Sensor and Simulation Notes
Note 86
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Some Further Considerations for the Circular Parallel-Plate Dipole

Capt Carl E. Baum
Air Force Weapons Laboratory

Abstract

A previous note contained calculations of various features of the circular parallel-plate dipole; this note continues these calculations. In this note various problems are considered, including the frequency response characteristics of a resistive rod used as the output resistor, the effect of the thickness of the sensor disk electrodes on the equivalent height, the effect of dielectric spacers on sensor capacitance and equivalent height, the electric field distortion near the edge of a conducting ground plane with a square edge, and the electric field distortion of a circular ground plane of nonzero thickness by considering it as an oblate spheroid.

I. Introduction

In a previous note[1] we have treated some of the design considerations for a circular parallel-plate dipole. These considerations included the capacitance and equivalent height of the sensor, the frequency response of the sensor associated with a resistor inserted between the plates and in series with a low impedance output cable, and the perturbation of the electric field produced by a surrounding dielectric shell.

In this note we consider several more problems associated with this type of sensor. First we consider the response of a uniform conducting dielectric rod as the output resistor. This can be compared with the response of a conducting tube in the previous note. Second we consider the error introduced into the equivalent height by the non zero thickness of the circular disk electrodes. Third we consider the errors introduced into the capacitance and equivalent height by the use of dielectric spacers. Finally we consider the error introduced into the incident field by the presence of a disk of finite thickness which might be used as a ground plane, either between the two electrodes of a differential sensor, or adjacent to a large conducting surface and forming one side of a single ended sensor.

II. Short Circuit Current for a Resistive Rod

In reference 1 we considered the output resistor as a cylindrical shell of radius Ψ_0 with surface resistance R_s. For $R_s >> Z$ it was found that the total current flowing along this sheet has a flat response to the incident wave up to frequencies such that the wavelength is of the order of Ψ_0 where Z is the wave impedance of the medium containing the incident wave. Z is given by[2]

$$Z = \sqrt{\frac{\mu}{\varepsilon}} \tag{1}$$

where μ and ε are the permeability and permittivity, respectively, of this medium; the conductivity of this medium was assumed to be zero. The medium inside the resistive shell was assumed to have the same electromagnetic parameters as the external medium.

While the total current on this cylindrical shell has a rather high frequency response there are still other problems to

1. Capt Carl E. Baum, Sensor and Simulation Note 80, The Circular Parallel-Plate Dipole, March 1969.

2. Rationalized MKSA units are used throughout.

consider. The current on the resistor must be related to the current into some other device which records this current or transports the current to some other position for eventual recording. The manner of connecting the resistor to some device such as a coaxial cable can significantly influence the high-frequency characteristics of the current into the cable. For example, there may be currents in the output cable associated with displacement current between the sensor plates and in the resistor structure. There are then problems to be considered concerning the geometry of how the resistor current gets into the output circuitry.

In figure 1 we show a possible geometry for the output resistor, including the connection to some output circuitry. For these calculations the resistor is taken as a uniform rod of length b, radius Ψ_o, permeability μ, permittivity ϵ_1, and conductivity σ. The propagation constant in the resistor is then

$$K_1 = \sqrt{-i\omega\mu(\sigma + i\omega\epsilon_1)} \tag{2}$$

The propagation constant in the medium between the plates is

$$K = \omega\sqrt{\mu\epsilon} \tag{3}$$

Note that this resistor is placed between the plates in a manner such that there is a narrow annular slot (with width small compared to Ψ_o) in the perfectly conducting sheet (the plane $z = 0$) around the resistor. The signal to the coaxial cable or other output circuitry passes through this narrow annular slot. One purpose of choosing this geometry is so that with this annular slot shorted out we can calculate the short circuit current at the slot as a boundary value problem in cylindrical coordinates. Such a calculation includes the displacement current in the resistor. In reference 1 we considered a resistive tube; with the present type of output connection one would have to add the displacement current inside the tube to the conduction current along the wall of the tube in order to find the short circuit current into the type of output connection being presently considered. In this note we only consider the short circuit current from the point of view of determining optimum resistor dimensions and conductivities. We do not here consider the source impedance presented to the output circuit.

Referring again to figure 1 note that the output circuitry occupies a volume with finite extent in the z direction. In reference 1 we considered the ideal case of two coaxial disks at $z = \pm b$. In a real sensor the signal has to come out somewhere. We assume for the present calculations that the signal is taken out near $z = 0$ on the z axis by including a conducting plate of finite thickness centered between the conducting disks. The y is pointing into the page.

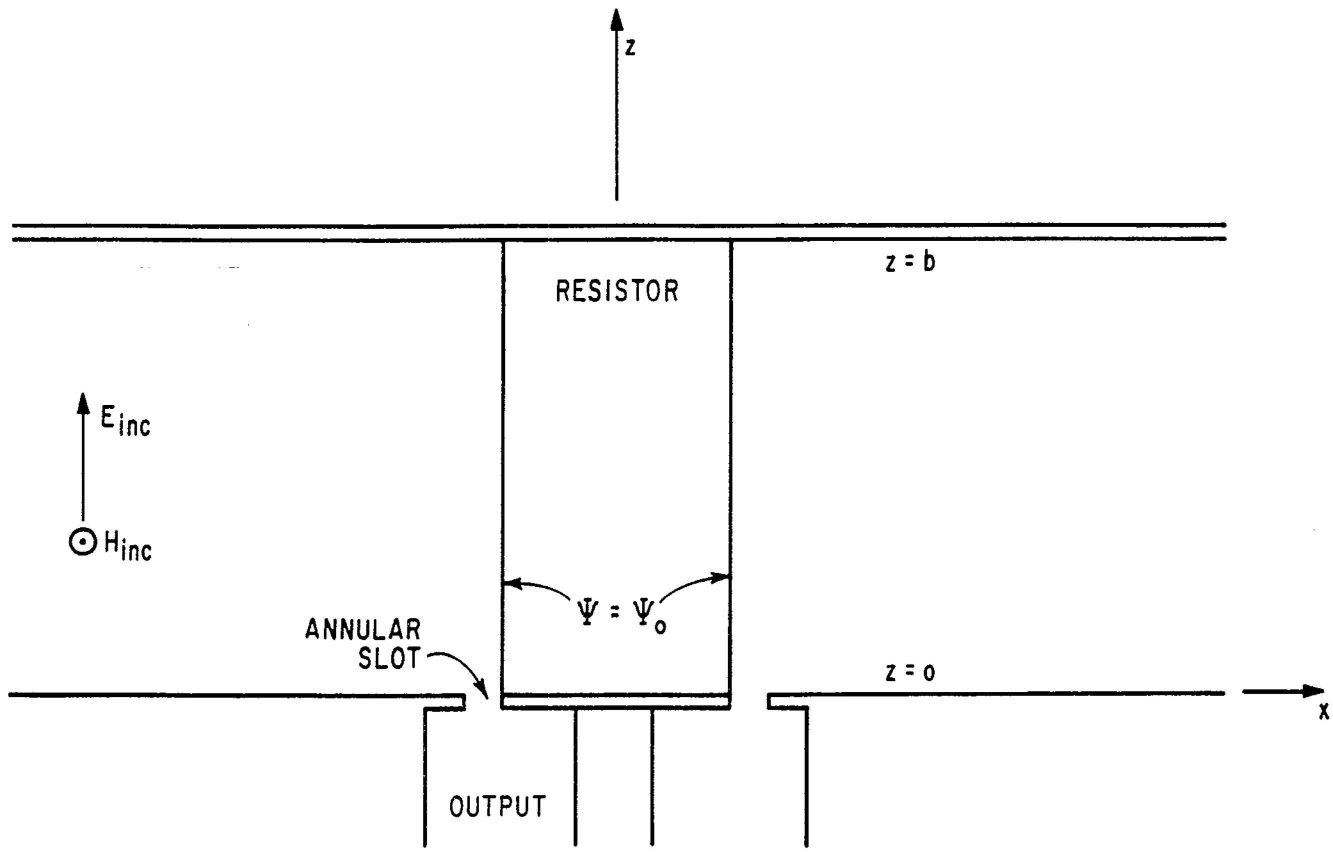


Figure 1. IDEALIZED RESISTOR WITH ANNULAR-SLOT OUTPUT

output circuitry would be inside this plate. Another case of interest is with a single disk at $z = b$ and an infinite ground plane at $z = 0$, in which case the output circuitry is below $z = 0$ as in figure 1. A differential sensor with two symmetrical disks has two resistors like the one shown in figure 1. We consider one resistor, the results applying equally well for two, as would be used in a differential sensor.

