SENSOR AND SIMULATION NOTES

NOTE 67

January 1969

Inductance of Periodic Distributed Inductors

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Abstract

The inductance per section of an idealized periodic distributed inductor
is calculated for two limiting cases. In one case, the distributed inductor
is contained between two perfectly conducting plates. In the other case, the
distributed inductor is isolated in space. In addition, some results are
presented on the magnitude of the magnetic field outside the distributed
inductor in the above two cases.

Acknowledgment

I. Introduction

In a previous note in this series[1] it was suggested that a distributed inductor be used as part of the termination of a transmission-line EMP simulator. The reasons for doing so were given in that note. In this note we wish to calculate the inductance of an idealized model of a structure that could be used to realize such a distributed inductor.

The structure whose inductance is to be computed has been described in the note referred to above, and it is shown schematically in figure 1. The structure shown in figure 1 would be most difficult to analyze exactly, but we can obtain a close approximation to its inductance per section by assuming the current to be distributed on sheets parallel to the xy and yz planes. Also we ignore the fact that the pitch of the wires will give rise to a small y component of current, so all currents will be assumed to flow parallel to the xy plane. Both assumptions, sheet currents and zero pitch, are conventional and accurate when making inductance calculations.[2] In addition, we assume the structure to be infinite in the x direction. Thus the structure is periodic in the x direction, and we need only consider one section of it.

The rest of the simulator will have an effect on the inductance of the structure shown in figure 1. We will study this effect by considering two limiting cases. In the first case the inductor is assumed to fit exactly between two perfectly conducting plates and in the second case the inductor is assumed to be isolated in space. If the top plate of the actual simulator is made up of parallel wires, the distributed inductor will have, because of the impossibility of inducing currents perpendicular to these wires, an inductance very close to the one we will calculate for the isolated inductor.

In this note we will assume quasi-static conditions to hold. This, of course, is the only case where an inductance for the distributed inductor can be meaningfully defined. It is left to the future to discuss what effect a structure like the one shown in figure 1 will have on the high-frequency components of the simulated EMP.

II. Mathematical Formulation

As stated in the Introduction, two limiting cases will be considered -- the case where the inductor fits exactly between two perfectly conducting plates and the case where the inductor is isolated in free space.

II A. Inductor Between Plates

The inductor of figure 1 will be idealized by assuming it to be made up of current sheets. The effect of the current sheets in the x and z directions are independent and will be considered separately in the following work. This is a useful procedure since the fields will be calculated by means of the magnetic vector potential, $\underline{A}$, and each Cartesian component of this vector satisfies a Poisson equation whose source term is the corresponding component of current, that is to say,

$$\nabla^2 A_x = - \mu_o J_x \tag{1}$$

$$\nabla^2 A_z = - \mu_o J_z \quad . \tag{2}$$

The inductance per section of the distributed inductor will be determined by calculating the magnetostatic energy per section and equating this to the circuit-theory expression for the energy of a current carrying inductor. In other words, we shall use a standard expression for magnetostatic energy,[3]

$$\frac{1}{2} \int_{\text{section}} \underline{A} \cdot \underline{K} \, dS = \frac{1}{2} LI^2 \quad ,$$

to write

$$L = \frac{1}{I^2} \int \underline{A} \cdot \underline{K} \, dS \quad , \tag{3}$$

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where $\underset{\sim}{K}$ is the sheet current flowing in a section of the inductor (note that the sheet currents in the z direction must be of strength 2K), and the integration in (3) is over all the sheets that make up a single section of the inductor.

The vector potential to be used in (3) could be calculated by using the free-space Green's function for equations (1) and (2) together with the known distribution of current. If the resultant expression for $\underset{\sim}{A}$ were then inserted in equation (3) a representation for L , analogous to Neumann's form for mutual-inductance calculations,[4] would be obtained in the form of an infinite sum (because of the infinite number of sections) of quadruple integrals. To obtain an expression suitable for numerical calculation this representation would then have to be greatly simplified analytically. We will circumvent this difficulty by dealing directly with equations (1) and (2) rather than with their solutions by means of integrals over the sources.

Figure 2A shows the source distribution corresponding to equation (1) and the coordinate system to be used in solving that equation. Because the structure is of infinite extent in the x direction and has perfectly conducting plates, A_x is periodic in both x and y . Therefore we need only consider the basic periodic section shown in figure 2A. In addition, from an examination of the symmetries of the problem it is clear that A_x is zero on the four walls of the basic section and also on the xy plane.

These facts, and the fact that A_x is continuous through the current sheet at z = b/2 allow us to write A_x in the form

$$A_x = \sum_{n,m} A_{n,m} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{h}\right) e^{-\Gamma_{n,m}(z-b/2)} \qquad z > b/2 \qquad (4)$$

$$A_x = \sum_{n,m} A_{n,m} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{h}\right) \frac{\sinh(\Gamma_{n,m}z)}{\sinh(\Gamma_{n,m}b/2)} \qquad |z| < b/2 \qquad (5)$$

where

$$\Gamma_{n,m}^2 = (n\pi/a)^2 + (m\pi/h)^2 \qquad . \qquad (6)$$

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By subtracting the y components of the $\underline{H}$ field at $z = b/2$, calculated by means of the two expressions (4) and (5), and then equating this difference to the sheet current K , it follows that

$$A_{n,m} = \left(\frac{4}{\pi}\right)^2 \frac{\mu_o K}{nm\Gamma_{n,m}\left(1 + \coth(\Gamma_{n,m}b/2)\right)} \tag{7}$$

if n and m are both odd, and that

$$A_{n,m} = 0$$

if either n or m is even (see appendix A).

When (7) is substituted in (4) or (5) and the result used in the integration of (3) over the surfaces $z = \pm b/2$, there arises an expression for the contribution to L from these surfaces (which we denote by L^x) that can be simplified to

$$L^x = \frac{1}{4} \left(\frac{4}{\pi}\right)^4 \left(\frac{K}{I}\right)^2 \mu_o a\, b\, h \sum_{n \text{ odd}} \sum_{m \text{ odd}} \frac{1 - e^{-\Gamma_{n,m}b}}{n^2 m^2 \Gamma_{n,m}b} \quad . \tag{8}$$

Now $K = NI/h$ where N is the number of turns in one section of the distributed inductor, so, in keeping with the notation of reference 1, we define a geometrical factor f_2 by

$$L = f_2(\alpha,\beta)L' \tag{9}$$

where

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$$L' = \mu_o N^2 ab/h \qquad (10)$$

$$\alpha = a/h \qquad (11)$$

$$\beta = b/h \qquad (12)$$

and obtain from (8), with an obvious notation,

$$f_2^x(\alpha,\beta) = \frac{64}{\pi^4} \sum_{n\ odd} \sum_{m\ odd} \frac{1 - e^{-\beta\pi\gamma_{n,m}}}{n^2 m^2 \gamma_{n,m} \pi\beta} \qquad (13)$$

where

$$\gamma_{n,m}^2 = (n/\alpha)^2 + m^2 \qquad .$$

We will now obtain the contribution to f_2 from the currents in the z direction. This contribution will be denoted by f_2^z. For simplicity in presentation we will revise the coordinate system and the basic periodic section by shifting them to the left by half a section from the positions they took in figure 2A and obtain the situation shown in figure 2B. The appropriate Green's function of equation (2) for this new section must be zero on the four walls, so by standard methods[5]

$$A_z(x,y,z) = 2\mu_o \int_{-b/2}^{b/2} \int_o^h G(x,y,z,a/2,y',z')K\ dy'dz' \qquad (14)$$

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where

$$G(x,y,z,x',y',z') = \frac{2}{ah} \sum_{n,m} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi x'}{a}\right) \sin\left(\frac{m\pi y}{h}\right) \sin\left(\frac{m\pi y'}{h}\right) \frac{e^{-\Gamma_{n,m}|z-z'|}}{\Gamma_{n,m}} \quad (15)$$

(see appendix A).

When (15) is substituted in (14) and the result is employed to evaluate the contribution to equation (3) from the z directed currents, an expression for f_2^z is obtained which may be easily simplified to

$$f_2^z(\alpha,\beta) = \frac{64}{\pi^2} \left(\frac{\beta}{\alpha}\right)^2 \sum_{n \text{ odd}} \sum_{m \text{ odd}} \left[\frac{1}{m^2(\beta\pi)^2 \gamma_{n,m}^2} - \frac{1 - e^{-\beta\pi\gamma_{n,m}}}{m^2(\beta\pi)^3 \gamma_{n,m}^3} \right] \quad . \quad (16)$$

Equations (13) and (16) may be combined, and in the resulting equation the terms may be regrouped to give

$$f_2(\alpha,\beta) = \frac{2}{\alpha}\left(\frac{2}{\pi}\right)^3 \sum_{n \text{ odd}} \frac{\tanh(n\pi\alpha/2)}{n^3} + \frac{2}{\beta}\left(\frac{2}{\pi}\right)^5 \sum_{n \text{ odd}} \sum_{m \text{ odd}} \frac{1 - e^{-\beta\pi\gamma_{n,m}}}{n^2 \gamma_{n,m}^3} \quad (17)$$

For α small

$$f_2(\alpha,\beta) \to \frac{2}{\alpha}\left(\frac{2}{\pi}\right)^3 \frac{\pi}{2} \alpha \sum_{n \text{ odd}} \frac{1}{n^3} = 1 \qquad \text{for any } \beta \quad . \quad (18)$$

For β small

$$f_2(\alpha,\beta) \to \frac{2}{\alpha}\left(\frac{2}{\pi}\right)^3 \sum_{n \text{ odd}} \frac{\tanh(n\pi\alpha/2)}{n^3} + 2\pi\left(\frac{2}{\pi}\right)^5 \sum_{n \text{ odd}} \sum_{m \text{ odd}} \frac{1}{n^2 \gamma_{n,m}^2}$$

$$= 2\left(\frac{2}{\pi}\right)^3 \sum_{n \text{ odd}} \left[\frac{\tanh(n\pi\alpha/2)}{\alpha n^3} + \frac{\alpha \tanh(n\pi/2\alpha)}{n^3} \right] \quad . \quad (19)$$

The right hand side of (19) is also equal to unity. This may be seen by considering the following relations

$$\sum_{n \text{ odd}} \frac{\tanh(n\pi\alpha/2)}{\alpha n^3} = \frac{\pi^3}{16} + \sum_{n \text{ odd}} \frac{\frac{\tanh(n\pi\alpha/2)}{\alpha} - \frac{\pi}{2}n}{n^3}$$

$$= \frac{\pi^3}{16} + \frac{1}{8i} \int_{C_1 + C_2} \frac{\alpha^{-1}\tanh(\frac{\pi}{2}z\alpha) - \frac{\pi}{2}z}{z^3} \tan(\frac{\pi}{2}z)dz \quad ,$$

where the contours C_1 and C_2 are shown in figure 3 and are thought of as being closed in the right and left half-planes, respectively. But the integral may also be evaluated by connecting C_1 and C_2 with contours at infinity in the upper and lower half-planes. This alternative method of evaluation leads to the equation,

$$\sum_{n \text{ odd}} \frac{\tanh(n\pi\alpha/2)}{\alpha n^3} = \frac{\pi^3}{16} - \sum_{n \text{ odd}} \frac{\alpha \tanh(n\pi/2\alpha)}{n^3} \quad ,$$

which shows that the right hand side of (19) is unity. In summary, $f_2(\alpha,\beta)$ approaches unity if either α or β approaches zero. Of course this fact is primarily a check on the algebra used in deriving f_2 since these limits are obvious from the physical point of view.

