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A Simplified Two-Dimensional Model for the Fields Above the Distributed-Source Surface Transmission Line

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Abstract

Associated with a distributed-source surface transmission line there are fields above the source and associated currents in the source. The source must then deliver charge and energy associated with these currents. The purpose of this note is to develop an approximate two-dimensional model for these fields, currents, and charges. Certain idealizations of the geometry are made and a step-function time dependence of the source (propagating at the speed of light with no loss in amplitude) is used.

I. Introduction

In a previous note[1] we proposed a type of simulator which, in its general class, can be termed a distributed-source surface transmission line. The general concept for such a simulator involves distributing an array of energy sources (e.g., charged capacitors) above the ground surface and then triggering the sources in a sequence so as to produce an electromagnetic wave propagating at approximately the speed of light over the ground surface. As illustrated in figure 1 this array of sources is approximated as a continuous sheet in order to simplify the calculations. The medium below the sources might be made conducting in order to allow a late-time vertical current density at the ground surface. Other sources and impedances might be included at each end ($x=0$ and $x=d$) to try to minimize the distortions due to these discontinuities in the structure.

In reference 1 we considered the fields below the distributed source for the uniform, planar geometry illustrated in figure 1. This was a two-dimensional calculation in that the width of the sheet source, w, was assumed infinite and the source was assumed independent of y. The sheet source is represented by specifying the tangential electric field, E_s, along the sheet at $z=h$ as having only an x component and varying only with x. With this tangential electric field being, in general, not identically zero there are then electromagnetic fields above the distributed source. Associated with the magnetic field just below the source there is a current in the source whereby the source delivers energy to the fields below the source. There is similarly a current in the source associated with the magnetic field just above the source. In this note we develop an approximate model for the fields above the source and the associated current in the source.

The model developed in this note is approximate because of some simplifications introduced in the geometry. The width, w, of the source in the y direction is assumed infinite in order to make the problem a two-dimensional one. The height, h, of the source above the ground is assumed small compared to the length, d. Thus, for this model for calculating the fields above the source, the source is assumed at the ground level. Finally, the exposed ground (for $x< 0$ and for $x>d$) is assumed perfectly conducting. This leaves us with the idealized geometry illustrated in figure 2. The purpose of these idealizations is, of course, to simplify the calculations somewhat. These idealizations also introduce inaccuracies into the model but some of the important features are preserved. In particular we can observe the buildup in the initial source currents as the source wave propagates at the speed of light in the +x direction. This phenomenon is associated with the fact that the source wave is propagating at the same speed as the propagation speed of the electromagnetic fields above the sheet source, i.e., the speed of light in vacuum, c. For the present note we take E_s as a step function wave propagating at speed, c, in the +x direction with constant amplitude, independent of x for $0<x<d$. Other time histories of E_s can be considered, as long as E_s is still a function only of retarded time, $t-x/c$, by using convolution techniques with the present results. Other forms of E_s can also be considered by introducing them at the appropriate point in the calculational procedure.

1. Capt Carl E. Baum, Sensor and Simulation Note 48, The Planar, Uniform Surface Transmission Line Driven from a Sheet Source, August 1967.