In designing a parallel-plate dipole of this type there are a few basic parameters normally established. These include the disk position ($z = b$) which establishes the equivalent height of the dipole, the sensor capacitance, and the resistance of the resistor at the sensor output. With these parameters and whether the sensor is to be single ended or differential established one then would like to maximize the frequency response of the sensor. In the present calculations we seek to make the short circuit current from the resistor be uniform with frequency to the maximum frequency possible for the case that the resistor is a uniform circular rod. Considering a single resistor of length b as in figure 1 we have a resistance

$$R_2 = \frac{b}{\pi \Psi_o^2 \sigma} \tag{4}$$

R_2 and b are both considered as fixed numbers while Ψ_o and σ can be varied to optimize the short circuit current from the resistor.

Instead of Ψ_o we then define a characteristic distance which is fixed by choice of R_2 and b as

$$\Psi_1 \equiv b \frac{Z}{R_2} = \frac{b}{R_2} \sqrt{\frac{\mu}{\varepsilon}} = \pi \Psi_o^2 \sigma \sqrt{\frac{\mu}{\varepsilon}} \tag{5}$$

Then define a normalized frequency as

$$\alpha \equiv K\Psi_1 = \omega\sqrt{\mu\varepsilon}\,\Psi_1 \tag{6}$$

Let

$$p \equiv \frac{\Psi_o}{\Psi_1} , \qquad \varepsilon_r = \frac{\varepsilon_1}{\varepsilon} \tag{7}$$

be dimensionless parameters. For fixed ε_r we wish to find the best value of p which optimizes the frequency response of the short circuit current as expressed in terms of α.

Some parameters of interest can now be written as

$$K\Psi_o = \alpha p$$

$$K_1\Psi_o = \alpha p \frac{K_1}{K} = \alpha p\left[\varepsilon_r - i \frac{\sigma}{\omega\varepsilon}\right]^{1/2} = \alpha pq \tag{8}$$

where we have defined

$$q \equiv \frac{K_1}{K} = \left[\varepsilon_r - i \frac{\sigma}{\omega\varepsilon}\right]^{1/2} = \left[\varepsilon_r - \frac{i}{\pi\alpha p^2}\right]^{1/2} \tag{9}$$

The wave impedance in the resistor is

$$Z_1 = \sqrt{\frac{i\omega\mu}{\sigma + j\omega\varepsilon_1}} \tag{10}$$

and we have the convenient relations

$$\frac{Z_1}{Z} = \frac{K}{K_1} = \frac{1}{q}$$

$$\frac{K_1}{Z_1} = -i(\sigma + i\omega\varepsilon_1)$$

$$Z_1K_1 = \omega\mu \tag{11}$$

With the above preliminaries taken care of we go to the boundary value problem. As in reference 1 we assume an incident plane wave between the two perfectly conducting surfaces ($z = 0$ and $z = b$) propagating in the $+x$ direction with only two field components in the form (with $e^{i\omega t}$ suppressed)

$$E_{z_{inc}} = E_o e^{-iKx} = E_o e^{-iK\Psi\cos(\phi)}$$

$$H_{y_{inc}} = -\frac{E_o}{Z} e^{-iKx} = -\frac{E_o}{Z} e^{-iK\Psi\cos(\phi)} \tag{12}$$

where we have a cylindrical coordinate system (Ψ, ϕ, z) related to cartesian coordinates (x, y, z) as

$$x = \Psi \cos(\phi) \ , \qquad y = \Psi \sin(\phi) \tag{13}$$

In cylindrical coordinates the incident wave has the expansions

$$E_{z_{inc}} = E_o\left[J_o(K\Psi) + 2\sum_{n=1}^{\infty}(-i)^n J_n(K\Psi)\cos(n\phi)\right]$$

$$H_{\Psi_{inc}} = -i\,\frac{E_o}{Z}\,2\sum_{n=1}^{\infty}(-i)^n\,\frac{J_n(K\Psi)}{K\Psi}\,\sin(n\phi) \tag{14}$$

$$H_{\phi_{inc}} = -i\,\frac{E_o}{Z}\left[J_o'(K\Psi) + 2\sum_{n=1}^{\infty}(-i)^n J_n'(K\Psi)\cos(n\phi)\right]$$

Primes on the Bessel functions denote the derivative with respect to the argument. Two components of the wave inside the resistor ($\Psi < \Psi_o$) are

$$E_{z_1} = E_o\left[a_o J_o(K_1\Psi) + 2\sum_{n=1}^{\infty}(-i)^n a_n J_n(K_1\Psi)\cos(n\phi)\right]$$

$$H_{\phi_1} = -i\,\frac{E_o}{Z_1}\left[a_o J_o'(K_1\Psi) + 2\sum_{n=1}^{\infty}(-i)^n a_n J_n'(K_1\Psi)\cos(n\phi)\right] \tag{15}$$

The reflected wave (for $\Psi > \Psi_o$) has two of its components given by

$$E_{z_2} = E_o\left[b_o H_o^{(2)}(K\Psi) + 2\sum_{n=1}^{\infty}(-i)^n b_n H_n^{(2)}(K\Psi)\cos(n\phi)\right]$$

$$H_{\phi_2} = -i\,\frac{E_o}{Z}\left[b_o H_o^{(2)\prime}(K\Psi) + 2\sum_{n=1}^{\infty}(-i)^n b_n H_n^{(2)\prime}(K\Psi)\cos(n\phi)\right] \tag{16}$$

This reflected wave is taken as a purely outward propagating wave as was also done for the example in reference 1. Thus the size of the conducting plates is assumed infinite making this a high-frequency or early-time calculation.

To find the coefficients in the field expansions first make E_z continuous at $\Psi = \Psi_o$ giving

$$a_n J_n(K_1\Psi_o) = J_n(K\Psi_o) + b_n H_n^{(2)}(K\Psi_o) \tag{17}$$

Making H_ϕ continuous at $\Psi = \Psi_o$ gives

$$\frac{1}{Z_1} a_n J_n'(K_1\Psi_o) = \frac{1}{Z}\left[J_n'(K\Psi_o) + b_n H_n^{(2)\prime}(K\Psi_o)\right] \tag{18}$$

which can be rewritten as

$$\frac{Z}{Z_1} a_n J_n'(K_1\Psi_o) = q a_n J_n'(K_1\Psi_o) = J_n'(K\Psi_o) + b_n H_n^{(2)\prime}(K\Psi_o) \tag{19}$$

Solving for a_n we have

$$a_n\left[qJ_n'(K_1\Psi_o)H_n^{(2)}(K\Psi_o) - J_n(K_1\Psi_o)H_n^{(2)\prime}(K\Psi_o)\right]$$

$$= J_n'(K\Psi_o)H_n^{(2)}(K\Psi_o) - J_n(K\Psi_o)H_n^{(2)\prime}(K\Psi_o)$$

$$= \frac{2i}{\pi K\Psi_o} \tag{20}$$

where we have used a Wronskian relation for the Bessel functions.

