NOTE 72

AN EQUIVALENT-CHARGE METHOD FOR DEFINING GEOMETRIES OF DIPOLE ANTENNAS

by

C.E. Baum
Air Force Weapons Laboratory

24 January 1969

Sensor and Simulation Notes
Note 72
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Capt Carl E. Baum
Air Force Weapons Laboratory

Abstract

In this note we discuss a technique for defining the geometry of a dipole antenna such that low-frequency parameters of the antenna, such as capacitance and mean charge separation distance, are readily calculable. This technique involves the assumption of an appropriate static charge distribution which we call an equivalent-charge distribution. Calculating the potential distribution from the equivalent charge one then chooses two appropriate equipotential surfaces as surfaces for the antenna conductors. With the antenna surfaces so chosen the static potential distribution for the antenna is known. Other parameters, such as the antenna charge and dipole moment, are calculated as integrals over the assumed equivalent-charge distribution.

I. Introduction

One of the problems in the design of a dipole antenna for radiating an electromagnetic pulse is to make various of its important electromagnetic parameters readily and accurately calculable. In two previous notes we have discussed some of these electromagnetic parameters, in particular those related to the high-frequency and low-frequency content of the radiated waveform.[1,2] For an axial and lengthwise symmetric dipole the high-frequency end of the frequency domain can be taken care of by including a biconical wave launcher as the central portion of the antenna. A symmetrical biconical wave launcher has well-known characteristics in the high-frequency limit as discussed in reference 2. In the low-frequency limit the important parameters of the dipole are the capacitance and the mean charge separation distance. However, for a typical fat cylindrical dipole it is not easy to calculate these parameters accurately, particularly if the biconical wave launcher is included as part of the structure.

One approach to calculating the capacitance and mean charge separation distance is to consider some particular dipole geometry in a static state with a charge separation on it giving a net dipole moment but with the total or net charge on the antenna equal to zero. Then by a solution of the Laplace equation the charge distribution and potentials can be determined and from these the capacitance and mean charge separation distance can be calculated. Such an approach requires a solution of the Laplace equation for the particular antenna geometry, and in many cases this can be a difficult numerical problem.

In this note we consider an alternative approach to this problem of calculating the low-frequency antenna parameters. This approach involves defining an antenna geometry in such a fashion that these parameters are already determined as part of the procedure used to define the geometry. In this approach we assume some particular static charge distribution in space such that the total charge is zero; we call this an equivalent-charge distribution. The next step is to calculate the potential (or voltage) distribution due to this equivalent-charge distribution. Consider two of the equipotential surfaces which define two separate closed volumes (one surface not enclosing the other surface); also consider only surfaces such that the equivalent charge is completely contained within the two volumes. Then by making all of both surfaces as conducting surfaces we have defined an antenna geometry, consisting of two conducting surfaces for which the potential distribution is known. As will be discussed, the charge on the antenna and the dipole moment of the antenna can be found as integrals over the equivalent-charge distribution. The antenna capacitance and mean charge separation distance are then known.

1. Capt Carl E. Baum, Sensor and Simulation Note 65, Some Limiting Low-Frequency Characteristics of a Pulse-Radiating Antenna, October 1968.
2. Capt Carl E. Baum, Sensor and Simulation Note 69, Design of a Pulse-Radiating Dipole Antenna as Related to High-Frequency and Low-Frequency Limits, January 1969.

Also in this note we consider a form of the equivalent-charge distribution such that near the center of the antenna the structure approximates a biconical wave launcher. Finally, as an example, we consider a family of antenna geometries defined by a particular equivalent-charge distribution.

II. Equivalent-Charge Method

For reference in the calculations and discussions which follow we have cylindrical (Ψ,ϕ,z) and spherical (r,θ,ϕ) coordinate systems and an example of an antenna geometry in figure 1. As illustrated in figure 1 the antenna has axial and lengthwise symmetry but this is a specialization which is introduced later. Initially, we consider the antenna geometry to be somewhat more general.

To begin the development of the equivalent-charge method for defining the antenna geometry assume some static charge distribution which we call the equivalent-charge density ρ_{eq}. Using $\vec{r}$ as the position vector at which the potential is calculated and $\vec{r}'$ as the position vector for ρ_{eq} (both $\vec{r}$ and $\vec{r}'$ having the same origin) we have the electric potential function as

$$\Phi = \int_{V'} \frac{\rho_{eq}(\vec{r}')dV'}{4\pi\epsilon_o |\vec{r}-\vec{r}'|} \tag{1}$$

Note that we assume the medium to have the same parameters as free space, ϵ_o being the permittivity of free space. V' is a volume completely containing ρ_{eq}. We also restrict the equivalent-charge density to a distribution satisfying

$$\int_{V'} \rho_{eq}(\vec{r}')dV' = 0 \tag{2}$$

so that the net equivalent charge is zero.

Now define equipotential surfaces by $\Phi = \Phi_+$ and $\Phi = \Phi_-$ where for convenience we choose $\Phi_+ > \Phi_-$. Call these surfaces S_+ and S_-, respectively. Next assume that Φ_+ and Φ_- have been chosen such that S_+ and S_- each describe a single closed surface, finite in extent, neither of which encloses the other. We do, however, allow the surfaces to touch at a finite number of isolated points at which the electric field becomes singular. Such a case occurs in the mathematical idealization of a biconical wave launcher where the two cones have a common apex. Of course, for a real antenna the two surfaces do not quite touch. Denote the volume enclosed by S_+ as V_+ and the volume enclosed by S_- as V_- where V_+ and V_- are now distinct volumes. Finally, assume that ρ_{eq} is completely contained in these two volumes, V_+ and V_-. This last assumption may restrict the form of ρ_{eq} and/or the values which are allowable for Φ_+ and Φ_-.

