Sensor and Simulation Notes

Note 73

February 1969

Pulse Radiation and Synthesis by an Infinite
Cylindrical Antenna

by

R. W. Latham and K. S. H. Lee
Northrop Corporate Laboratories
Pasadena, California

Abstract

The radiated field is calculated for the case where a gap generator
in an infinite cylindrical antenna is excited by a step function of voltage.
The gap voltage waveform required to generate a radiation field decaying
exponentially in time is also determined.

Acknowledgment

I. Introduction

An airborne antenna is being built to simulate the electromagnetic pulse due to a nuclear explosion.[1] Such an antenna could take various shapes. One possible shape is a long, slender, axi-symmetric structure with a conical source region.[2] The energy fed into this structure at the apex of the conical_portion will propagate in a TEM manner until it strikes the transition region between the cone and the remainder of the long antenna. From this it follows that, even in the radiation zone, for very early times the time dependence of the fields will be simply related to the time dependence of the source.

In this note we will try to develop some feeling for what happens in the next interval of time, i.e. the interval between the instant when the leading edge of the pulse reaches the end of the conical portion and the instant when this leading edge reaches the end of the entire antenna structure. In order to do this we idealize the entire source region by assuming it to be a gap generator, i.e. we assume a longitudinal tangential electric field distribution over a very narrow circumferential slot. This idealization will be more reasonable the smaller the conical source region is, compared to the entire length of the structure. We make a further idealization by assuming the remainder of the antenna to be a circular cylinder, perfectly conducting and infinitely long. The real antenna will probably be somewhat tapered and resistive; so our results will be inaccurate to the extent that they do not take these possibilities into account. Nevertheless, we feel that our results will not differ in their general nature from those for the far more difficult real problem. The assumption of infinite length is no limitation at all in the time interval of interest (before the reflections from the ends of the antenna have any effect). The geometrical configuration of the ideal antenna that we will study is shown in figure 1.

In Section II we consider the problem of calculating the waveform of the radiated pulse when the gap in the idealized antenna described above is excited by a voltage that is a prescribed function of time, and treat in detail the case where the voltage is a step function of time.

3

In Section III we consider the inverse problem of determining what voltage at the gap would be needed to radiate a waveform of specified shape, and treat in detail the case where the radiated waveform jumps instantaneously to some finite value and then decays exponentially.

II. Transient Radiation Field of An Infinite Cylindrical Antenna

In this section, we shall obtain the time variation of the radiation field of an infinite cylindrical antenna excited by a step-function voltage across an circumferential gap of infinitesimal width. The method of solution, although quite simple, involves numerical integration of an integral. Our point of departure is the time-harmonic far-zone electric field which is, for $\theta \neq 0$, given by[3]

$$\widetilde{E}_\theta(r,\theta,\omega) = \frac{\widetilde{V}(\omega)}{i\pi} \frac{1}{\sin\theta \, H_o^{(1)}(ka\sin\theta)} \frac{e^{ikr}}{r} \quad , \qquad (1)$$

where $\widetilde{V}$ has the dimension of volt per unit frequency; $\widetilde{E}_\theta$ volt per meter per unit frequency; $k = \omega/c$; $H_o^{(1)}$ is a Hankel function of the first kind of order zero. The meaning of r, θ, and a is given in figure 1, which depicts the geometry of the problem under consideration. In terms of the Laplace-transform variable p , where $p = -i\omega$, equation (1) becomes

$$\frac{r\widetilde{E}_\theta}{v_o} = \frac{e^{-pr/c}}{2\sin\theta \, K_o(pa\sin\theta/c)} \frac{\widetilde{V}(p)}{v_o} \quad , \qquad (2)$$

where the relation $H_o^{(1)}(ix) = -(2i/\pi)K_o(x)$ has been used, and v_o is a constant having the dimension of voltage. For a step-function voltage, i.e., $v(t) = v_o U(t)$, we have

$$\widetilde{V}(p) = v_o/p$$

and, hence, the inverse Laplace transform is

$$\frac{rE_\theta(r,\theta,t)}{v_o} = \frac{1}{2 \sin \theta} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{(t - r/c)p}}{pK_o(pa \sin \theta/c)} \, dp$$

$$= \frac{1}{2 \sin \theta} \frac{1}{2\pi i} \int_C \frac{e^{\zeta q \csc \theta}}{\zeta K_o(\zeta)} \, d\zeta \quad , \tag{3}$$

where $q = (ct - r)/a$, and the path C is shown in figure 2.

In deforming the path C the following properties of $K_o(\zeta)$ should be borne in mind (figure 2)[4]:

(a) $K_o(\zeta)$ has no zeros for which $|\arg \zeta| \le \pi/2$.

(b) $K_o(\zeta)$ has no zeros for which $\pi/2 < \arg \zeta \le \pi$ and $-\pi \le \arg \zeta < -\pi/2$.

(c) $K_o(\zeta)$ has a branch cut along the negative real axis in the ζ-plane.

(d) $K_o(\zeta) \sim \zeta^{-1/2} e^{-\zeta} \sqrt{\pi/2}$ for large $|\zeta|$ and $|\arg \zeta| < 3\pi/2$.

(e) $K_o(\zeta) \sim -\ln(\Gamma\zeta/2)$ as $\zeta \to 0$, where Γ = exponential of Euler's constant = $1.7810\cdots$.

(f) $K_o(|\xi|e^{\pm\pi i}) = K_o(|\xi|) \mp \pi i \, I_o(|\xi|)$.