The short circuit current across the annular slot at $\Psi = \Psi_o$ is given by

$$I = -\int_o^{2\pi} H_\phi\Big|_{\Psi=\Psi_o} \Psi_o d\phi = i2\pi\Psi_o \frac{E_o}{Z_1} a_o J_o'(K_1\Psi_o)$$

$$= -\frac{4E_o}{Z_1 K} J_o'(K_1\Psi_o)\left[qJ_o'(K_1\Psi_o)H_o^{(2)}(K\Psi_o) - J_o(K_1\Psi_o)H_o^{(2)\prime}(K\Psi_o)\right]^{-1} \tag{21}$$

If we expand the Bessel functions for small arguments (small in magnitude) we find for small $|K\Psi_o|$ and $|K_1\Psi_o|$ the asymptotic relation

$$I \simeq -E_o \pi \Psi_o^2 (\sigma + i\omega\varepsilon_1) \tag{22}$$

This, of course, is the sum of the low-frequency conduction and displacement currents in the resistor. Note that this asymptotic form in equation 22 is not the asymptotic form for low frequency ω but for small $|K\Psi_o|$ and $|K_1\Psi_o|$ so that the leading term of the Bessel function expansions is used. If an asymptotic form for small ω is used other terms related to the resistor size (which might be thought of as an inductive effect) also enter in. An appropriate normalizing current is then

$$I_o \equiv -E_o \pi \Psi_o^2 \sigma \tag{23}$$

so that in normalized form we have

$$\frac{I}{I_o} = \frac{4J_o'(K_1\Psi_o)}{\pi\Psi_o^2\sigma Z_1 K}\left[qJ_o'(K_1\Psi_o)H_o^{(2)}(K\Psi_o) - J_o(K_1\Psi_o)H_o^{(2)'}(K\Psi_o)\right]^{-1} \tag{24}$$

In terms of the normalized parameters α, p, and q defined previously we have

$$\frac{4}{\pi\Psi_o^2\sigma Z_1 K} = \frac{4Z}{\Psi_1 Z_1 K} = \frac{4q}{\alpha} \tag{25}$$

so that we have

$$\frac{I}{I_o} = \frac{4q}{\alpha} J_o'(\alpha pq)\left[qJ_o'(\alpha pq)H_o^{(2)}(\alpha p) - J_o(\alpha pq)H_o^{(2)'}(\alpha p)\right]^{-1} \tag{26}$$

In figures 2 and 3 the magnitude and phase of I/I_o are plotted as functions of the normalized frequency α for $\varepsilon_r = 1$ and $\varepsilon_r = 10$ and for several values of the parameter p. Note in figure 2 for $\varepsilon_r = 1$ that based on the magnitude of I/I_o the best frequency response curve has $p \simeq 0.4$. In figure 2 for $\varepsilon_r = 10$ the best case is with $p \simeq 0.15$. For these cases the magnitude of the frequency response is roughly constant to the highest normalized frequency which is roughly $\alpha \simeq 7$.

