note that the values of d/h in figure 4 are different from those in figures 2 and 3.

IV. Electric Field below the Lower Plate

In an actual simulator it may be that electronic equipment is located beneath the lower plate, i.e. the plate of greater breadth. In placing such equipment in this position it might be assumed that the field of a simulated EMP is contained wholly within the transmission line and that none of it leaks around the ends of the lower plate into the region where it could interfere with the electronic equipment. In the case of a finite lower plate a portion of the field does of course penetrate into the region beneath the lower plate. In this section of the note the magnitude of the electric field just beneath the center of the lower plate $(z_A = 0 + i0-)$ is compared with that just above the center of the lower plate $(z_B = 0 + i0+)$.

The magnitude of the electric field is given by

$$|E| = \left|\frac{dz}{dw}\right|^{-1} \tag{21}$$

which, in this case, is

$$|E| = \left|A \frac{(\mathrm{sn}\ w - c_1)(\mathrm{sn}\ w - c_2)}{(\mathrm{sn}\ w + \nu)^2}\right| \tag{22}$$

where the constant A relates to the specific geometry of the parallel plates and does not depend on w . Under the conformal transformation (1) the points z_A and z_B correspond respectively to $w_A = - K(m_1)$ and $w_B = - K(m_1) + iK(m)$. Upon substituting these values of w into (22) the ratio between the magnitude of the electric field at z_A and z_B is

$$\frac{|E_A|}{|E_B|} = \frac{(1 + c_1)(1 + c_2)(\sqrt{m_1}\ \nu - 1)^2}{(1 + \sqrt{m_1}\ c_1)(1 + \sqrt{m_1}\ c_2)(\nu - 1)^2} \quad . \tag{23}$$

Plots of $|E_A|/|E_B| \times 100\%$ versus a/h with d/h as a parameter are presented in figure 8.

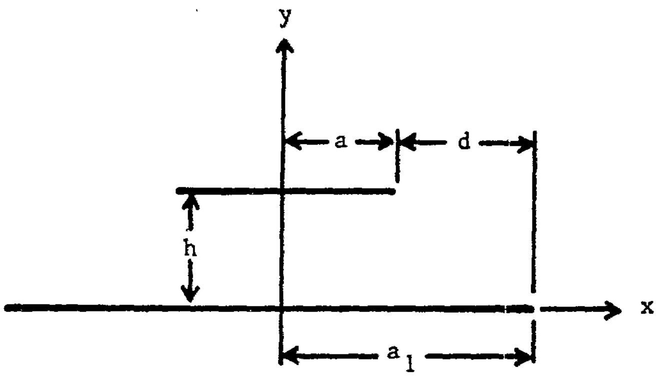

Figure 1. Two Parallel Plates of Unequal Breadths.