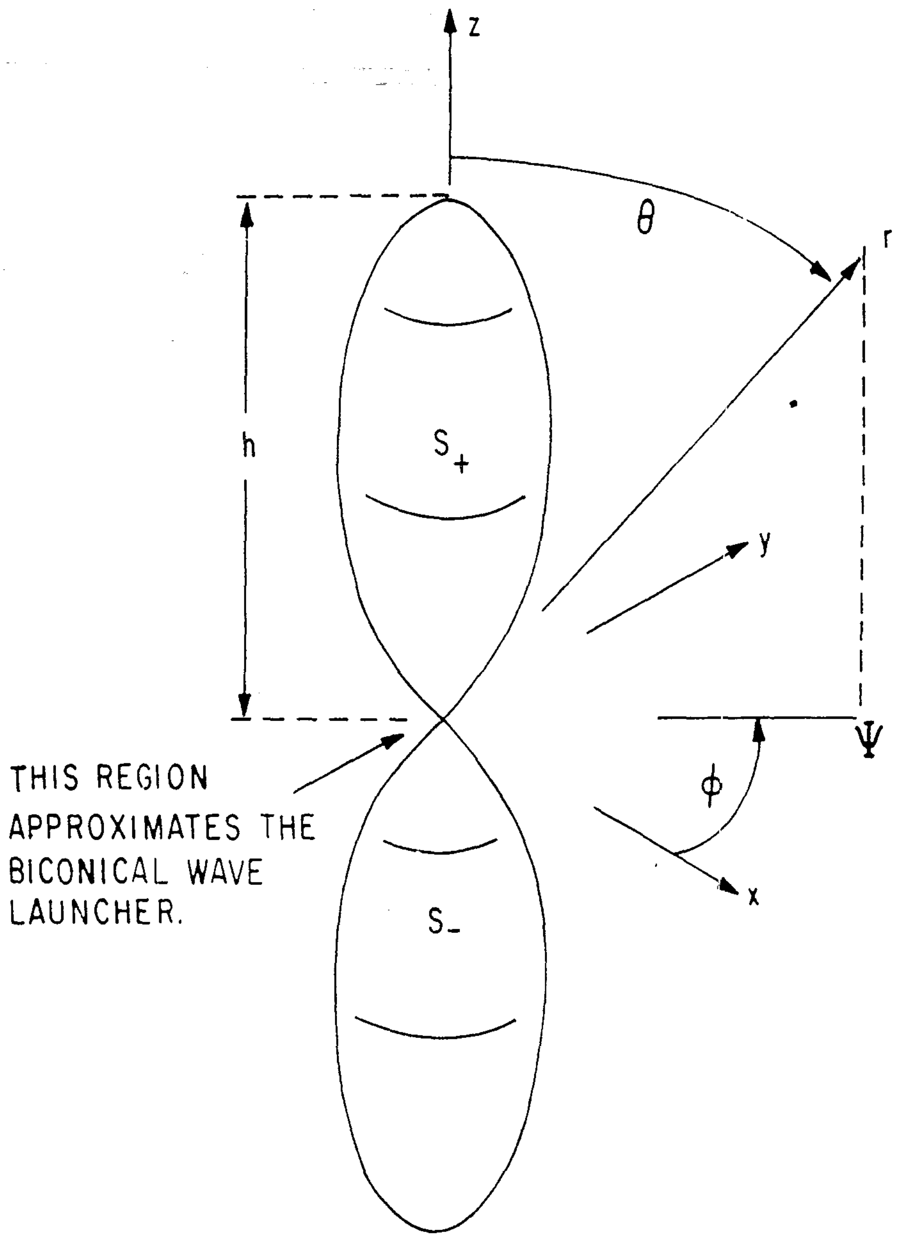

FIGURE 1. ANTENNA SURFACES WITH COORDINATE SYSTEMS

Having defined the surfaces S_+ and S_- suppose we place perfect conductors on both of these surfaces so that both surfaces are completely perfectly conducting. Both surfaces were chosen as equipotential surfaces so that if we maintain the two conducting surfaces at potentials Φ_+ and Φ_-, respectively, then the original potential distribution is undisturbed. Now none of the equivalent charge is in the exterior volume, i.e., all space less V_+ and V_-. Thus, the potential distribution in the exterior volume satisfies the Laplace equation and assumes the potentials Φ_+ and Φ_- on S_+ and S_-, respectively, and goes to zero as $|\vec{r}| \to \infty$ (from equation 1). Ignoring ρ_{eq} for the moment, then if we maintain the conducting surfaces S_+ and S_- at potentials Φ_+ and Φ_-, respectively, and the potential zero at infinity, the potential in the exterior volume must have the same solution as before, i.e., that calculated by equation 1. Thus, we use the assumed ρ_{eq} as an artifice to calculate the potential distribution of equation 1, to which we fit conducting surfaces on equipotentials to determine the antenna geometry. Then as long as all the equivalent charge is inside (or on) the antenna surfaces S_+ and S_-, the equivalent charge can be removed and the two surfaces can be used as conducting boundaries fitting a now known solution of the Laplace equation.

Now if S_+ and S_- are given potentials Φ_+ and Φ_-, respectively, we can calculate the charge on each surface by noting that the displacement vector is normal to each surface and is equal in magnitude to the surface charge density on each of the conducting surfaces. The displacement vector can also be calculated from Φ as

$$\vec{D} = -\epsilon_o \nabla\Phi \tag{3}$$

But from equation 1 and Gauss' theorem we can calculate the surface integral of $\vec{D}$ as the volume integral of ρ_{eq}. Thus, the charge on S_+ is given by

$$Q_a = \int_{S_+} \vec{D}\cdot\vec{n}\, dS = \int_{V_+} \rho_{eq}(\vec{r}\,')\, dV' \tag{4}$$

where $\vec{n}$ is the outward-pointing normal of the surface. The charge on S_- is just $-Q_a$ since the net equivalent charge is zero as in equation 2; any equivalent charge not within or on S_+ is within or on S_-. Thus, from ρ_{eq} we have not only the potential distribution but also the charge on each part of the antenna. Defining the antenna voltage as

$$V_a \equiv \Phi_+ - \Phi_- \tag{5}$$

we have the antenna capacitance as

$$C_a = \frac{Q_a}{V_a} = \frac{1}{\Phi_+ - \Phi_-} \int_{V_+} \rho_{eq}(r')\, dV' \tag{6}$$

With ρ_{eq} chosen and Φ_+ and Φ_- chosen we have then both fixed the antenna geometry (implicitly from equation 1) and have a formula (equation 6) for calculating the antenna capacitance.

Now we need to calculate the mean charge separation distance for the antenna. To do this we start with some static charge density ρ and calculate the potential distribution as

$$\Phi = \int_{V'} \frac{\rho(\vec{r}\,')\,dV'}{4\pi\epsilon_o|\vec{r}-\vec{r}\,'|} \tag{7}$$

Here ρ may be either ρ_{eq} or the charge on our antenna with surfaces S_+ and S_- with potentials Φ_+ and Φ_-, respectively. In the latter case one might express equation 7 as surface integrals over S_+ and S_- using a surface charge density ρ_s giving

$$\Phi = \int_{S_+ + S_-} \frac{\rho_s(\vec{r}\,')\;dS'}{4\pi\epsilon_o|\vec{r}-\vec{r}\,'|} \tag{8}$$

Defining

$$r \equiv |\vec{r}| \qquad , \qquad r' \equiv |\vec{r}\,'| \tag{9}$$

and defining ψ as the angle between $\vec{r}$ and $\vec{r}\,'$ so that

$$\vec{r}\cdot\vec{r}\,' = rr'\cos(\psi) \tag{10}$$

we can write

$$|\vec{r}-\vec{r}\,'|^2 = r^2 + r'^2 - 2\vec{r}\cdot\vec{r}\,' = r^2 + r'^2 - 2rr'\cos(\psi) \tag{11}$$

Then as $r \to \infty$ we have the asymptotic form

$$\begin{aligned}\frac{1}{|\vec{r}-\vec{r}\,'|} &= \frac{1}{r}\left[1 + \frac{r'^2}{r^2} - 2\frac{r'}{r}\cos(\psi)\right]^{-1/2} \\ &= \frac{1}{r}\left[1 - 2\frac{r'}{r}\cos(\psi) + 0(r^{-2})\right]^{-1/2} \\ &= \frac{1}{r} + \frac{r'}{r^2}\cos(\psi) + 0(r^{-3})\end{aligned} \tag{12}$$

Thus as $r \to \infty$ equation 7 has the asymptotic form

$$\Phi = \frac{1}{4\pi\varepsilon_o r} \int_{V'} \rho(\vec{r}\,')\, dV' + \frac{1}{4\pi\varepsilon_o r^2} \int_{V'} r'\cos(\psi)\rho(\vec{r}\,')\, dV' + O(r^{-3}) \tag{13}$$

The first integral is zero for both the equivalent charge by assumption and for the charge on the antenna with surfaces S_+ and S_- by our above discussion. Rewriting the second integral we have as $r \to \infty$

$$\Phi = \frac{\vec{r}}{4\pi\varepsilon_o r^3} \cdot \int_{V'} \vec{r}\,'\rho(\vec{r}\,')\, dV' + O(r^{-3}) \tag{14}$$

Define the unit vector in the $\vec{r}$ direction as $\vec{e}_r$ and note that the above integral is the electric dipole moment, i.e.,

$$\vec{p} \equiv \int_{V'} \vec{r}\,'\,\rho(\vec{r}\,')\, dV' \tag{15}$$

If a surface charge density is used then the electric dipole moment is expressed as a surface integral (over $S_+ + S_-$). We then have as $\vec{r} \to \infty$

$$\Phi = \frac{1}{4\pi\varepsilon_o r^2}\, \vec{e}_r \cdot \vec{p} + O(r^{-3}) \tag{16}$$

The electric dipole moment then gives the r^{-2} term in the expansion of the potential for large r. The potential at large r (and thus exterior to V_+ and V_-) is identical for the two cases: the equivalent-charge formulation as in equation 1 and the antenna with surfaces S_+ and S_- at potentials Φ_+ and Φ_-, respectively. Then $\vec{e}_r \cdot \vec{p}$ must be the same for both cases. Since the direction vector $\vec{e}_r$ is arbitrary and can assume any direction then $\vec{p}$ must be the same for both cases. Therefore, we can calculate the static electric dipole moment for our dipole antenna as an integral over the equivalent-charge density giving

$$\vec{p} = \int_{V'} \vec{r}\,'\rho_{eq}(\vec{r}\,')\, dV' \tag{17}$$

If desired one can continue the asymptotic expansion of Φ for large r obtaining the coefficients of r^{-3}, r^{-4}, etc. Since Φ is the same for both ρ_{eq} and the antenna at large r then these coefficients of the higher order terms must also be the same for both cases and can then be calculated for the antenna by using ρ_{eq}. For our present purposes, however, we are only concerned with the electric dipole moment because this gives us the late-time dipole moment of a pulse-radiating dipole as discussed in references 1 and 2 provided V_a (as in equation 5) is the late-time antenna voltage. The late-time dipole moment is important for the low-frequency content of the radiated waveform.

Having the electric dipole moment of our antenna as in equation 17 we can now calculate the mean charge separation distance of the antenna as

$$\vec{h}_a \equiv \frac{\vec{p}}{Q_a} = \frac{\int_{V'} \vec{r}'\rho_{eq}(\vec{r}')\, dV'}{\int_{V_+} \rho_{eq}(\vec{r}')\, dV'} \tag{18}$$

Note that the volume of integration for the integral in the numerator is $V' = V_+ + V_-$ thereby including all of ρ_{eq} while the denominator only includes that portion of ρ_{eq} which is contained in V_+ (possibly including the corresponding boundary surfaces in each case). In reference 2 we showed that $\vec{h}_a$ was the same as $\vec{h}_{eq}$, the equivalent height of the dipole antenna when considered as a low-frequency electric field sensor. Thus, a dipole antenna defined by the equivalent-charge method also has a readily calculable equivalent height, a desirable feature for an electric field sensor.