The first sum on the right hand side of (17) may, for the sake of more rapid calculation, be rewritten as

$$\frac{2}{\alpha}\left(\frac{2}{\pi}\right)^3 \sum_{n \text{ odd}} \frac{\tanh(n\pi\alpha/2)}{n^3} = 1 - 2\left(\frac{2}{\pi}\right)^3 \alpha \sum_{n \text{ odd}} \frac{\tanh(n\pi/2\alpha)}{n^3}$$

$$= 1 - 2\left(\frac{2}{\pi}\right)^3 \alpha \left[\sum_{n \text{ odd}} \frac{1}{n^3} - \sum_{n \text{ odd}} \frac{1 - \tanh(n\pi/2\alpha)}{n^3} \right]$$

$$= 1 - 2\left(\frac{2}{\pi}\right)^3 \alpha \left[\frac{7}{8}\zeta(3) - 2 \sum_{n \text{ odd}} \frac{1}{n^3(e^{n\pi/\alpha} + 1)} \right] \quad (20)$$

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where ζ is the tabulated Riemann Zeta function.[6]

Part of the double sum in (17) is not very rapidly convergent. This part of the sum may be transformed by making use of the following relations.

$$G(c) \equiv \sum_{n \text{ odd}} \frac{1}{(c^2+n^2)^{3/2}} = \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{1}{\left[c^2 + (2n+1)^2\right]^{3/2}}$$

$$= \frac{1}{4\pi} \sum_{\nu=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-i\nu\tau} \, d\tau}{\left[c^2 + (\tau/\pi+1)^2\right]^{3/2}}$$

by the Poisson summation formula.[7] So

$$G(c) = \frac{1}{2\pi} \sum_{\nu=-\infty}^{\infty} (-)^{\nu} \int_{0}^{\infty} \frac{\cos \nu s}{\left[c^2 + (s/\pi)^2\right]^{3/2}} \, ds$$

$$= \frac{1}{2c^2} + \frac{\pi}{c} \sum_{\nu=1}^{\infty} (-)^{\nu} \nu \, K_1(c\pi\nu) \tag{21}$$

from reference 8, where K_1 is the modified Bessel function of the second kind and first order. Equation (21) is a rapidly converging representation of G .

With the use of equations (20) and (21), equation (17) may be rewritten as

$$f_2(\alpha,\beta) = 1 - \frac{16\alpha}{\pi^3} \left[\frac{7}{8} \zeta(3) - 2 \sum_{n \text{ odd}} \frac{1}{n^3(e^{n\pi/\alpha}+1)} \right] + \frac{2}{3\pi\beta}$$

$$+ \left(\frac{4}{\pi}\right)^4 \frac{\alpha}{4\beta} \sum_{n \text{ odd}} \sum_{\text{all } m} \frac{(-)^m m}{n^3} K_1(nm\pi/\alpha)$$

$$- \frac{2}{\beta} \left(\frac{2}{\pi}\right)^5 \sum_{n \text{ odd}} \sum_{m \text{ odd}} \frac{e^{-\beta\pi\gamma_{n,m}}}{n^2\gamma_{n,m}^3} \tag{22}$$

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This is the representation of f_2 that was used in the numerical computations.

By making use of equations (4) and (14) and the relation

$$H_y = \frac{1}{\mu_0} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \qquad ,$$

it is easy, but rather lengthy and uninstructive, to derive the following representation for $H_y(a/2,h/2,z)$ (see appendix A),

$$H_y(a/2,h/2,z) = \frac{8K}{\pi^2} \sum_{n \ odd} \sum_{m \ odd} (-)^{\frac{n+m}{2}} \frac{m(1-e^{-\beta\pi\gamma_{n,m}})}{n\gamma_{n,m}^2} e^{-\pi\gamma_{n,m}(z/h-\beta/2)} \qquad (23)$$

where the arguments of H_y correspond to figure 2A, and $z \geq b/2$. For $z = b/2$ equation (23) converges slowly but may be rewritten as

$$H_y(a/2,h/2,b/2) = \frac{2K}{\pi} \sum_{n \ odd} \frac{(-)^{\frac{n+1}{2}}}{n \cosh(n\pi/2\alpha)}$$

$$- \frac{8K}{\pi^2} \sum_{n \ odd} \sum_{m \ odd} (-)^{\frac{n+m}{2}} \frac{me^{-\beta\pi\gamma_{n,m}}}{n\gamma_{n,m}^2} \qquad . \qquad (24)$$

Equations (23) and (24) are suitable for numerical evaluation and were used to compute the H field data presented in Section III.

II B. Inductor in Space

If the distributed inductor is not enclosed between perfectly conducting plates, but instead is assumed to be isolated in space, the analysis follows closely the one presented in Section II A. The only difference is that the y coordinate is now unrestricted. This results in the appearance in the equations of Fourier integrals in y rather than the Fourier series that

arose in Section II A. The algebra of the problem is again rather lengthy but it is so closely analogous to the previous work_that it is pointless to repeat it here. Instead, we will write without further ado the final form for f_2 corresponding to equation (22) of the previous case. The detailed algebra may be found in appendix B.

$$f_2(\alpha,\beta) = 1 - \left(\frac{2}{\pi}\right)^3 \alpha \left[\frac{7}{8} \zeta(3) - \sum_{n \ odd} \frac{e^{-n\pi/\alpha}}{n^3}\right] + \frac{\alpha^2}{6\pi\beta}$$

$$- \left(\frac{2}{\pi}\right)^4 \frac{\alpha}{\beta} \sum_{n \ odd} \frac{K_1(n\pi/\alpha)}{n^3}$$

$$- \left(\frac{2}{\pi}\right)^3 \frac{1}{\beta} \int_0^\infty \sin^2 x \left[\sum_{n \ odd} \frac{e^{-2\beta\delta_{nx}}}{n^2\delta_{nx}^3}\right] dx \qquad (25)$$

where

$$\delta_{nx}^2 = (n\pi/2\alpha)^2 + x^2 \qquad .$$

Here, as in the previous case, f_2 approaches unity if either α or β approaches zero.

The representations for H_y analogous to equations (23) and (24) of the previous case are

$$H_y(a/2,h/2,z) = \left(\frac{2}{\pi}\right)^2 K \int_0^\infty x \sin x \left[\sum_{n \ odd} \frac{(-)^{\frac{n+1}{2}}}{n} \frac{(1-e^{-2\beta\delta_{nx}})}{\delta_{nx}^2} e^{-2(z/h-\beta/2)\delta_{nx}}\right] dx \quad (26)$$

and

$$H_y(a/2,h/2,b/2) = \frac{2K}{\pi} \sum_{n \ odd} \frac{(-)^{\frac{n+1}{2}} e^{-n\pi/2}}{n} - \left(\frac{2}{\pi}\right)^2 K \int_0^\infty x \sin x \left[\sum_{n \ odd} \frac{(-)^{\frac{n+1}{2}}}{n} \frac{e^{-2\beta\delta_{nx}}}{\delta_{nx}^2}\right] dx \quad (27)$$

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III. Numerical Results

The values of inductance per section of the idealized distributed inductor shown in figure 2 have been computed for several values of α and β. For the case where the distributed inductor is between two perfectly conducting plates, equation (22) was used. For the case where the inductor is in free space, equation (25) was used. Because of the rapid convergence of these two expressions it was possible to calculate the values of inductance to an accuracy of several significant figures. The values in the tables are rounded to the number of figures given. Tables 1 through 5 contain the inductance data on the inductor between plates and tables 6 through 10 contain the data on the inductor in free space. Some of this data is also shown graphically in figures 4 through 7.

Some representative values of the external magnetic field were also calculated. Equation (23) was used for the inductor between plates and equation (26) was used for the inductor in free space. The fields calculated are displayed in tables 11 and 12 and in figures 8 and 9. It should be noted that the arguments of H_y given in the titles of these figures and curves correspond to the coordinate system shown in figure 2A. It should also be noted that the external field is fairly insensitive to the thickness of the distributed inductor in the z direction, and so results are given for only one typical value of β.

TABLE 1

$f_2(\alpha,\beta)$, INDUCTOR BETWEEN-PLATES

β \ α	.05	.10	.15	.20	.25
.05	.9780	.9640	.9542	.9466	.9405
.10	.9755	.9561	.9407	.9280	.9174
.15	.9746	.9528	.9341	.9181	.9043
.20	.9742	.9510	.9304	.9122	.8959
.25	.9739	.9500	.9281	.9083	.8902
.30	.9737	.9493	.9265	.9055	.8862
.35	.9736	.9488	.9254	.9036	.8831
.40	.9735	.9484	.9246	.9021	.8808
.45	.9735	.9481	.9239	.9009	.8790
.50	.9734	.9478	.9234	.8999	.8776
.55	.9733	.9477	.9229	.8992	.8764
.60	.9733	.9475	.9226	.8985	.8754
.65	.9733	.9474	.9223	.8980	.8745
.70	.9732	.9472	.9220	.8975	.8738
.75	.9732	.9471	.9218	.8971	.8732
.80	.9732	.9471	.9216	.8968	.8726
.85	.9732	.9470	.9214	.8964	.8721
.90	.9732	.9469	.9212	.8962	.8717
.95	.9731	.9468	.9211	.8959	.8713
1.00	.9731	.9468	.9210	.8957	.8709
1.05	.9731	.9467	.9209	.8955	.8706
1.10	.9731	.9467	.9208	.8953	.8703
1.15	.9731	.9466	.9207	.8951	.8701
1.20	.9731	.9466	.9206	.8950	.8698
1.25	.9731	.9466	.9205	.8948	.8696