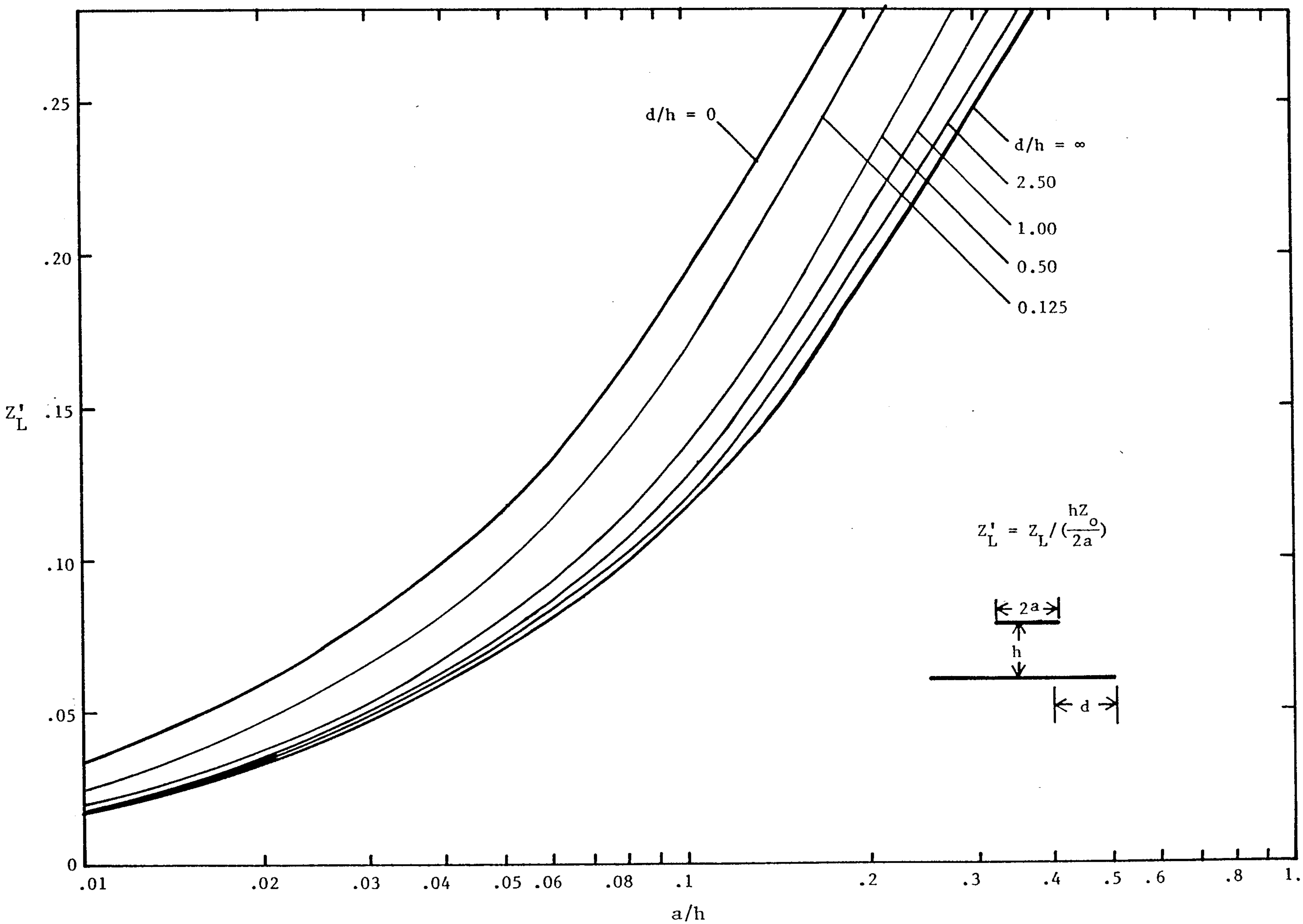

Figure 2. Z_L' versus a/h with d/h as a parameter $(.01 \le a/h \le 1.0)$.