As a special case of interest we can use the equivalent-charge method to define an axially and lengthwise symmetric dipole antenna. Simply constrain ρ_{eq} to be independent of ϕ' and to be odd in z' so that we have an equivalent charge density of the form

$$\rho_{eq}(\Psi', z') = -\rho_{eq}(\Psi', -z') \tag{19}$$

Then Φ in equation 1 is also independent of ϕ and odd in z. Choose the antenna surfaces by a single parameter $\Phi_s > 0$ such that

$$\Phi_s = \Phi_+ = -\Phi_- \tag{20}$$

giving an antenna voltage from equation 5 as

$$V_a = 2\Phi_s \tag{21}$$

From equation 20 the antenna surfaces are now symmetric in z and independent of ϕ. The antenna capacitance is

$$C_a = \frac{1}{2\Phi_s} \int_{V_+} \rho_{eq}(\vec{r}\,')\, dV' \tag{22}$$

and the mean charge separation distance in equation 18 can be rewritten as

$$\vec{h}_a = h_a \vec{e}_z \tag{23}$$

where

$$h_a = |\vec{h}_a| \tag{24}$$

and

$$h_a = \frac{\int_{V'} z' \rho_{eq}(\vec{r}\,')\, dV'}{\int_{V_+} \rho_{eq}(\vec{r}\,')\, dV'} = \frac{2\int_{V_+} z' \rho_{eq}(\vec{r}\,')\, dV'}{\int_{V_+} \rho_{eq}(\vec{r}\,')\, dV'} \tag{25}$$

and where $\vec{e}_z$ is the unit vector in the z direction. Thus for an axially and lengthwise symmetric dipole antenna the equivalent-charge method has somewhat simpler forms for the important equations.

III. Inclusion of Biconical Wave Launcher as Part of Dipole Geometry

For the case of the axially and lengthwise symmetric dipole antenna we would like to include a symmetrical biconical wave launcher near the origin ($\vec{r} = \vec{0}$). As it will turn out one way to accomplish this is to have the equivalent charge near the origin in the form of a line charge density on the z axis with a step discontinuity at $z' = 0$.

Consider the equivalent charge as a line charge density $\lambda(z')$ on the z axis and odd in z' so that

$$\lambda(z') = -\lambda(-z') \tag{26}$$

Then the potential function is

$$\Phi = \int_{-\infty}^{\infty} \frac{\lambda(z')\, dz'}{4\pi\varepsilon_o |\vec{r}-\vec{r}\,'|} = \frac{1}{4\pi\varepsilon_o} \int_{-\infty}^{\infty} \frac{\lambda(z')\, dz'}{[(z-z')^2+\psi^2]^{1/2}}$$

$$= \frac{1}{4\pi\varepsilon_o} \int_0^{\infty} \left\{ [(z-z')^2+\psi^2]^{-1/2} - [(z+z')^2+\psi^2]^{-1/2} \right\} \lambda(z')\, dz' \tag{27}$$

where we have taken advantage of the assumption that λ is odd in z'. Note that we let the integral over z' extend from $-\infty$ to $+\infty$ for simplicity in the present calculations. For the case of a finite sized antenna $\lambda(z')$ goes to zero for z' larger than some particular value of z', say $z_o > 0$.

Now for $z' > 0$ let λ be a constant $\lambda_o > 0$. For $z' = 0$ let $\lambda = 0$. This gives a step discontinuity in λ of magnitude $2\lambda_o$ at $z' = 0$. Let

$$\eta_1 = \frac{z'-z}{\Psi} \quad , \qquad \eta_2 = \frac{z'+z}{\Psi} \tag{28}$$

so that we have

$$\Phi = \lim_{\nu\to\infty} \frac{\lambda_o}{4\pi\epsilon_o}\left\{\int_{-\frac{z}{\Psi}}^{\nu} [1+\eta_1^2]^{-1/2}\, d\eta_1 - \int_{\frac{z}{\Psi}}^{\nu} [1+\eta_2^2]^{-1/2}\, d\eta_2\right\}$$

$$= \frac{\lambda_o}{4\pi\epsilon_o}\int_{-\frac{z}{\Psi}}^{\frac{z}{\Psi}} [1+\eta^2]^{-1/2}\, d\eta$$

$$= \frac{\lambda_o}{2\pi\epsilon_o} \operatorname{arcsinh}\left(\frac{z}{\Psi}\right) \tag{29}$$

Note that we have introduced ν as the upper limit on the integrals and let $\nu\to\infty$ so that the resulting integral could be evaluated in the sense of a principal value. Otherwise, the integral would be divergent as the limits on z' go to $\pm\,\infty$.

We now see for this special form of λ that the equipotentials are circular cones centered on the z axis since Φ is a function only of z/Ψ. Now we have

$$\frac{z}{\Psi} = \cot(\theta) \tag{30}$$

Then using identities for arcsinh and for the trigonometric functions the potential function has the forms

$$\Phi = \frac{\lambda_o}{2\pi\epsilon_o} \ln\left[\frac{z}{\Psi} + \left(\left(\frac{z}{\Psi}\right)^2 + 1\right)^{1/2}\right]$$

$$= \frac{\lambda_o}{2\pi\epsilon_o} \ln\left[\cot\left(\frac{\theta}{2}\right)\right] \tag{31}$$

Setting $\Phi = \Phi_s$ to determine the upper antenna surface, define θ_o as that value of θ for this surface so that

$$\Phi_s = \frac{\lambda_o}{2\pi\varepsilon_o} \ln \left[\cot\left(\frac{\theta_o}{2}\right)\right] \tag{32}$$

For the lower antenna surface set $\Phi = -\Phi_s$ which makes $\theta = \pi - \theta_o$ on the lower surface. Thus, we have defined a symmetrical biconical structure, infinite in extent.

In this section we have considered a special form of $\lambda(z')$ such that the antenna surfaces were biconical and infinite in extent. For a finite-sized antenna we need some form of equivalent-charge distribution which is confined to a finite region of space. Note with a line charge density $\lambda(z')$ that Φ tends to $+\infty$ or $-\infty$ as the line charge is approached, depending on whether λ is positive or negative at the position of interest. However, near a step discontinuity in $\lambda(z')$ of the form used in this section one can approach the line charge on a path with constant θ (except for 0 or π) and Φ does not diverge. Then one might expect that near such a step discontinuity in a more general line charge distribution the equipotentials would exhibit a similar behavior near the discontinuity. By this procedure one can define an antenna surface which asymptotically approaches a biconical structure as the coordinate origin is approached. This is illustrated in the next section by assuming another form for $\lambda(z')$ while retaining the step discontinuity at $z' = 0$.

For convenience we define some parameters based on the results of this section. Let $\lambda_o > 0$ be the value of $\lambda(z')$ for $z' > 0$ as $z' \to 0$. With Φ_s chosen to define the antenna surfaces we can let θ_o be defined by equation 32. Also we define

$$\Theta_o \equiv \tan\left(\frac{\theta_o}{2}\right) \tag{33}$$

Then we can use θ_o and Θ_o as parameters, when convenient, in calculating families of antenna surfaces based on equivalent-charge distributions which have forms near the coordinate origin like that used in this section.

IV. Example: Equivalent-Charge Distribution Chosen as a Particular Line Charge Density

For an axially and lengthwise symmetric antenna geometry which has a biconical shape near the coordinate origin, consider an equivalent line charge $\lambda(z')$ (on the z axis) given by

$$\lambda(z') = \begin{cases} \lambda_o & \text{for } 0 < z' \leq z_o \\ -\lambda_o & \text{for } 0 > z' \geq -z_o \\ 0 & \text{for } z' = 0 \\ 0 & \text{for } |z'| > z_o \end{cases} \tag{34}$$

where $z_o > 0$ and $\lambda_o > 0$.

A natural way to normalize the time variable in this equation is to define the time of propagation across the radius of the cylinder to be

$$t_a = a/c \quad ,$$

and to set

$$\tau = t/t_a \quad .$$

In this normalized time domain, equation (16) becomes simply

$$v(\tau) = 2\int_0^{\tau-1} \frac{f(\tau')d\tau'}{\sqrt{(\tau-\tau')^2-1}} \quad . \tag{17}$$

We apply (17) first to the simple case of a radiated step function,

$$f(\tau) = v_o U(\tau) \quad .$$

The gap voltage needed to generate this field will be given by

$$\frac{v(\tau)}{2v_o} = \int_0^{\tau-1} \frac{U(\tau')d\tau'}{\sqrt{(\tau-\tau')^2-1}}$$

$$= \int_1^{\tau} \frac{U(\tau-x)dx}{\sqrt{x^2-1}} \qquad (x = \tau - \tau')$$

$$= U(\tau-1)\cosh^{-1}(\tau) \tag{18}$$

Substitute

$$\eta = \frac{z'-z}{\Psi} \tag{41}$$

giving

$$\frac{4\pi\epsilon_o}{\lambda_o}\Phi = \int_{-\frac{z}{\Psi}}^{\frac{z_o-z}{\Psi}} [1+\eta^2]^{-1/2}\, d\eta - \int_{\frac{-z_o-z}{\Psi}}^{-\frac{z}{\Psi}} [1+\eta^2]^{-1/2}\, d\eta$$

$$= \operatorname{arcsinh}\left(\frac{z_o-z}{\Psi}\right) - \operatorname{arcsinh}\left(-\frac{z}{\Psi}\right) - \operatorname{arcsinh}\left(-\frac{z}{\Psi}\right) + \operatorname{arcsinh}\left(\frac{-z_o-z}{\Psi}\right)$$

$$= 2\operatorname{arcsinh}\left(\frac{z}{\Psi}\right) - \operatorname{arcsinh}\left(\frac{z+z_o}{\Psi}\right) - \operatorname{arcsinh}\left(\frac{z-z_o}{\Psi}\right) \tag{42}$$