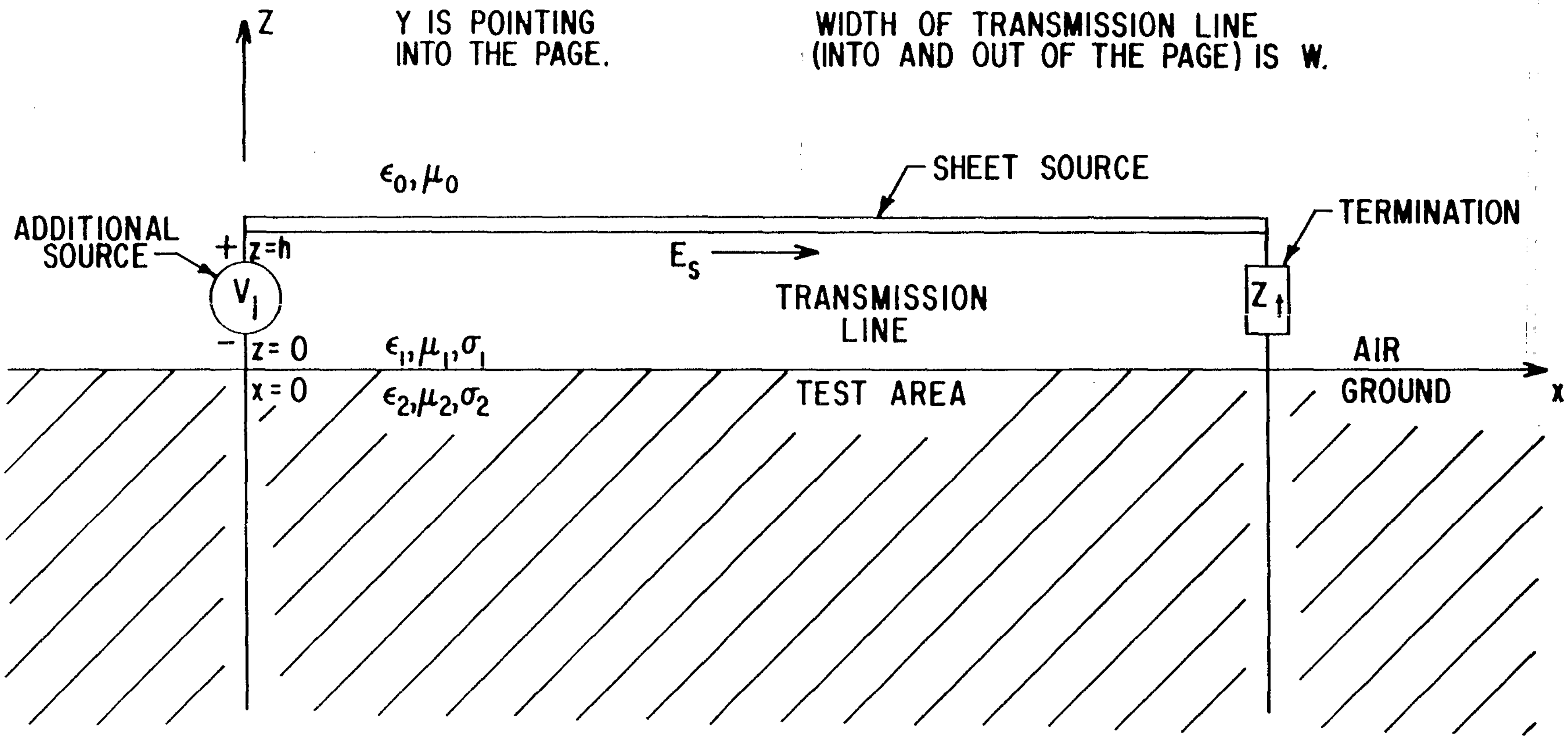


FIGURE 1. THE DISTRIBUTED-SOURCE, PLANAR, UNIFORM SURFACE TRANSMISSION LINE

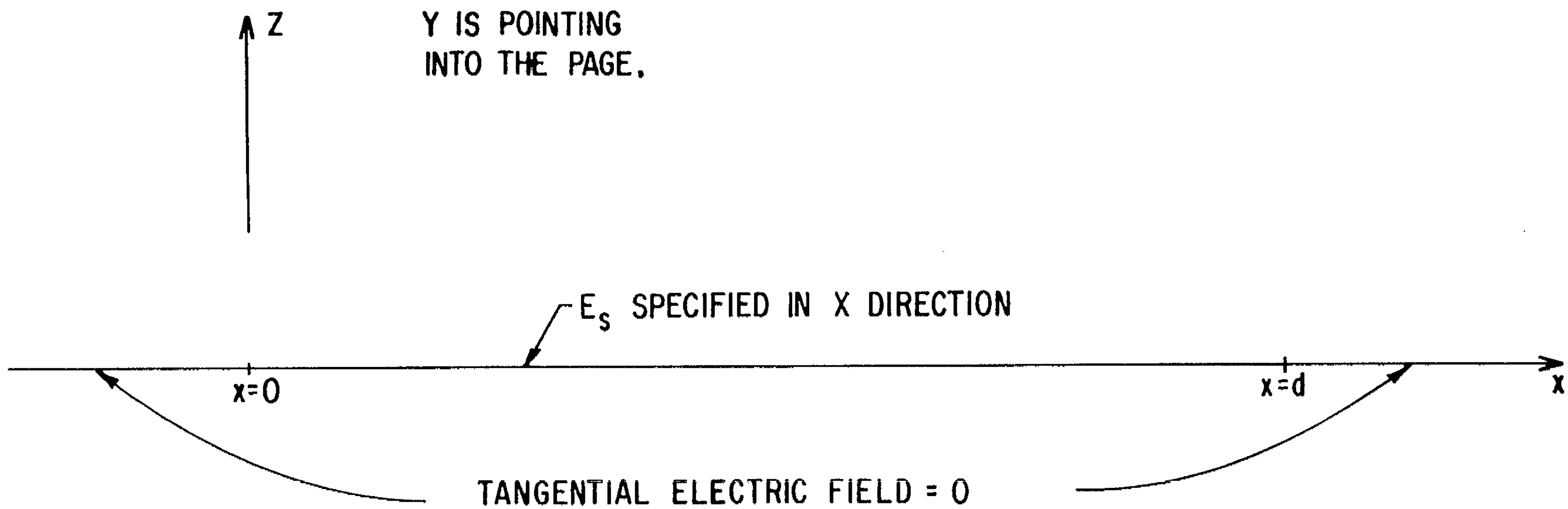


FIGURE 2. SIMPLIFIED GEOMETRY FOR THE DISTRIBUTED SOURCE FOR CALCULATING THE FIELDS ABOVE IT

In this note we first consider the electromagnetic fields associated with a specified tangential electric field distribution on a plane and then specialize this to the case that the tangential electric field has only an x component and is independent of y. Besides the present application of these results to the fields above the distributed source, they have other possible applications to the fields below the distributed source including such things as the field distortion at the ends of the distributed-source and the effect of the discreteness of the source on the field uniformity, provided that the medium below the distributed source is a single homogeneous, isotropic medium.