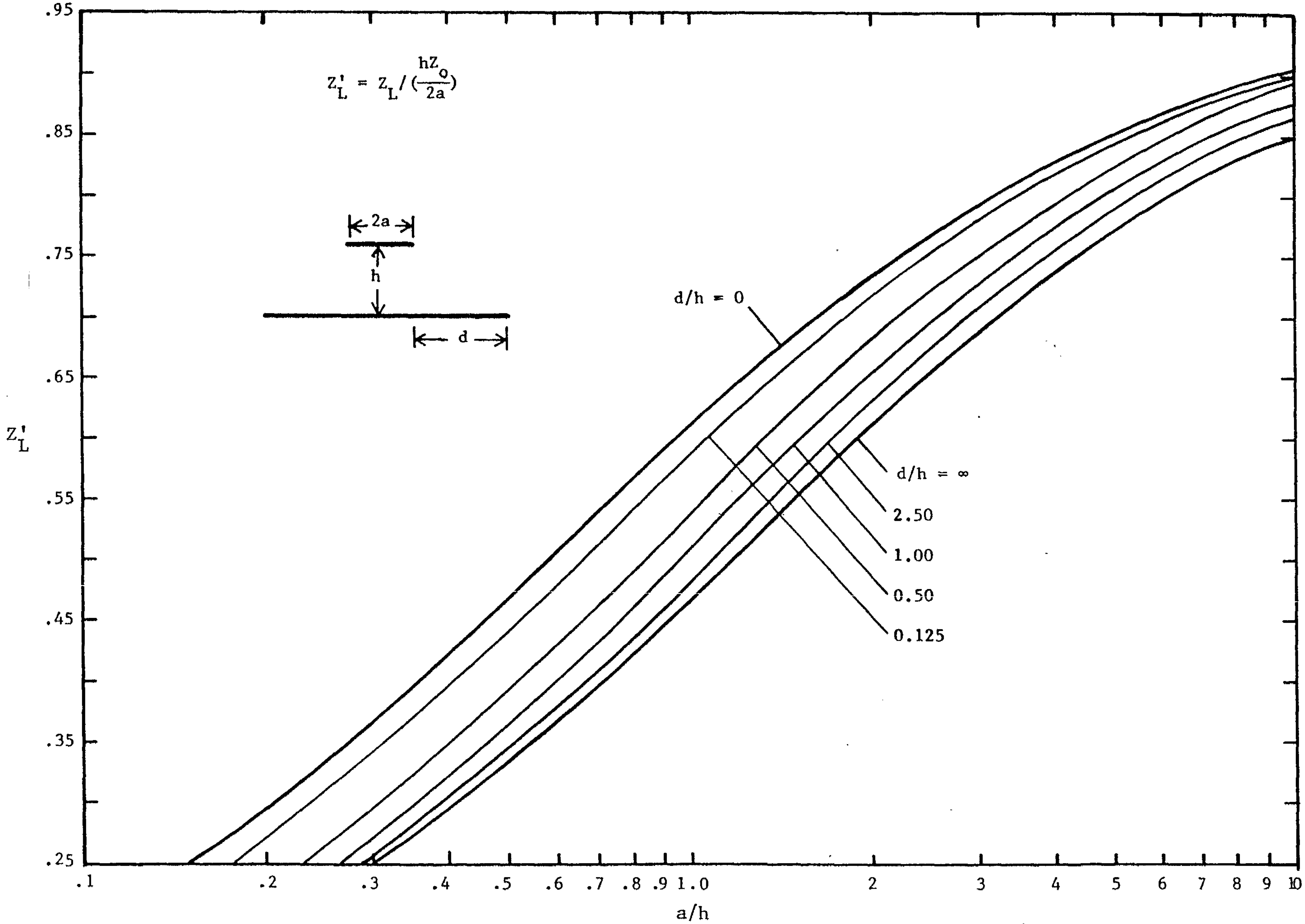

Figure 3. Z_L' versus a/h with d/h as a parameter $(.1 \leq a/h \leq 10)$.

