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Electrically-Small Cylindrical Loops for Measuring the Magnetic Field Perpendicular to the Cylinder Axis

Capt Carl E. Baum
Air Force Weapons Laboratory

Abstract

This note considers some loop designs for measuring the magnetic field perpendicular to the axis of a circular cylinder which contains the sensor. The length of the cylinder is assumed much larger than its radius and the sensor geometry is assumed constant along the cylinder axis so that two-dimensional approximations can be used to calculate the electrical parameters of the sensor. We first consider a surface current density on the surface of the cylinder and find the maximum figure of merit such a sheet current can have. Then we consider various distributions of discrete sensor conductors which approximate such a current sheet and calculate the resulting electrical parameters.

I. Introduction

In designing loops to measure the magnetic field one needs to consider the volume in which the sensor must fit. One common shape for this volume is a circular cylinder. As one example consider a circular hole drilled into the earth into which one wishes to place a sensor to measure the magnetic field at various distances into the hole. As a second example consider an instrumentation rocket on which one wishes to mount some loops within a circular cylinder of about the same radius as the rocket body. In both of these cases one would like to design the sensors to utilize the available geometric volume as efficiently as possible in terms of the parameters of sensor performance.

For the present calculations we assume that our sensor must fit inside some circular cylinder of radius a and length ℓ as illustrated in figure 1. In the present note we consider sensor designs for measuring the magnetic field perpendicular to the axis of the cylinder. In order to simplify the problem somewhat we assume $\ell \gg a$ so that we can treat the problem as a two-dimensional one with fields, currents, etc., independent of z. Frequencies of interest are assumed low enough that the sensors can be considered electrically small and analyzed with quasi-static approximations.

In previous notes[1,2] we have considered the parameters of electrically-small loops. In this note we apply the concepts of equivalent volume and figure of merit to the problem of optimizing the distribution of conductors on a cylindrical surface. We first consider the optimum surface current distribution for measuring a component of the magnetic field perpendicular to the cylinder axis. This distribution is then approximated by discrete currents of various magnitudes on circular wires in various positions adjacent to the surface of the circular cylinder.

For the cylindrical loop designs considered the equivalent area and inductance are calculated. From these the equivalent volume is calculated. The circular cylinder of radius a and length ℓ in which the sensor is contained has a geometric volume given by

$$V_g = \pi a^2 \ell \tag{1}$$

1. Capt Carl E. Baum, Sensor and Simulation Note 38, Parameters for Some Electrically-Small Electromagnetic Sensors, March 1967.

2. Capt Carl E. Baum, Sensor and Simulation Note 74, Parameters for Electrically-Small Loops and Dipoles Expressed in Terms of Current and Charge Distributions, January 1969.

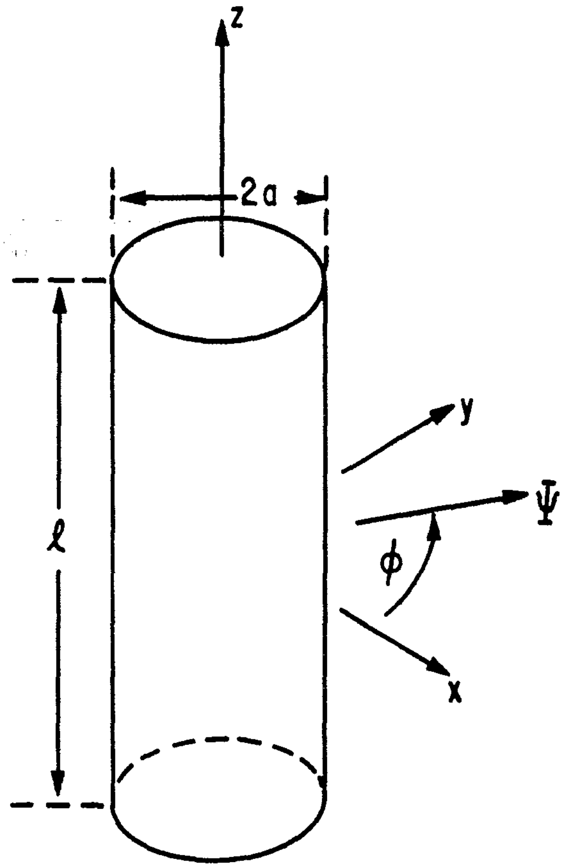


FIGURE I. GEOMETRY OF SENSOR VOLUME

The ratio of the equivalent volume to this geometric volume defines the figure of merit used in this note.

II. Optimum Distribution of Loop Current on the Cylindrical Surface

Now assume some static current density $\vec{J}$ in the cylindrical volume defined by $\Psi \leq a$ and assume that $\vec{J}$ is independent of z so that the problem is a two-dimensional one. Assume that the medium both inside and outside the cylinder has the same permeability μ. Further assume that the equivalent area $\vec{A}_{eq}$ is parallel to $\vec{e}_x$, the unit vector in the x direction, so that the loop is sensitive to only the x component of the magnetic field. In reference 2 (equation 23) we show that the figure of merit of a loop can be written using integrals over the current density $\vec{J}$ as

$$\eta = \frac{\pi}{V_g} \frac{\left\{\int_{V_g} \vec{r}\,' \times \vec{J}(\vec{r}\,')dV'\right\} \cdot \left\{\int_{V_g} \vec{r}\,' \times \vec{J}(\vec{r}\,')dV'\right\}}{\int_{V_g}\int_{V_g} \frac{\vec{J}(\vec{r}\,') \cdot \vec{J}(\vec{r})}{|\vec{r} - \vec{r}\,'|} dV'dV} \tag{2}$$

where $\vec{r}\,'$ and $\vec{r}$ are position vectors with corresponding volume integrals denoted by dV' and dV respectively. The integrals in the numerator come from the equivalent area given by

$$\vec{A}_{eq} = \frac{1}{2I} \int_{V_g} \vec{r}\,' \times \vec{J}(\vec{r}\,')dV' \tag{3}$$

which we constrain to have only an x component. The current at the sensor terminals is denoted by I. Note that with $\vec{J}$ independent of z certain of the volume integrals in equations 2 and 3 must be limited in z by $-\ell/2 \leq z \leq \ell/2$ so that the integrals can be finite. Certain of the parameters such as the equivalent area can also be considered on a per-unit-length basis. Also currents have to be included at $z = \pm\ell/2$ in order to close the loop turns at the end of the sensor and maintain a divergenceless current density.

Now we have assumed that $\vec{J}$ is independent of z to give a two-dimensional problem. The divergence equation for the current density becomes

$$\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} = 0 \tag{4}$$

Thus we can set $J_x = J_y = 0$ and satisfy the requirement of a divergenceless $\vec{J}$. Now since J_x and J_y satisfy equation 4 then J_x and J_y are solenoidal in a plane perpendicular to the z axis, and since currents are confined to $\Psi \leq a$ then such a current distribution corresponds to loop turns with equivalent areas (or magnetic dipole moments) with only z components. Thus only J_z contributes to an $\vec{A}_{eq}$ with only an x component. J_x and J_y are not needed. Note that this argument does not include the ends of the sensor where J_z must stop.

The sensor inductance gives the double volume integral in the denominator on the right side of equation 2. The inductance is given by

$$L = \frac{\mu}{4\pi I^2} \int_{V_g} \int_{V_\infty} \frac{\vec{J}(\vec{r}\,') \cdot \vec{J}(\vec{r})}{|\vec{r} - \vec{r}\,'|} \, dV' dV$$

$$= L_x + L_y + L_z \tag{5}$$

where

$$L_x \equiv \frac{\mu}{4\pi I^2} \int_{V_g} \int_{V_\infty} \frac{J_x(\vec{r}\,') J_x(\vec{r})}{|\vec{r} - \vec{r}\,'|} \, dV' dV \tag{6}$$

and similarly for L_y and L_z. Note that one of the volume integrals is taken over the volume $\Psi < a$ (called V_∞) to give the two-dimensional vector potential, independent of z, assuming the currents independent of z. The other volume integral is only over V_g to give a finite inductance for the portion of the current distribution limited to $-\ell/2 \leq z \leq \ell/2$. Note that $L_x + L_y$ is the inductance of the current distribution with components J_x and J_y satisfying equation 4. The magnetic energy $(1/2)(L_x + L_y)I^2$ associated with this current distribution is strictly positive unless $J_x = J_y = 0$, in which case this energy is zero. Thus we minimize L for a fixed J_z distribution by setting $J_x = J_y = 0$.