II. Fields Associated with a Two-Dimensional Distribution of the Tangential Electric Field on a Plane

We first consider a tangential electric field distribution on the z=0 plane with components E'_x and E'_y, both of which may be functions of x' and y', the coordinates for the given tangential electric field distribution. Figure 3A illustrates the coordinates of the observation point or the position at which we wish to calculate the fields. The distance from a source point (x', y', 0) on the z=0 plane to the observation point is given by

$$R \equiv [(x-x')^2 + (y-y')^2 + z^2]^{1/2} \tag{1}$$

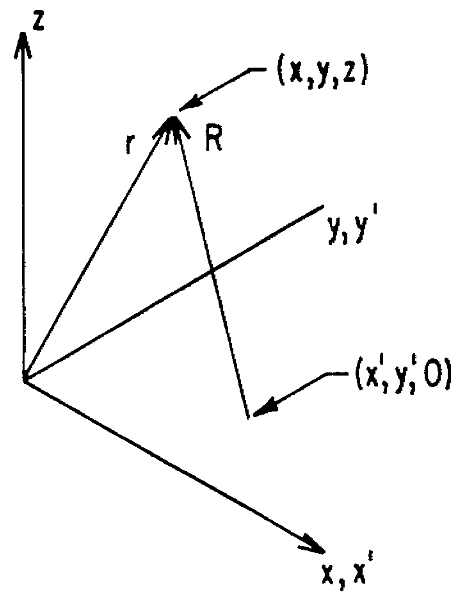
There are the three unit vectors $\vec{e}_x$, $\vec{e}_y$, $\vec{e}_z$ for the three coordinate directions. The unit vector in the direction from the source point to the observation point is

$$\vec{e}_R \equiv \frac{x-x'}{R} \vec{e}_x + \frac{y-y'}{R} \vec{e}_y + \frac{z}{R} \vec{e}_z \tag{2}$$