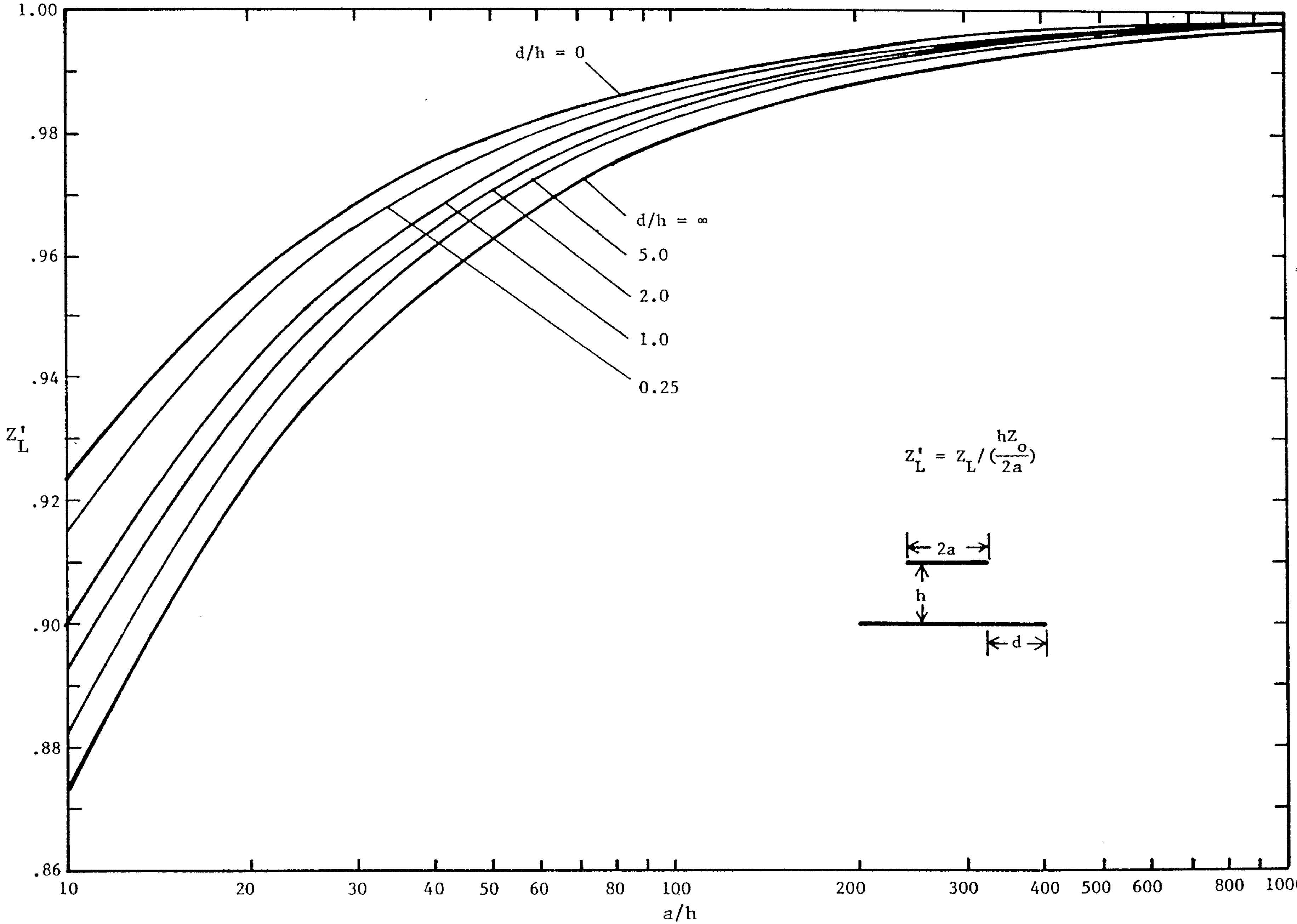

Figure 4. Z'_L versus a/h with d/h as a parameter $(10 \leq a/h \leq 1000)$.