Since J_x and J_y do not contribute to A_{eq} and can only add to the inductance we set $J_x = J_y = 0$ because these two current density components can then only decrease the figure of merit η. Of course, at the ends of a real sensor at $z = \pm\ell/2$ there must be currents with other than a z component in order to close the loop turns. However, for the present calculations we ignore these end effects and only consider the two-dimensional problem.

We now turn our attention to a particular form of J_z. We consider a surface current density $\vec{J}_s$ on the surface $\Psi = a$. There is only the one component J_{s_z} and this component is independent of z. Make a Fourier expansion of J_{s_z} in ϕ as

$$J_{s_z} = J_{s_o}\left\{b_o + \sum_{k=1}^{\infty} [a_k \sin (k\phi) + b_k \cos (k\phi)]\right\} \tag{7}$$

where J_{s_o} is some convenient constant with units amperes/meter. For reference consider figure 2 for the coordinates for this two-dimensional model. We want no net current in the z direction so that this current distribution can be constructed from closed loop turns (which connect at $z = \pm\ell/2$). Thus we require

$$\int_o^{2\pi} J_{s_z} a d\phi = 0 \tag{8}$$

which requires

$$b_o = 0 \tag{9}$$

Note that since J_{s_z} is a real function of ϕ then all the a_k and b_k are real.

Now consider the equivalent area of the loop. We can use equation 3 provided we include currents on the surfaces $z = \pm\ell/2$ so as to close the loop turns at the ends of the sensor. (In the formulation of equation 3 it turns out that the contribution from the currents on $z = \pm\ell/2$ equals that from the currents on $\Psi = a$.) As an alternate and simpler approach for our special case first define a surface turns density on $\Psi = a$ as

$$n_s(\phi) \equiv \frac{1}{I} J_{s_z}(\phi) \tag{10}$$

Each complete loop turn has two "wires" on the surface $\Psi = a$, each wire at some particular ϕ. Since we are interested in the x component of the equivalent area we can consider the contribution from each loop turn as ℓ times the difference in the values of y for each wire. For each wire we then have a contribution of ℓy to the x component of the equivalent area. Integrating over the surface turns density then gives

$$A_{eq_x} = \ell \int_o^{2\pi} n_s(\phi) y(\phi) a d\phi = a^2 \ell \int_o^{2\pi} n_s(\phi) \sin (\phi) d\phi$$

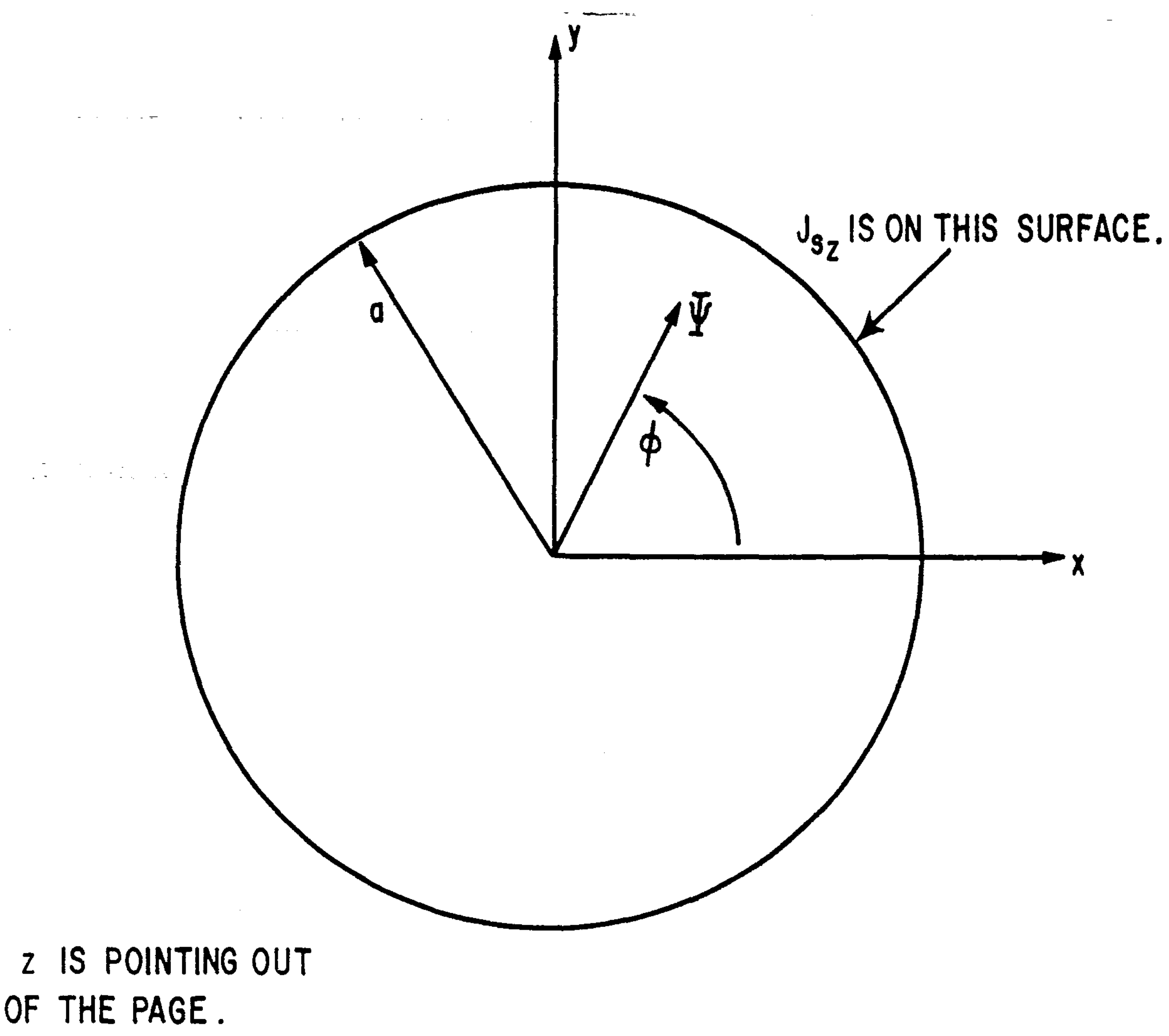


FIGURE 2. GEOMETRY OF TWO — DIMENSIONAL CURRENT SHEET

$$= \frac{a^2 \ell}{I} \int_0^{2\pi} J_{s_z}(\phi) \sin(\phi) d\phi \tag{11}$$

Substitute for J_{s_z} in this equation its Fourier expansion from equation 7. Due to the orthogonality of the trigonometric functions over the interval $0 \le \phi \le 2\pi$ the only term in the expansion which contributes to the integral is the term containing $\sin(\phi)$. Thus we have

$$A_{eq_x} = \frac{\pi a^2 \ell}{I} J_{s_o} a_1 \tag{12}$$

Note that a_1 is the only coefficient which contributes to A_{eq_x}. Similarly for the y component of $\vec{A}_{eq}$ we have

$$A_{eq_y} = \ell \int_0^{2\pi} n_s(\phi)[-x(\phi)] a d\phi = \frac{-a^2 \ell}{I} \int_0^{2\pi} J_{s_z} \cos(\phi) d\phi$$

$$= \frac{-\pi a^2 \ell}{I} J_{s_o} b_1 \tag{13}$$

Since we constrain $\vec{A}_{eq}$ to have only an x component we then must have

$$b_1 = 0 \tag{14}$$

In order to calculate the loop inductance first consider the vector potential. One way to calculate the vector potential is to use

$$\vec{A}(\vec{r}) = \frac{\mu}{4\pi} \int_{S_\infty} \frac{\vec{J}_s(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} dS' \tag{15}$$

where the surface integral is over the surface S_∞ defined by $\Psi = a$. In the present note we calculate the vector potential by a solution of the Laplace equation

$$\nabla^2 \vec{A} = -\mu \vec{J} \tag{16}$$

which for our single z component of the current density reduces to

$$\nabla^2 A_z = -\mu J_z \tag{17}$$

Making A_z continuous at $\Psi = a$ we have an expansion for $\Psi \leq a$ given by

$$A_z = \sum_{k=1}^{\infty} \left(\frac{\Psi}{a}\right)^k [\alpha_k \sin(k\phi) + \beta_k \cos(k\phi)] \tag{18}$$

and for $\Psi \geq a$ given by

$$A_z = \sum_{k=1}^{\infty} \left(\frac{\Psi}{a}\right)^{-k} [\alpha_k \sin(k\phi) + \beta_k \cos(k\phi)] \tag{19}$$

where we have included those terms in the expansion needed to match the expansion of J_{s_z} with $b_o = 0$ and with proper behavior for small and large Ψ.