From (a) and (d) one can easily deduce that (3) gives

$$\frac{rE_\theta}{v_o} = 0 \quad , \quad \text{if} \quad ct < r - a \sin \theta \ .$$

For $ct > r - a \sin \theta$, that is, for $q > -\sin \theta$, we deform the path C into the left half ζ-plane as shown in figure 2. By virtue of (d) and (e) the integrals over the large half circle Γ and the small circle γ tend to zero as $R \to \infty$ and $\delta \to 0$, respectively. Thus, by invoking (b) we can write (3) as

$$\frac{rE_\theta}{v_o} = -\frac{1}{2 \sin \theta} \cdot \frac{1}{2\pi i} \left[\int_{L_+} + \int_{L_-} \right] \frac{e^{\zeta q \csc \theta}}{\zeta K_o(\zeta)} \, d\zeta \quad , \quad \text{for} \quad q > -\sin \theta$$

6

which, by virtue of (f), becomes

$$\frac{rE_\theta}{v_o} = \frac{1}{2 \sin \theta} \int_0^\infty \frac{e^{-\xi q \csc \theta} I_o(\xi)}{K_o^2(\xi) + \pi^2 I_o^2(\xi)} \frac{d\xi}{\xi} \quad , \qquad \text{for } q > - \sin \theta \quad . \quad (4)$$

In figures 3 and 4, rE_θ/v_o is plotted against T, where $T = q + 1$, for $\theta = \pi/2$, $\pi/3$, $\pi/4$ and $\pi/6$. The solid curves in these figures were obtained by numerically integrating (4).

Large time behavior of rE_θ/v_o

To obtain the asymptotic form of the integral in (4) for $q \csc \theta \to \infty$, we first break the integral into three parts:

$$\int_0^\infty \frac{e^{-\kappa x} I_o(x)}{K_o^2(x) + \pi^2 I_o^2(x)} \frac{dx}{x} = \int_0^M \frac{e^{-\kappa x}}{\pi^2 + \ln^2(\Gamma x/2)} \frac{dx}{x}$$

$$+ \int_0^M e^{-\kappa x} \left[\frac{I_o(x)}{K_o^2(x) + \pi^2 I_o^2(x)} - \frac{1}{\pi^2 + \ln^2(\Gamma x/2)} \right] \frac{dx}{x}$$

$$+ \int_M^\infty \frac{e^{-\kappa x} I_o(x)}{K_o^2(x) + \pi^2 I_o^2(x)} \frac{dx}{x} \quad , \qquad (5)$$

where $\kappa = q \csc \theta$, and M is so chosen that $\kappa^{-1} \ll M \ll 2/\Gamma$. Such an M is clearly possible when $\kappa \to \infty$. For very large κ, the main contribution to the integral on the left-hand side of (5) clearly comes from the first integral on the right-hand side, which we now proceed to evaluate asymptotically. We write

$$\int_0^M \frac{e^{-\kappa x}}{\pi^2 + \ln^2(\Gamma x/2)} \frac{dx}{x} = \int_0^M \frac{dx}{x} \frac{e^{-\kappa x}}{\ln^2(\Gamma x/2)} \left\{1 + \pi^2/\ln^2(\Gamma x/2)\right\}^{-1}$$

$$\sim \int_0^{M\Gamma/2} \frac{e^{-(\frac{2\kappa}{\Gamma})u}}{u \ln^2 u} \, du \quad , \quad \text{as} \quad \kappa \to \infty \quad . \quad (6)$$

Integration by parts gives

$$\int_0^{M\Gamma/2} \frac{e^{-(\frac{2\kappa}{\Gamma})u}}{u \ln^2 u} \, du = -\frac{e^{-\kappa M}}{\ln(M\Gamma/2)} - \int_0^{\kappa M} \frac{e^{-y} dy}{\ln y - \ln(2\kappa/\Gamma)} \quad . \quad (7)$$

The integral on the right-hand side can be evaluated as follows:

$$\int_0^{\kappa M} \frac{e^{-y} dy}{\ln y - \ln(2\kappa/\Gamma)} = -\frac{1}{\ln(2\kappa/\Gamma)} \int_0^{\kappa M} e^{-y} \left\{1 - \frac{\ln y}{\ln(2\kappa/\Gamma)}\right\}^{-1} dy$$

$$\sim \frac{-1}{\ln(2\kappa/\Gamma)} \quad , \quad \text{as} \quad \kappa \to \infty \quad . \quad (8)$$

Collecting the results in (6), (7) and (8) we have

$$\int_0^\infty \frac{e^{-\kappa x} I_o(x)}{K_o^2(x) + \pi^2 I_o^2(x)} \frac{dx}{x} \sim \frac{1}{\ln(2\kappa/\Gamma)} \quad , \quad \text{as} \quad \kappa \to \infty \quad .$$

Using this result we obtain from (4)

$$\frac{r E_\theta}{v_o} \sim \frac{1}{2 \sin \theta} \frac{1}{\ln\left(\frac{2T}{\Gamma \sin \theta}\right)} \quad , \quad \text{as} \quad T \to \infty \quad . \quad (9)$$

8

Equation (9) is plotted in figure 4 in broken lines.