Now set $\Phi = \pm\ \Phi_s$. This gives an implicit relationship between z and Ψ which gives the shape of the antenna surface. Since the antenna is symmetric in z we can consider only the portion for $z > 0$. Then setting $\Phi = \Phi_s$ and substituting from equations 32 and 33 and rewriting arcsinh in logarithmic form we have

$$-2\ \ln(\Theta_o) = 2\ \ln\left[\frac{z}{\Psi}+\left(\left(\frac{z}{\Psi}\right)^2+1\right)^{1/2}\right] - \ln\left[\frac{z+z_o}{\Psi}+\left(\left(\frac{z+z_o}{\Psi}\right)^2+1\right)^{1/2}\right] - \ln\left[\frac{z-z_o}{\Psi}+\left(\left(\frac{z-z_o}{\Psi}\right)^2+1\right)^{1/2}\right] \tag{43}$$

which can also be written as

$$\Theta_o^{-2} = \frac{\left[\frac{z}{\Psi}+\left(\left(\frac{z}{\Psi}\right)^2+1\right)^{1/2}\right]^2}{\left[\frac{z+z_o}{\Psi}+\left(\left(\frac{z+z_o}{\Psi}\right)^2+1\right)^{1/2}\right]\left[\frac{z-z_o}{\Psi}+\left(\left(\frac{z-z_o}{\Psi}\right)^2+1\right)^{1/2}\right]}$$

$$= \frac{[z+(z^2+\Psi^2)^{1/2}]^2}{[z+z_o+((z+z_o)^2+\Psi^2)^{1/2}][z-z_o+((z-z_o)^2+\Psi^2)^{1/2}]} \tag{44}$$

Using Θ_o as the parameter for the antenna geometry we can then find Ψ as a function of z for any particular Θ_o for $0 < \Theta_o < 1$ thereby determining the antenna shape.

In yet another form the equation defining the antenna geometry can be written

$$\Theta_o^{-2} = \frac{\left[\frac{z}{z_o} + \left(\left(\frac{z}{z_o}\right)^2 + \left(\frac{\Psi}{z_o}\right)^2\right)^{1/2}\right]^2}{\left[\frac{z}{z_o} + 1 + \left(\left(\frac{z}{z_o} + 1\right)^2 + \left(\frac{\Psi}{z_o}\right)^2\right)^{1/2}\right]\left[\frac{z}{z_o} - 1 + \left(\left(\frac{z}{z_o} - 1\right)^2 + \left(\frac{\Psi}{z_o}\right)^2\right)^{1/2}\right]} \tag{45}$$

Let $z = h$ where the antenna surface intersects the z axis for $z > 0$. Setting $z = h$ and $\Psi = 0$ in equation 45 gives

$$\Theta_o^{-2} = \frac{\left(\frac{h}{z_o}\right)^2}{\left(\frac{h}{z_o} + 1\right)\left(\frac{h}{z_o} - 1\right)} = \frac{\left(\frac{h}{z_o}\right)^2}{\left(\frac{h}{z_o}\right)^2 - 1} \, . \tag{46}$$

Solving for h/z_o we have

$$\frac{h}{z_o} = [1 - \Theta_o^2]^{-1/2} \tag{47}$$

or in terms of θ_o we have

$$\frac{h}{z_o} = \left[1 - \tan^2\left(\frac{\theta_o}{2}\right)\right]^{-1/2} = \left[1 - \frac{1 - \cos(\theta_o)}{1 + \cos(\theta_o)}\right]^{-1/2}$$

$$= \left[\frac{1 + \cos(\theta_o)}{2\cos(\theta_o)}\right]^{1/2} = \left[\frac{\sec(\theta_o) + 1}{2}\right]^{1/2} \tag{48}$$

For the present example h is the half length of the antenna since the maximum z for this antenna surface occurs on the z axis as can be seen in the later plots of the antenna shape. This can be seen analytically by calculating $\partial\Phi/\partial\Psi$ for $z = h > z_o$ and noting that this derivative is strictly negative for $\Psi > 0$. For a fixed $z = h$ then $\Phi = \Phi_s$ only at $\Psi = 0$ making this point a maximum in z for the surface.

Having the antenna half length as a function of the antenna shape parameter (θ_o or Θ_o) we can now relate the other antenna parameters to h instead of z_o. From equation 37 the antenna capacitance, in normalized form, can be written as

$$\frac{C_a}{\varepsilon_o h} = -\pi \frac{z_o}{h}\left[\ln(\Theta_o)\right]^{-1} = -\pi \frac{\left[1-\Theta_o^2\right]^{1/2}}{\ln(\Theta_o)} \tag{49}$$

The mean charge separation distance can be written in normalized form as

$$\frac{h_a}{h} = \frac{z_o}{h} = [1-\Theta_o^2]^{1/2} \tag{50}$$

These two forms are used in reference 2 as convenient parameters for considering the low-frequency characteristics of a pulse-radiating dipole antenna. In the case that the generator capacitance is much larger than the antenna capacitance we also have another useful parameter from reference 2 as

$$f'_\infty \equiv \frac{1}{4\pi}\frac{h_a}{h}\frac{C_a}{\varepsilon_o h} = -\frac{1}{4}\frac{1-\Theta_o^2}{\ln(\Theta_o)} \tag{51}$$

In terms of Θ_o the low-frequency antenna parameters then have rather simple forms.

Another point of interest is the shape of the antenna surface near the coordinate origin. To see this rewrite equation 42 for $\Phi = \Phi_s$ as

$$\frac{4\pi\varepsilon_o}{\lambda_o}\Phi_s = -2\ln(\Theta_o) = 2\,\text{arcsinh}\left(\frac{z}{\Psi}\right) - \text{arcsinh}\left(\frac{z_o+z}{\Psi}\right) + \text{arcsinh}\left(\frac{z_o-z}{\Psi}\right) \tag{52}$$

Converting to logarithmic form and then algebraic form, as in equation 44, gives

$$\Theta_o^{-1} = \left[\frac{z}{\Psi} + \left(\left(\frac{z}{\Psi}\right)^2 + 1\right)^{1/2}\right]\left[\frac{z_o-z+((z_o-z)^2+\Psi^2)^{1/2}}{z_o+z+((z_o+z)^2+\Psi^2)^{1/2}}\right]^{1/2} \tag{53}$$

Now let z go to zero. The second factor on the right side of equation 53 goes to 1. Thus, we have

$$\lim_{z\to 0}\left[\frac{z}{\Psi} + \left(\left(\frac{z}{\Psi}\right)^2 + 1\right)^{1/2}\right] = \Theta_o^{-1} \tag{54}$$

Rewriting z/Ψ in terms of θ as in equation 30 and similarly with Θ_o from equation 33 we have

$$\lim_{z\to 0}\cot\left(\frac{\theta}{2}\right) = \cot\left(\frac{\theta_o}{2}\right), \quad \lim_{z\to 0}\theta = \theta_o \tag{55}$$

$$\tau' = \tau - 1 - x$$

to get

$$\frac{v(\tau)}{2v_o} = e^{-\beta(\tau-1)} \int_o^{\tau-1} \frac{e^{\beta x} dx}{\sqrt{(1 + x)^2 - 1}} \quad .$$

Thus

$$\frac{v(\tau)}{2v_o} = e^{-\beta(\tau-1)} \sum_{n=o}^{\infty} \frac{\beta^n}{n!} \int_o^{\tau-1} \frac{x^n dx}{\sqrt{(1 + x)^2 - 1}} \quad .$$

For small $\beta(\tau - 1)$, the first few terms of this series suffice, and so we may write

$$\frac{v(\tau)}{2v_o} \approx e^{-\beta(\tau-1)} \left[(1 - \beta)\cosh^{-1}(\tau) + \beta\sqrt{\tau^2 - 1} + + \right] \quad .$$

which, for very small $\tau - 1$, becomes

$$\frac{v(\tau)}{2v_o} \approx \sqrt{2(\tau - 1)} \quad . \tag{23}$$

Equation (23) is plotted in figure 5, for comparison with the exact results.