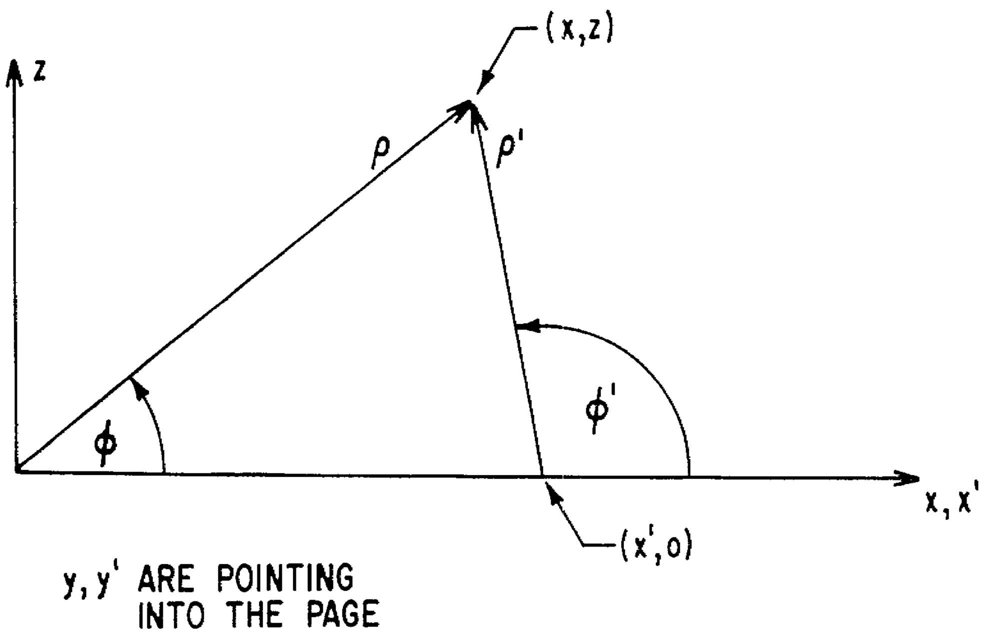
The permittivity, permeability, and conductivity (ϵ, μ, and σ, respectively) of the medium for z>0 are assumed to be scalars and to be independent of the coordinates. The results are formulated for the fields for z>0, but by use of appropriate symmetry relationships they can be applied to the fields below the z = 0 plane with the same restrictions on the medium there. We first formulate the Laplace transform of the fields and for this case ϵ, μ, and σ can be functions of the Laplace transform variable, s (or $j\omega$ for frequency domain considerations). Then we consider the time domain case with $\epsilon = \epsilon_o$, $\mu = \mu_o$, $\sigma = 0$ for which case the medium corresponds to free space.[2]

As our starting point we use an equation in Smythe[3] which gives the vector potential as an integral over the electric field, $\vec{E}'$, on the z=0 plane as

2. All units are rationalized MKSA.
3. W. R. Smythe, Static and Dynamic Electricity, 2nd ed., 1950, p.496, eqn. 14.19(1).



A. THREE - DIMENSIONAL CASE



B. TWO - DIMENSIONAL CASE

FIGURE 3. COORDINATE SYSTEMS

$$\tilde{\vec{A}} = -\frac{1}{2\pi s}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\gamma R+1}{R^2}\,[(\vec{e}_z \times \tilde{\vec{E}}'(x',y'))\times \vec{e}_R]e^{-\gamma R}dx'dy' \tag{3}$$