TABLE 2

$f_2(\alpha,\beta)$, INDUCTOR—BETWEEN PLATES

β \ α	.30	.35	.40	.45	.50
.05	.9353	.9309	.9270	.9236	.9205
.10	.9083	.9003	.8931	.8868	.8810
.15	.8921	.8812	.8714	.8625	.8544
.20	.8813	.8681	.8561	.8451	.8350
.25	.8738	.8587	.8449	.8321	.8203
.30	.8683	.8517	.8363	.8221	.8088
.35	.8641	.8463	.8297	.8142	.7996
.40	.8609	.8421	.8244	.8078	.7922
.45	.8583	.8387	.8201	.8026	.7860
.50	.8562	.8359	.8166	.7982	.7808
.55	.8545	.8336	.8136	.7946	.7765
.60	.8531	.8317	.8112	.7915	.7729
.65	.8519	.8300	.8090	.7889	.7696
.70	.8508	.8286	.8072	.7866	.7668
.75	.8499	.8274	.8056	.7846	.7644
.80	.8491	.8263	.8042	.7828	.7623
.85	.8484	.8254	.8030	.7813	.7604
.90	.8479	.8245	.8019	.7799	.7587
.95	.8472	.8237	.8009	.7787	.7571
1.00	.8467	.8231	.8000	.7775	.7558
1.05	.8463	.8224	.7992	.7765	.7545
1.10	.8459	.8219	.7984	.7756	.7534
1.15	.8455	.8214	.7978	.7748	.7524
1.20	.8451	.8209	.7972	.7740	.7514
1.25	.8448	.8205	.7966	.7733	.7505

TABLE 3

$$f_2(\alpha,\beta), \text{ INDUCTOR BETWEEN PLATES}$$

β \ α	.55	.60	.65	.70	.75
.05	.9177	.9151	.9127	.9106	.9087
.10	.8758	.8709	.8665	.8624	.8587
.15	.8470	.8402	.8339	.8281	.8228
.20	.8258	.8172	.8092	.8019	.7951
.25	.8094	.7993	.7899	.7811	.7730
.30	.7965	.7850	.7743	.7643	.7550
.35	.7860	.7733	.7614	.7504	.7400
.40	.7775	.7637	.7508	.7387	.7274
.45	.7704	.7557	.7418	.7288	.7167
.50	.7644	.7488	.7342	.7204	.7075
.55	.7593	.7430	.7277	.7132	.6995
.60	.7550	.7380	.7220	.7069	.6926
.65	.7512	.7337	.7171	.7014	.6865
.70	.7479	.7299	.7127	.6965	.6812
.75	.7450	.7265	.7089	.6922	.6764
.80	.7425	.7236	.7055	.6884	.6722
.85	.7402	.7209	.7025	.6850	.6684
.90	.7382	.7186	.6998	.6819	.6650
.95	.7364	.7165	.6974	.6792	.6619
1.00	.7348	.7145	.6952	.6767	.6591
1.05	.7333	.7128	.6932	.6744	.6565
1.10	.7319	.7112	.6913	.6723	.6542
1.15	.7307	.7098	.6897	.6704	.6521
1.20	.7295	.7084	.6881	.6687	.6501
1.25	.7285	.7072	.6867	.6670	.6483

TABLE 4

$f_2(\alpha,\beta)$, INDUCTOR BETWEEN PLATES

β \ α	.80	.85	.90	.95	1.00
.05	.9068	.9052	.9036	.9022	.9009
.10	.8552	.8520	.8491	.8463	.8438
.15	.8178	.8132	.8089	.8050	.8013
.20	.7888	.7829	.7775	.7724	.7677
.25	.7655	.7584	.7519	.7458	.7402
.30	.7463	.7383	.7307	.7237	.7172
.35	.7303	.7213	.7129	.7050	.6977
.40	.7168	.7069	.6977	.6891	.6810
.45	.7053	.6946	.6846	.6753	.6666
.50	.6953	.6840	.6733	.6633	.6540
.55	.6867	.6747	.6634	.6528	.6429
.60	.6792	.6665	.6547	.6436	.6332
.65	.6725	.6594	.6470	.6354	.6245
.70	.6667	.6530	.6401	.6281	.6168
.75	.6615	.6474	.6341	.6216	.6099
.80	.6568	.6423	.6286	.6158	.6037
.85	.6526	.6377	.6237	.6105	.5981
.90	.6489	.6337	.6193	.6057	.5930
.95	.6455	.6299	.6153	.6014	.5884
1.00	.6424	.6266	.6116	.5975	.5842
1.05	.6396	.6235	.6083	.5939	.5803
1.10	.6370	.6207	.6052	.5906	.5768
1.15	.6346	.6181	.6024	.5876	.5736
1.20	.6324	.6157	.5998	.5848	.5706
1.25	.6304	.6135	.5974	.5822	.5678

TABLE 5

$f_2(\alpha,\beta)$, INDUCTOR BETWEEN PLATES

β \ α	1.05	1.10	1.15	1.20	1.25
.05	.8996	.8985	.8974	.8964	.8955
.10	.8414	.8392	.8371	.8352	.8334
.15	.7979	.7947	.7917	.7889	.7863
.20	.7633	.7592	.7554	.7518	.7485
.25	.7349	.7300	.7254	.7211	.7170
.30	.7111	.7054	.7001	.6951	.6905
.35	.6909	.6845	.6785	.6729	.6676
.40	.6735	.6664	.6598	.6537	.6479
.45	.6584	.6507	.6436	.6369	.6306
.50	.6452	.6370	.6293	.6221	.6154
.55	.6336	.6249	.6168	.6091	.6019
.60	.6234	.6142	.6056	.5975	.5900
.65	.6143	.6047	.5957	.5872	.5793
.70	.6061	.5961	.5868	.5780	.5697
.75	.5989	.5885	.5788	.5696	.5610
.80	.5923	.5816	.5715	.5621	.5532
.85	.5864	.5754	.5650	.5553	.5461
.90	.5810	.5697	.5591	.5491	.5397
.95	.5761	.5646	.5537	.5434	.5338
1.00	.5716	.5598	.5487	.5383	.5284
1.05	.5676	.5555	.5442	.5335	.5235
1.10	.5638	.5516	.5400	.5292	.5190
1.15	.5604	.5479	.5362	.5252	.5148
1.20	.5572	.5446	.5327	.5215	.5109
1.25	.5542	.5415	.5294	.5180	.5073

TABLE 6

$f_2(\alpha,\beta)$, INDUCTOR IN SPACE

β \ α	.05	.10	.15	.20	.25
.05	.9890	.9820	.9771	.9733	.9702
.10	.9878	.9780	.9703	.9640	.9587
.15	.9873	.9764	.9671	.9591	.9521
.20	.9871	.9755	.9652	.9561	.9480
.25	.9870	.9750	.9641	.9541	.9451
.30	.9869	.9746	.9633	.9528	.9431
.35	.9868	.9744	.9627	.9518	.9416
.40	.9868	.9742	.9623	.9510	.9404
.45	.9867	.9740	.9619	.9504	.9395
.50	.9867	.9739	.9617	.9500	.9388
.55	.9867	.9738	.9615	.9496	.9382
.60	.9867	.9737	.9613	.9493	.9377
.65	.9866	.9737	.9611	.9490	.9373
.70	.9866	.9736	.9610	.9488	.9369
.75	.9866	.9736	.9609	.9486	.9366
.80	.9866	.9735	.9608	.9484	.9363
.85	.9866	.9735	.9607	.9482	.9361
.90	.9866	.9735	.9606	.9481	.9358
.95	.9866	.9734	.9605	.9480	.9356
1.00	.9866	.9734	.9605	.9479	.9355
1.05	.9866	.9734	.9604	.9478	.9353
1.10	.9866	.9733	.9604	.9477	.9352
1.15	.9865	.9733	.9603	.9476	.9350
1.20	.9865	.9733	.9603	.9475	.9349
1.25	.9865	.9733	.9602	.9474	.9348

TABLE 7

$f_2(\alpha,\beta)$, INDUCTOR IN SPACE

β \ α	.30	.35	.40	.45	.50
.05	.9677	.9655	.9635	.9618	.9602
.10	.9542	.9501	.9466	.9434	.9405
.15	.9460	.9406	.9357	.9312	.9272
.20	.9407	.9341	.9280	.9225	.9175
.25	.9369	.9293	.9224	.9160	.9101
.30	.9341	.9258	.9182	.9110	.9043
.35	.9320	.9232	.9148	.9070	.8997
.40	.9304	.9210	.9122	.9038	.8960
.45	.9292	.9193	.9100	.9012	.8929
.50	.9281	.9180	.9083	.8991	.8903
.55	.9273	.9168	.9068	.8973	.8881
.60	.9265	.9158	.9056	.8957	.8863
.65	.9259	.9150	.9045	.8944	.8847
.70	.9254	.9143	.9036	.8932	.8833
.75	.9250	.9137	.9028	.8922	.8821
.80	.9246	.9131	.9021	.8914	.8810
.85	.9242	.9127	.9015	.8906	.8800
.90	.9239	.9122	.9009	.8899	.8792
.95	.9236	.9119	.9004	.8893	.8784
1.00	.9234	.9115	.9000	.8887	.8777
1.05	.9231	.9112	.8996	.8882	.8771
1.10	.9229	.9109	.8992	.8877	.8765
1.15	.9227	.9107	.8989	.8873	.8760
1.20	.9226	.9104	.8986	.8869	.8755
1.25	.9224	.9102	.8983	.8866	.8751

TABLE 8

$f_2(\alpha, \beta)$, INDUCTOR IN SPACE

β \ α	.55	.60	.65	.70	.75
.05	.9588	.9575	.9563	.9552	.9542
.10	.9378	.9354	.9331	.9311	.9291
.15	.9234	.9200	.9168	.9138	.9110
.20	.9128	.9084	.9044	.9006	.8971
.25	.9046	.8995	.8947	.8901	.8859
.30	.8981	.8923	.8868	.8817	.8768
.35	.8929	.8864	.8803	.8746	.8692
.40	.8886	.8816	.8750	.8687	.8628
.45	.8850	.8775	.8704	.8637	.8574
.50	.8820	.8741	.8666	.8595	.8527
.55	.8794	.8712	.8633	.8558	.8486
.60	.8773	.8686	.8604	.8526	.8451
.65	.8754	.8664	.8579	.8498	.8420
.70	.8737	.8645	.8557	.8473	.8392
.75	.8723	.8628	.8538	.8451	.8368
.80	.8710	.8613	.8521	.8431	.8346
.85	.8698	.8600	.8505	.8414	.8326
.90	.8688	.8588	.8491	.8398	.8309
.95	.8679	.8577	.8479	.8384	.8293
1.00	.8671	.8568	.8468	.8371	.8278
1.05	.8663	.8559	.8457	.8360	.8265
1.10	.8656	.8551	.8448	.8349	.8253
1.15	.8650	.8543	.8440	.8339	.8242
1.20	.8644	.8536	.8432	.8330	.8232
1.25	.8640	.8530	.8424	.8322	.8222