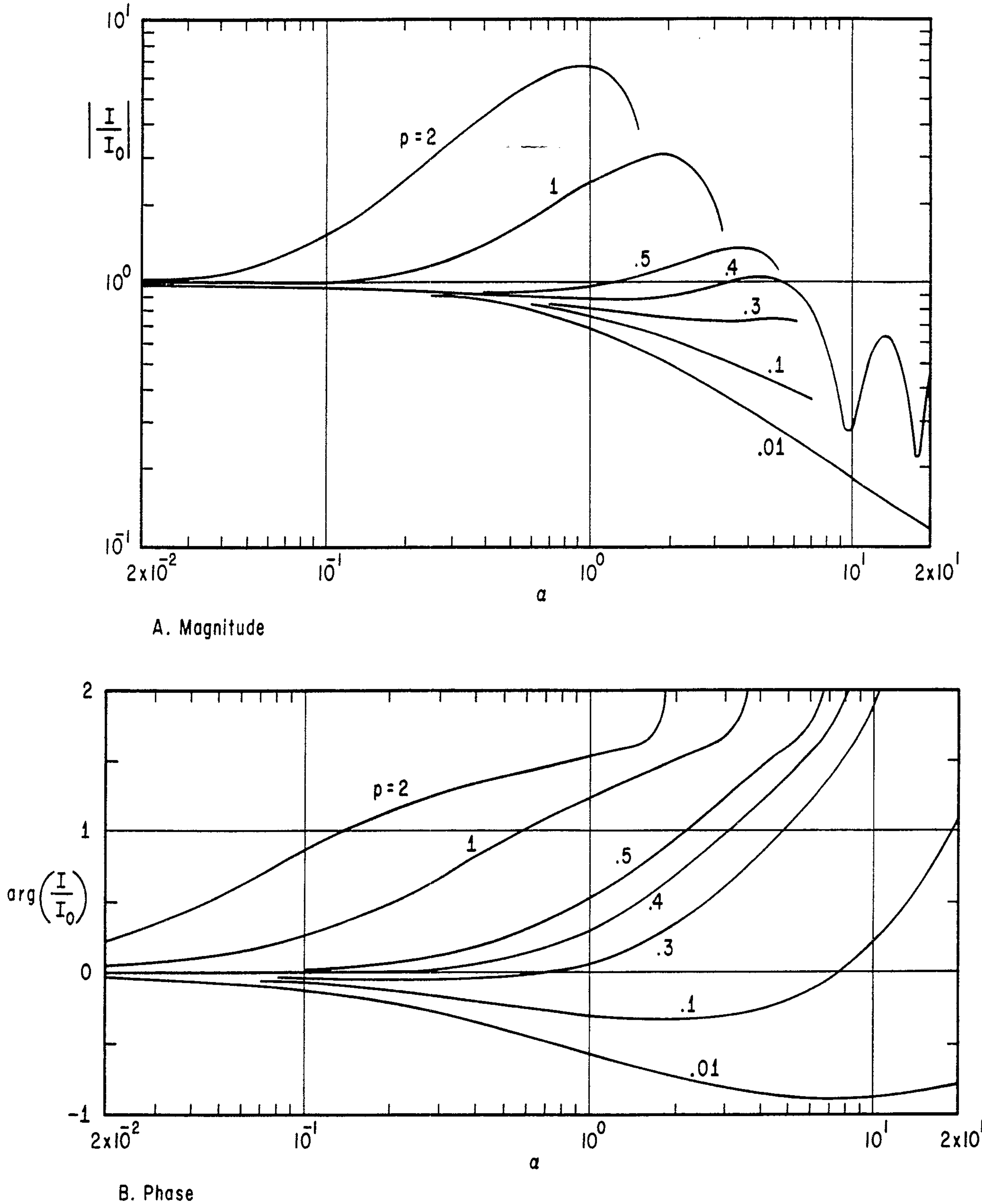


Figure 2. SHORT CIRCUIT CURRENT : $\epsilon_r = 1$

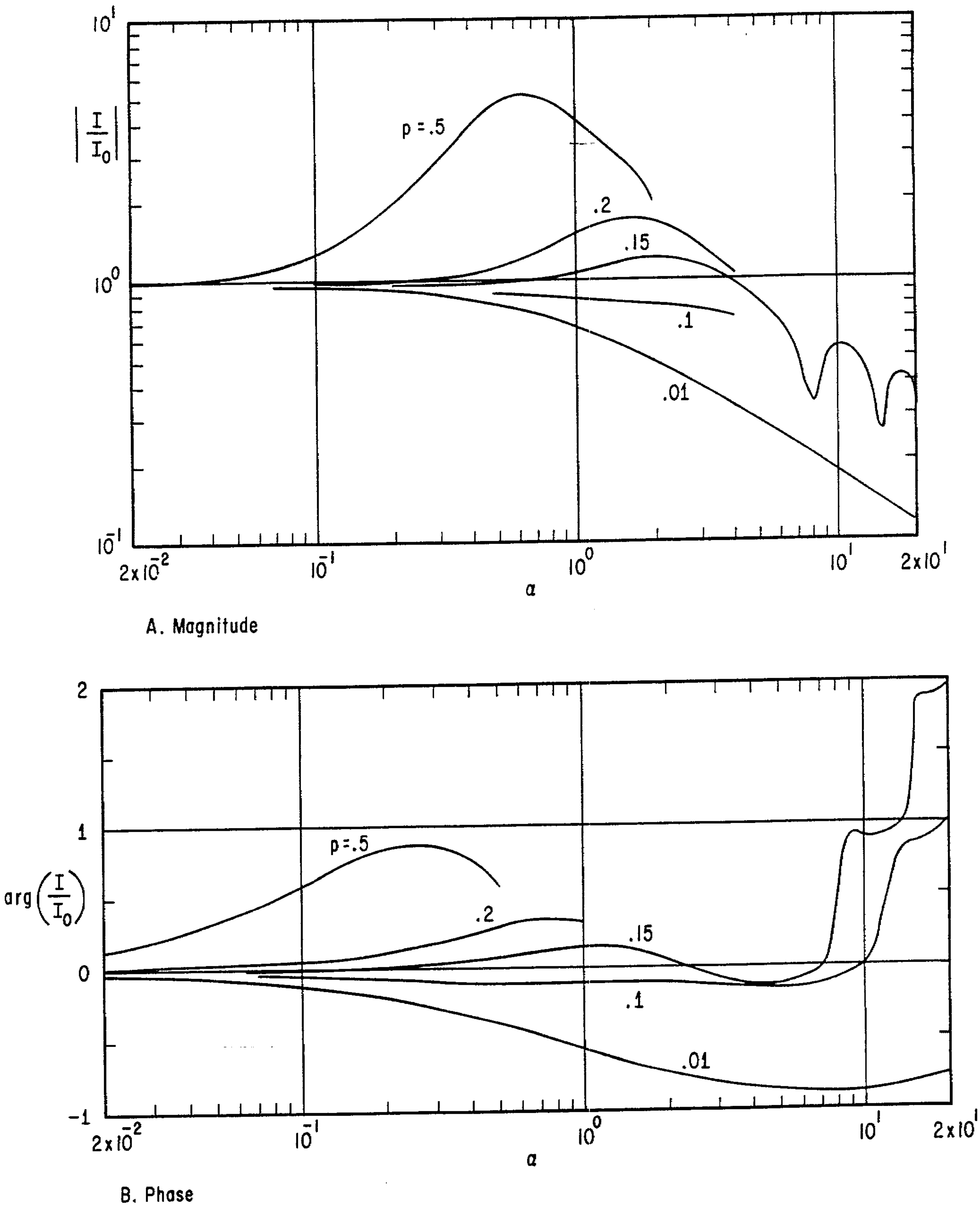


Figure 3. SHORT CIRCUIT CURRENT : ϵ_r = 10

For a particular value of p the resistor radius and conductivity can be found from

$$\Psi_o = p\Psi_1 = pb \frac{\vec{z}}{R_2} \tag{27}$$

and

$$\sigma = \frac{b}{\pi\Psi_o^2 R_2} = \frac{R_2}{\pi p^2 b z^2} \tag{28}$$

Note that as R_2 is increased the optimum Ψ_o is decreased and the corresponding ω for maximum frequency response is increased.

As a rough physical interpretation there are at least two effects which make the frequency response nonflat at high frequencies. There is the displacement current as shown in equation 22. The displacement current tends to increase the current magnitude at high frequencies. For fixed R_2 the effect of the displacement current is decreased by increasing σ and correspondingly decreasing Ψ_o. The second effect is that as the frequency is increased and wavelengths in the resistor become of the order of Ψ_o (in magnitude) the current is not uniform in the resistor and skin depth limitations enter the problem. This decreases the magnitude of the net current flowing through the resistor. The optimum value of p, and thus the optimum value of Ψ_o for fixed b and R_2, is chosen to roughly compensate these two effects to maximize the frequency response.

The Bessel functions of real and complex arguments were calculated using a computer code discussed in a previous note.[3]

III. Error Introduced into the Equivalent Height by Non Zero Thickness of the Disk Electrodes

One of the fundamental parameters of the circular parallel-plate dipole is its equivalent height. In this section we consider the effect of finite thickness of the one or two circular disk electrodes on the equivalent height. This problem is considered from a quasi static or low frequency viewpoint.

3. Richard C. Lindberg, Mathematics Note 1, BESSEL: A Subroutine for the Generation of Bessel Functions with Real or Complex Arguments, October 1966.

In figure 4 we show a cross section view of a circular parallel-plate dipole with spacing 2b between the disks (inside faces) and with each disk of thickness w; the disk radius is a. In another note we have shown that the equivalent height $\vec{h}_{eq}$ is the same as the mean charge separation distance $\vec{h}_a$. Since the two thick disks are both symmetric about the z axis we have only a z component of the equivalent height and can write

$$\vec{h}_{eq} = h_{eq}\vec{e}_z = \vec{h}_a = h_a\vec{e}_z \tag{29}$$

where $\vec{e}_z$ is a unit vector in the z direction and we have defined

$$h_{eq} \equiv |\vec{h}_{eq}| \ , \qquad h_a \equiv |\vec{h}_a| \tag{30}$$

To calculate the equivalent height we calculate the mean charge separation distance using the quasi static or low-frequency charge distribution on the antenna with no incident field and with a positive charge Q on the upper electrode (z = b to z = b + w) and with -Q on the lower electrode.

Let the upper electrode have voltage +V and the lower electrode -V. The capacitance of the sensor is

$$C = \frac{Q}{2V} \tag{31}$$

Using symmetry about the z = 0 plane the equivalent height is calculated from

$$h_{eq} = h_a = \frac{2}{Q}\int_{V_+} \rho z dV_+ \tag{32}$$

where V_+ is the volume of the upper electrode and ρ is the charge density. Our objective is to find approximate lower and upper bounds for h_{eq} which we call h_1 and h_2 respectively so that

$$h_1 \leq h_{eq} \leq h_2 \tag{33}$$

4. Capt Carl E. Baum, Sensor and Simulation Note 69, Design of a Pulse-Radiating Dipole Antenna as Related to High-Frequency and Low-Frequency Limits, January 1969.