Small time behavior of rE/v_o

We shall now estimate (4) when $q \to -\sin\theta$, i.e., when $T \to 1 - \sin\theta$. Let $\nu = q\csc\theta + 1$. Then

$$I = \int_0^\infty \frac{e^{-xq\csc\theta}I_o(x)}{K_o^2(x) + \pi^2 I_o^2(x)} \frac{dx}{x} = \int_0^\infty \frac{e^{-\left(\frac{\nu-1}{\nu}\right)y} I_o(y/\nu)}{K_o^2(y/\nu) + \pi^2 I_o^2(y/\nu)} \frac{dy}{y} \quad .$$

We wish now to evaluate I when $\nu \to 0$. Let δ be so chosen that $1 \gg \delta \gg \nu$. We then write

$$I = \int_0^\delta (\cdots)dy + \int_\delta^\infty (\cdots)dy = I_1 + I_2 \quad .$$

In evaluating I_2 we may use the asymptotic forms for $I_o(y/\nu)$ and $K_o(y/\nu)$. Thus

$$I_2 \sim \frac{1}{\pi}\sqrt{\frac{2}{\nu\pi}} \int_\delta^\infty \frac{e^{-y}}{\sqrt{y}} dy = \frac{\sqrt{2}}{\pi} \frac{1}{\sqrt{\nu}} \operatorname{erfc}(\delta)$$

$$\sim \frac{\sqrt{2}}{\pi} \frac{1}{\sqrt{\nu}} \quad , \qquad \text{for } \delta \ll 1 \quad . \tag{10}$$

In evaluating I_1 we write

$$I_1 = \int_0^{\delta/\nu} \frac{e^{-z(\nu-1)}I_o(z)}{K_o^2(z) + \pi^2 I_o^2(z)} \frac{dz}{z} = \int_0^M (\cdots)dz + \int_M^{\delta/\nu} (\cdots)dz \quad ,$$

where $M < 2/\Gamma$. Now it is clear that I_1 tends to a finite value as

9

$\nu \to 0$. Hence, the expression given by (10) is the only dominant term of (4) when ν is very small, and so we have

$$\frac{rE_\theta}{v_o} \sim \frac{1}{\pi} \frac{1}{\sqrt{2} \sin \theta} \frac{1}{\sqrt{T - 1 + \sin \theta}} \quad , \quad \text{as } T \to 1 - \sin \theta \quad . \quad (11)$$

Equation (11) is plotted in figure 3 in broken lines.

The fact that rE_θ/v_o tends to infinity as $T \to 1 - \sin \theta$ is attributable to the infinitesimal size of the gap. If the gap were assumed to have finite width, rE_θ/v_o would increase from zero as T increases from $1 - \sin \theta$. One can understand this square-root singularity in the early times by recalling that the input admittance of an antenna with an excitation gap of zero width is infinite.[*] The curves in figures 3 and 4, except for the portions corresponding to $T < 1$ or so, should describe quite well the time behavior of the far field radiated by an airborne antenna of the shape described in the Introduction and excited by a step-function voltage.

---

[*]For more information concerning this point the interested reader may refer to the paper by T. T. Wu, "Transient Response of A Dipole Antenna," J. Math. Phys., Vol. 2, p. 892, 1961.

III. Synthesis

To study the possibility of radiating a pulse with a specified wave-form by appropriately controlling the gap voltage, we return to equation (2), which we rewrite here for convenience.

$$\widetilde{E}_\theta(r,\theta,p) = \frac{e^{-pr/c}\,\widetilde{V}(p)}{2r\,\sin\theta\,K_o(pa\,\sin\theta/c)}$$

This equation simply relates the Laplace transform of the radiated electric field to the Laplace transform of the gap voltage. There is no necessity to think of the gap voltage as the specified quantity. One could, with equal justification, specify the waveform of the electric field at some particular angle, θ , and determine the gap voltage necessary to generate such a field. The angle at which we choose to specify the radiated field is arbitrary, but $\theta = \pi/2$ seems a convenient and typical one. We will specialize our equations in the remainder of this section to this typical angle, and so we may write

$$\widetilde{V}(p) = 2rK_o(pa/c)\widetilde{E}_\theta(r,\pi/2,p)e^{pr/c} \qquad . \qquad (12)$$

Now if, in the time domain, we write

$$E_\theta(r,\pi/2,t) = \frac{f(t - r/c)}{r} \qquad t > r/c$$

$$= 0 \qquad t < r/c \qquad ,$$

it follows from the shifting theorem of Laplace transform theory that

$$\widetilde{E}_\theta(r, \pi/2, p) = \frac{F(p)}{r} e^{-pr/c} \tag{13}$$

where

$$F(p) = \int_0^\infty e^{-px} f(x) dx \qquad .$$

The substitution of (13) in to (12) gives

$$\widetilde{V}(p) = 2K_0(pa/c)F(p) \qquad . \tag{14}$$

We take the inverse Laplace transform of this equation by invoking the convolution theorem and the inverse transform[5]

$$L^{-1}\left[K_0(pa/c)\right] = \frac{U(t - a/c)}{\sqrt{t^2 - (a/c)^2}} \qquad . \tag{15}$$

Applying the convolution theorem[6] to (14), and using (15), we obtain

$$v(t) = 2\int_0^t \frac{U(t - t' - a/c)}{\sqrt{(t - t')^2 - (a/c)^2}} f(t') dt'$$

$$= 2\int_0^{t-a/c} \frac{f(t') \, dt'}{\sqrt{(t - t')^2 - (a/c)^2}} \qquad . \tag{16}$$

where $v(t)$ has the dimension of voltage.