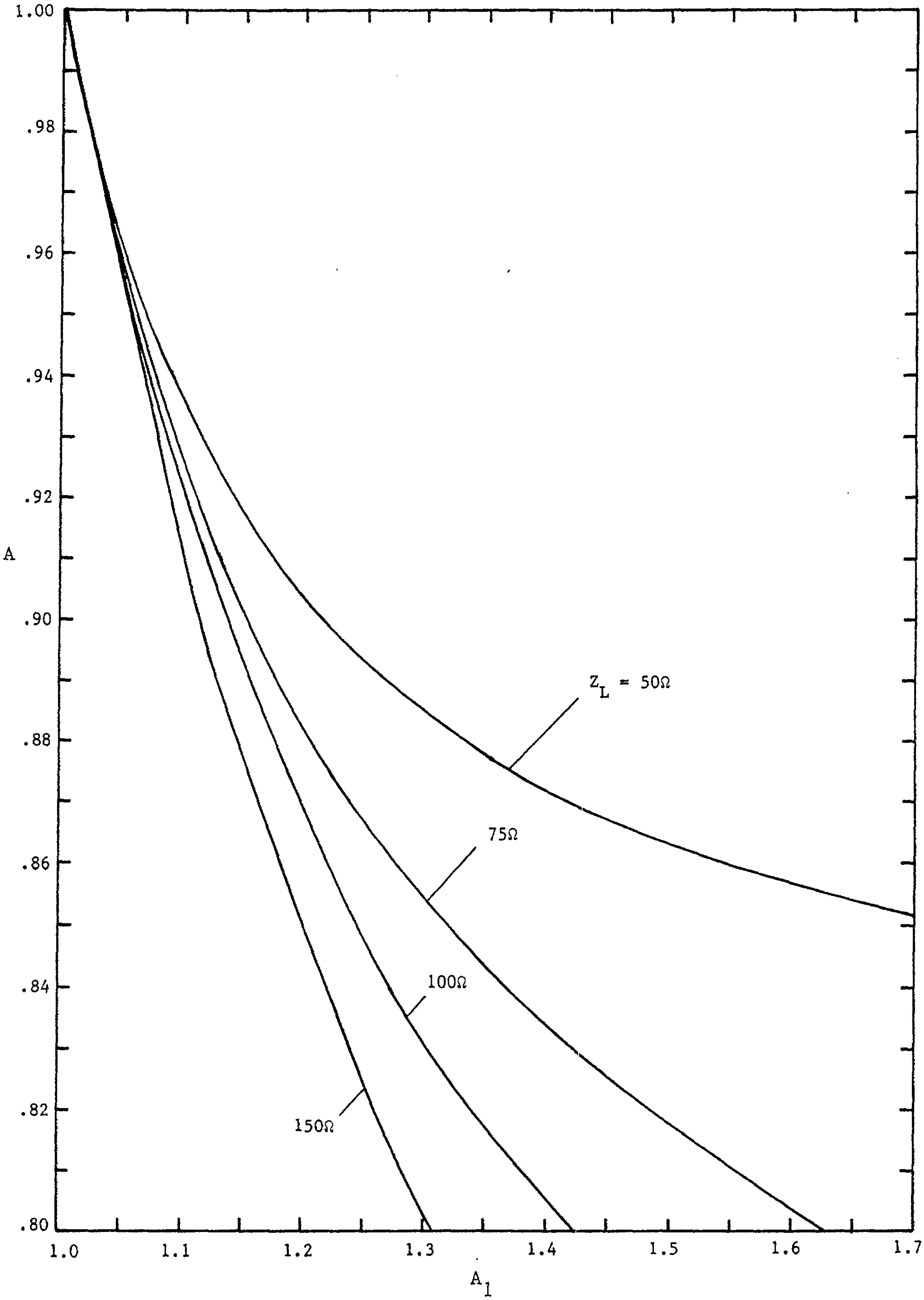

Figure 5. A versus A_1 with Z_L as a parameter $(1.0 \le A_1 \le 1.7)$.