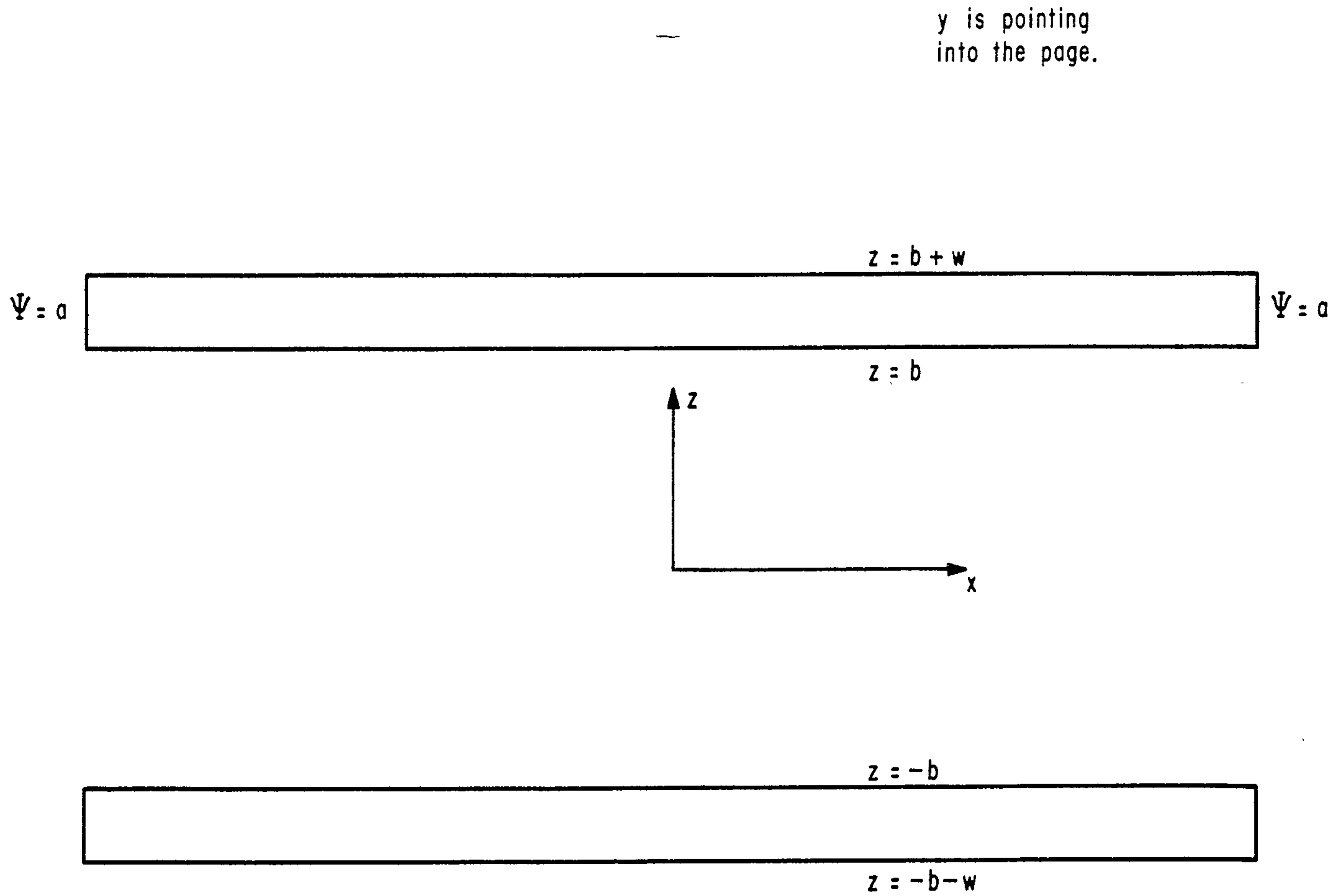


Figure 4. TWO EQUAL THICK COAXIAL CIRCULAR DISKS

Of course we would like $h_2 - h_1$ to be as small as possible to minimize the error in designing such a sensor for a particular h_{eq}.

Clearly a lower bound for h_{eq} is just

$$h_1 = 2b \tag{34}$$

since all the charge is located at $|z| \geq b$. We could also use $2(b + w)$ for an upper limit on h_{eq} but using the capacitance formulas in reference 1 we can get a significantly tighter approximate upper bound. First note that the surface charge density on the inner face $(z = b)$ of the upper electrode has a lower bound as

$$\rho_s \geq \varepsilon \frac{V}{b} \tag{35}$$

where ε is the permittivity. This is because $-V/b$ is the average vertical electric field (average over z) between the electrodes $(|z| \leq b,\ |\Psi| \leq a)$ and the maximum field magnitude for each Ψ occurs on the electrodes. The field enhancement is particularly strong near the edge of the electrodes $(\Psi = a)$. Let Q_1 be the total charge on the inner face $(z = b)$ of the upper electrode. A lower bound for Q_1 is then

$$Q_1 > \varepsilon \frac{V}{b} \pi a^2 \tag{36}$$

This simply says that the charge on the inside of the disk is larger than the charge if the field were uniform because of the fringing fields. Another way to look at this is to look at the electric field lines which terminate on charges on the inside surfaces of the two disks $(z = \pm b)$. These field lines bulge outwards, some extending beyond $\Psi = a$; the average cross section area (at fixed z with $|z| < b$) through which these field lines pass is greater than πa^2. This makes the capacitance associated with the inside surfaces greater than $\varepsilon \pi a^2/(2b)$.

Let Q_2 be defined by

$$Q_2 \equiv Q - Q_1 \tag{37}$$

From equations 31 and 36 we have the inequality

$$Q_2 < Q - \varepsilon \frac{V}{b} \pi a^2 = 2VC - \varepsilon \frac{V}{b} \pi a^2 \tag{38}$$

Now to obtain an upper bound for h_{eq} we have

$$h_{eq} < \frac{2}{Q}[Q_1 b + Q_2 (b + w)] \tag{39}$$

This is greater than h_{eq} because Q_2 is distributed over values of z between b and b + w. The average z associated with Q_2 is then less than b + w.

From equation 36 we can define

$$Q_3 \equiv Q_1 - \varepsilon \frac{V}{b} \pi a^2 > 0 \tag{40}$$

For Q_1 and Q_2 we then have

$$\begin{aligned} Q_1 &= \varepsilon \frac{V}{b} \pi a^2 + Q_3 \\ Q_2 &= Q - \varepsilon \frac{V}{b} \pi a^2 - Q_3 \end{aligned} \tag{41}$$

The inequality in equation 39 can then be written as

$$h_{eq} < \frac{2}{Q}\{[\varepsilon \frac{V}{b} \pi a^2 + Q_3]b + [Q - \varepsilon \frac{V}{b} \pi a^2 - Q_3](b + w)\} \tag{42}$$

or

$$\begin{aligned} h_{eq} &< \frac{2}{Q}\{Qb + [Q - \varepsilon \frac{V}{b} \pi a^2 - Q_3]w\} \\ &= 2b + 2\left[1 - \frac{\varepsilon V}{Qb} \pi a^2 - \frac{Q_3}{Q}\right]w \\ &= 2b + 2\left[1 - \frac{\varepsilon \pi a^2}{2bC} - \frac{Q_3}{Q}\right]w \end{aligned} \tag{43}$$

Since Q_3 is positive we can drop this term while still maintaining the inequality. Thus for our upper bound we take

$$h_2 = 2b + 2w\left[1 - \frac{1}{C}\,\frac{\varepsilon\pi a^2}{2b}\right] \tag{44}$$

Note for large a/b that C approaches $\varepsilon\pi a^2/(2b)$. We now have

$$h_2 - h_1 = 2w\left[1 - \frac{1}{C}\,\frac{\varepsilon\pi a^2}{2b}\right] \tag{45}$$

and for large a/b this is much less than 2w. A convenient way to write the bounds on h_{eq} is as

$$0 \le \frac{h_{eq}}{2b} - 1 \le \frac{w}{b}\left[1 - \frac{1}{C}\,\frac{\varepsilon\pi a^2}{2b}\right] \equiv B \tag{46}$$

B can be considered a relative error bound in calculating h_{eq}.

To get an estimate for B and h_2 we can use an approximate capacitance formula discussed in reference 1. This is the Kirchoff approximation which includes the non zero disk thickness. This approximation is

$$C \simeq \varepsilon\left\{\frac{\pi a^2}{2b} + a\{\ln[\frac{8\pi a}{b}(1 + \frac{w}{2b})] - 1 + \frac{2\pi w}{b}\ln[1 + \frac{2b}{w}]\}\right\} \tag{47}$$

Then we have

$$B \simeq \frac{w}{b}\,\frac{\ln[\frac{8\pi a}{b}(1 + \frac{w}{2b})] - 1 + \frac{2\pi w}{b}\ln[1 + \frac{2b}{w}]}{\frac{\pi a}{2b} + \ln[\frac{8\pi a}{b}(1 + \frac{w}{2b})] - 1 + \frac{2\pi w}{b}\ln[1 + \frac{2b}{w}]} \tag{48}$$

For small w/b and for large a/b this is roughly

$$B \simeq \frac{2w}{\pi a}[\ln(\frac{8\pi a}{b}) - 1] \tag{49}$$

IV. Error Introduced into the Capacitance and Equivalent Height by Dielectric Spacers

Assume that dielectric spacers are used to hold the two disks or disk plus large ground plane at a fixed separation, i.e. the spacers establish b. Let these spacers have a dielectric constant ε_s and let them have shapes which are independent of z between the plates ($|z| < b$, $|\Psi| < a$). Let the total cross section area of these spacers at each fixed z (with $|z| < b$) equal A_s. The electric field between the plates is approximately a uniform E_z for large a/b if positions near the disk edges are not considered. The increase in capacitance due to these spacers is then

$$C_s \simeq (\varepsilon_s - \varepsilon) \frac{A_s}{2b} \tag{50}$$

The relative increase in the capacitance is defined by

$$f_s \equiv \frac{C_s}{C} \tag{51}$$

If we approximate C by $\varepsilon\pi a^2/(2b)$ for large a/b then this fractional increase is roughly

$$f_s \simeq \left[\frac{\varepsilon_s}{\varepsilon} - 1\right]\frac{A_s}{\pi a^2} \tag{52}$$