A natural way to normalize the time variable in this equation is to define the time of propagation across the radius of the cylinder to be

$$t_a = a/c \quad ,$$

and to set

$$\tau = t/t_a \quad \cdot$$

In this normalized time domain, equation (16) becomes simply

$$v(\tau) = 2 \int_0^{\tau-1} \frac{f(\tau')d\tau'}{\sqrt{(\tau - \tau')^2 - 1}} \quad . \tag{17}$$

We apply (17) first to the simple case of a radiated step function,

$$f(\tau) = v_o U(\tau) \quad \cdot$$

The gap voltage needed to generate this field will be given by

$$\frac{v(\tau)}{2v_o} = \int_0^{\tau-1} \frac{U(\tau')d\tau'}{\sqrt{(\tau - \tau')^2 - 1}}$$

$$= \int_1^{\tau} \frac{U(\tau - x)dx}{\sqrt{x^2 - 1}} \qquad (x = \tau - \tau')$$

$$= U(\tau - 1)\cosh^{-1}(\tau) \tag{18}$$

13

The remainder of this section is devoted to the more difficult task of determining the gap voltage needed to generate a radiation field whose $f(\tau)$ is defined by

$$- \quad f(\tau) = v_o U(\tau) e^{-\beta\tau} \quad .$$

Equation (17) gives, in this case,

$$\frac{v(\tau)}{2v_o} = \int_o^{\tau-1} \frac{U(\tau')e^{-\beta\tau'}d\tau'}{\sqrt{(\tau - \tau')^2 - 1}} \quad .$$

$$= U(\tau - 1) \int_o^{\tau-1} \frac{e^{-\beta\tau'}d\tau'}{\sqrt{(\tau - \tau')^2 - 1}} \tag{19}$$

It has not been possible to express this integral in terms of a finite number of well-known or tabulated functions. Equation (19) has been evaluated numerically and plotted as a function of τ in figure 5 for several values of β between zero and five. When β is exactly zero equation (18) is applicable. It should be noted that if, in the real space and time domains, the radius of the cylinder is of the order of a foot while the decay time of the exponential is of the order of a microsecond, then β in the above equations will be of the order of 10^{-3}. It should also be noted that, if the real-time decay parameter is denoted by β_r , then

$$\beta_r t = \beta\tau \quad .$$

In figure 6 we have plotted the function defined by equation (19) as a function

14

of $\beta\tau$ (or $\beta_r t$) for a few small values of β that may be of practical interest in certain instances.

We now return to equation (19) and develop some analytical representations of the function defined by that equation. These representations may be more useful for certain ranges of the variables, τ and β, than the graphical data displayed in figures 5 and 6.

Assuming that we restrict our attention to the range $\tau > 1$, we may write equation (19) as

$$\frac{v(\tau)}{2v_o} = \int_o^{\tau-1} \frac{e^{-\beta\tau'}d\tau'}{\sqrt{(\tau - \tau')^2 - 1}} \qquad . \tag{20}$$

We now make use of Watson's lemma which states that,[7] asymptotically for large α

$$\int_o^A e^{-\alpha x}f(x)dx \sim \sum_{n=o}^{\infty} \frac{f^{(n)}(0)}{\alpha^{n+1}} \qquad ,$$

where A is some finite constant. Applying this lemma to equation (20) we may write the asymptotic expansion for large β

$$\frac{v(\tau)}{2v_o} \sim \sum_{n=o}^{\infty} \frac{1}{\beta^{n+1}} \frac{d^n}{d\tau'^n} \left[\frac{1}{\sqrt{(\tau - \tau')^2 - 2}} \right] \Bigg|_{\tau' = 0} \qquad . \tag{21}$$

But it can be seen that, for the function in question,

$$\frac{d}{d\tau'} = - \frac{d}{d\tau} \qquad ,$$

so we may use this in (21) and allow τ' to go to zero within the differentiation to get

$$\frac{v(\tau)}{2v_o} \sim \frac{1}{\beta} \sum_{n=o}^{\infty} \left(\frac{-1}{\beta}\right)^n \frac{d^n}{d\tau^n} \left[\frac{1}{\sqrt{\tau^2 - 1}}\right] \qquad . \qquad \qquad (22)$$

The first two terms of this asymptotic series give

$$\frac{v(\tau)}{2v_o} \sim \frac{1}{\beta(\tau^2 - 1)^{1/2}} + \frac{\tau}{\beta^2(\tau^2 - 1)^{3/2}}$$

As is clear from its derivation, this expression will be valid for

$$\beta \gg 1$$

$$\tau > 1 \qquad ,$$

but empirically it is also valid as long as

$$\beta\tau \gg 1 \qquad .$$

We now re-examine equation (19) for the case

$$\beta(\tau - 1) \ll 1$$

by setting, in the integral,

$$\tau' = \tau - 1 - x$$

to get

$$\frac{v(\tau)}{2v_o} = e^{-\beta(\tau-1)} \int_o^{\tau-1} \frac{e^{\beta x} dx}{\sqrt{(1+x)^2 - 1}}$$

Thus

$$\frac{v(\tau)}{2v_o} = e^{-\beta(\tau-1)} \sum_{n=o}^{\infty} \frac{\beta^n}{n!} \int_o^{\tau-1} \frac{x^n dx}{\sqrt{(1+x)^2 - 1}}$$

For small $\beta(\tau - 1)$, the first few terms of this series suffice, and so we may write

$$\frac{v(\tau)}{2v_o} \approx e^{-\beta(\tau-1)} \left[(1 - \beta)\cosh^{-1}(\tau) + \beta\sqrt{\tau^2 - 1} + + \right]$$