TABLE 9

$f_2(\alpha, \beta)$, INDUCTOR IN SPACE

β \ α	.80	.85	.90	.95	1.00
.05	.9533	.9524	.9516	.9508	.9501
.10	.9273	.9256	.9240	.9226	.9212
.15	.9084	.9060	.9037	.9016	.8996
.20	.8938	.8907	.8878	.8850	.8824
.25	.8820	.8783	.8748	.8715	.8683
.30	.8723	.8680	.8639	.8601	.8565
.35	.8642	.8593	.8548	.8505	.8464
.40	.8573	.8520	.8470	.8423	.8378
.45	.8514	.8457	.8403	.8351	.8302
.50	.8463	.8402	.8344	.8289	.8237
.55	.8418	.8354	.8292	.8234	.8178
.60	.8380	.8312	.8247	.8186	.8127
.65	.8345	.8275	.8207	.8143	.8081
.70	.8315	.8242	.8171	.8104	.8040
.75	.8288	.8212	.8139	.8070	.8003
.80	.8264	.8186	.8110	.8039	.7970
.85	.8242	.8162	.8085	.8011	.7940
.90	.8223	.8140	.8061	.7985	.7912
.95	.8205	.8121	.8040	.7962	.7887
1.00	.8189	.8103	.8020	.7941	.7865
1.05	.8174	.8087	.8002	.7921	.7844
1.10	.8161	.8072	.7986	.7904	.7824
1.15	.8148	.8058	.7971	.7887	.7807
1.20	.8137	.8045	.7957	.7872	.7790
1.25	.8126	.8034	.7944	.7858	.7775

TABLE 10

$f_2(\alpha, \beta)$, INDUCTOR IN SPACE

β \ α	1.05	1.10	1.15	1.20	1.25
.05	.9494	.9488	.9482	.9476	.9470
.10	.9199	.9186	.9175	.9163	.9153
.15	.8977	.8959	.8942	.8926	.8911
.20	.8800	.8777	.8755	.8734	.8714
.25	.8654	.8626	.8599	.8574	.8550
.30	.8531	.8499	.8468	.8439	.8411
.35	.8426	.8389	.8355	.8322	.8290
.40	.8335	.8295	.8256	.8220	.8185
.45	.8256	.8212	.8170	.8131	.8093
.50	.8187	.8139	.8094	.8051	.8011
.55	.8125	.8075	.8027	.7981	.7938
.60	.8071	.8018	.7967	.7918	.7872
.65	.8022	.7966	.7913	.7862	.7813
.70	.7979	.7920	.7864	.7811	.7760
.75	.7940	.7879	.7821	.7765	.7712
.80	.7904	.7841	.7781	.7723	.7668
.85	.7872	.7807	.7745	.7685	.7628
.90	.7843	.7776	.7712	.7651	.7592
.95	.7816	.7747	.7682	.7619	.7558
1.00	.7791	.7721	.7654	.7590	.7528
1.05	.7769	.7697	.7629	.7563	.7499
1.10	.7748	.7675	.7605	.7538	.7473
1.15	.7729	.7655	.7583	.7515	.7449
1.20	.7712	.7636	.7563	.7493	.7426
1.25	.7695	.7618	.7545	.7474	.7405

TABLE 11

EXTERNAL FIELD, $-H_y(a/2,h/2,z)/K_s$ AS A FUNCTION OF z

INDUCTOR BETWEEN PLATES AND $\beta = .3$

z/h-β/2 \ α	.2	.4	.6	.8	1.0
0.0	5.40×10^{-4}	3.05×10^{-2}	1.19×10^{-1}	2.34×10^{-1}	3.45×10^{-1}
0.2	1.20×10^{-4}	8.67×10^{-3}	3.45×10^{-2}	6.62×10^{-2}	9.50×10^{-2}
0.4	1.40×10^{-5}	2.45×10^{-3}	1.32×10^{-2}	2.94×10^{-2}	4.61×10^{-2}
0.6	9.95×10^{-7}	5.48×10^{-4}	4.32×10^{-3}	1.16×10^{-2}	2.01×10^{-2}
0.8	5.48×10^{-8}	1.10×10^{-4}	1.33×10^{-3}	4.34×10^{-3}	8.45×10^{-3}
1.0	2.64×10^{-9}	2.11×10^{-5}	3.98×10^{-4}	1.60×10^{-3}	3.50×10^{-3}
1.2	1.19×10^{-10}	3.96×10^{-6}	1.18×10^{-4}	5.89×10^{-4}	1.44×10^{-3}
1.4	5.11×10^{-12}	7.35×10^{-7}	3.49×10^{-5}	2.16×10^{-4}	5.93×10^{-4}
1.6	2.15×10^{-13}	1.36×10^{-7}	1.03×10^{-5}	7.89×10^{-5}	2.44×10^{-4}
1.8	8.94×10^{-15}	2.51×10^{-8}	3.04×10^{-6}	2.89×10^{-5}	1.00×10^{-4}
2.0	3.68×10^{-16}	4.62×10^{-9}	8.95×10^{-7}	1.03×10^{-5}	4.12×10^{-5}

TABLE 12

EXTERNAL FIELD, $-H_y(a/2,h/2,z)/K$, AS A FUNCTION OF z

INDUCTOR IN SPACE AND $\beta = .3$

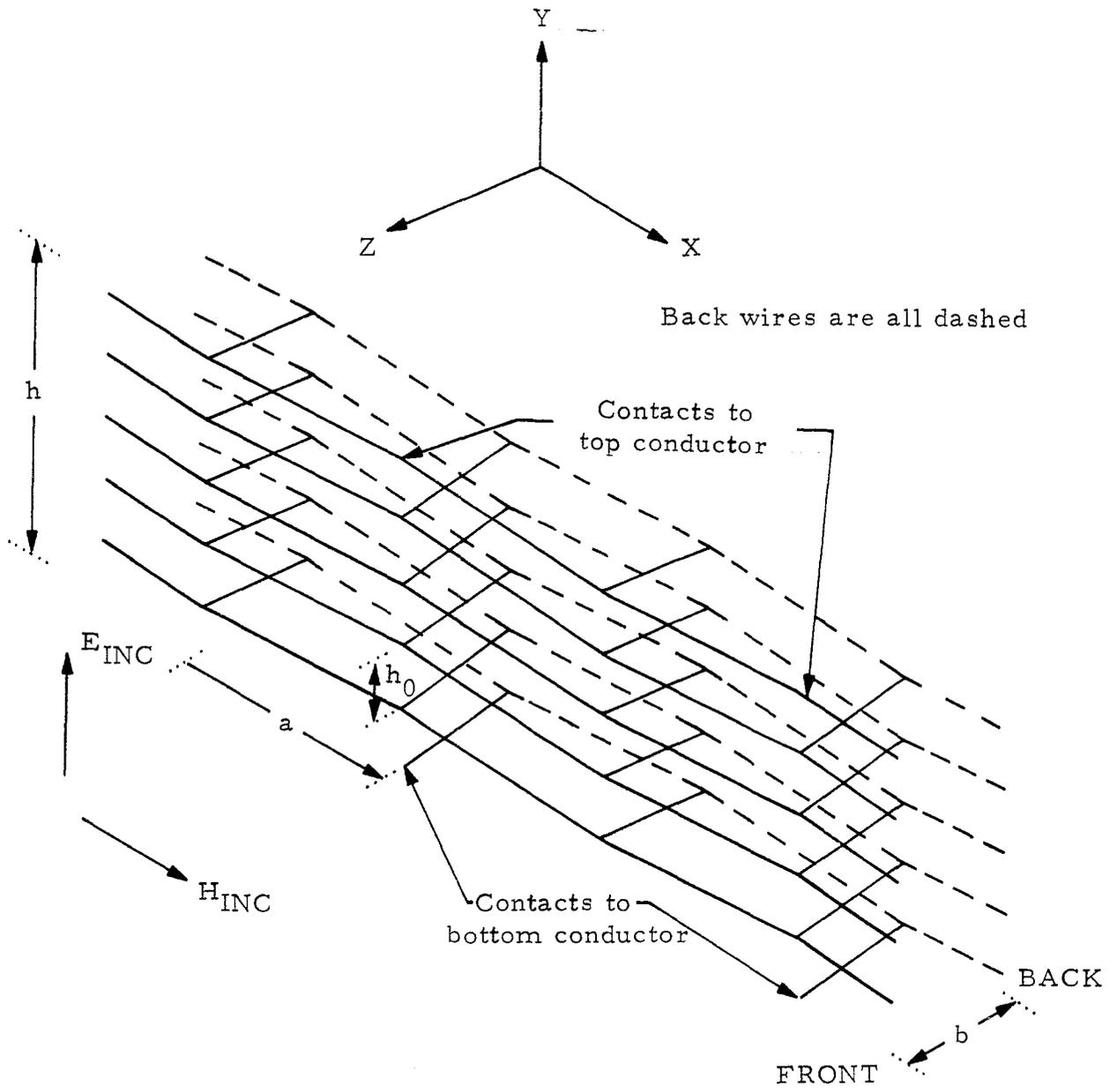| z/h-β/2 \ α	.2	.4	.6	.8	1.0
0.0	2.24×10^{-4}	9.84×10^{-3}	3.27×10^{-2}	5.77×10^{-2}	7.95×10^{-2}
0.2	5.99×10^{-5}	4.34×10^{-3}	1.73×10^{-2}	3.37×10^{-2}	4.92×10^{-2}
0.4	6.99×10^{-6}	1.23×10^{-3}	6.66×10^{-3}	1.51×10^{-2}	2.44×10^{-2}
0.6	4.98×10^{-7}	2.74×10^{-4}	2.20×10^{-3}	6.09×10^{-3}	1.10×10^{-3}
0.8	2.74×10^{-8}	5.53×10^{-5}	6.85×10^{-4}	2.36×10^{-3}	4.88×10^{-3}
1.0	1.32×10^{-9}	1.06×10^{-5}	2.10×10^{-4}	9.11×10^{-4}	2.17×10^{-3}
1.2	5.93×10^{-11}	2.01×10^{-6}	6.41×10^{-5}	3.54×10^{-4}	9.72×10^{-4}
1.4	2.56×10^{-12}	3.78×10^{-7}	1.97×10^{-5}	1.39×10^{-4}	4.41×10^{-4}
1.6	1.08×10^{-13}	7.09×10^{-8}	6.07×10^{-6}	5.50×10^{-5}	2.03×10^{-4}
1.8	4.48×10^{-15}	1.33×10^{-8}	1.89×10^{-6}	2.20×10^{-5}	9.45×10^{-5}
2.0	1.85×10^{-16}	2.52×10^{-9}	5.93×10^{-7}	8.90×10^{-6}	4.45×10^{-5}

Figure 1. Distributed inductor: perspective view

Figure 2A. Coordinate system used to contemplate A_x, inductor between plates

Figure 2B. Shifted coordinate system used to compute A_z, inductor between plates