The use of dielectric spacers also affects the equivalent height of the sensor. The mean charge separation distance, as discussed in reference 4, is fundamentally the dipole moment of the sensor divided by the amount of charge transferred between the sensor terminals. This charge is significant here because as charge is transferred from one disk to the other there is a polarization current in the opposite direction in the dielectric spacers. The dipole moment has only a z component given by

$$p_z = 2b(Q - Q_s) \tag{53}$$

where the charge transferred between the electrodes is

$$Q = 2V(C + C_s) \tag{54}$$

and the charge associated with the dielectric polarization is

$$Q_s = 2VC_s \tag{55}$$

Note for these calculations the two disks are assumed to have zero thickness and to be at potentials ±V. The dipole moment is then

$$p_z = 4bVC \tag{56}$$

and the equivalent height is

$$h_{eq} = h_a = \frac{p_z}{Q} = 2b \frac{C}{C + C_s}$$

$$= 2b\left[1 + \frac{C_s}{C}\right]^{-1}$$

$$= 2b[1 + f_s]^{-1} \tag{57}$$

For small f_s we have

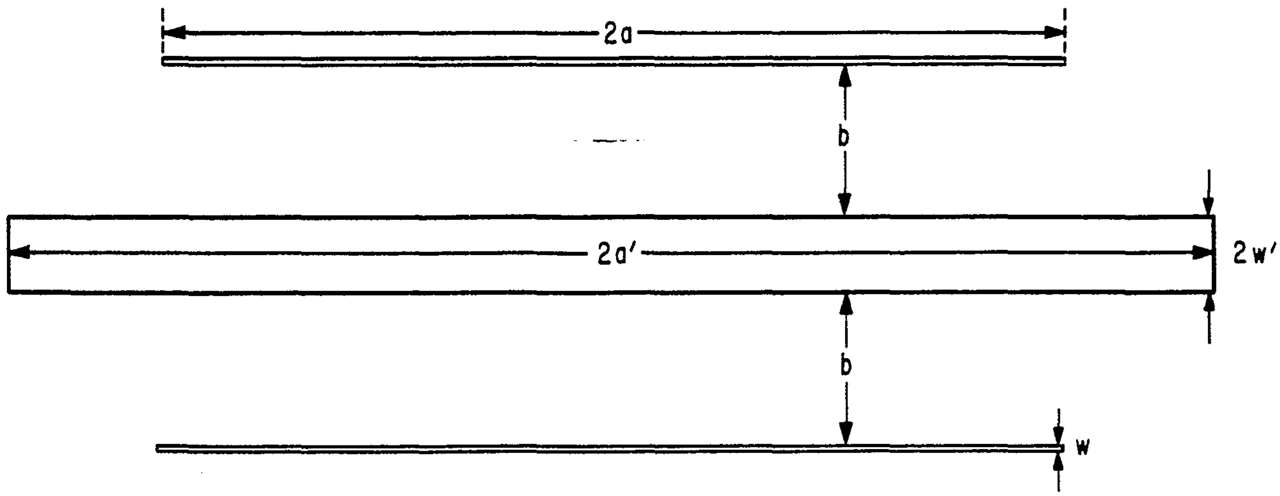
$$h_{eq} \simeq 2b[1 - f_s] \tag{58}$$

The decrease in the equivalent height is then about $2bf_s$ and f_s is the approximate fractional decrease of the equivalent height.

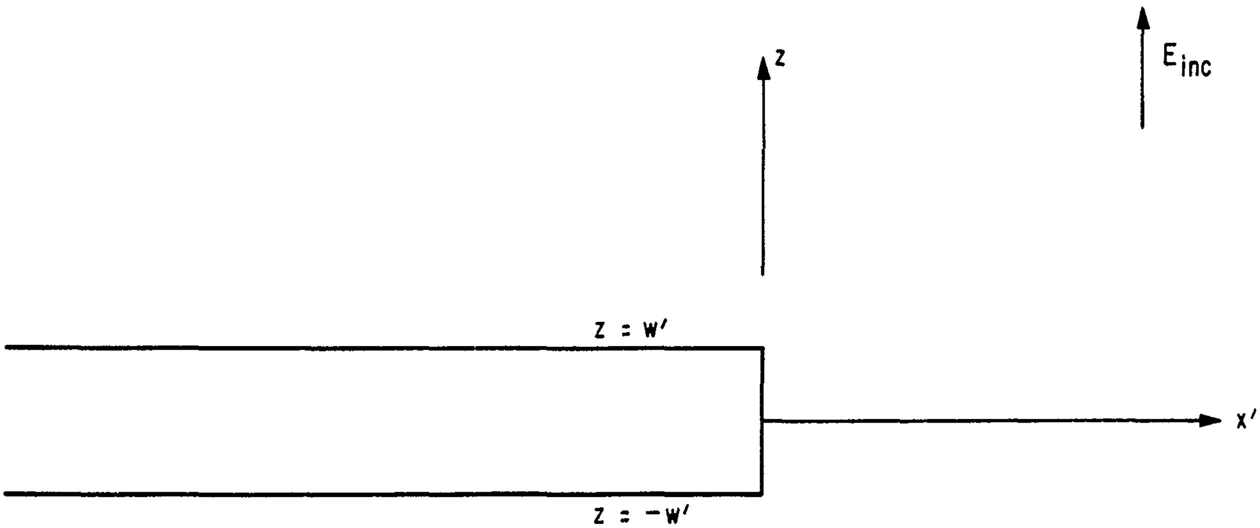
The decrease in the equivalent height is then attributable to the dipole moment induced in the dielectric spacers which subtracts from the dipole moment due to the charge transferred between the disks. The z component of the dipole moment in the dielectric is just the dielectric volume $2bA_s$ times the z component of the polarization vector $(\varepsilon_s - \varepsilon)E_z$ in the dielectric or $(\varepsilon_s - \varepsilon)2bA_sE_z$; this is just $-2bQ_s$.

V. Electric Field Distortion Near the Edge of a Thick Ground Plane with a Square Edge

Another source of error in a parallel-plate dipole is the non zero thickness of a ground plane. Such a ground plane could be a conducting plate of thickness 2w' centered between the two disks of a differential sensor or it could be a conducting plate of thickness w' which is placed on a larger conducting ground plane as part of a single ended sensor. Figure 5 illustrates this situation. In figure 5A there is illustrated a cross section of a differential sensor with such a central ground plane which might contain cables to conduct the signal to the periphery of the sensor. Figure 5B shows a region near the edge of the ground plane with a square edge.



A. Ground plane with circular disks



B. Edge of ground plane with coordinates as a two-dimensional problem

Figure 5. GROUND PLANE WITH SQUARE EDGE

In this section we consider the distortion of the incident electric field near the edge of such a ground plane at low frequencies so that we can use a quasi static approach. The ground plane is considered as a circular disk of non zero thickness 2w' and radius a' with a' >> w'. Near the edge of this disk we approximate the geometry as two dimensional by neglecting the curvature of the edge. As shown in figure 5B we set up a coordinate system based on the edge. Only two coordinates x', z are of interest with x' = 0 on the edge and z = 0 centered halfway in the disk; z = w' is the top surface of the disk. The boundary conditions are that the electric potential be a constant (which we take to be zero) on the disk surface and on the positive x' axis. We are of course considering the case where the static incident electric field is uniform and can be written as

$$\vec{E}_{inc} = E_o \vec{e}_z \tag{59}$$

We only consider the region $z \geq 0$ for $x \geq 0$ and $z \geq w'$ for $x < 0$. The sensor electrodes are assumed absent for these calculations.

Define a complex variable as

$$\chi \equiv x' + jz \tag{60}$$

An appropriate conformal transformation for this geometry is[5]

$$\chi = \frac{w'}{\pi}[(\psi^2 - 1)^{1/2} + \operatorname{arccosh}(\psi)] \tag{61}$$

where ψ is another complex variable which we write as

$$\psi \equiv u + jv \tag{62}$$

The electric potential function is a constant times v since for large $|\psi|$ we have the asymptotic relation

$$\chi = \frac{w'}{\pi} \psi + O(\ln(\psi)) \tag{63}$$

The electric potential function is then $-w'E_o v/\pi$.