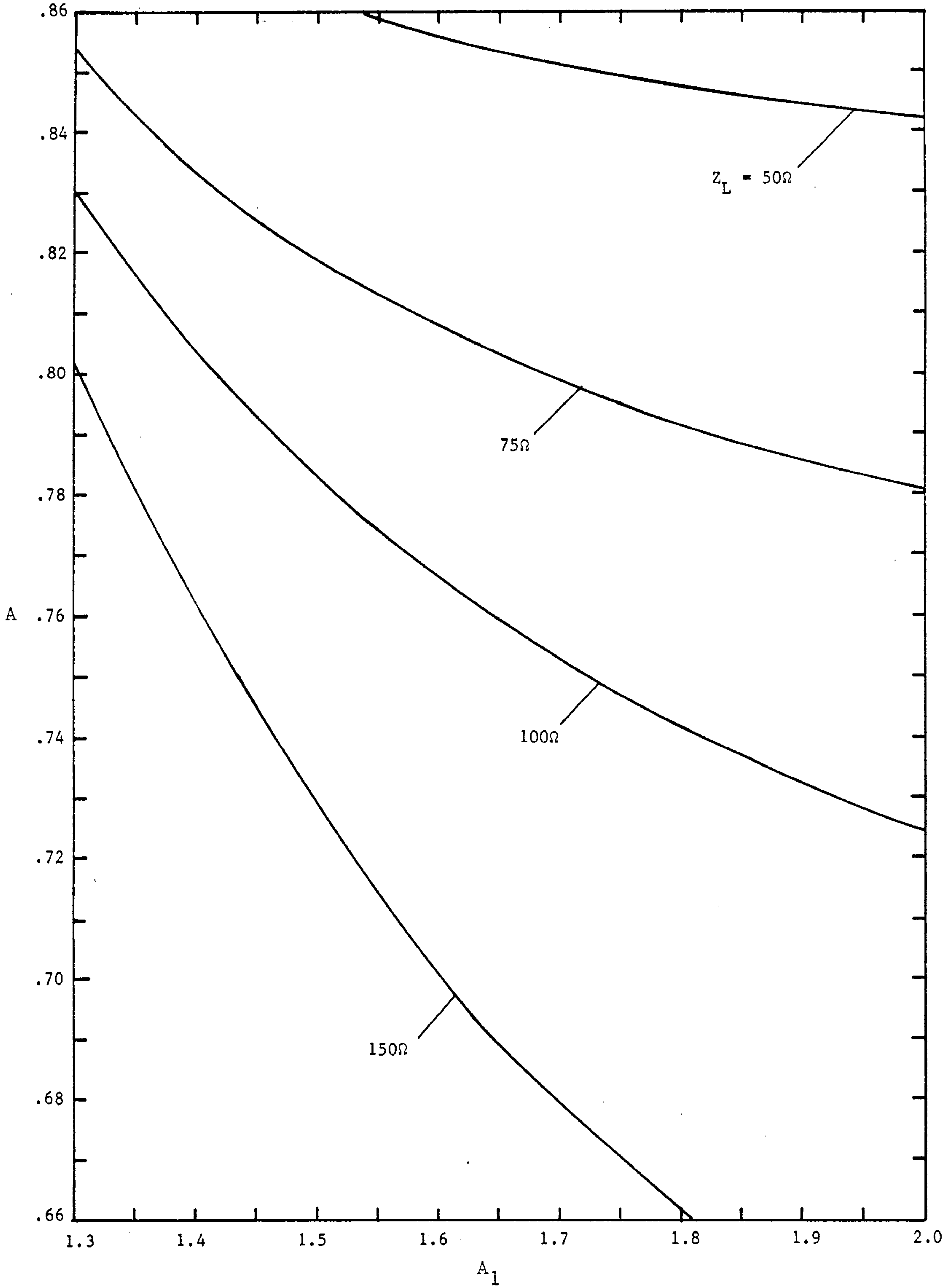

Figure 6. A versus A_1 with Z_L as a parameter $(1.3 \leq A_1 \leq 2.0)$.