Now $\vec{J}$ is only present at $\Psi = a$ where it has the form of a surface current density. The boundary condition at $\Psi = a$ is given by

$$\left.\frac{\partial A_z}{\partial \Psi}\right|_{\Psi=a-} - \left.\frac{\partial A_z}{\partial \Psi}\right|_{\Psi=a+} = \mu J_{s_z} \tag{20}$$

This gives relations between the coefficients as

$$\begin{aligned} \frac{k}{a}\alpha_k + \frac{k}{a}\alpha_k &= \mu J_{s_o} a_k \\ \frac{k}{a}\beta_k + \frac{k}{a}\beta_k &= \mu J_{s_o} b_k \end{aligned} \tag{21}$$

which give

$$\alpha_k = \frac{a}{2k}\mu J_{s_o} a_k \;, \qquad \beta_k = \frac{a}{2k}\mu J_{s_o} b_k \tag{22}$$

The loop inductance (as discussed in reference 2) is given by

$$L = \frac{1}{I^2} \int_{V_g} \vec{A}(\vec{r}) \cdot \vec{J}(\vec{r}) dV \tag{23}$$

which can be written for our case as

$$L = \frac{\ell}{I^2} \int_o^{2\pi} A_z \Big|_{\Psi=a} J_{s_z} a d\phi \tag{24}$$

Substituting for A_z and J_{s_z} gives

$$L = \frac{\pi a^2 \ell}{2I^2} \mu J_{s_o}^2 \sum_{k=1}^{\infty} \left[\frac{a_k^2}{k} + \frac{b_k^2}{k} \right] \tag{25}$$

The equivalent volume is

$$V_{eq} = \frac{\mu}{L} \vec{A}_{eq} \cdot \vec{A}_{eq} \tag{26}$$

The figure of merit is then

$$\eta \equiv \frac{V_{eq}}{V_g} = 2a_1^2 \left\{ \sum_{k=1}^{\infty} \left[\frac{a_k^2}{k} + \frac{b_k^2}{k} \right] \right\}^{-1} \tag{27}$$

Since we wish to maximize η then clearly we must have $a_1 \neq 0$. Since we can choose one of the non zero coefficients arbitrarily (thereby fixing J_{s_o} in equation 7) we choose

$$a_1 \equiv 1 \tag{28}$$

giving

$$\eta = 2 \left\{ 1 + b_1 + \sum_{k=2}^{\infty} \left[\frac{a_k^2}{k} + \frac{b_k^2}{k} \right] \right\}^{-1} \tag{29}$$

Now we have $a_k^2/k \geq 0$ for $k \geq 2$ and $b_k^2/k \geq 0$ for $k \geq 1$. Clearly, then, η is maximized by setting

$$\begin{aligned} a_k &= 0 \qquad \text{for } k \geq 2 \\ b_k &= 0 \qquad \text{for } k \geq 1 \end{aligned} \tag{30}$$

The maximum figure of merit for this type of current distribution, in the two-dimensional approximation, is given by

$$\eta_{max} = 2 \tag{31}$$

The corresponding surface current distribution on $\Psi = a$ is given by

$$\vec{J}_s = \vec{e}_z J_{s_o} \sin(\phi) \tag{32}$$

This sinusoidal distribution of the current with respect to ϕ can also be considered as a uniform distribution of the current or loop turns with respect to x. The equivalent area for this distribution, using equation 12, is just

$$\vec{A}_{eq} = \frac{\pi a^2 \ell J_{s_o}}{I} \vec{e}_x \tag{33}$$

The inductance for this distribution is given from equation 25 by

$$L = \frac{\mu \pi a^2 \ell J_{s_o}^2}{2I^2} \tag{34}$$

III. <u>Distribution of Loop Turns at Discrete Positions Adjacent to the Cylindrical Surface</u>

Having found the optimum surface current distribution on $\Psi = a$ and the corresponding maximum figure of merit, we next consider distributions of currents on a finite number of conductors which approximate this surface current distribution. These conductors are assumed to have circular cross sections and might be circular wires, coaxial cables, etc. Let the conductor radius be r_o; let these conductors be arranged parallel to the z axis and be centered on a circular cylinder with radius given by $\Psi = b$ with the relation

$$b + r_o \equiv a \tag{35}$$

so that the conductors just fit inside a circular cylinder given by $\Psi = a$ as considered in the previous section. For convenience we define a parameter

$$\gamma \equiv \frac{r_o}{b} \tag{36}$$

Two pairs of these conductors are illustrated in figure 3. Note that each pair can be characterized by a certain value of ϕ, say ϕ_m; the two wires in this pair are centered on $\phi = \pm\phi_m$ so that the conductor distribution is symmetric in y. We also assume that in this typical conductor pair the current is I_m in the upper conductor(s) (positive y) and $-I_m$ in the lower conductor(s) where the positive current convention is in the +z direction. Each conductor pair can be considered as one or more loop turns where we let N_m be the effective number of loop turns with then a current of I_m/N_m per turn. Note that in the case of coaxial cables the center conductor can be used as additional conductors to increase the effective number of loop turns. An example of this type of loop is the moebius strip loop.[3,4] The cartesian coordinates of the center of the conductors for the typical conductor pair are (x_m, y_m) and $(x_m, -y_m)$. Define some normalized coordinates as

$$X_m \equiv \frac{x_m}{b} = \cos\ (\phi_m)\ , \qquad Y_m \equiv \frac{y_m}{b} = \sin\ (\phi_m) \tag{37}$$

The length of the sensor in the z direction is just ℓ as was the case in the previous section. Each conductor pair or turn group with N_m turns is assumed to have the conductors or turns connected as required at each end of the sensor. The equivalent area for each turn group is in the x direction and has the magnitude