which, for very small $\tau - 1$, becomes

$$\frac{v(\tau)}{2v_o} \approx \sqrt{2(\tau - 1)} \qquad . \tag{23}$$

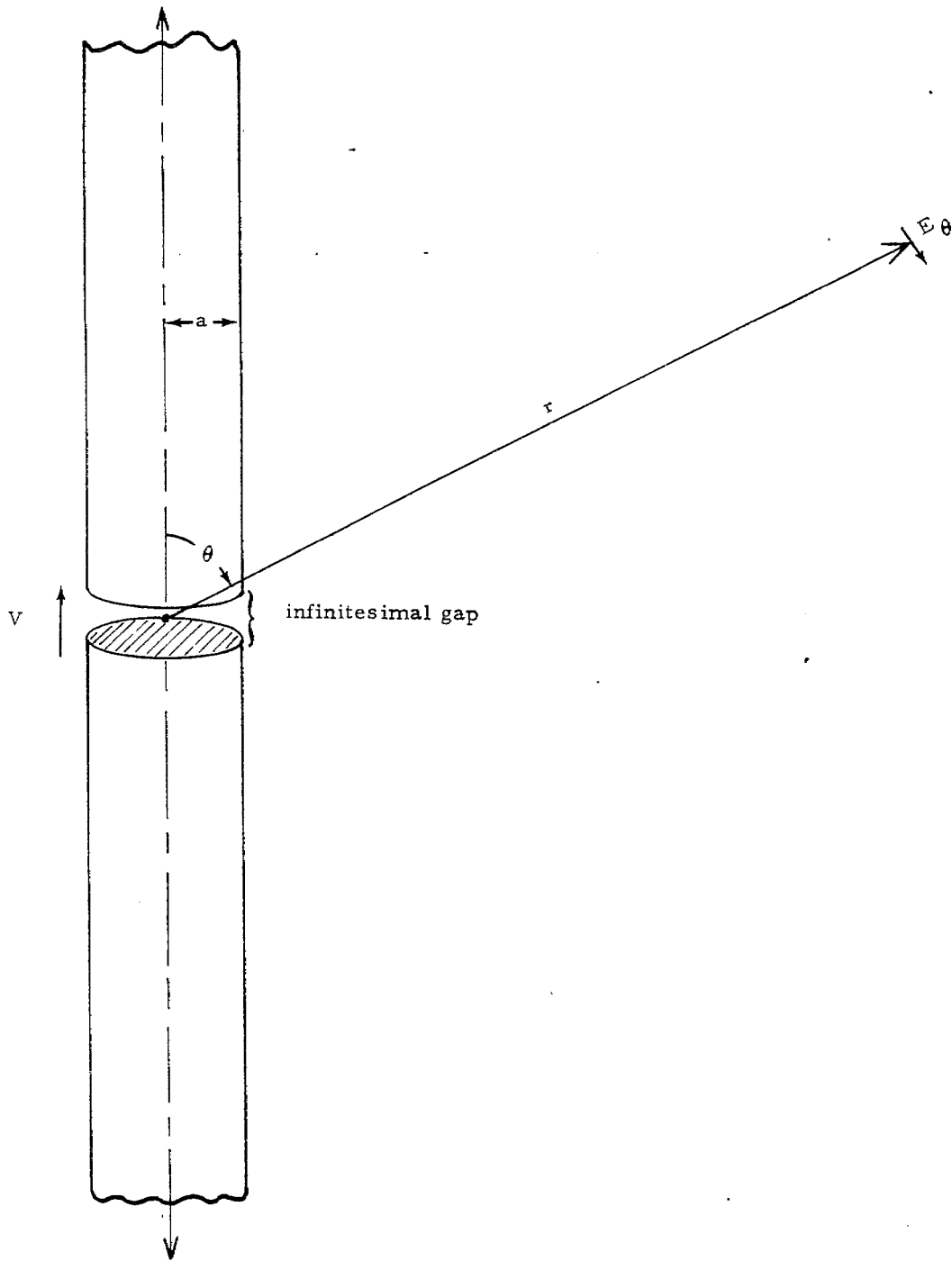Equation (23) is plotted in figure 5, for comparison with the exact results.