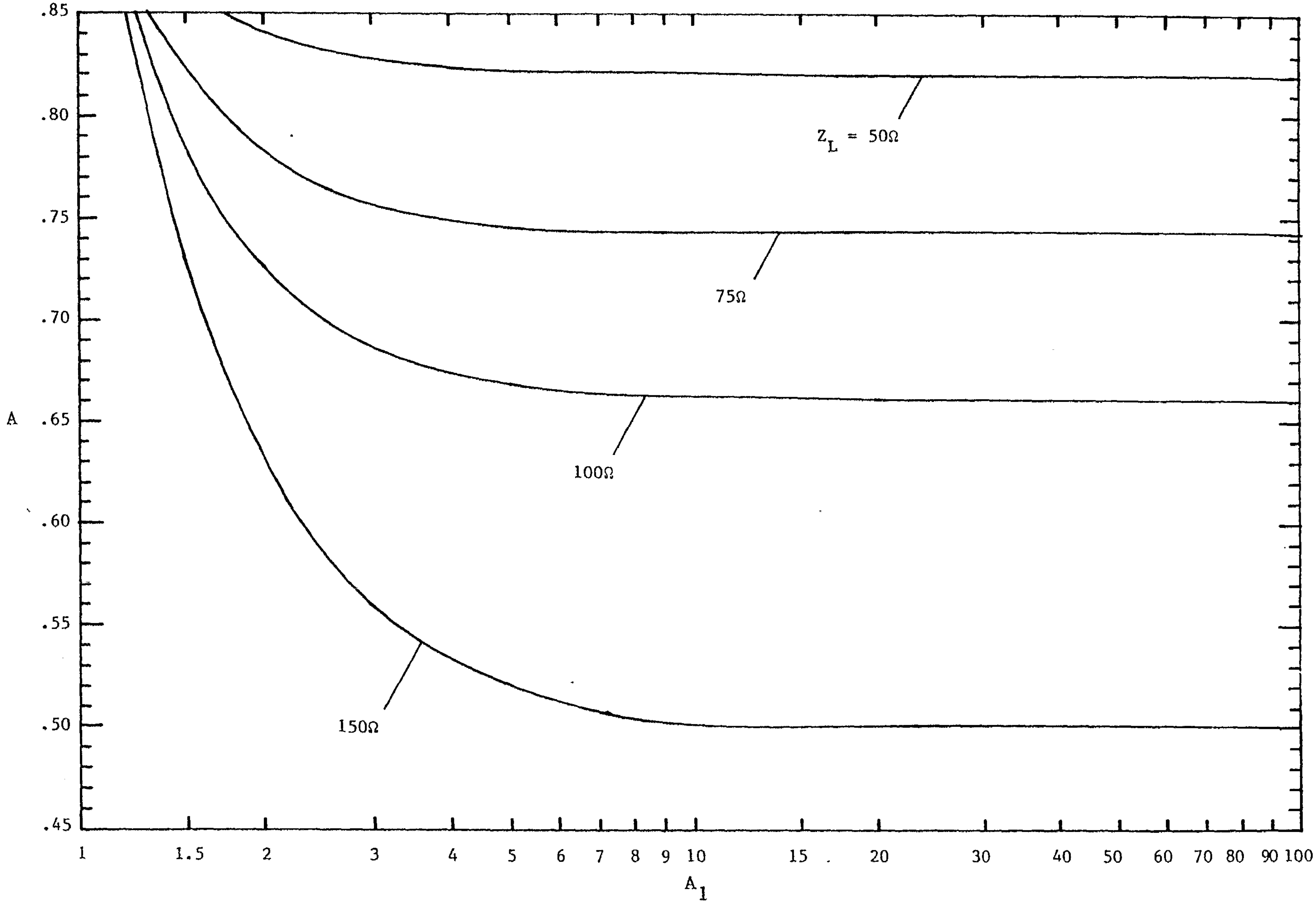

Figure 7. A versus A_1 with Z_L as a parameter $(1 \le A_1 \le 100)$.