$$A_{eq_m} \simeq 2\ell y_m N_m = 2\ell b N_m Y_m \tag{38}$$

3. Lt Carl E. Baum, Sensor and Simulation Note 7, Characteristics of the Moebius Strip Loop, December 1964.

4. Lt Carl E. Baum. Sensor and Simulation Note 25, The Multiple Moebius Strip Loop, August 1966.

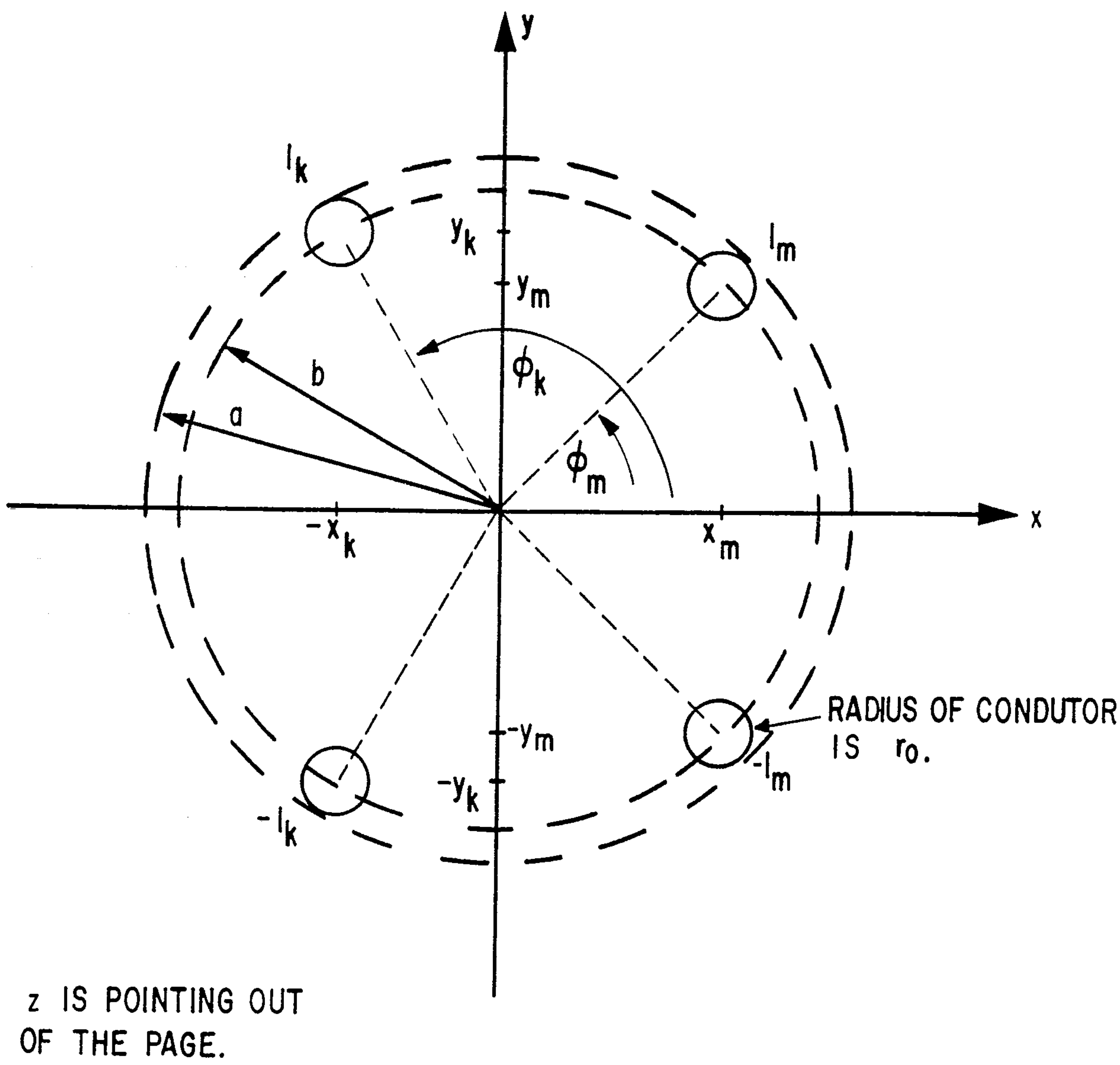


FIGURE 3. COORDINATES FOR LOOP CONDUCTORS

The equivalent area of the sensor will depend on how the various turn groups are interconnected. The reader should note that N_m is defined for the individual turn group; depending on the manner of interconnecting the turn groups this definition may or may not be directly relatable to the definition of turns density for the sensor as a whole as, for example, given by equation 10.

Next consider the self inductance of each turn group and the mutual inductance between turn groups. Ignore the presence of the other conductors and assume that the mth turn group can be represented as two perfectly conducting circular wires. Further assume that we can approximate the magnetic field distribution as a two-dimensional distribution, independent of z. Then for the field due to the current in the mth turn group we can use the conformal transformation[5]

$$w = \ln\left[\frac{\dfrac{\xi - x_m}{y_m} + j}{\dfrac{\xi - x_m}{y_m} - j}\right] \tag{39}$$

where

$$\xi = x + jy \ , \qquad w = u + jv \tag{40}$$

Since we assume that we have $r_o << b$ (or $\gamma << 1$) we have located the singularities in equation 39 on the conductor centers as a good approximation. Expanding equation 39 gives

$$u = \frac{1}{2}\ln\left[\frac{\left(\dfrac{x - x_m}{y_m}\right)^2 + \left(1 + \dfrac{y}{y_m}\right)^2}{\left(\dfrac{x - x_m}{y_m}\right)^2 + \left(1 - \dfrac{y}{y_m}\right)^2}\right] \tag{41}$$

5. Lt Carl E. Baum, Sensor and Simulation Note 27, Impedances and Field Distributions for Symmetrical Two Wire and Four Wire Transmission Line Simulators, October 1966.

$$v = \text{arc tan}\left[\frac{2\,\dfrac{x - x_m}{y_m}}{\left(\dfrac{x - x_m}{y_m}\right)^2 + \left(\dfrac{y}{y_m}\right)^2 - 1}\right] \tag{42}$$

From reference 5 we have the geometric impedance factor for a symmetrical two wire transmission line as

$$f_g = \frac{1}{\pi}\,\text{arc cosh}\left(\frac{y_m}{r_o}\right) = \frac{1}{\pi}\,\text{arc cosh}\left(\frac{Y_m}{\gamma}\right) \tag{43}$$

where we use the dimensions appropriate to the mth turn group. For $y_m >> r_o$ (which we assume) we have

$$f_g \simeq \frac{1}{\pi}\ln\left(\frac{2y_m}{r_o}\right) = \frac{1}{\pi}\ln\left(\frac{2Y_m}{\gamma}\right) \tag{44}$$

The inductance of a length ℓ of this mth turn group is then just

$$L_m = N_m^2\mu\ell f_g \simeq N_m^2\,\frac{\mu\ell}{\pi}\,\ln\left(\frac{2Y_m}{\gamma}\right) \tag{45}$$

For convenience we define a normalized self inductance as

$$L_m' \equiv \frac{L_m}{\mu\ell N_m^2} = f_g \simeq \frac{1}{\pi}\ln\left(\frac{2Y_m}{\gamma}\right) \tag{46}$$

Now consider the interaction between two of the turn groups, say the mth and the kth with $m \neq k$. Assume that the spacing between the conductors in adjacent turn groups is much larger than r_o. The change in the potential function u between the two positions of the kth turn group, i.e. (x_k, y_k) and $(x_k, -y_k)$, is given by

$$\Delta u = \ln\left[\frac{\left(\dfrac{x_k - x_m}{y_m}\right)^2 + \left(1 + \dfrac{y_k}{y_m}\right)^2}{\left(\dfrac{x_k - x_m}{y_m}\right)^2 + \left(1 - \dfrac{y_k}{y_m}\right)^2}\right]$$

$$= \ln \left[\frac{(X_k - X_m)^2 + (Y_k + Y_m)^2}{(X_k - X_m)^2 + (Y_k - Y_m)^2} \right] \tag{47}$$

The change in v in a path around one of the conductors in the mth turn group (around I_m only) is just

$$\Delta v = 2\pi \tag{48}$$

Including the length of the turn groups and the number of turns in each turn group the mutual inductance between the kth and mth turn groups (for $k \neq m$) is just

$$M_{k,m} = M_{m,k} \simeq N_k N_m \mu \ell \frac{\Delta u}{\Delta v}$$

$$= N_k N_m \frac{\mu \ell}{2\pi} \ln \left[\frac{(X_k - X_m)^2 + (Y_k + Y_m)^2}{(X_k - X_m)^2 + (Y_k - Y_m)^2} \right] \tag{49}$$

Define a normalized mutual inductance for $k \neq m$ as

$$M'_{k,m} = M'_{m,k} \equiv \frac{M_{k,m}}{\mu \ell N_k N_m} \simeq \frac{1}{2\pi} \ln \left[\frac{(X_k - X_m)^2 + (Y_k + Y_m)^2}{(X_k - X_m)^2 + (Y_k - Y_m)^2} \right] \tag{50}$$

As earlier defined the bulk current in the group of conductors forming the mth turn group is I_m and the current per turn is I_m/N_m. Now let V_m be the voltage per turn so that $V_m N_m$ is the voltage for the mth turn group. Then for frequencies sufficiently low that we can use the quasi static inductances we can write the voltage for the mth turn group (in the absence of an incident magnetic field) as

$$V_m N_m = L_m \frac{\partial}{\partial t} \left(\frac{I_m}{N_m} \right) + \sum_{\substack{k=1 \\ k \neq m}}^{M} M_{m,k} \frac{\partial}{\partial t} \left(\frac{I_k}{N_k} \right) \tag{51}$$

where we assume that there are M turn groups and for convenience we assume that their relative positions are ordered as

$$0 < \phi_1 < \phi_2 < \ldots < \phi_M < \pi \tag{52}$$

i.e., the turn groups are numbered in order of increasing ϕ. Using the normalized inductances equation 51 can be rewritten as

$$V_m = \mu \ell \left\{ L'_m \frac{\partial I_m}{\partial t} + \sum_{\substack{k=1 \\ k \neq m}}^{M} M'_{m,k} \frac{\partial I_k}{\partial t} \right\} \tag{53}$$

Now assume all currents have the same time history, except perhaps for different amplitudes. This makes all the voltages have the same time histories, except again for possible different amplitudes. Alternatively one can constrain the voltages to have the same time history (except for amplitude) and infer the same for the currents, except for additive constants of integration which we can also assume to be zero. Suppose that we define a reference voltage V_o and a reference current I_o which have the same waveforms as the V_m and I_m, and which are related by

$$V_o \equiv \mu \ell \frac{\partial I_o}{\partial t} \tag{54}$$

Then we can define voltage and current coefficients by

$$V_m \equiv v_m V_o \ , \qquad \frac{\partial I_m}{\partial t} \equiv i_m \frac{\partial I_o}{\partial t} \tag{55}$$

Then equation 53 can be rewritten using these coefficients as

$$v_m = L'_m i_m + \sum_{\substack{k=1 \\ k \neq m}}^{M} M'_{m,k} i_k \tag{56}$$

If we define for convenience of notation

$$M_{m,m} \equiv L_m \ , \qquad M'_{m,m} \equiv L'_m \tag{57}$$

then equation 54 can be written as

$$v_m = \sum_{k=1}^{M} M'_{m,k} i_k \tag{58}$$

This can be regarded as a matrix equation relating the M component coefficient vectors $\vec{v}$ and $\vec{i}$ (with components v_m and i_m, respectively) by the M by M normalized inductance matrix $(M'_{m,k})$.