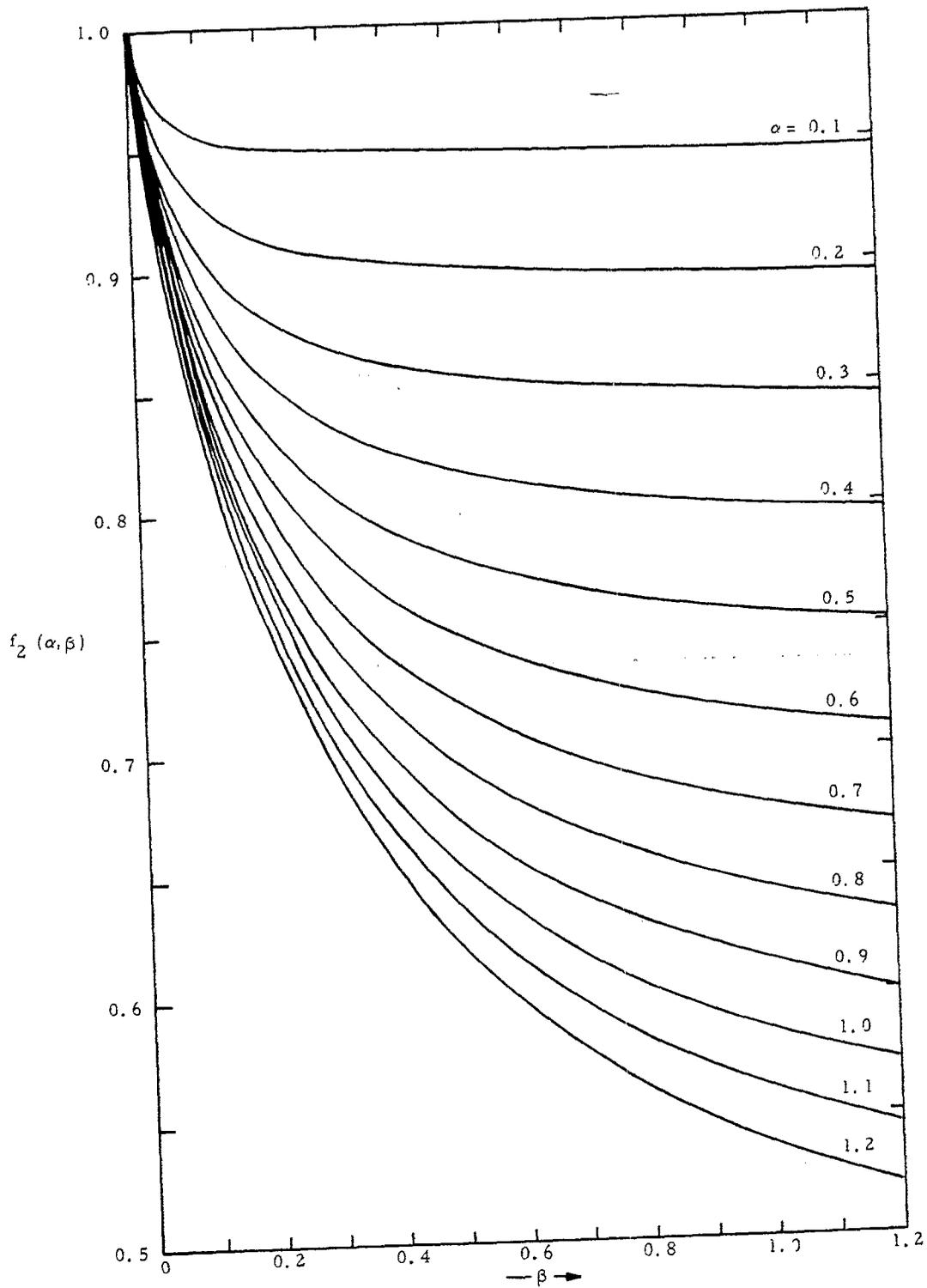Figure 3. Integration paths used in proving that $f(\alpha, 0) = 1$
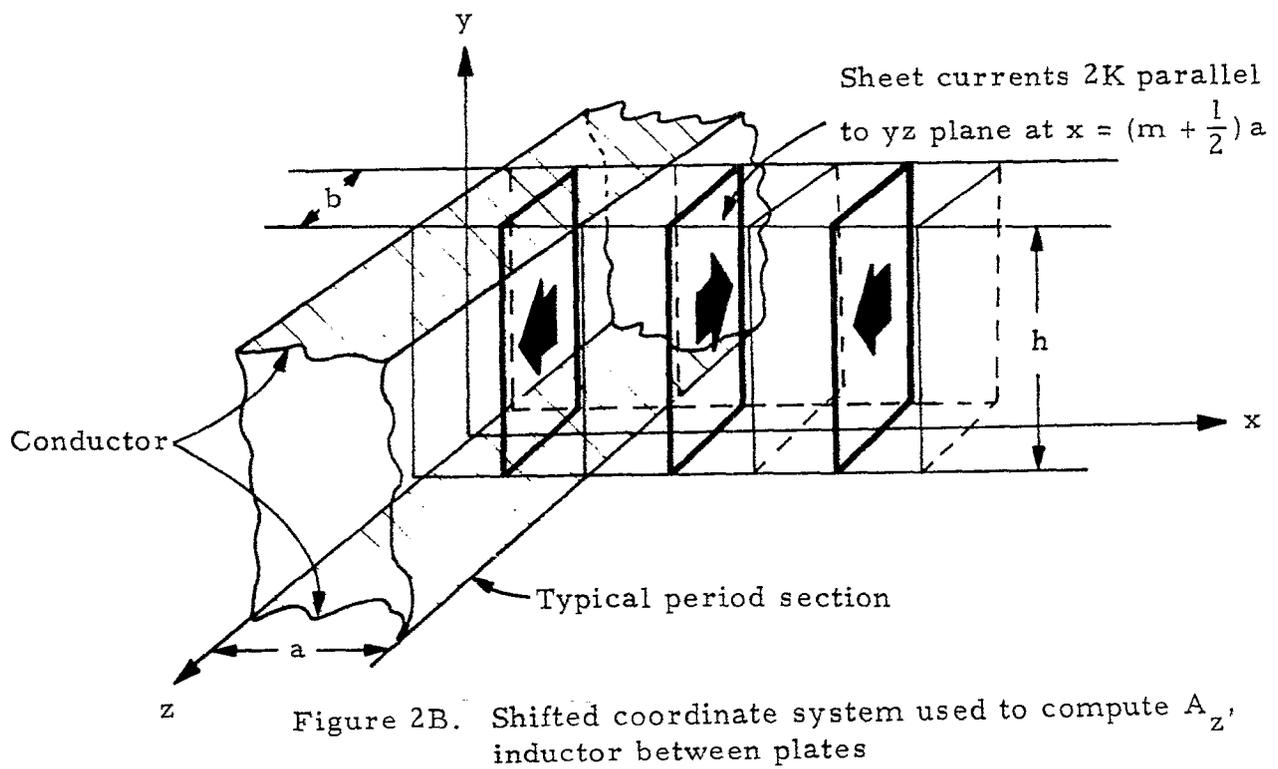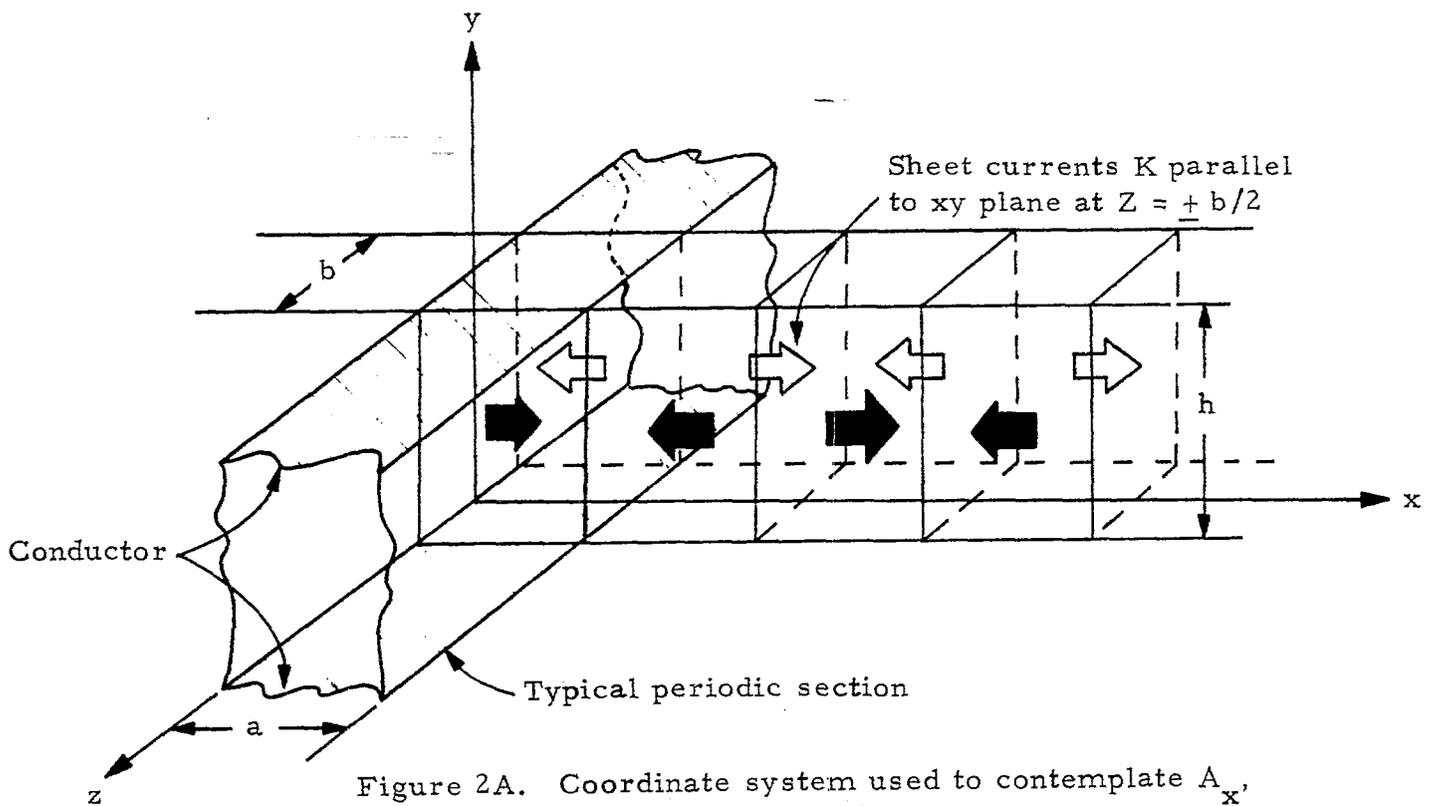
Figure 4. $f_2 (\alpha, \beta)$, Inductor between plates