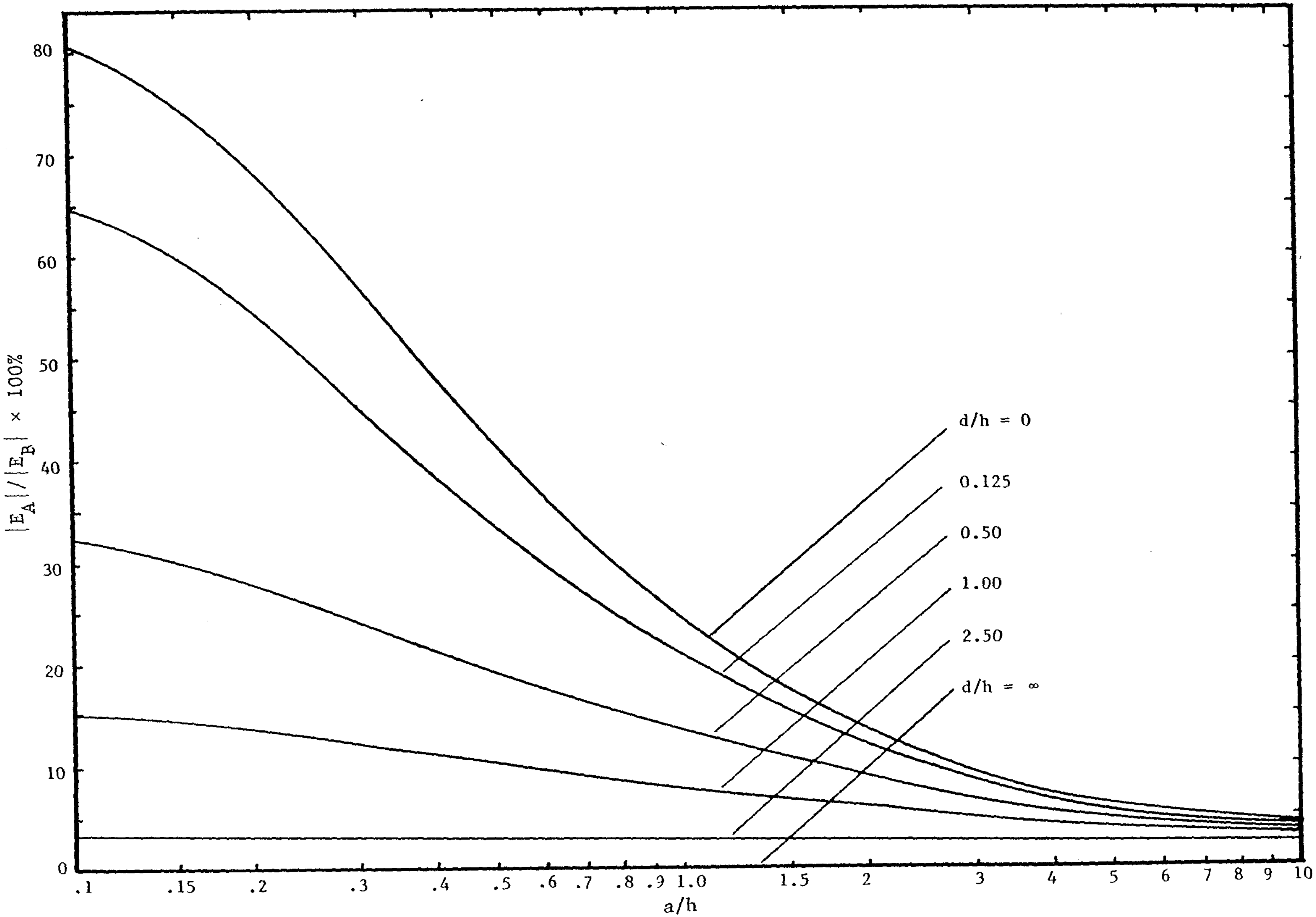

Figure 8. $|E_A|/|E_B| \times 100\%$ versus a/h with Z_L as a parameter.

References

1. Carl E. Baum, Sensor and Simulation Note 21, "Impedances and Field Distributions for Parallel-Plate Transmission-Line Simulators," June 1966.
2. A. E. H. Love, "Some Electrostatic Distributions in Two Dimensions," Proc. London Math. Soc. <u>22</u>, pp. 337-369, 1923.
3. Milton Abromowitz and Irene A. Stegun, Editors, <u>Handbook of Mathematical</u> Functions, National Bureau of Standards, AMS-55, June 1964.
4. F. Assadourian and E. Rimai, "Simplified Theory of Microstrip Transmission Systems," Proc. of IRE, December 1952, pp. 1651-1657.