There are various ways the turn groups might be connected to each other and to the sensor terminals. For any particular form of interconnection one can calculate the distribution of currents and/or voltages. With this information one can calculate the equivalent area, inductance, equivalent volume, and figure of merit of the sensor. In the next two sections we consider two types of sensor design based respectively on series and parallel connection of the turn groups.

IV. Special Cases Using Series Connection of Turn Groups

Now assume all the turn groups are connected in series and constrain

$$\left.\begin{aligned} N_m &\equiv 1 \\ i_m &\equiv 1 \\ X_m &\equiv 1 - \frac{2m}{M+1} \end{aligned}\right\} \text{ for } m = 1, 2, \ldots, M \tag{59}$$

This defines loops with single turns which are uniformly spaced in the x direction. This uniform spacing of equal currents with respect to x is used as an approximation to the optimum surface current density on $\Psi = a$ given by equation 32 and as discussed in section II. As M is increased this discrete current distribution should more closely approximate the continuous current sheet discussed in section II.

For this series connection the equivalent area is the sum of the equivalent areas for the individual turns from equation 38 giving

$$A_{eq} \simeq 2\ell b \sum_{m=1}^{M} Y_m = 2\ell b \sum_{m=1}^{M} \left(1 - X_m^2\right)^{1/2} \tag{60}$$

where X_m is defined above. Another interesting quantity is the average equivalent area per turn which can be written in normalized form by dividing by ℓa to give

$$\frac{A_{eq}}{M\ell a} = \frac{2}{M(1 + \gamma)} \sum_{m=1}^{M} \left(1 - x_m^2\right)^{1/2} \tag{61}$$

This is plotted in figure 4 for various values of M. For large M we can use equation 33 with a replaced by b and with J_{s_o} given by IM/2b so that as $M \to \infty$

$$A_{eq} \to \frac{\pi b \ell M}{2} \tag{62}$$

and

$$\frac{A_{eq}}{M\ell a} \to \frac{\pi b}{2a} = \frac{\pi}{2} (1 + \gamma)^{-1} \tag{63}$$

The voltage at the loop terminals in the absence of an incident magnetic field is just the sum of the turn voltages. Recalling the definitions in equations 55 we have a loop voltage given by

$$V = \sum_{m=1}^{M} v_m V_o = L \frac{dI_o}{dt} \tag{64}$$

where I_o is taken as the current at the sensor terminals and L is the sensor inductance. Using equation 54 we have

$$\frac{L}{\mu\ell} = \sum_{m=1}^{M} v_m \tag{65}$$

Then substituting from equation 56 we have

$$\frac{L}{\mu\ell M^2} = \frac{1}{M^2} \sum_{m=1}^{M} \left\{L'_m + \sum_{\substack{k=1 \\ k\neq m}}^{M} M'_{m,k}\right\} \tag{66}$$

where the normalized inductances are found in equations 46 and 50. This result is plotted in figure 5.