17

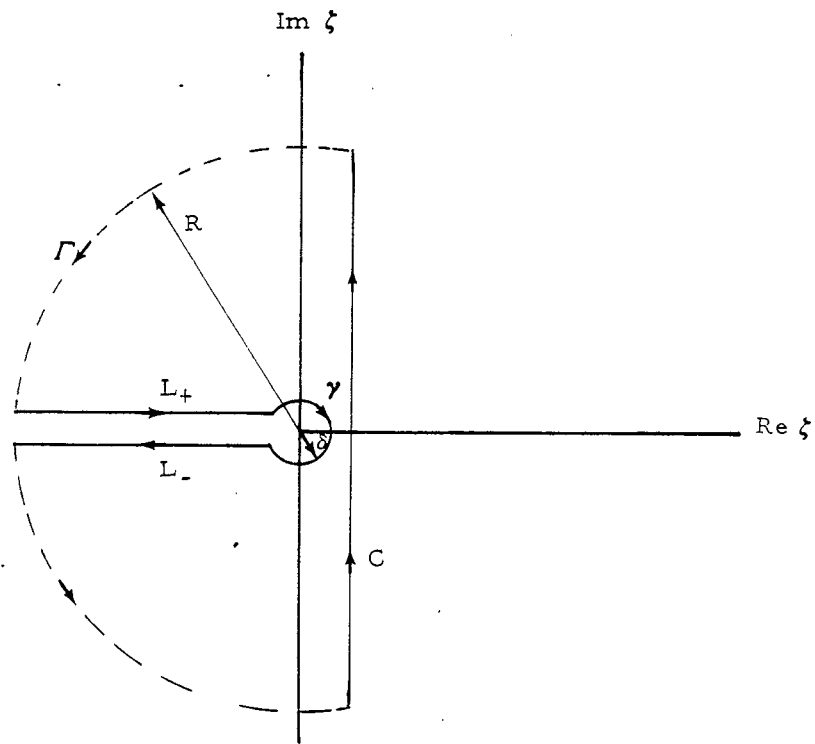Figure 1. Infinite cylindrical antenna with a gap generator.
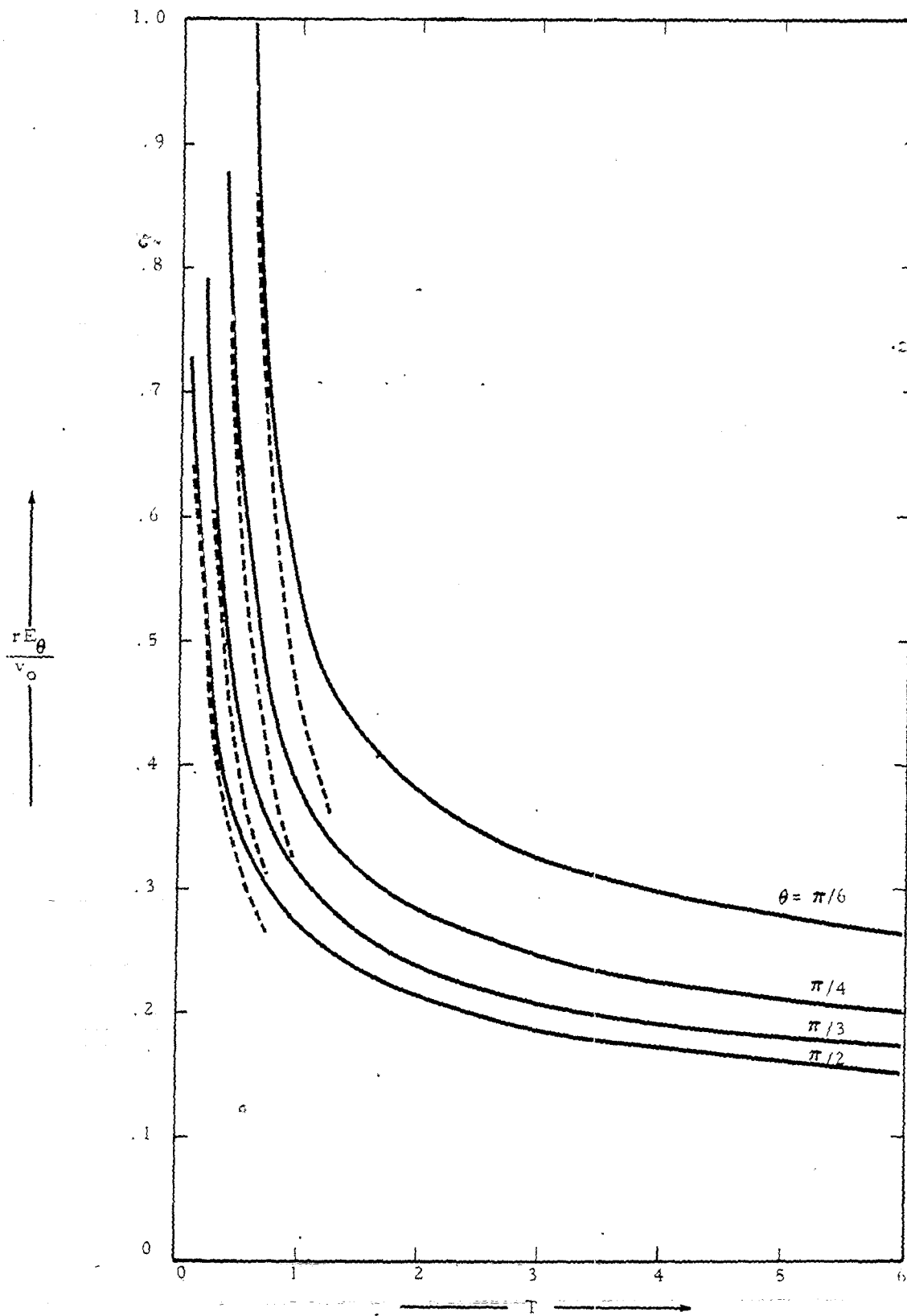
Figure 2. Deformation of path of integration.

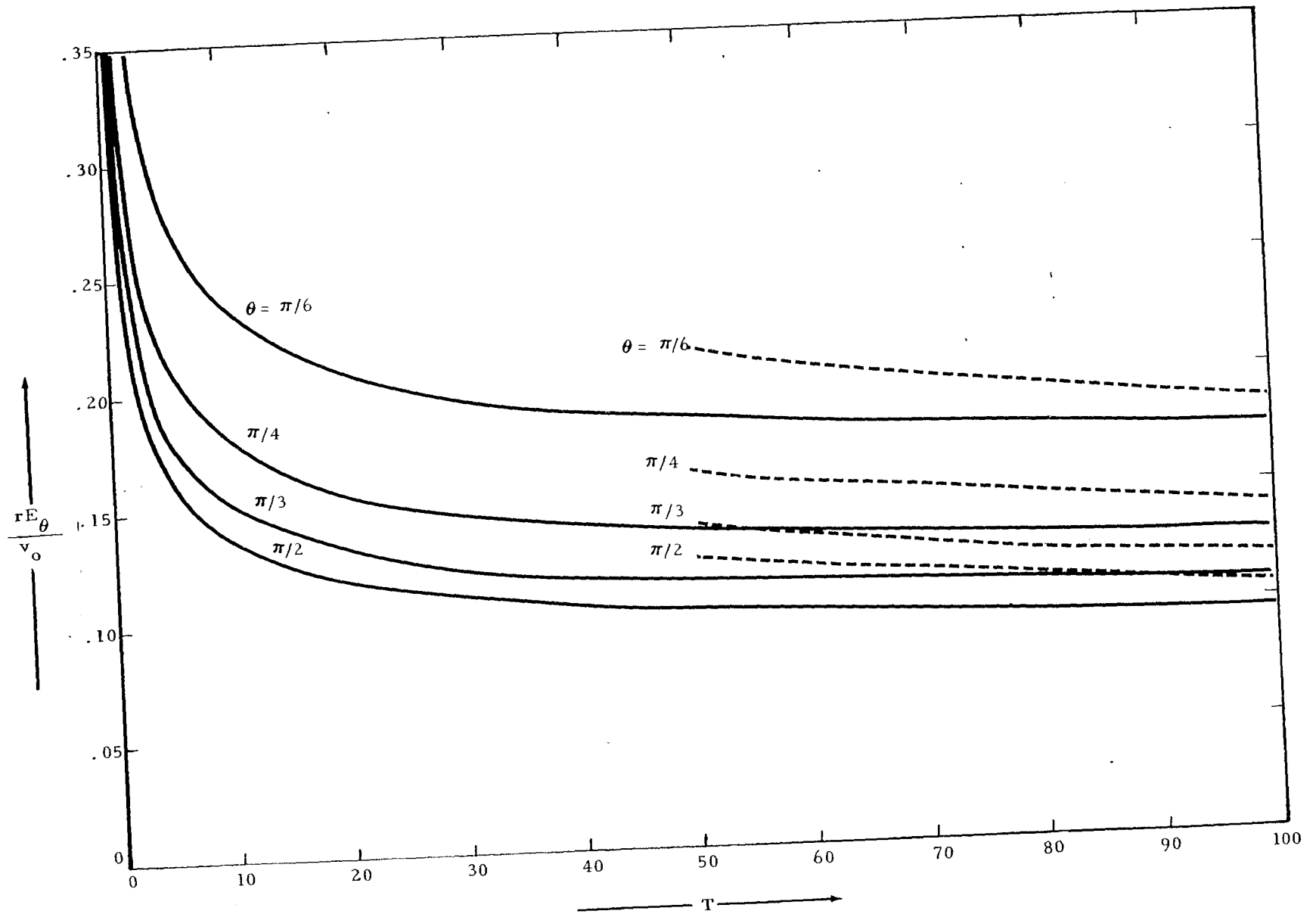Figure 3. Radiation field for a step-function voltage.