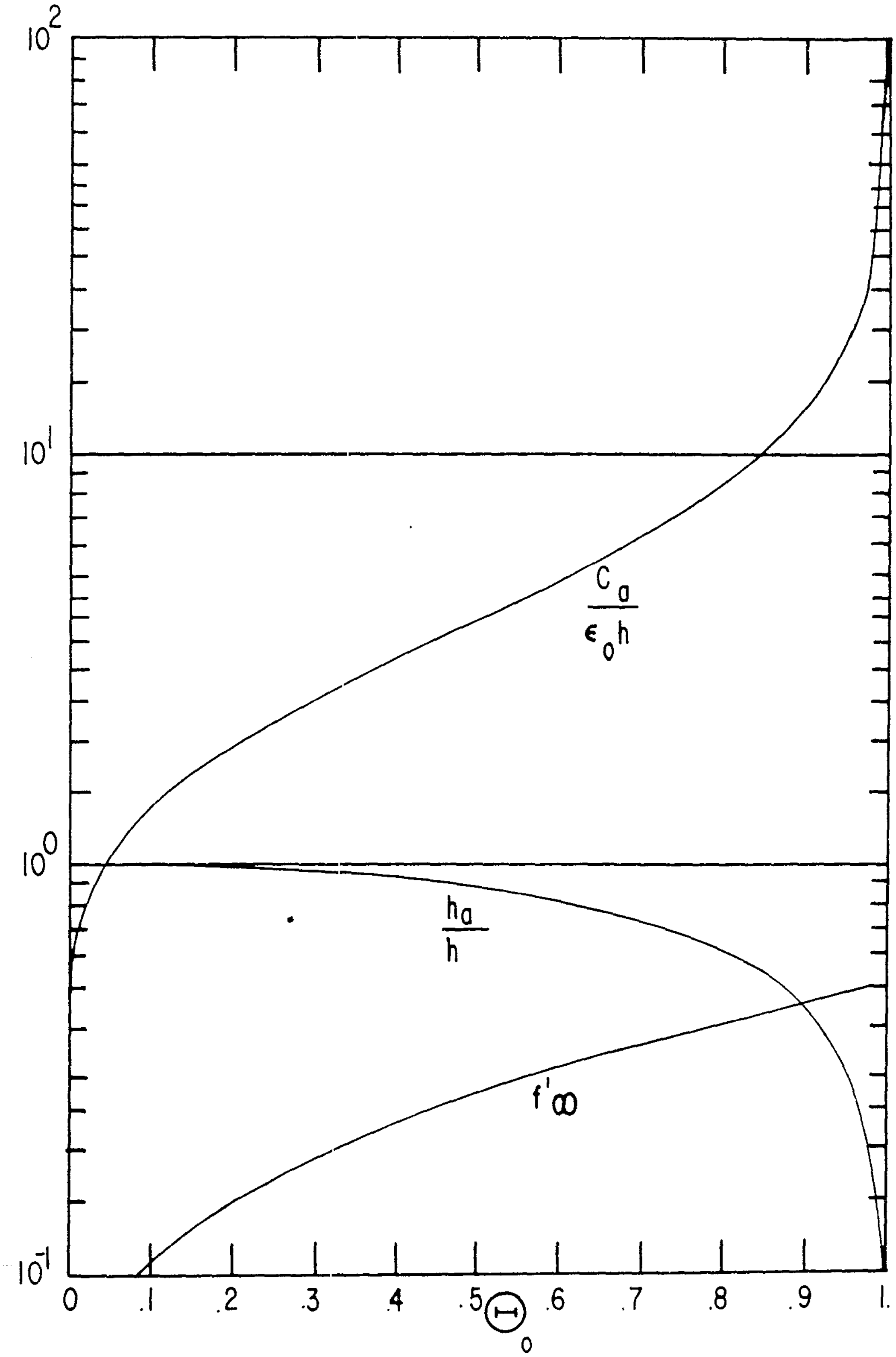

FIGURE 2. ANTENNA PARAMETERS AS FUNCTIONS OF Θ_0

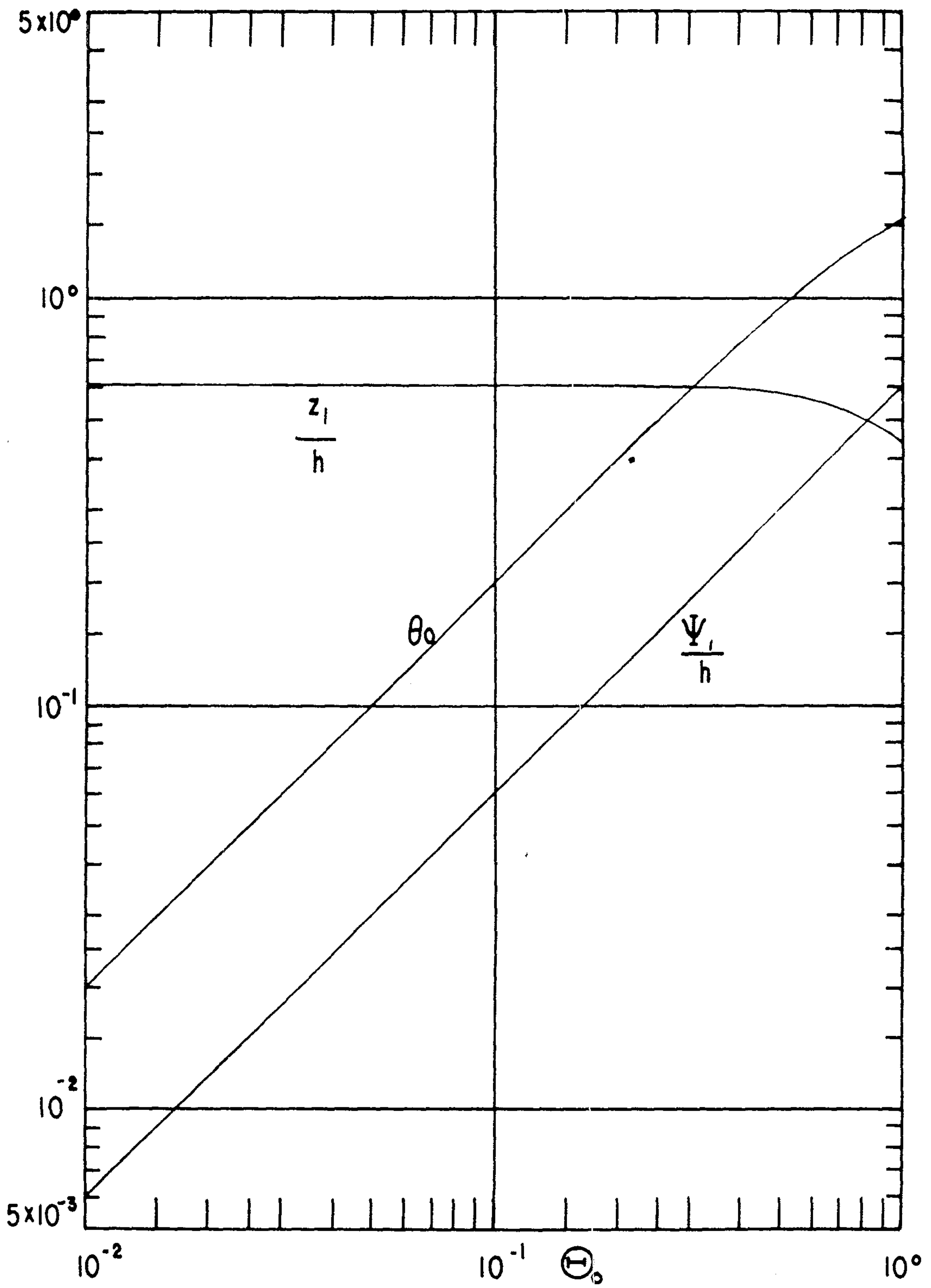

FIGURE 3. CONE ANGLE, MAXIMUM ANTENNA RADIUS, AND POSITION OF MAXIMUM AS FUNCTIONS OF Θ_0

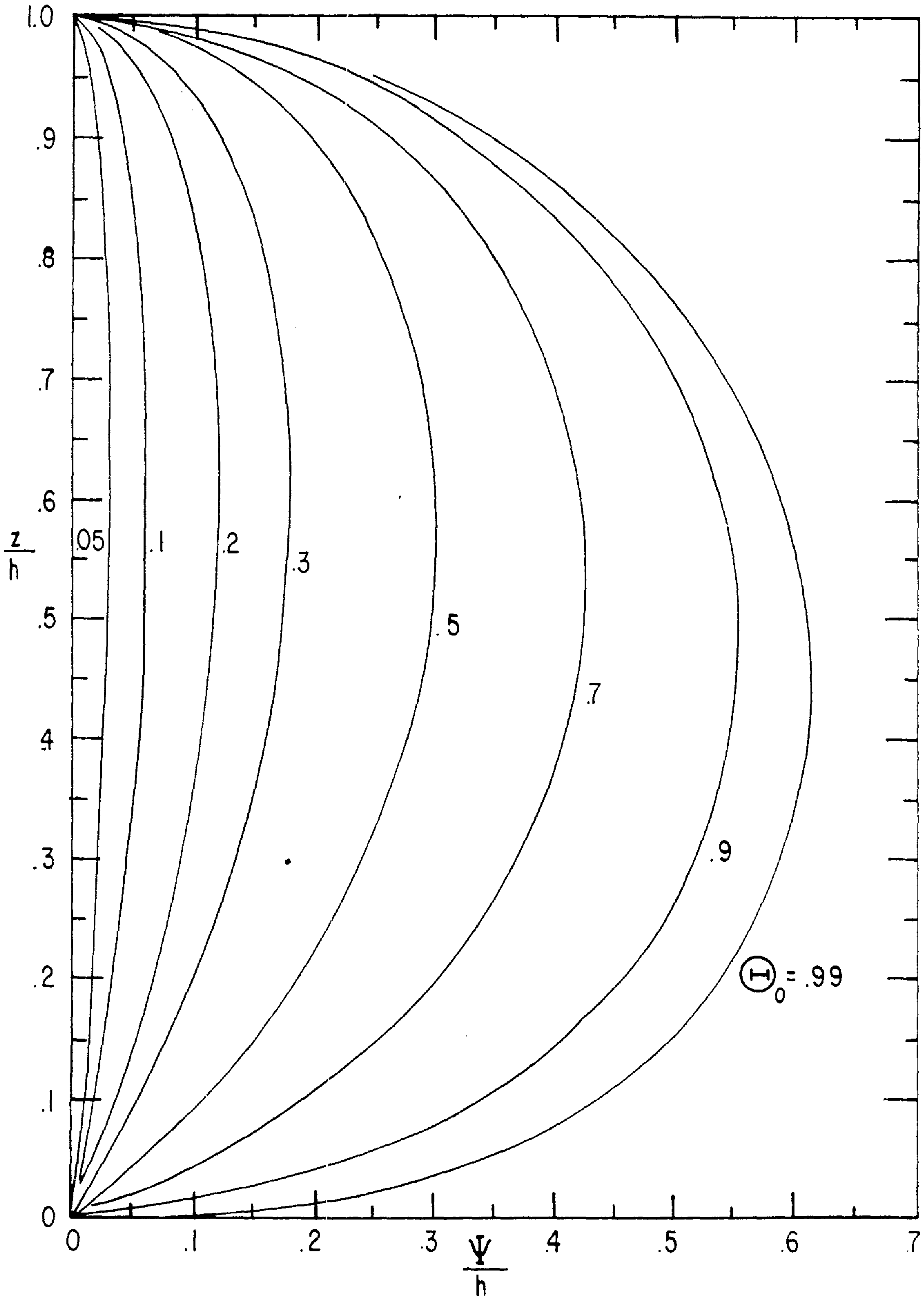

FIGURE 4. ANTENNA CONTOURS FOR VARIOUS Θ_0

Θ_o	$\frac{C_a}{\varepsilon_o h}$	$\frac{h_a}{h}$	f'_∞	Θ_o	$\frac{C_a}{\varepsilon_o h}$	$\frac{h_a}{h}$	f'_∞
0	0	1	0				
.01	.682	1.000	.054	.51	4.013	.860	.275
.02	.803	1.000	.064	.52	4.104	.854	.279
.03	.896	1.000	.071	.53	4.196	.848	.283
.04	.975	.999	.078	.54	4.291	.842	.287
.05	1.047	.999	.083	.55	4.389	.835	.292
.06	1.115	.998	.089	.56	4.489	.828	.296
.07	1.178	.998	.094	.57	4.592	.822	.300
.08	1.240	.997	.098	.58	4.698	.815	.305
.09	1.299	.996	.103	.59	4.807	.807	.309
.10	1.358	.995	.107	.60	4.920	.800	.313
.11	1.415	.994	.112	.61	5.036	.792	.318
.12	1.471	.993	.116	.62	5.156	.785	.322
.13	1.527	.992	.120	.63	5.280	.777	.326
.14	1.582	.990	.125	.64	5.409	.768	.331
.15	1.637	.989	.129	.65	5.542	.760	.335
.16	1.692	.987	.133	.66	5.680	.751	.340
.17	1.747	.985	.137	.67	5.824	.742	.344
.18	1.802	.984	.141	.68	5.973	.733	.348
.19	1.857	.982	.145	.69	6.128	.724	.353
.20	1.913	.980	.149	.70	6.290	.714	.357
.21	1.968	.978	.153	.71	6.459	.704	.362
.22	2.024	.975	.157	.72	6.637	.694	.367
.23	2.080	.973	.161	.73	6.822	.683	.371
.24	2.137	.971	.165	.74	7.018	.673	.376
.25	2.194	.968	.169	.75	7.223	.661	.380
.26	2.252	.966	.173	.76	7.440	.650	.385
.27	2.310	.963	.177	.77	7.669	.638	.389
.28	2.369	.960	.181	.78	7.912	.626	.394
.29	2.429	.957	.185	.79	8.171	.613	.399
.30	2.489	.954	.189	.80	8.447	.600	.403
.31	2.550	.951	.193	.81	8.743	.586	.408
.32	2.612	.947	.197	.82	9.061	.572	.413
.33	2.675	.944	.201	.83	9.404	.558	.417
.34	2.739	.940	.205	.84	9.777	.543	.422
.35	2.803	.937	.209	.85	10.183	.527	.427
.36	2.869	.933	.213	.86	10.629	.510	.432
.37	2.936	.929	.217	.87	11.123	.493	.436
.38	3.003	.925	.221	.88	11.673	.475	.441
.39	3.072	.921	.225	.89	12.292	.456	.446
.40	3.142	.917	.229	.90	12.997	.436	.451
.41	3.214	.912	.233	.91	13.811	.415	.456
.42	3.287	.908	.237	.92	14.766	.392	.461
.43	3.361	.903	.241	.93	15.912	.368	.465
.44	3.436	.898	.246	.94	17.322	.341	.470
.45	3.513	.893	.250	.95	19.125	.312	.475
.46	3.592	.888	.254	.96	21.548	.280	.480
.47	3.673	.883	.258	.97	25.074	.243	.485
.48	3.755	.877	.262	.98	30.945	.199	.490
.49	3.839	.872	.266	.99	44.096	.141	.495
.50	3.925	.866	.271	1.00	∞	.000	.500

Table 1. Antenna Parameters as Functions of Θ_o

Θ_o	θ_o (radians)	$\frac{\Psi_1}{h}$	$\frac{z_1}{h}$
.010	.0200	.0060	.6179
.020	.0400	.0120	.6179
.030	.0600	.0180	.6179
.040	.0800	.0240	.6178
.050	.0999	.0300	.6177
.060	.1199	.0360	.6175
.070	.1398	.0420	.6173