The figure of merit is just

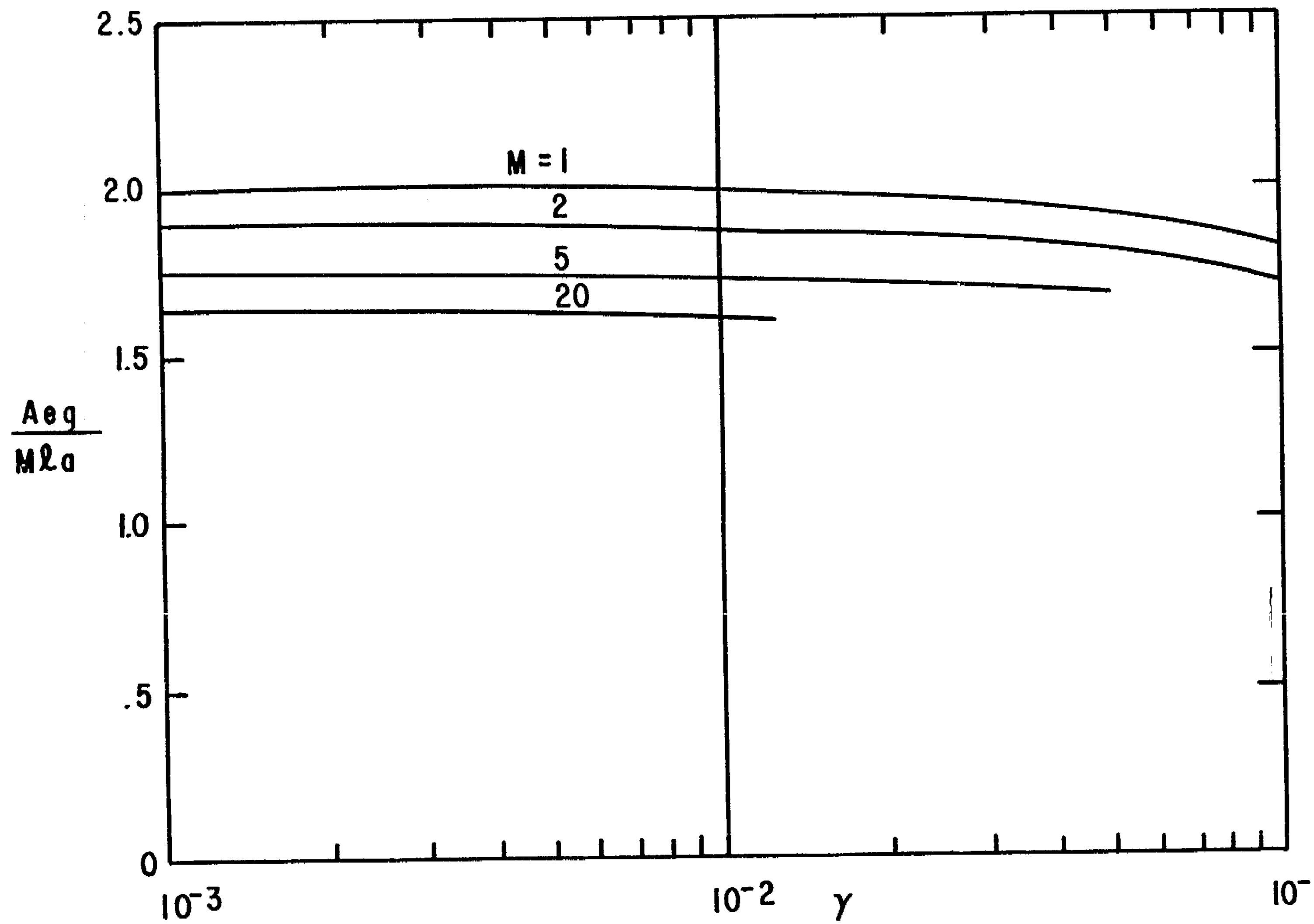


Figure 4. EQUIVALENT AREA FOR CASES OF SERIES CONNECTION

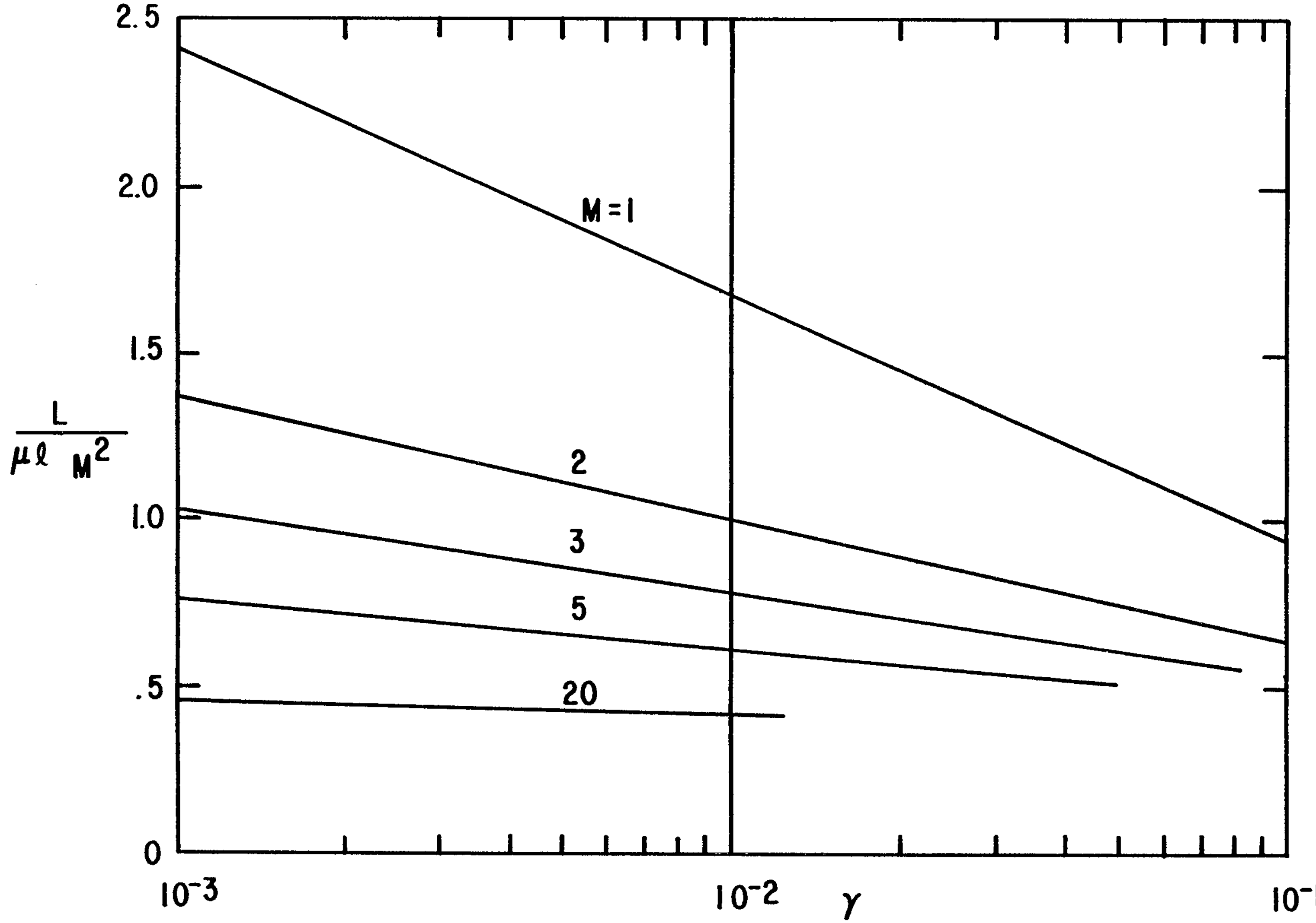


FIGURE 5. INDUCTANCE FOR CASES OF SERIES CONNECTION

$$\eta \equiv (\pi a^2 \ell)^{-1} \frac{\mu A_{eq}^2}{L} = \frac{1}{\pi}\left[\frac{A_{eq}}{M\ell a}\right]^2 \left[\frac{L}{\mu \ell M^2}\right]^{-1} \tag{67}$$

This is plotted in figure 6. As one would expect η is increased by increasing M.

V. Special Cases Using Parallel Connection of Turn Groups

For these special cases assume that all the turn groups have the same equivalent area so that the open circuit voltages for each turn group are the same from an incident magnetic field with a uniform x component. Then connect these turn groups in parallel so that the sensor has an equivalent area given by

$$A_{eq} = A_{eq_m} \simeq 2\ell b N_m Y_m \qquad \text{for } m = 1, 2, \ldots, M \tag{68}$$

In the absence of an incident magnetic field $V_m N_m$ is the voltage for each turn group and thus also the voltage at the sensor terminals. Thus we have

$$V_m N_m = V_k N_k \qquad \text{for } k, m = 1, 2, \ldots, M \tag{69}$$

From the first of equations 55 we also have

$$v_m N_m = v_k N_k \qquad \text{for } k, m = 1, 2, \ldots, M \tag{70}$$

and from equation 68 we have

$$Y_m N_m = Y_k N_k \qquad \text{for } k, m = 1, 2, \ldots, M \tag{71}$$

Thus we can define

$$v_m \equiv Y_m \qquad \text{for } m = 1, 2, \ldots, M \tag{72}$$

Then from equation 56 the i_m can be found from the v_m provided the X_m are specified.

The current at the sensor terminals is just

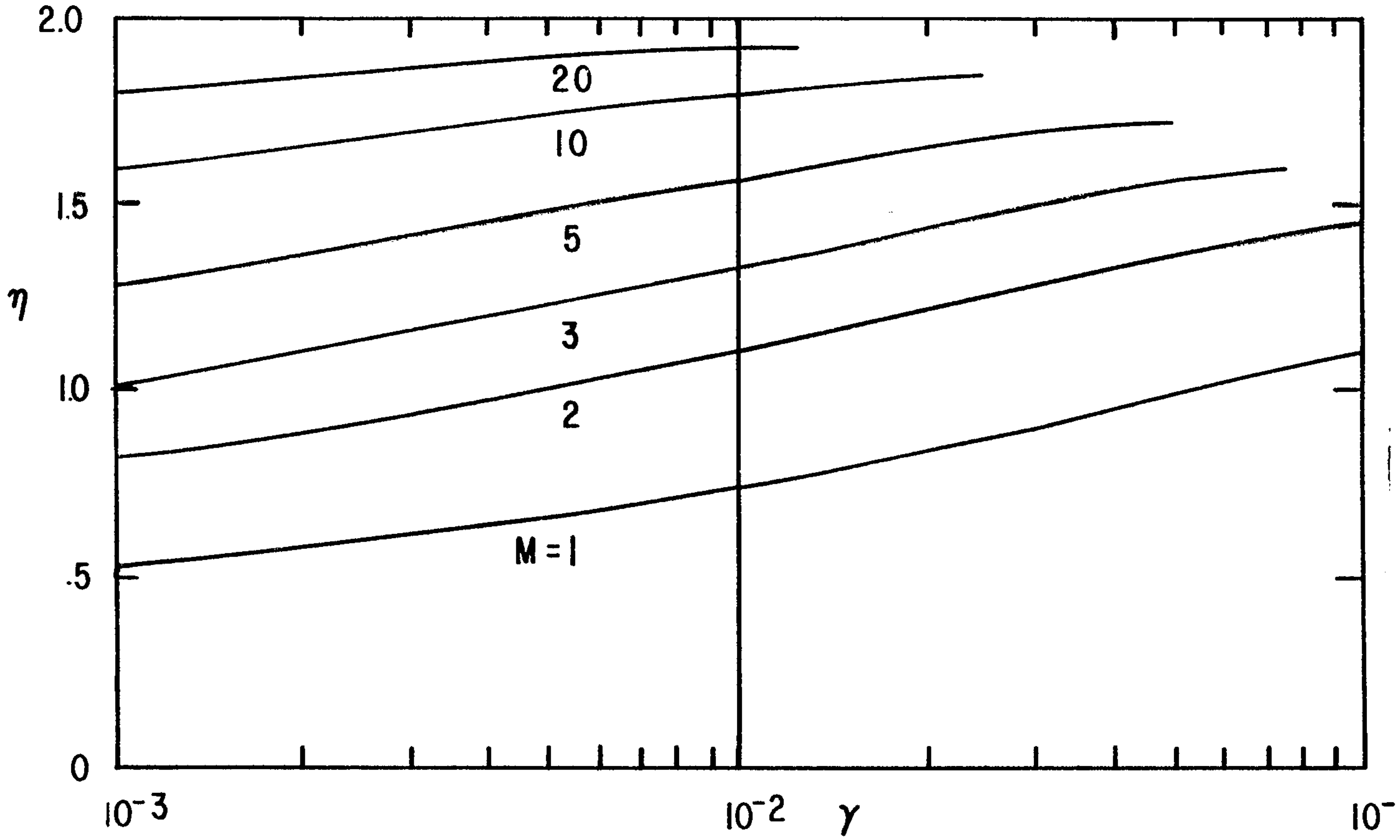


FIGURE 6. FIGURE OF MERIT FOR CASES OF SERIES CONNECTION

$$I = \sum_{m=1}^{M} \frac{I_m}{N_m} \tag{73}$$

From equation 71 we can define some reference number of turns N_o such that

$$N_m \equiv \frac{N_o}{Y_m} \qquad \text{for } m = 1, 2, \ldots, M \tag{74}$$

The sensor inductance L comes from

$$V_m N_m = v_m N_m V_o = L \frac{dI}{dt} = L \frac{\partial I_o}{\partial t} \sum_{m=1}^{M} \frac{i_m}{N_m} \tag{75}$$

which can be written as

$$v_m N_m = N_o = \frac{L}{\mu \ell} \sum_{m=1}^{M} \frac{i_m Y_m}{N_o} \tag{76}$$

or

$$\frac{L}{\mu \ell N_o^2} = \left\{ \sum_{m=1}^{M} i_m Y_m \right\}^{-1} \tag{77}$$

where the i_m are found from a solution of the equations

$$Y_n = L'_n i_n + \sum_{\substack{k=1 \\ k \neq n}}^{M} M'_{n,k} i_k \qquad \text{for } n = 1, 2, \ldots, M \tag{78}$$