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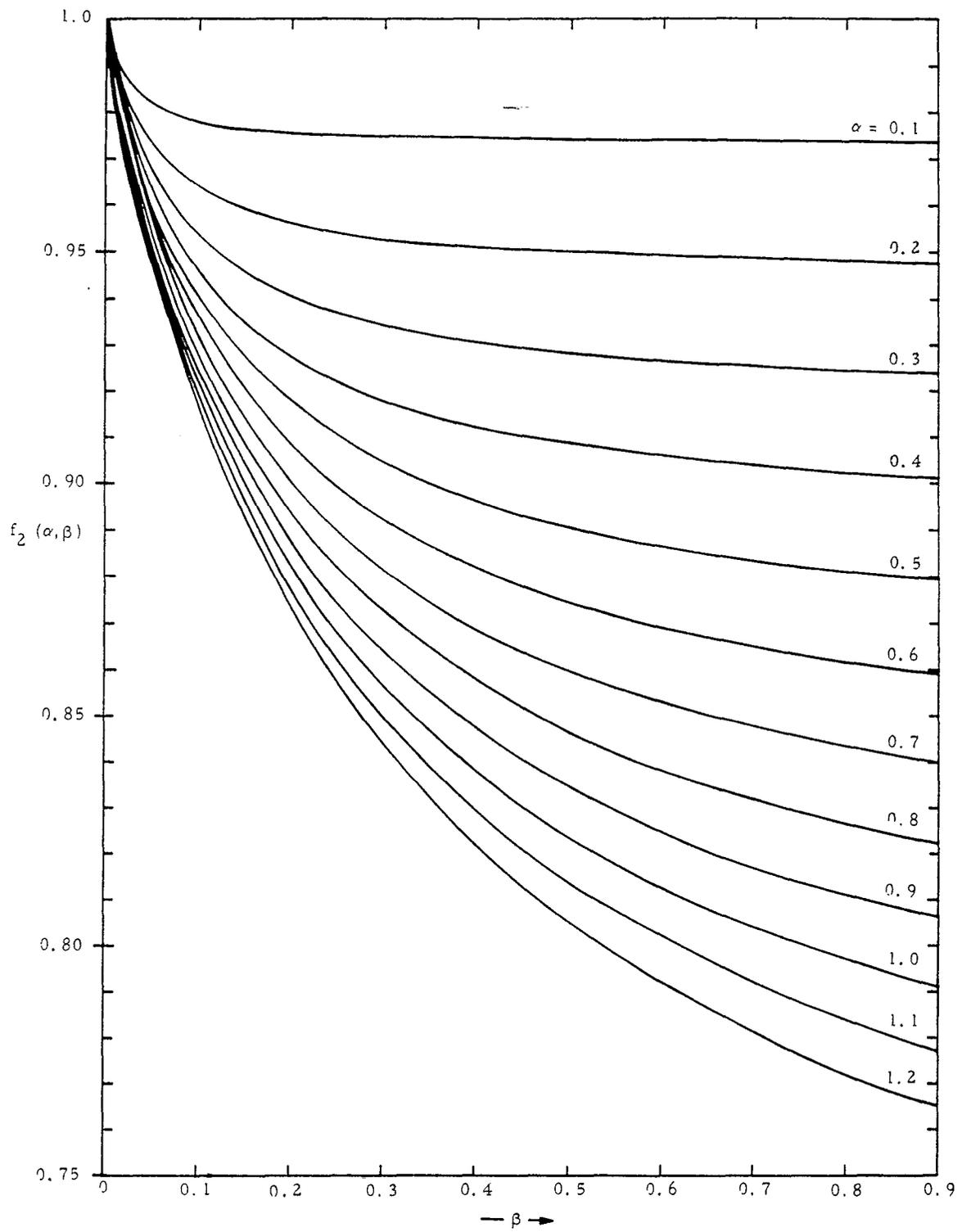
Figure 5. $f_2 (\alpha, \beta)$, Inductor in space
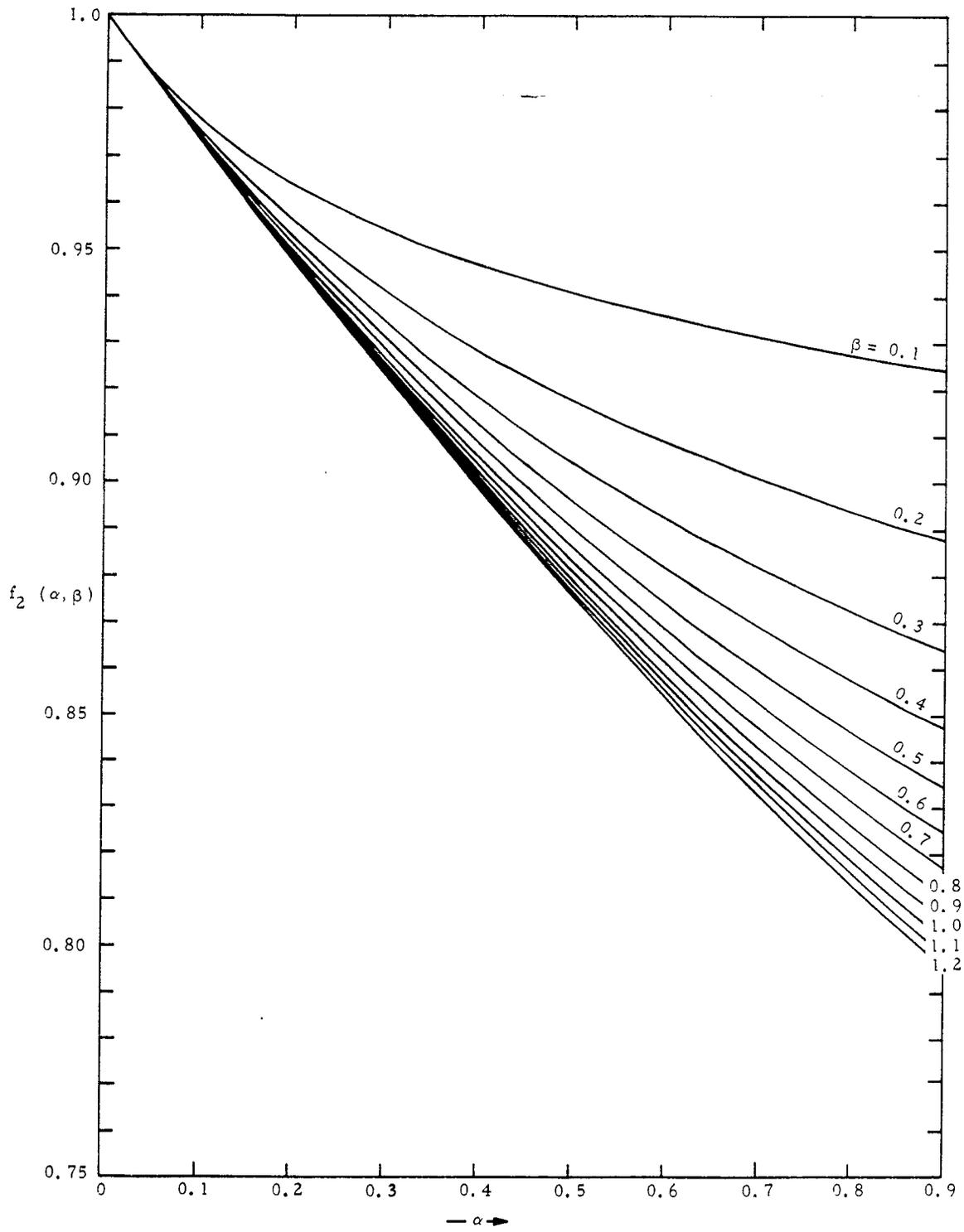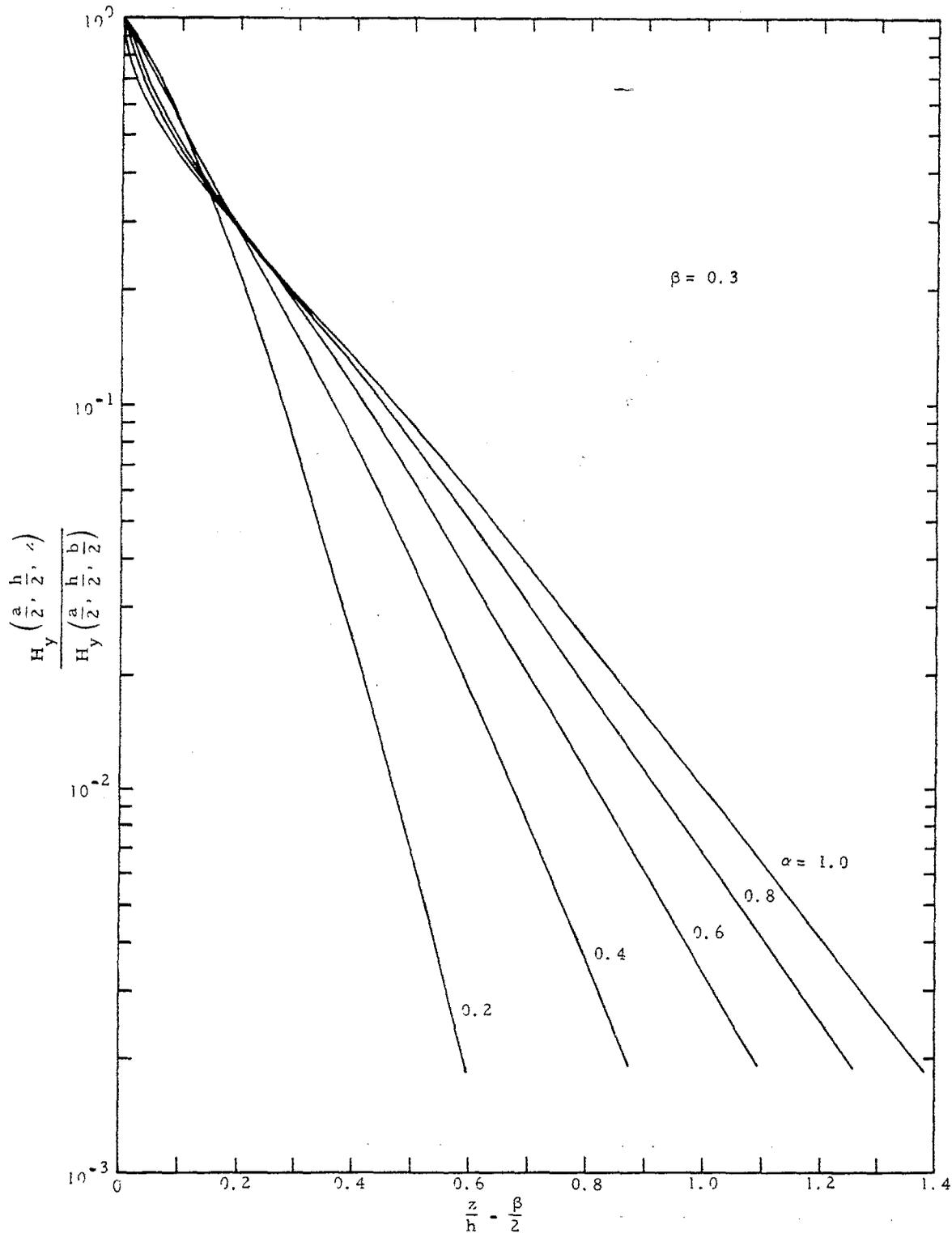
29
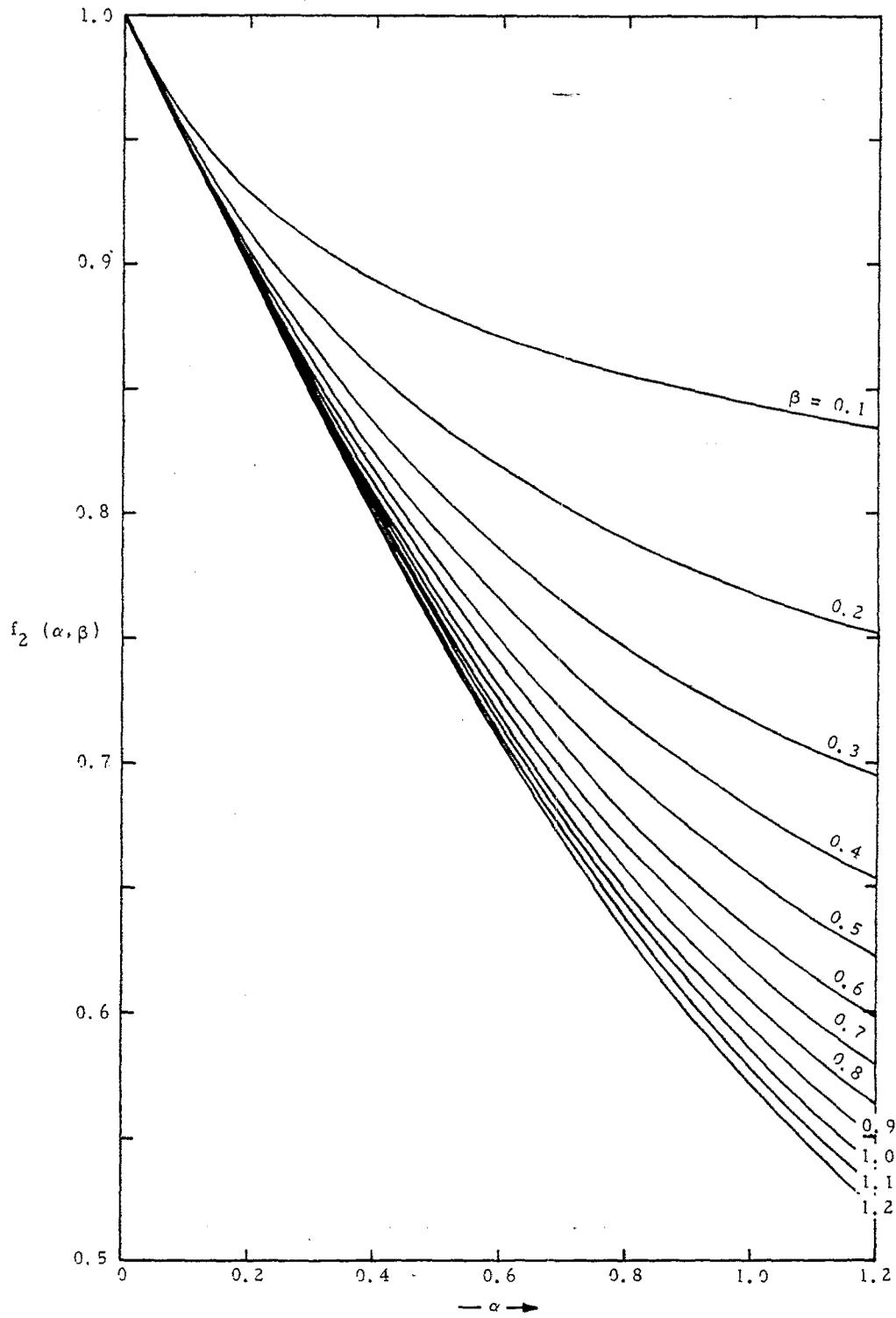
Figure 6. f_2 (α, β), inductor between plates