5. R. V. Churchill, Complex Variables and Applications, 2nd ed., McGraw Hill, 1960, p. 291.

Along the top surface of the disk ($z = w'$, $x' < 0$) we have

$$x' = \frac{w'}{\pi}\left[-(u^2 - 1)^{1/2} + \text{arccosh}(-u)\right] \tag{64}$$

while $v = 0$ and $u < -1$. The electric field just above this surface has only a z component and can be calculated as

$$E_z = -\frac{w'E_o}{\pi}\left(\frac{\partial x'}{\partial u}\right)^{-1} = -E_o\left[-(u^2 - 1)^{-1/2} u - (u^2 - 1)^{-1/2}\right]^{-1}$$

$$= E_o\left[\frac{u - 1}{u + 1}\right]^{1/2} \tag{65}$$

Now for $u << -1$ so that $x' << -w'$ we have

$$x' \simeq \frac{w'}{\pi} u \tag{66}$$

giving

$$E_z \simeq E_o\left[\frac{1 - \frac{w'}{\pi x'}}{1 + \frac{w'}{\pi x'}}\right]^{1/2} \simeq E_o[1 - \frac{w'}{\pi x'}] \tag{67}$$

The relative field enhancement due to the ground plane thickness is about $w'/(\pi|x'|)$. This gives one some idea about the distortion included in the field at the sensor electrodes at some distance $|x'|$ from the edge of the ground plane. Define this relative field enhancement (for $x' < 0$) as Ξ_1 which for $|x'| >> w'$ (but still small compared to a') is approximately given by

$$\Xi_1 \simeq -\frac{w'}{\pi x'} = \frac{w'}{\pi|x'|} \tag{68}$$

VI. Electric Field Distortion for a Thick Ground Plane Approximated as an Oblate Spheroid

We turn now from the field distortion near the edge of the ground plane to the overall effect of the ground plane thickness on the sensor accuracy. Consider the static electric field distribution around the ground plane without the presence of the disk electrodes. The ground plane of radius a' and thickness 2w' excludes the electric field from its volume and thereby enhances the electric field at some positions in the vicinity of the ground plane. Referring to figure 5A the sensor disk electrodes are each spaced from the nearest surface of the ground plane by the distance b. The field enhancement due to the ground plane then increases the equivalent height to something greater than 2b (in the differential case). Thus we consider the enhancement of the incident electric field by the ground plane as an estimate of the error introduced into the calculation of the equivalent height (i.e. 2b). For this calculation we approximate the conducting ground plane as an oblate spheroid.

In figure 6 oblate spheroidal coordinates are illustrated for some cross section on a plane containing the z and x axes. This set of coordinates, for which we use ξ, ζ, ϕ, is defined by[6]

$$
\begin{aligned}
x &\equiv r_o \cosh(\xi)\,\sin(\zeta)\,\cos(\phi) = \Psi\cos(\phi) \\
y &\equiv r_o \cosh(\xi)\,\sin(\zeta)\,\sin(\phi) = \Psi\sin(\phi) \qquad (69) \\
z &\equiv r_o \sinh(\xi)\,\cos(\zeta)
\end{aligned}
$$

where Ψ, ϕ, z is our cylindrical coordinate system as before and r_o is just a convenient constant with dimension meters. We also have

$$\Psi = r_o \cosh(\xi)\,\sin(\zeta) \qquad (70)$$

The scale factors for this orthogonal curvilinear coordinate system are

$$
\begin{aligned}
h_\xi &= h_\zeta = r_o[\cosh^2(\xi) - \sin^2(\zeta)]^{1/2} \qquad (71) \\
h_\phi &= r_o \cosh(\xi)\,\sin(\zeta) = \Psi
\end{aligned}
$$

The general solution of the Laplace equation

6. Moon and Spencer, Field Theory for Engineers, D. Van Nostrand, 1961, chapter 10.

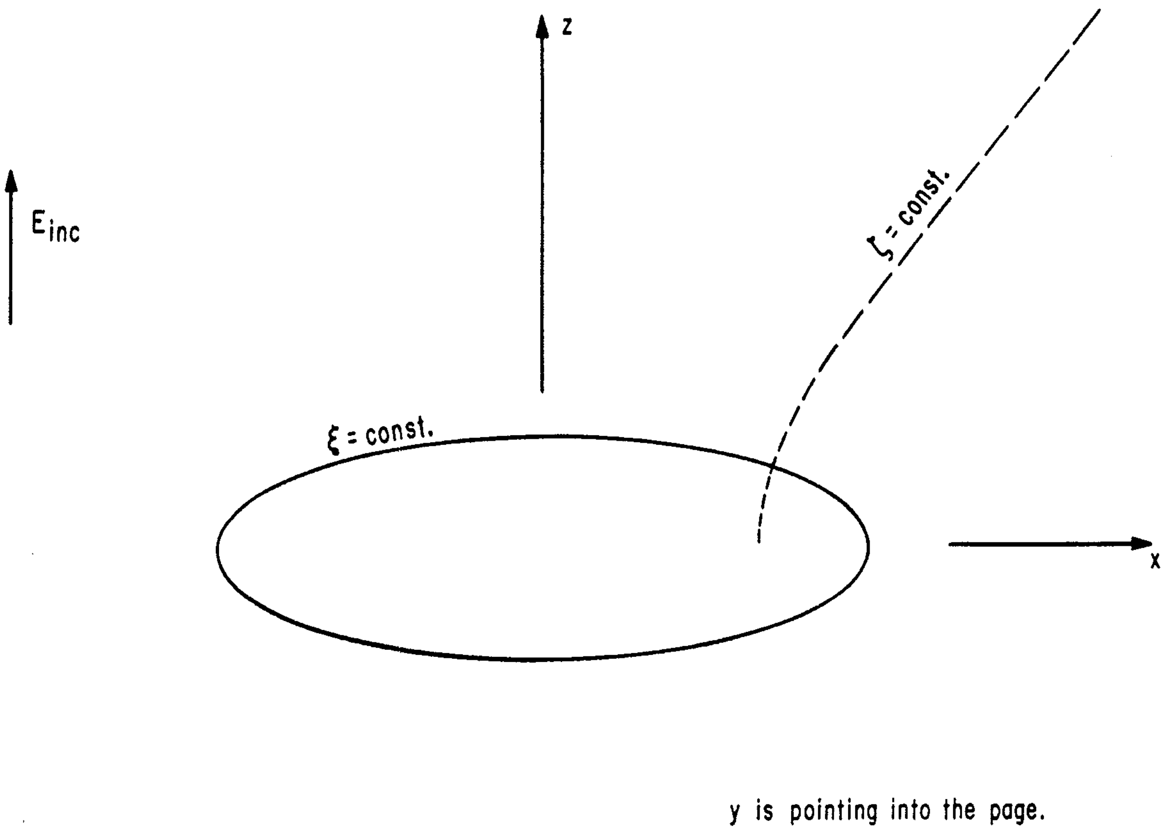


Figure 6. GROUND PLANE APPROXIMATED AS AN OBLATE SPHEROID: CROSS SECTION VIEW

$$\nabla^2\Phi = 0 \quad (72)$$

is of the form

$$\Phi = \Phi_o \sum_{n,m} \beta_{n,m} \begin{Bmatrix} P_n^m(i \sinh(\xi)) \\ Q_n^m(i \sinh(\xi)) \end{Bmatrix} \begin{Bmatrix} P_n^m(\cos(\zeta)) \\ Q_n^m(\cos(\zeta)) \end{Bmatrix} \begin{Bmatrix} \cos(m\phi) \\ \sin(m\phi) \end{Bmatrix} \quad (73)$$

where two terms in braces implies that some linear combination is used. Note that in general n and m need not be integers, but in the present case they will be integers. P and Q are Legendre functions of the first and second kind respectively and we follow the definitions for general complex arguments and for the cut in the complex plane (real axis between -1 and 1) as defined in a standard reference work.[7] The problem discussed here is also found in reference 6.