The equivalent area can be written as

$$A_{eq} \simeq 2\ell b N_o = 2\ell a N_o (1 + \gamma)^{-1} \tag{79}$$

The figure of merit is given by

$$\eta \equiv (\pi a^2 \ell)^{-1} \frac{\mu A_{eq}^2}{L} = \frac{1}{\pi}\left[\frac{A_{eq}}{N_o \ell a}\right]^2 \left[\frac{L}{\mu \ell N_o^2}\right]^{-1}$$

$$= \frac{4}{\pi(1+\gamma)^2}\left[\frac{L}{\mu \ell N_o^2}\right]^{-1} \tag{80}$$

Consider three cases of this type of sensor as illustrated in figure 7.

<u>Case 1</u>:

As illustrated in figure 7A for this case we choose

$$M \equiv 1 \, , \qquad \phi_1 \equiv \frac{\pi}{2} \tag{81}$$

Then from equations 78 and 46 we have

$$i_1 = \frac{1}{L_1'} \simeq \pi\left[\ln\left(\frac{2}{\gamma}\right)\right]^{-1} \tag{82}$$

From equation 74 we have

$$N_o = N_1 \tag{83}$$

The sensor inductance is then

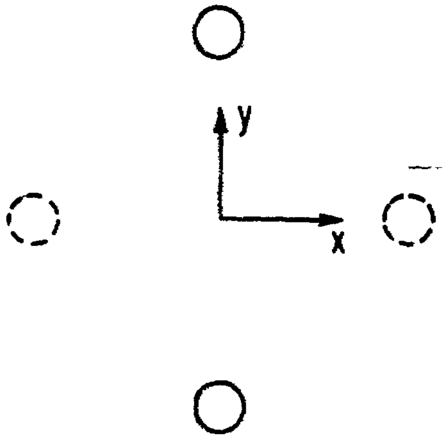
$$\frac{L}{\mu \ell N_1^2} = \frac{1}{i_1} = L_1' \simeq \frac{1}{\pi} \ln\left(\frac{2}{\gamma}\right) \tag{84}$$

The equivalent area is

$$A_{eq} \simeq 2\ell b N_1 \tag{85}$$

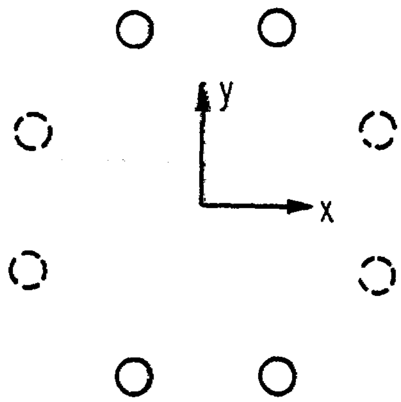
The figure of merit is

$$\eta \simeq \frac{4}{(1+\gamma)^2}\left[\ln\left(\frac{2}{\gamma}\right)\right]^{-1} \tag{86}$$



A. CASE 1 WITH M = 1

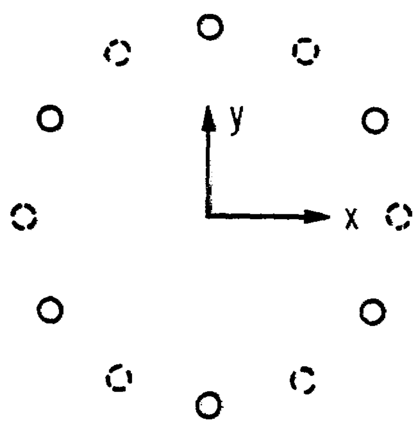
z IS POINTING OUT OF THE PAGE.



B. CASE 2 WITH M = 2

SOLID CIRCLES INDICATE POSITION OF SENSOR CONDUCTORS.

DASHED CIRCLES INDICATE POSITION FOR PLACING CONDUCTORS FOR SIMILAR SENSOR FOR ORTHOGONAL FIELD COMPONENT.



C. CASE 3 WITH M = 3

FIGURE 7. THREE CASES OF PARALLEL CONNECTION WITH ORTHOGONAL LOOP POSITIONS ALSO INDICATED

The inductance is plotted in figure 8; the figure of merit is plotted in figure 9.

Note for this particular sensor geometry that the conductors are positioned at $\phi = \pm\pi/2$ and have an equivalent area with only an x component. As illustrated in figure 7A one can place the same type of sensor with conductors at $\phi = 0, \pi$. This second sensor has an equivalent area with only a y component. These two sensors then can be used to measure two orthogonal magnetic field components. Considered together the two sensors have conductors uniformly spaced with increments of $\pi/2$ in ϕ.

Case 2:

As illustrated in figure 7B for this second case we choose

$$M \equiv 2 , \qquad \phi_1 \equiv \frac{3\pi}{8} , \qquad \phi_2 \equiv \frac{5\pi}{8} \tag{87}$$

This gives

$$Y_1 = Y_2 = v_1 = v_2 = \sin\left(\frac{3\pi}{8}\right) \simeq .9239$$

$$X_1 = -X_2 = \cos\left(\frac{3\pi}{8}\right) \simeq .3827$$

$$N_1 = N_2 = Y_1 N_o \tag{88}$$

The normalized inductances from equations 46 and 50 are

$$L_1' = L_2' \simeq \frac{1}{\pi} \ln\left(\frac{2Y_1}{\gamma}\right)$$

$$M_{1,2}' = M_{2,1}' \simeq \frac{1}{2\pi} \ln\left[\frac{4X_1^2 + 4Y_1^2}{4X_1^2}\right] = -\frac{1}{\pi} \ln(X_1) \tag{89}$$

Due to symmetry $i_1 = i_2$ so that from equation 78 we have

$$i_1 = i_2 = \frac{Y_1}{L_1' + M_{1,2}} \simeq \pi Y_1 \left[\ln\left(\frac{2Y_1}{\gamma X_1}\right)\right]^{-1} \tag{90}$$