$$\frac{rE_\theta}{v_o}$$

Figure 4. Radiation field for a step-function voltage.

$$\frac{v(\tau)}{2v_o} = U(\tau -1)\int_0^{\tau-1} \frac{e^{-\beta\tau'}d\tau'}{\sqrt{(\tau-\tau')^2 -1}}$$

$$\longrightarrow \sqrt{2(\tau - 1)}$$

$$\longrightarrow \frac{1}{\beta\sqrt{\tau^2 -1}} \quad \text{as } \beta \longrightarrow \infty$$
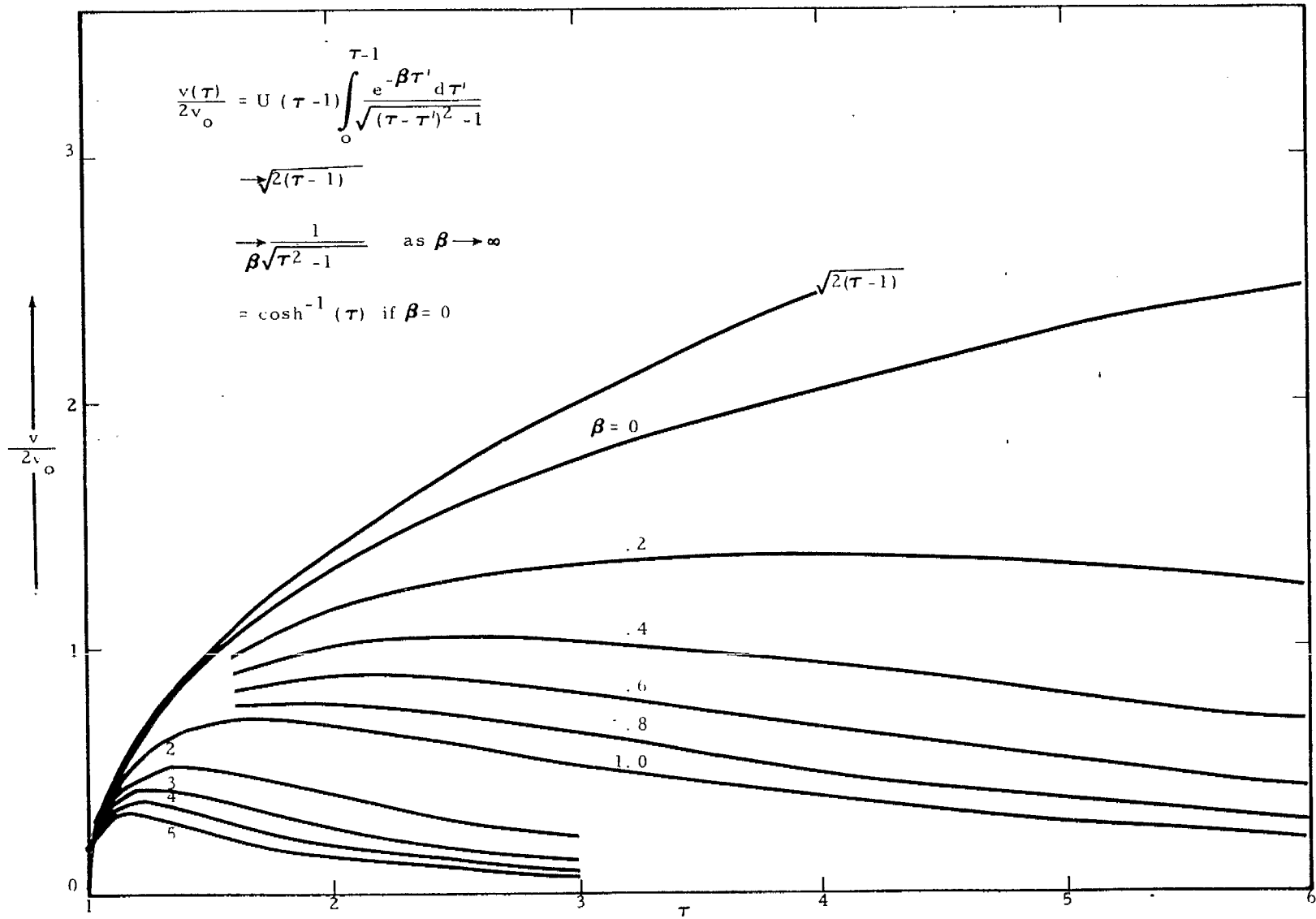
$$= \cosh^{-1}(\tau) \quad \text{if } \beta = 0$$