.080	.1597	.0481	.6171
.090	.1795	.0541	.6168
.100	.1993	.0601	.6165
.125	.2487	.0751	.6156
.150	.2978	.0901	.6145
.175	.3465	.1052	.6132
.200	.3948	.1202	.6117
.225	.4426	.1353	.6100
.250	.4900	.1504	.6081
.275	.5367	.1655	.6060
.300	.5829	.1806	.6037
.325	.6285	.1958	.6011
.350	.6733	.2109	.5984
.375	.7175	.2261	.5954
.400	.7610	.2413	.5922
.425	.8037	.2566	.5888
.450	.8457	.2718	.5852
.475	.8869	.2871	.5814
.500	.9273	.3025	.5773
.525	.9669	.3178	.5730
.550	1.0057	.3332	.5684
.575	1.0437	.3487	.5636
.600	1.0808	.3642	.5586
.625	1.1172	.3797	.5533
.650	1.1528	.3953	.5478
.675	1.1875	.4109	.5420
.700	1.2215	.4266	.5360
.725	1.2546	.4423	.5296
.750	1.2870	.4581	.5230
.775	1.3186	.4740	.5161
.800	1.3495	.4899	.5089
.825	1.3796	.5059	.5014
.850	1.4090	.5220	.4935
.875	1.4377	.5382	.4853
.900	1.4656	.5544	.4768
.910	1.4766	.5610	.4733
.920	1.4875	.5675	.4697
.930	1.4983	.5741	.4661
.940	1.5090	.5806	.4624
.950	1.5195	.5872	.4586
.960	1.5300	.5938	.4548
.970	1.5403	.6004	.4509
.980	1.5506	.6071	.4469
.990	1.5607	.6137	.4429

Table 2. Cone Angle, Maximum Antenna Radius, and Position of Maximum as Functions of Θ_o

z/h	$\Theta_o = .01$ $\theta_o^o = .0200$ Ψ/h	$\Theta_o = .02$ $\theta_o^o = .0400$ Ψ/h	$\Theta_o = .03$ $\theta_o^o = .0600$ Ψ/h	$\Theta_o = .04$ $\theta_o^o = .0800$ Ψ/h
0	0	0	0	0
.02	.0004	.0008	.0012	.0016
.04	.0008	.0016	.0023	.0031
.06	.0012	.0023	.0034	.0046
.08	.0015	.0030	.0045	.0060
.10	.0018	.0037	.0055	.0073
.12	.0022	.0043	.0064	.0086
.14	.0025	.0049	.0073	.0098
.16	.0028	.0055	.0082	.0110
.18	.0030	.0060	.0091	.0121
.20	.0033	.0066	.0098	.0131
.22	.0036	.0071	.0106	.0141
.24	.0038	.0076	.0113	.0151
.26	.0040	.0080	.0120	.0160
.28	.0042	.0084	.0127	.0169
.30	.0044	.0088	.0133	.0177
.32	.0046	.0092	.0138	.0184
.34	.0048	.0096	.0144	.0192
.36	.0050	.0099	.0149	.0198
.38	.0051	.0102	.0153	.0204
.40	.0053	.0105	.0158	.0210
.42	.0054	.0108	.0162	.0215
.44	.0055	.0110	.0165	.0220
.46	.0056	.0112	.0163	.0224
.48	.0057	.0114	.0171	.0228
.50	.0058	.0116	.0174	.0232
.52	.0059	.0117	.0176	.0234
.54	.0059	.0118	.0178	.0237
.56	.0060	.0119	.0179	.0239
.58	.0060	.0120	.0180	.0240
.60	.0060	.0120	.0180	.0241
.62	.0060	.0121	.0181	.0241
.64	.0060	.0120	.0180	.0240
.66	.0060	.0120	.0180	.0240
.68	.0060	.0119	.0179	.0238
.70	.0059	.0118	.0177	.0236
.72	.0059	.0117	.0175	.0233
.74	.0058	.0115	.0172	.0229
.76	.0057	.0113	.0169	.0225
.78	.0055	.0110	.0165	.0220
.80	.0054	.0107	.0160	.0214
.82	.0052	.0104	.0155	.0207
.84	.0050	.0099	.0149	.0199
.86	.0048	.0095	.0142	.0189
.88	.0045	.0089	.0134	.0178
.90	.0042	.0083	.0124	.0166
.92	.0038	.0076	.0113	.0151
.94	.0033	.0067	.0100	.0133
.96	.0028	.0055	.0083	.0110
.98	.0020	.0040	.0060	.0079
1	0	0	0	0