Assume an incident potential of the form

$$\Phi_{inc} = -E_o z = -E_o r_o \sinh(\xi) \cos(\zeta) \quad (74)$$

The Legendre functions of interest are

$$P_1^o(\cos(\zeta)) = \cos(\zeta) \ , \qquad P_1^o(i \sinh(\xi)) = i \sinh(\xi)$$

$$\begin{aligned} Q_1^o(i \sinh(\xi)) &= \frac{i \sinh(\xi)}{2} \ln\left[\frac{i \sinh(\xi) + 1}{i \sinh(\xi) - 1}\right] - 1 \\ &= \sinh(\xi) \operatorname{arccot}[\sinh(\xi)] - 1 \\ &= \sinh(\xi) \arctan\left[\frac{1}{\sinh(\xi)}\right] - 1 \end{aligned} \quad (75)$$

The incident potential is then

$$\Phi_{inc} = E_o r_o i P_1^o(i \sinh(\xi)) P_1^o(\cos(\zeta)) \quad (76)$$

7. Abramowitz and Stegun, ed., Handbook of Mathematic Functions, AMS 55, National Bureau of Standards, 1964, chapter 8.

Referring to figure 6 note that surfaces of constant ξ are oblate ellipsoids of the form

$$\frac{\Psi^2}{r_o^2 \cosh^2(\xi)} + \frac{z^2}{r_o^2 \sinh^2(\xi)} = 1 \tag{77}$$

Place a perfectly conducting surface on $\xi = \xi_o$. We use this spheroid as a rough approximation to the ground plane geometry. This oblate spheroid has a radius or maximum extent in Ψ of

$$r_1 = r_o \cosh(\xi_o) \tag{78}$$

and a half thickness or maximum extent in z of

$$r_2 = r_o \sinh(\xi_o) \tag{79}$$

In using this conducting ellipsoid to roughly approximate a ground plane one might roughly use $r_1 \simeq a'$ and $r_2 \simeq w'$ where a' and w' are the radius and half thickness respectively of the conducting disk as in figure 5. We are interested in thin ground planes so that $r_2/r_1 << 1$ and

$$\frac{r_2}{r_1} = \tanh(\xi_o) \simeq \xi_o \tag{80}$$

With Φ_{inc} as in equation 76 we can make $\Phi = 0$ on $\xi = \xi_o$ by using only two terms from the expansion in equation 73 as

$$\Phi = -E_o r_o [-iP_1^o(i \sinh(\xi)) + \beta Q_1^o(i \sinh(\xi))] P_1^o(\cos(\zeta)) \tag{81}$$

Note from the last of equations 75 that as $\xi \to \infty$

$$Q_1^o(i \sinh(\xi)) = O((\sinh(\xi))^{-2}) \tag{82}$$

so that $\Phi \to \Phi_{inc}$ for large ξ. Setting $\Phi = 0$ on $\xi = \xi_o$ gives

$$\beta = i \frac{P_1^o(i \sinh(\xi_o))}{Q_1^o(i \sinh(\xi_o))}$$

$$= - \frac{\sinh(\xi_o)}{\sinh(\xi_o) \operatorname{arccot}[\sinh(\xi_o)] - 1} \qquad (83)$$

The potential is then

$$\Phi = -E_o r_o \left\{ \sinh(\xi) - \frac{\sinh(\xi_o)[\sinh(\xi)\operatorname{arccot}[\sinh(\xi)] - 1]}{\sinh(\xi_o)\operatorname{arccot}[\sinh(\xi_o)] - 1} \right\} \cos(\zeta)$$

$$= -E_o z \left\{ 1 - \frac{\sinh(\xi_o)}{\sinh(\xi)} \frac{\sinh(\xi)\operatorname{arccot}[\sinh(\xi)] - 1}{\sinh(\xi_o)\operatorname{arccot}[\sinh(\xi_o)] - 1} \right\} \qquad (84)$$

The electric field has two components and is given by

$$\vec{E} = - \frac{\vec{e}_\xi}{h_\xi} \frac{\partial \Phi}{\partial \xi} - \frac{\vec{e}_\zeta}{h_\zeta} \frac{\partial \Phi}{\partial \zeta} \qquad (85)$$

The maximum electric field magnitude occurs on $\xi = \xi_o$ where it has only a ξ component given by

$$E_\xi \Big|_{\xi=\xi_o} = \frac{E_o \cos(\zeta)}{[\cosh^2(\xi_o) - \sin^2(\zeta)]^{1/2}} \left\{ \cosh(\xi_o) \right.$$

$$- \frac{\sinh(\xi_o)}{\sinh(\xi_o)\operatorname{arccot}[\sinh(\xi_o)] - 1} [\cosh(\xi_o)\operatorname{arccot}[\sinh(\xi_o)]$$

$$\left. - \tanh(\xi_o)] \right\}$$

$$= \frac{E_o \cosh(\xi_o)\cos(\zeta)}{[\cosh^2(\xi_o)-\sin^2(\zeta)]^{1/2}} \frac{1-\tanh^2(\xi_o)}{1-\sinh(\xi_o)\operatorname{arccot}[\sinh(\xi_o)]}$$

$$= E_o\left[1 + \frac{\sinh^2(\xi_o)}{\cos^2(\zeta)}\right]^{-1/2} \cosh(\xi_o)\frac{1-\tanh^2(\xi_o)}{1-\sinh(\xi_o)\operatorname{arccot}[\sinh(\xi_o)]}$$

$$= \frac{E_o}{\cosh(\xi_o)}\left[1 + \frac{\sinh^2(\xi_o)}{\cos^2(\zeta)}\right]^{-1/2} \{1-\sinh(\xi_o)\operatorname{arccot}[\sinh(\xi_o)]\}^{-1} \tag{86}$$

This has its maximum value at $\zeta = 0$ (the z axis) where we have

$$E_{max} = \frac{E_o}{\cosh^2(\xi_o)}\{1 - \sinh(\xi_o)\operatorname{arccot}[\sinh(\xi_o)]\}^{-1} \tag{87}$$

For small ξ_o we have

$$E_{max} = \frac{E_o}{\cosh^2(\xi_o)}\{1 - \sinh(\xi_o)[\frac{\pi}{2} - \sinh(\xi_o) + O(\xi_o^3)]\}^{-1}$$

$$= E_o\left\{1 - \frac{\pi}{2}\frac{\sinh(\xi_o)}{\cosh^2(\xi_o)} + O(\xi_o^4)\right\}^{-1}$$

$$= E_o\{1 - \frac{\pi}{2}\xi_o + O(\xi_o^3)\}^{-1} \tag{88}$$

or as $\xi_o \to 0$

$$E_{max} = E_o\{1 + \frac{\pi}{2}\xi_o\} + O(\xi_o^2) \tag{89}$$

Call the relative increase in the maximum field Ξ_2; this is then for small ξ_o roughly

$$\Xi_2 \simeq \frac{\pi}{2} \xi_o \simeq \frac{\pi}{2} \frac{r_2}{r_1} \tag{90}$$

From this one can estimate how thin the ground plane should be for a given radius in order to hold the sensor error to some desired small number by using the rough approximations $w' \simeq r_2$ and $a' \simeq r_1$.

VII. Summary

In this note we have found some optimum conditions for the design of a resistive dielectric rod to maximize the frequency response of the short circuit current from the rod used as an output resistor.

Most of the note deals with various effects which detract from the sensor accuracy. In some cases error bounds are found and in others approximations to the relative error are found. These effects include thickness of the sensor electrodes and ground plane and use of dielectric spacers. All these error calculations are based on solutions of the Laplace equation and thus only apply for frequencies such that wavelengths are significantly larger than sensor dimensions.

There are of course numerous sources for small errors to enter the basic parameters of a circular parallel-plate dipole and other types of electromagnetic sensors as well. Perhaps some future notes can further treat such error problems.

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