Figure 5. Gap voltage to radiate a pulse of the form $r^{-1} v_o \, U(\tau - r/a)e^{-(\tau -r/a)}$
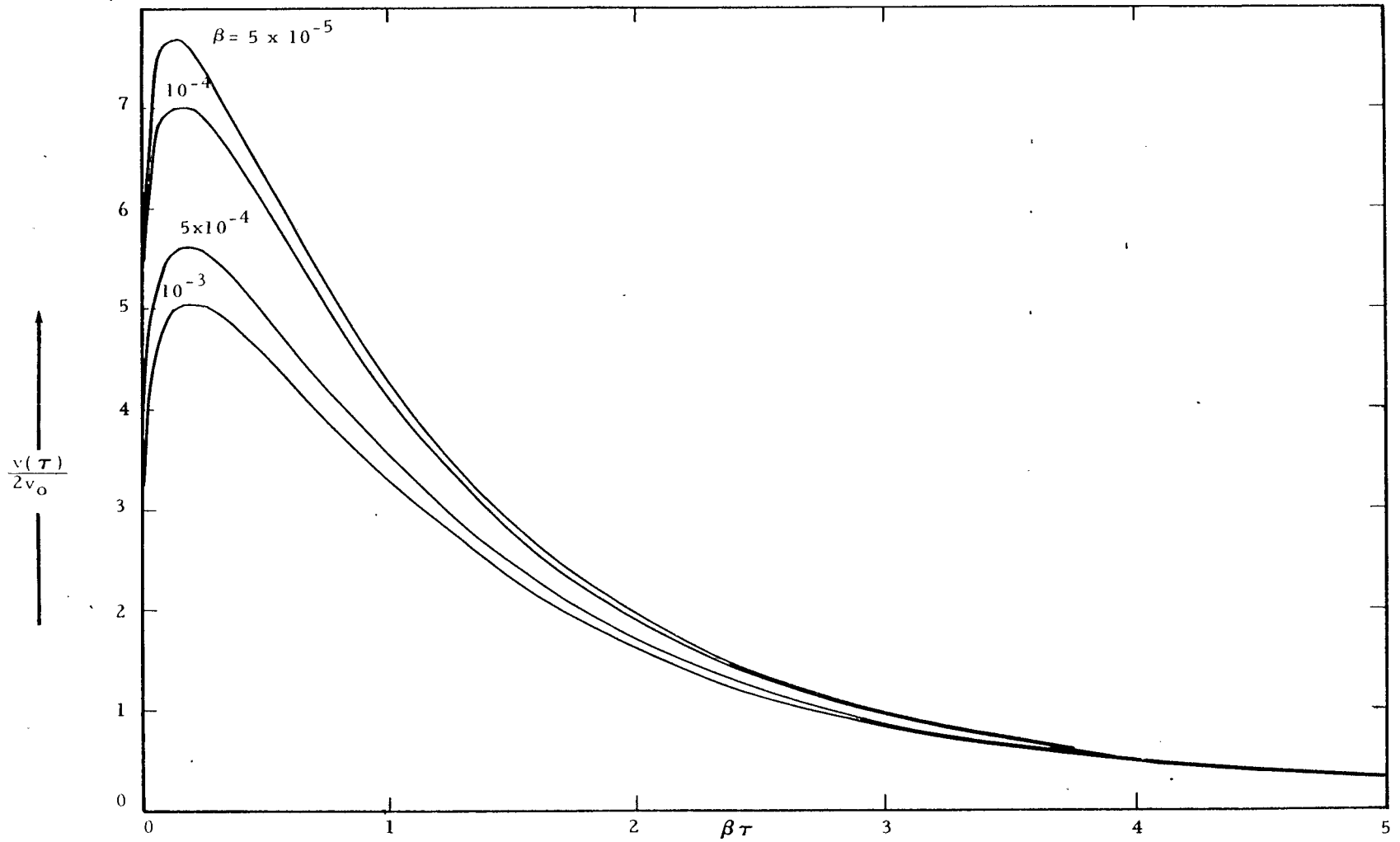
Figure 6. Necessary gap voltage for small β.

References

1. Carl E. Baum, Sensor and Simulation Note 65, "Some Limiting Low-Frequency Characteristics of a Pulse-Radiating Antenna," October 1968.

2. Carl E. Baum, Sensor and Simulation Note 69, "Design of A Pulse-Radiating Antenna As Related to High-Frequency And Low-Frequency Limits," January 1969.

3. Charles H. Papas, "On the Infinitely Long Cylindrical Antenna," J. Appl. Phys., Vol. 20, pp. 437-440, May 1949.

4. G. N. Watson, Theory of Bessel Functions, Cambridge University Press, Cambridge, 1944.

5. Milton Abromowitz and Irene A. Stegun, Editors, Handbook of Mathematical Functions, National Bureau of Standards, AMS-55, June 1964, p. 1028.

6. Ruel V. Churchill, Operational Mathematics, McGraw-Hill, New York, 1958, pp. 35-41.

7. Harold Jeffreys, Asymptotic Approximations, Clarendon Press, Oxford, 1962, pp. 17-18.