Table 3a. Antenna Contours for Various Θ_o

z/h	$\Theta_o = .05$, $\theta_o = .0999$ Ψ/h	$\Theta_o = .06$, $\theta_o = .1199$ Ψ/h	$\Theta_o = .07$, $\theta_o = .1398$ Ψ/h	$\Theta_o = .08$, $\theta_o = .1597$ Ψ/h
0	0	0	0	0
.02	.0020	.0024	.0028	.0032
.04	.0039	.0047	.0055	.0062
.06	.0057	.0069	.0080	.0092
.08	.0074	.0089	.0104	.0120
.10	.0091	.0109	.0128	.0146
.12	.0107	.0129	.0150	.0172
.14	.0122	.0147	.0172	.0195
.16	.0137	.0164	.0192	.0219
.18	.0151	.0181	.0212	.0241
.20	.0164	.0197	.0230	.0262
.22	.0177	.0212	.0247	.0283
.24	.0189	.0227	.0264	.0302
.26	.0200	.0240	.0280	.0320
.28	.0211	.0253	.0295	.0337
.30	.0221	.0265	.0309	.0354
.32	.0231	.0277	.0322	.0369
.34	.0240	.0288	.0335	.0383
.36	.0248	.0298	.0347	.0397
.38	.0256	.0307	.0357	.0409
.40	.0263	.0316	.0367	.0420
.42	.269	.0323	.0377	.0431
.44	.0275	.0331	.0385	.0440
.46	.0281	.0336	.0392	.0449
.48	.0285	.0342	.0399	.0457
.50	.0290	.0347	.0405	.0463
.52	.0293	.0351	.0410	.0469
.54	.0296	.0354	.0414	.0473
.56	.0298	.0357	.0417	.0477
.58	.0300	.0359	.0419	.0480
.60	.0301	.0360	.0421	.0481
.62	.0301	.0360	.0421	.0481
.64	.0301	.0360	.0420	.0481
.66	.0299	.0358	.0419	.0479
.68	.0298	.0356	.0416	.0476
.70	.0295	.0353	.0412	.0471
.72	.0291	.0349	.0407	.0466
.74	.0287	.0343	.0401	.0458
.76	.0281	.0337	.0393	.0450
.78	.0275	.0330	.0384	.0439
.80	.0267	.0321	.0373	.0427
.82	.0259	.0310	.0361	.0413
.84	.0248	.0298	.0347	.0397
.86	.0237	.0284	.0330	.0378
.88	.0223	.0268	.0311	.0356
.90	.0207	.0248	.0289	.0330
.92	.0188	.0226	.0263	.0300
.94	.0166	.0199	.0232	.0264
.96	.0138	.0165	.0193	.0219
.98	.0099	.0119	.0138	.0158
1	0	0	0	0

Table 3b. Antenna Contours for Various Θ_o

	$\Theta_o = .09$, $\theta_o = .1795$	$\Theta_o = .1$, $\theta_o = .1993$	$\Theta_o = .2$, $\theta_o = .3948$	$\Theta_o = .3$, $\theta_o = .5829$
z/h	Ψ/h	Ψ/h	Ψ/h	Ψ/h
0	0	0	0	0
.02	.0036	.0040	.0082	.0129
.04	.0070	.0078	.0160	.0251
.06	.0103	.0115	.0234	.0367
.08	.0135	.0149	.0306	.0478
.10	.0164	.0183	.0373	.0583
.12	.0193	.0215	.0439	.0683
.14	.0220	.0246	.0501	.0778
.16	.0247	.0275	.0560	.0867
.18	.0272	.0303	.0615	.0952
.20	.0296	.0329	.0669	.1032
.22	.0319	.0354	.0719	.1109
.24	.0340	.0378	.0767	.1180
.26	.0361	.0401	.0812	.1248
.28	.0380	.0422	.0855	.1311
.30	.0398	.0443	.0896	.1370
.32	.0415	.0462	.0933	.1425
.34	.0431	.0480	.0968	.1477
.36	.0446	.0496	.1001	.1524
.38	.0460	.0512	.1031	.1568
.40	.0473	.0525	.1059	.1608
.42	.0484	.0538	.1085	.1646
.44	.0495	.0550	.1107	.1678
.46	.0505	.0561	.1128	.1708
.48	.0513	.0570	.1146	.1732
.50	.0521	.0579	.1162	.1754
.52	.0527	.0586	.1175	.1773
.54	.0532	.0591	.1186	.1787
.56	.0536	.0596	.1194	.1797
.58	.0539	.0599	.1199	.1804
.60	.0541	.0601	.1202	.1807
.62	.0541	.0601	.1202	.1805
.64	.0540	.0600	.1200	.1800
.66	.0538	.0598	.1194	.1791
.68	.0535	.0594	.1187	.1776
.70	.0530	.0588	.1175	.1757
.72	.0523	.0581	.1159	.1733
.74	.0515	.0572	.1142	.1705
.76	.0505	.0561	.1119	.1670
.78	.0494	.0548	.1093	.1629
.80	.0480	.0533	.1062	.1583
.82	.0464	.0516	.1026	.1529
.84	.0446	.0496	.0986	.1466
.86	.0424	.0472	.0938	.1396
.88	.0400	.0445	.0884	.1314
.90	.0372	.0413	.0820	.1218
.92	.0338	.0375	.0746	.1108
.94	.0298	.0330	.0657	.0974
.96	.0247	.0274	.0544	.0807
.98	.0177	.0197	.0391	.0579
1	0	0	0	0

Table 3c. Antenna Contours for Various Θ_o

z/h	$\Theta_o = .4$, $\theta_o = .7610$ Ψ/h	$\Theta_o = .5$, $\theta_o = .9273$ Ψ/h	$\Theta_o = .6$, $\theta_o = 1.0808$ Ψ/h	$\Theta_o = .7$, $\theta_o = 1.2215$ Ψ/h
0	0	0	0	0
.02	.0185	.0257	.0356	.0508
.04	.0359	.0495	.0678	.0943
.06	.0523	.0716	.0969	.1323
.08	.0678	.0922	.1234	.1655
.10	.0824	.1113	.1476	.1951
.12	.0961	.1290	.1697	.2215
.14	.1090	.1457	.1900	.2452
.16	.1212	.1611	.2086	.2665
.18	.1326	.1755	.2258	.2859
.20	.1434	.1889	.2416	.3034
.22	.1535	.2014	.2562	.3194
.24	.1630	.2130	.2695	.3339
.26	.1719	.2238	.2818	.3470
.28	.1801	.2337	.2930	.3589
.30	.1878	.2429	.3034	.3696
.32	.1950	.2514	.3127	.3794
.34	.2016	.2592	.3213	.3880
.36	.2076	.2662	.3289	.3957
.38	.2132	.2726	.3358	.4025
.40	.2182	.2784	.3418	.4084
.42	.2227	.2835	.3472	.4134
.44	.2268	.2880	.3518	.4176
.46	.2303	.2919	.3556	.4210
.48	.2334	.2952	.3587	.4236
.50	.2360	.2979	.3611	.4254
.52	.2380	.3000	.3629	.4264
.54	.2396	.3014	.3639	.4266
.56	.2407	.3023	.3642	.4261
.58	.2412	.3025	.3638	.4247
.60	.2413	.3021	.3626	.4226
.62	.2409	.3010	.3608	.4196
.64	.2399	.2993	.3582	.4159
.66	.2383	.2970	.3547	.4112
.68	.2361	.2939	.3506	.4057
.70	.2334	.2901	.3455	.3993
.72	.2300	.2855	.3396	.3919
.74	.2259	.2802	.3328	.3834
.76	.2212	.2739	.3250	.3739
.78	.2156	.2667	.3160	.3631
.80	.2092	.2585	.3060	.3512
.82	.2019	.2493	.2946	.3377
.84	.1935	.2387	.2818	.3227
.86	.1840	.2266	.2674	.3057
.88	.1730	.2130	.2510	.2867
.90	.1604	.1972	.2322	.2650
.92	.1456	.1789	.2104	.2398
.94	.1280	.1571	.1846	.2102
.96	.1060	.1300	.1526	.1736
.98	.0760	.0931	.1092	.1241
1	0	0	0	0