Figure 7. f_2 (α, β), inductor in space

Figure 8. Relative external field, inductor between plates
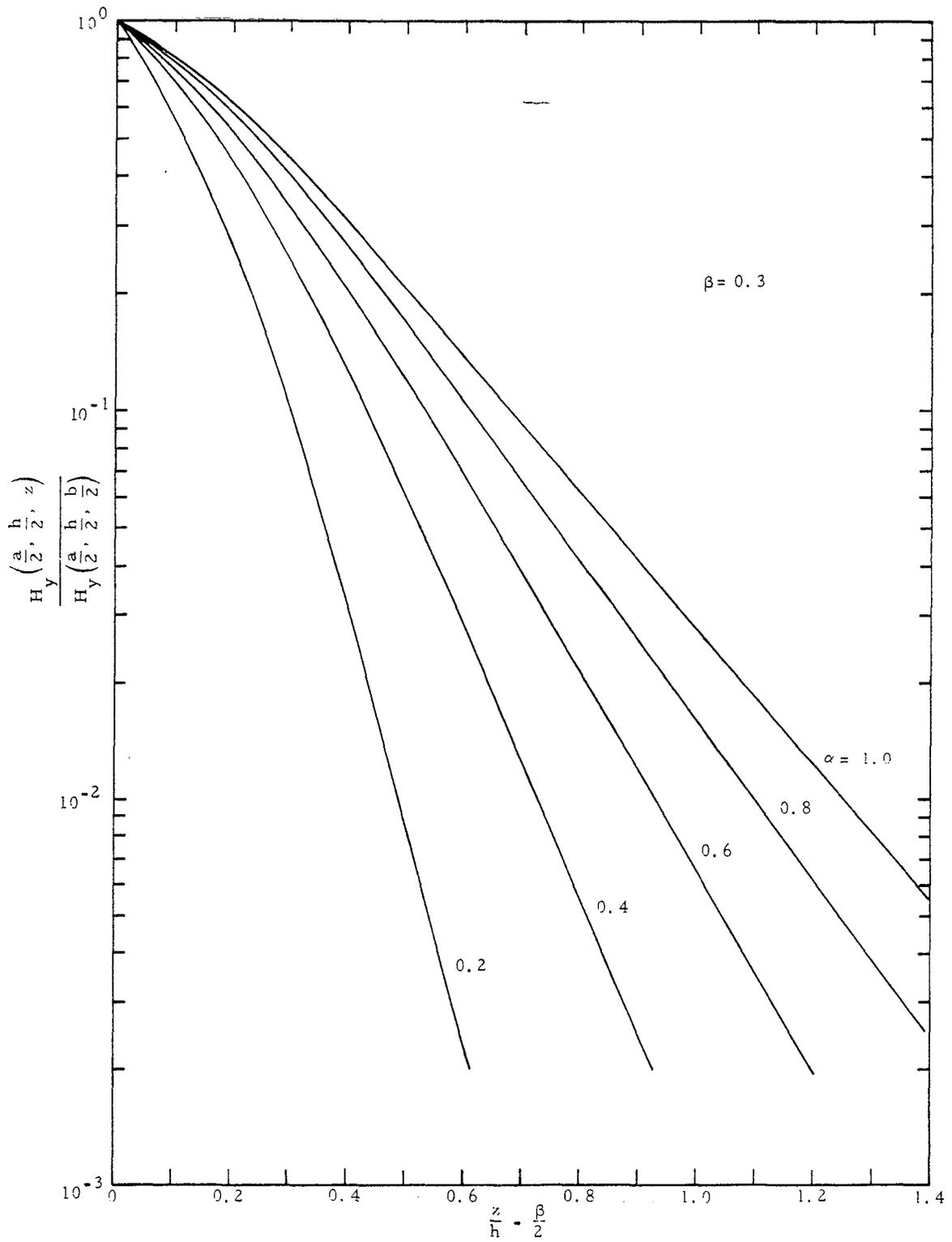
Figure 9.   Relative external field, inductor in space

Appendix A

In this appendix we present some of the detailed algebra to justify certain of the equations in Section II A. These details were omitted previously in order to make a concise presentation of the work but it is perhaps well to include them in an appendix, if only for the sake of completeness. In particular we present some of the missing steps in deriving equations (7), (8), (15), (16), (17), (23), and (24).

From (4) and (5):

$$K = H_y(x,y,b/2 - 0) - H_y(x,y,b/2 + 0)$$

$$= \frac{1}{\mu_o} \left[\frac{\partial A_x(x,y,b/2 - 0)}{\partial z} - \frac{\partial A_x(x,y,b/2 + 0)}{\partial z} \right]$$

$$= \frac{1}{\mu_o} \sum_{p,q} A_{pq} \sin \frac{p\pi x}{a} \sin \frac{q\pi y}{h} \Gamma_{p,q} \left[\coth(\Gamma_{p,q} b/2) + 1 \right]$$

multiplying both sides of this equation by

$$\sin \frac{n\pi x}{a} \sin \frac{m\pi y}{h}$$

and integrating over the rectangle $(0 < x < a; 0 < y < h)$ one obtains equation (7) for the coefficients $A_{n,m}$ and so one can write for the front current sheet

$$\underline{A} \cdot \underline{K} = \left(\frac{4}{\pi}\right)^2 K^2 \sum_{n,m \text{ odd}} \left[\frac{\sin \frac{n\pi x}{a} \sin \frac{m\pi y}{h}}{nm\Gamma_{n,m}\left[1 + \coth(\Gamma_{n,m} b/2)\right]} \right] .$$

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By integrating this expression over the square mentioned above and doubling
the result to account for the back current sheet one obtains the result

$$L^x{}_I{}^2 = 2\left(\frac{4K}{\pi}\right)^2 \sum_{n,m \text{ odd}} \frac{4ah}{n^2 m^2 \pi^2} \frac{1}{\Gamma_{n,m}\left[1 + \coth(\Gamma_{n,m} b/2)\right]} \quad .$$

Now since

$$\frac{1}{1 + \coth x} = \frac{1 - e^{-2x}}{2}$$

equation (8) follows immediately.