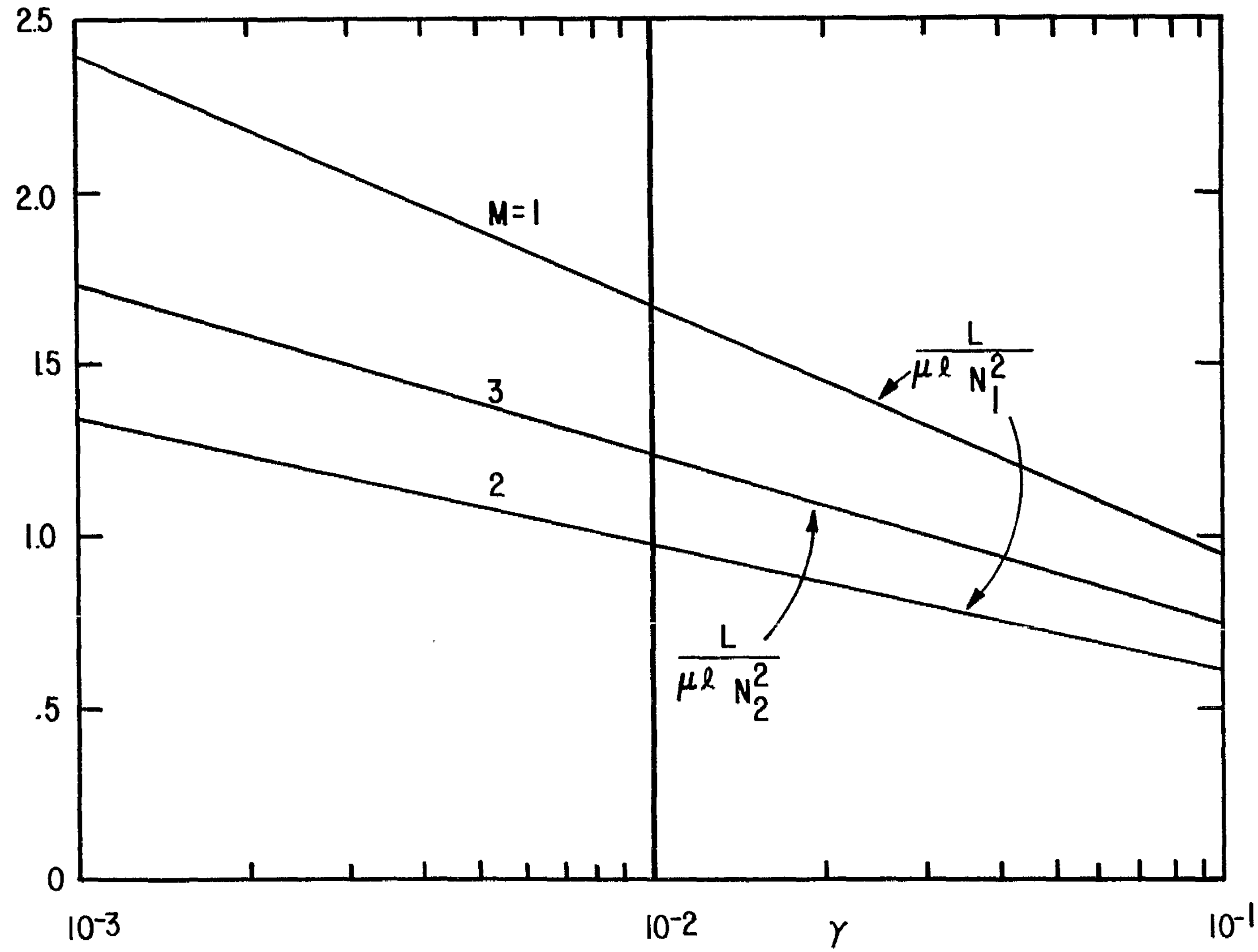


FIGURE 8. INDUCTANCE FOR CASES OF PARALLEL CONNECTION

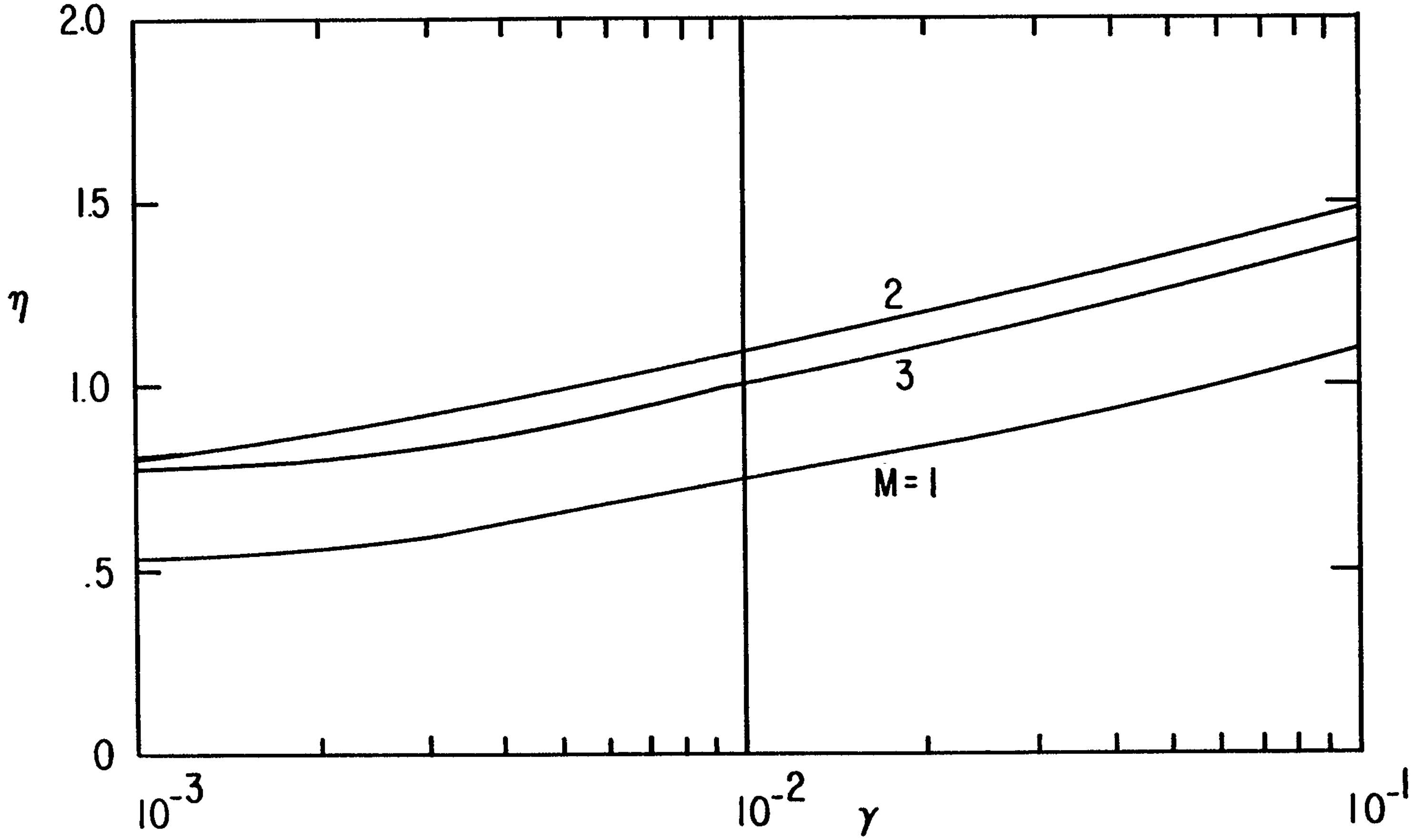


FIGURE 9 FIGURE OF MERIT FOR CASES OF PARALLEL CONNECTION

The equivalent area is

$$A_{eq} \simeq 2\ell b Y_1 N_1 \tag{91}$$

and the inductance is

$$\frac{L}{\mu \ell N_1^2} = \left(\frac{N_o}{N_1}\right)^2 (2 i_1 Y_1)^{-1} = \frac{Y_1}{2 i_1} = \frac{1}{2}[L_1' + M_{1,2}']$$

$$\simeq \frac{1}{2\pi} \ln\left(\frac{2Y_1}{\gamma X_1}\right) \tag{92}$$

The figure of merit is

$$\eta = \frac{4}{\pi(1 + \gamma)^2} \left(\frac{N_o}{N_1}\right)^2 \left[\frac{L}{\mu \ell N_1^2}\right]^{-1} \simeq \frac{8Y_1^2}{(1 + \gamma)^2} \left[\ln\left(\frac{2Y_1}{\gamma X_1}\right)\right]^{-1} \tag{93}$$

The inductance and figure of merit are plotted in figures 8 and 9 respectively.

For this sensor geometry the conductors are positioned at $\phi = \pm 3\pi/8$ and $\phi = \pm 5\pi/8$ giving an equivalent area with only an x component. As illustrated in figure 7B one can place the same type of sensor rotated by $\pi/2$ so that the conductors are located at $\phi = \pm\pi/8$ and $\phi = \pm 7\pi/8$. This second sensor has only a y component for its equivalent area. Considered together, these two sensors can be used to measure two orthogonal components of the magnetic field. These two sensors together have their conductors uniformly spaced in ϕ in increments of $\pi/4$.

<u>Case 3</u>:

For this third case, as illustrated in figure 7C, we choose

$$M \equiv 3 , \qquad \phi_1 \equiv \frac{\pi}{6} , \qquad \phi_2 \equiv \frac{\pi}{2} , \qquad \phi_3 \equiv \frac{5\pi}{6} \tag{94}$$

which gives

$$Y_1 = Y_3 = v_1 = v_3 = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$