Table 3d. Antenna Contours for Various Θ_o

	$\Theta_o = .8$ $\theta_o = 1.3495$	$\Theta_o = .9$ $\theta_o = 1.4656$	$\Theta_o = .91$ $\theta_o = 1.4766$	$\Theta_o = .92$ $\theta_o = 1.4875$
z/h	Ψ/h	Ψ/h	Ψ/h	Ψ/h
0	0	0	0	0
.02	.0772	.1335	.1425	.1523
.04	.1369	.2115	.2216	.2324
.06	.1850	.2668	.2770	.2877
.08	.2250	.3097	.3198	.3302
.10	.2590	.3448	.3547	.3648
.12	.2885	.3744	.3841	.3939
.14	.3143	.3998	.4092	.4188
.16	.3372	.4219	.4311	.4404
.18	.3575	.4413	.4503	.4594
.20	.3757	.4585	.4673	.4761
.22	.3920	.4736	.4822	.4909
.24	.4066	.4871	.4955	.5039
.26	.4196	.4989	.5072	.5155
.28	.4313	.5094	.5175	.5256
.30	.4417	.5186	.5265	.5344
.32	.4510	.5266	.5344	.5421
.34	.4591	.5335	.5411	.5487
.36	.4662	.5393	.5467	.5541
.38	.4723	.5442	.5514	.5586
.40	.4774	.5480	.5551	.5622
.42	.4817	.5510	.5579	.5648
.44	.4850	.5530	.5598	.5666
.46	.4875	.5542	.5608	.5674
.48	.4891	.5545	.5609	.5674
.50	.4899	.5539	.5602	.5665
.52	.4898	.5525	.5586	.5648
.54	.4889	.5502	.5562	.5622
.56	.4873	.5470	.5530	.5588
.58	.4847	.5431	.5488	.5545
.60	.4813	.5382	.5438	.5494
.62	.4771	.5325	.5379	.5433
.64	.4719	.5258	.5311	.5363
.66	.4658	.5182	.5233	.5283
.68	.4588	.5096	.5145	.5194
.70	.4508	.5000	.5047	.5095
.72	.4418	.4892	.4938	.4984
.74	.4317	.4773	.4817	.4861
.76	.4204	.4642	.4684	.4727
.78	.4078	.4498	.4538	.4578
.80	.3938	.4338	.4376	.4414
.82	.3782	.4162	.4198	.4234
.84	.3610	.3967	.4001	.4035
.86	.3417	.3751	.3783	.3815
.88	.3200	.3509	.3539	.3568
.90	.2954	.3237	.3263	.3290
.92	.2672	.2924	.2948	.2972
.94	.2339	.2557	.2578	.2598
.96	.1930	.2108	.2125	.2142
.98	.1379	.1505	.1517	.1529
1	0	0	0	0

Table 3e. Antenna Contours for Various Θ_o

	$\Theta_o = .93$ $\theta_o = 1.4983$	$\Theta_o = .94$ $\theta_o = 1.5090$	$\Theta_o = .95$ $\theta_o = 1.5195$	$\Theta_o = .96$ $\theta_o = 1.5300$
z/h	Ψ/h	Ψ/h	Ψ/h	Ψ/h
0	0	0	0	0
.02	.1631	.1750	.1880	.2022
.04	.2437	.2557	.2683	.2814
.06	.2988	.3102	.3221	.3344
.08	.3410	.3519	.3632	.3748
.10	.3752	.3858	.3965	.4075
.12	.4039	.4141	.4245	.4350
.14	.4285	.4384	.4484	.4585
.16	.4499	.4594	.4691	.4789
.18	.4686	.4779	.4872	.4967
.20	.4851	.4941	.5032	.5123
.22	.4996	.5084	.5172	.5262
.24	.5125	.5210	.5296	.5383
.26	.5238	.5321	.5405	.5489
.28	.5337	.5419	.5500	.5582
.30	.5424	.5503	.5583	.5663
.32	.5498	.5576	.5654	.5732
.34	.5562	.5638	.5715	.5791
.36	.5616	.5690	.5764	.5839
.38	.5659	.5732	.5804	.5877
.40	.5693	.5764	.5835	.5906
.42	.5718	.5787	.5856	.5925
.44	.5733	.5801	.5868	.5936
.46	.5740	.5806	.5872	.5938
.48	.5739	.5803	.5867	.5931
.50	.5728	.5791	.5854	.5916
.52	.5710	.5771	.5832	.5893
.54	.5682	.5742	.5801	.5861
.56	.5646	.5704	.5763	.5820
.58	.5602	.5658	.5715	.5771
.60	.5549	.5603	.5658	.5713
.62	.5486	.5540	.5593	.5646
.64	.5415	.5467	.5518	.5569
.66	.5334	.5384	.5434	.5483
.68	.5243	.5291	.5340	.5387
.70	.5141	.5188	.5235	.5281
.72	.5029	.5074	.5119	.5163
.74	.4905	.4948	.4991	.5034
.76	.4768	.4810	.4851	.4891
.78	.4618	.4657	.4697	.4735
.80	.4452	.4490	.4527	.4564
.82	.4270	.4306	.4341	.4376
.84	.4069	.4102	.4135	.4168
.86	.3846	.3877	.3908	.3939
.88	.3597	.3626	.3654	.3683
.90	.3317	.3343	.3369	.3394
.92	.2995	.3019	.3042	.3065
.94	.2619	.2639	.2659	.2679
.96	.2158	.2175	.2191	.2208
.98	.1540	.1552	.1563	.1575
1	0	0	0	0

Table 3f. Antenna Contours for Various Θ_o

	$\Theta_o = .97$ $\theta_o = 1.5403$	$\Theta_o = .98$ $\theta_o = 1.5506$	$\Theta_o = .99$ $\theta_o = 1.5607$
z/h	Ψ/h	Ψ/h	Ψ/h
0	0	0	0
.02	.2176	.2342	.2520
.04	.2952	.3095	.3243
.06	.3470	.3600	.3733
.08	.3866	.3986	.4109
.10	.4187	.4301	.4416
.12	.4456	.4564	.4674
.14	.4687	.4790	.4895
.16	.4887	.4986	.5087
.18	.5062	.5158	.5254
.20	.5216	.5308	.5402
.22	.5351	.5441	.5531
.24	.5470	.5557	.5645
.26	.5574	.5659	.5744
.28	.5665	.5747	.5830
.30	.5743	.5824	.5904
.32	.5810	.5889	.5967
.34	.5867	.5943	.6019
.36	.5913	.5987	.6062
.38	.5949	.6022	.6094
.40	.5976	.6047	.6117
.42	.5994	.6063	.6132
.44	.6003	.6070	.6137
.46	.6004	.6069	.6134
.48	.5995	.6059	.6123
.50	.5979	.6041	.6103
.52	.5953	.6014	.6074
.54	.5920	.5979	.6037
.56	.5878	.5935	.5992
.58	.5827	.5882	.5937
.60	.5767	.5821	.5874
.62	.5698	.5750	.5802
.64	.5620	.5671	.5721
.66	.5533	.5582	.5630
.68	.5435	.5482	.5529
.70	.5327	.5372	.5417
.72	.5207	.5251	.5295
.74	.5076	.5118	.5160
.76	.4932	.4972	.5012
.78	.4774	.4812	.4850
.80	.4601	.4637	.4673
.82	.4411	.4445	.4479
.84	.4201	.4233	.4265
.86	.3969	.3999	.4029
.88	.3711	.3739	.3766
.90	.3420	.3445	.3470
.92	.3088	.3110	.3133
.94	.2699	.2718	.2738
.96	.2223	.2239	.2255
.98	.1586	.1597	.1608
1	0	0	0

Table 3g. Antenna Contours for Various Θ_o

V. Summary

In the present note we have then described a technique whereby one can define geometries of dipole antennas with readily calculable low-frequency parameters. This technique, which we call an equivalent-charge method, involves assuming an appropriate static equivalent-charge distribution. From this charge distribution one calculates the potential distribution and chooses two appropriate equipotential surfaces as surfaces for the antenna conductors. Then the charge on one of the antenna surfaces and the dipole moment of the antenna are calculated by appropriate integrals over the assumed equivalent-charge distribution. From these parameters one can calculate the antenna capacitance and mean charge separation distance.

While the antenna charge and dipole moment can be readily calculated as integrals over the assumed equivalent-charge distribution, the position of the antenna conductors on chosen equipotential surfaces may be more difficult to calculate. This point is illustrated by the example of a dipole geometry used in section IV.

One interesting type of dipole geometry is one with axial and lengthwise symmetry. This can be obtained by choosing the equivalent-charge distribution odd in z' and independent of ϕ', and by choosing $\Phi = \Phi_s$ and $\Phi = -\Phi_s$ for the antenna surfaces. For such an antenna geometry it is possible to include a biconical section near the center of the antenna. One way to do this is to make the equivalent-charge distribution have the form of a line charge density on the z axis near $z' = 0$ and make this line charge density have a step discontinuity at $z' = 0$.

We would like to thank A2C Richard T. Clark for the numerical calculations.