We will now sketch the derivation of the Green's function to be used
in equation (14). By definition this function satisfies

$$\nabla^2 G(x,y,z,a/2,y',z') = -\ \delta(x - a/2)\delta(y - y')\delta(z - z') \qquad \text{(A-1)}$$

$$G = 0 \qquad x = 0,a$$
$$y = 0,h$$
$$z \rightarrow \pm \infty$$

Since G satisfies Laplaces equation for $z > z'$ or $z < z'$ we
may write

$$G(x,y,z,a/2,y',z') = \sum_{n,m} g_{n,m} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} e^{-\Gamma_{n,m}\left|z-z'\right|} \qquad \text{(A-2)}$$

If one integrates equation (A-1) with respect to z from $z' - \varepsilon$ to $z' + \varepsilon$,

he obtains

$$2\varepsilon(\nabla^2_{xy}G) + \frac{\partial G}{\partial z}(x,y,z'+\varepsilon,a/2y',z') - \frac{\partial G}{\partial z}(x,y,z'-\varepsilon,a/2,y',z') = - \delta(x-a/2)\delta(y-y')$$

Now if one substitutes (A-2) in this equation, allows ε to approach zero, multiplies by $\sin(n\pi x/a)\sin(m\pi y/h)$ and integrates over the square $(0 < x < a; 0 < y < h)$, he obtains

$$g_{n,m} = \sin\frac{n\pi}{2} \sin\frac{m\pi y}{h} \frac{2}{ah\Gamma_{n,m}} \quad .$$

By substituting these coefficients back in (A-2) one obtains the representation for G given by equation (15). Now carrying out the integral of equation (14)

$$A_z(a/2,y,z) = 2\mu_o \int_{-b/2}^{b/2} \int_o^h \frac{2}{ah} \sum_m \sin\frac{m\pi y}{h} \sin\frac{m\pi y'}{h} \frac{e^{-\Gamma_{n,m}|z-z'|}}{\Gamma_{n,m}} K\, dy'dz'$$

$$= \frac{16hK}{\pi a} \sum_{m\ odd} \frac{\sin(m\pi y/h)}{\Gamma_{n,m}} \left(1 - e^{-\Gamma_{n,m}b/2} \cosh\Gamma_{n,m}z\right) \quad . \qquad (A-3)$$

By multiplying this equation by $2K$ and integrating over the rectangle $(0 < y < h; -b/2 < z < b/2)$ one may write

$$L^z I^2 = \frac{32K^2\mu_o}{\pi a} \sum_{m\ odd} \frac{2h}{m\pi} \cdot \frac{1}{m\Gamma^2_{n,m}} \left[b - \frac{2\sinh(\Gamma_{n,m}b/2)e^{-\Gamma_{n,m}b/2}}{\Gamma_{n,m}} \right]$$

from which, by employing the definitions (10), (11), and (12), one may easily obtain equation (16). It is then easy to see, from (13) and (16) that an

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expression for $f_2(\alpha,\beta)$ is

$$f_2(\alpha,\beta) = f_2^y(\alpha,\beta) + f_2^x(\alpha,\beta)$$

$$= \frac{64}{\pi^2} \left(\frac{\beta}{\alpha}\right)^2 \sum_{n,m \text{ odd}} \left[\frac{1}{m^2(\beta\pi)^2 \gamma_{n,m}^2} - \frac{\left(1 - e^{-\beta\pi\gamma_{n,m}}\right)}{m^2\gamma_{n,m}(\beta\pi)^3} \left(\frac{1}{\gamma_{n,m}^2} - \frac{\alpha^2}{n^2}\right) \right] \qquad \text{(A-4)}$$

Now by using the formula

$$\sum_{n \text{ odd}} \frac{1}{n^2 + x^2} = \frac{\pi}{4x} \tanh \frac{\pi x}{2}$$

the first term in the sum of (A-4) may be performed analytically with respect to n while the second term may be condensed to give, as a final result, equation (17).

In order to calculate H_y one must compute

$$H_y = \frac{1}{\mu_o} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \qquad \text{(A-5)}$$

where, from (4) and (7)

$$\frac{\partial A_x}{\partial z} = \frac{\partial}{\partial z} \sum_{n,m \text{ odd}} \left[\frac{4}{\pi^2} \frac{\mu_o K}{nm\Gamma_{n,m}\left[1+\coth(\Gamma_{n,m}b/2)\right]} \right] \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{h} e^{-\Gamma_{n,m}(z-b/2)}$$

while from (14) and (15)

$$\frac{\partial A_z}{\partial x} = \frac{\partial}{\partial x} \frac{16\mu_o Kh}{\pi a} \sum_{n,m \text{ odd}} \frac{\sin(n\pi x/a)\sin(m\pi y/h)}{\Gamma_{n,m}} \left(1 - e^{-\Gamma_{n,m}b/2} \cosh \Gamma_{n,m}z\right)$$

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recalling that, because of the differing coordinate systems we must set $x = a/2$ in the first of these while $x = a$ in the second, we may combine them according to (A-5) to get (23) while, for the case $z = b/2$, (24) follows in a manner analogous to that used to derive (17) from (16).

Appendix B

If there are no plates present, the sheet currents generate fields that may be calculated by employing a Fourier integral in the y variable along with a Fourier series in the x variable, where the axes are parallel to those shown in figure 2. We choose as the coordinate system for calculating A_x one similar to that shown in figure 2A except that we now assume the xz plane to be in the center of the current sheets. This plane is thus a plane of symmetry and we may write the equations analogous to (4) and (5) as

$$A_x = \sum_n \int_0^\infty dk \, A_n(k) \sin\left(\frac{n\pi x}{a}\right) \cos ky \, e^{-\xi_{nk}(z-b/2)} \qquad z > b/2$$

$$A_x = \sum_n \int_0^\infty dk \, A_n(k) \sin\left(\frac{n\pi x}{a}\right) \frac{\sinh(\xi_{nk}z)}{\sinh(\xi_{nk}b/2)} \qquad |z| < b/2 \qquad (B-1)$$

where

$$\xi_{nk}^2 = (n\pi/a)^2 + k^2$$

Equating the difference in the z derivatives (at $z = b/2$) of the two expressions for A_x, to μ_0 times the sheet current gives:

$$\mu_0 K = \sum_n \int_0^\infty dk \, A_n(k) \sin\left(\frac{n\pi x}{a}\right) \cos ky \, \xi_{nk}\left[\coth(\xi_{nk}b/2) + 1\right] \qquad ,$$

from which, by multiplying by

$$\sin(m\pi x/a)\cos k'y$$

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and integrating over the rectangle $(0 < x < a; -h/2 < y < h/2)$ one may determine

$$A_m(k') = \frac{8\mu_o K}{\pi^2} \frac{\sin(k'h/2)(1 - e^{-\xi_{mk}b})}{k'\xi_{mk}} \qquad (m \text{ odd})$$

$$= 0 \qquad\qquad (m \text{ even})$$

One can now substitute this in (B-1), multiply by K, and again integrate over the rectangle mentioned above to obtain

$$L^x I^2 = \frac{32K^2 abh}{3} \sum_{n \text{ odd}} \int_o^\infty dk \left(\frac{\sin(kh/2)}{kh}\right)^2 \frac{(1 - e^{-\xi_{nk}b})}{\xi_{nk}b \, n^2} \qquad (B-2)$$

To calculate $L^z I^2$ we may integrate $2A_z K$ over a rectangle of z-directed sheet current where A_z is again given by (14) but

$$G(a/2,y,z; a/2,y'z') = \sum_{n \text{ odd}} \int_o^\infty dk \frac{e^{ik(y-y')}}{2\pi} \frac{e^{-\xi_{nk}|z-z'|}}{\xi_{nk}a} \qquad .$$

This procedure leads at once to

$$L^z I^2 = \frac{16K^2 h^2 b}{\pi a} \sum_{n \text{ odd}} \int_o^\infty dk \left(\frac{\sin kh/2}{kh/2}\right)^2 \frac{1}{\xi_{nk}^2} \left\{1 - \frac{1 - e^{-\xi_{nk}b}}{\xi_{nk}b}\right\} \qquad (B-3)$$

We now add (B-2) and (B-3), change the variable of integration from k to $(kh/2)$, which we call x, and use the definitions of Section 2A to obtain

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$$f_2(\alpha,\beta) = \frac{8}{\pi^3} \sum_{n \text{ odd}} \int_0^\infty dx \left(\frac{\sin x}{x}\right)^2 \frac{1}{n^2 + \left(\frac{2x\alpha}{\pi}\right)^2}$$

$$+ \frac{8}{\pi^3} \sum_{n \text{ odd}} \int_0^\infty dx \left(\frac{\sin x}{x}\right)^2 \frac{1 - e^{-2\beta\delta_{nx}}}{\beta\delta_{nx}} \left(\frac{1}{n^2} - \frac{\pi^2}{(n\pi)^2 + (2x\alpha)^2}\right) \qquad \text{(B-4)}$$

where

$$\delta_{nx}^2 = (n\pi/2\alpha)^2 + x^2$$

The only gaps now remaining in the demonstration of expression (25), which for numerical reasons is a more suitable representation than (B-4), are the two derivations

$$\int_0^\infty dx \left(\frac{\sin x}{x}\right)^2 \frac{1}{n^2 + (2x\alpha/\pi)^2} = \frac{1}{n^2} \int_0^\infty dx \, \sin^2 x \left(\frac{1}{x^2} - \frac{1}{x^2 + (n\pi/2\alpha)^2}\right)$$

$$= \frac{\pi}{2n^2} - \frac{1}{2n^2} \int_0^\infty \frac{1 - \cos 2x}{x^2 + (n\pi/2\alpha)^2} \, dx$$

$$= \frac{\pi}{2n^2} - \frac{1}{2n^2} \left(\frac{2\alpha}{n\pi}\right) \int_0^\infty \frac{1 - \cos(n\pi u/\alpha)}{u^2 + 1} \, du$$

$$= \frac{\pi}{2n^2} - \frac{\alpha}{2n^3} \left(1 - e^{-\frac{n\pi}{\alpha}}\right)$$

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and

$$\int_0^\infty dx \left(\frac{\sin x}{x}\right)^2 \frac{1}{\delta_{nx}} \left[\frac{1}{2} - \frac{\pi^2}{(n\pi)^2 + (2x\alpha)^2}\right] = \int_0^\infty dx \frac{\sin^2 x}{n^2 \delta_{nx}^3}$$

$$= \frac{1}{2n^2} \int_0^\infty \frac{1 - \cos 2x}{(x^2 + (n\pi/2\alpha)^2)^{3/2}} dx$$

$$= \frac{1}{2n^2} \left[\left(\frac{2\alpha}{n\pi}\right)^2 - \frac{2\alpha}{n\pi} K_1(n\pi/\alpha)\right]$$

where K_1 is a modified Bessel function.

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