The tilde, ˜, is used to denote the Laplace transform of the quantity with respect to time. Zero initial conditions are assumed and/or this transform can be interpreted as a two-sided Laplace transform. The propagation constant, γ, is given by

$$\gamma \equiv \sqrt{s\mu(\sigma+s\varepsilon)} \equiv jk \tag{4}$$

where k is also termed the propagation constant and is useful for frequency-domain considerations as

$$k = \sqrt{\omega\mu(\omega\varepsilon-j\sigma)} \tag{5}$$

The result of equation 3 is reached by considering the tangential electric field on the z=0 plane to be produced by an appropriate distribution of magnetic dipoles and then finding the vector potential as an integral over this magnetic-dipole distribution. Note that we have allowed a nonzero σ in the above formulation thereby allowing conduction currents in the medium of interest. In this type of formulation the vector potential is an integral over only the source currents. This is equivalent to allowing the permittivity to be a complex function of frequency. Equation 3 is an exact result, matching the tangential electric field on the z=0 plane and having the proper behavior in the limit of large z.

The scalar potential (for z>0) from the magnetic dipole distribution (on the z=0 plane) is zero so that the electric field for z>0 is related to the vector potential as

$$\vec{E} = -\frac{\partial\vec{A}}{\partial t}\ ,\quad \tilde{\vec{E}} = -s\tilde{\vec{A}} \tag{6}$$

giving

$$\tilde{\vec{E}} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\gamma R+1}{R^2}\,[(\vec{e}_z \times \tilde{\vec{E}}')\times \vec{e}_R]e^{-\gamma R}\,dx'dy' \tag{7}$$

The magnetic field (for z>0) is related to the vector potential as

$$\vec{B} = \mu\vec{H} = \nabla \times \vec{A}\quad ,\qquad \tilde{\vec{B}} = \mu\tilde{\vec{H}} = \nabla \times \tilde{\vec{A}} \tag{8}$$

giving

$$\tilde{\vec{B}} = -\frac{1}{s}\nabla\times\tilde{\vec{E}} = -\frac{1}{2\pi s}\left\{\nabla\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\gamma R+1}{R^2}\,[(\vec{e}_z \times \tilde{\vec{E}}')\times \vec{e}_R]\,dx'dy'\right\} \tag{9}$$

Expanding the integrand in vector components we have

$$\vec{e}_z \times \vec{\tilde{E}}' = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ 0 & 0 & 1 \\ \tilde{E}'_x & \tilde{E}'_y & \tilde{E}'_z \end{vmatrix} = -\tilde{E}'_y \vec{e}_x + \tilde{E}'_x \vec{e}_y \tag{10}$$

Using equation 2 for $\vec{e}_R$ we have

$$(\vec{e}_z \times \vec{\tilde{E}}') \times \vec{e}_R = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ -\tilde{E}'_y & \tilde{E}'_x & 0 \\ \frac{x-x'}{R} & \frac{y-y'}{R} & \frac{z}{R} \end{vmatrix}$$

$$= \frac{z}{R} \tilde{E}'_x \vec{e}_x + \frac{z}{R} \tilde{E}'_y \vec{e}_y - \left[\frac{x-x'}{R} \tilde{E}'_x + \frac{y-y'}{R} \tilde{E}'_y \right] \vec{e}_z \tag{11}$$

Note that only the tangential components of the electric field on the z=0 plane appear in the integrands in equations 7 and 9. For the x and y components of the electric field we then have

$$\tilde{E}_{x,y} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\gamma R+1}{R^2} \frac{z}{R} e^{-\gamma R} \tilde{E}'_{x,y} \, dx'dy' \tag{12}$$

where the subscript x, y is used to indicate that either the x or y component can be used as long as it is used consistently throughout the equation. The z component is given by

$$\tilde{E}_z = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\gamma R+1}{R^2} e^{-\gamma R} \left[\frac{x-x'}{R} \tilde{E}'_x + \frac{y-y'}{R} \tilde{E}'_y \right] dx'dy' \tag{13}$$

Next consider the magnetic field which we calculate from

$$\vec{\tilde{B}} = -\frac{1}{s} \nabla \times \vec{\tilde{E}} = -\frac{1}{s} \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \tilde{E}_x & \tilde{E}_y & \tilde{E}_z \end{vmatrix} \tag{14}$$

Thus one can obtain $\vec{\tilde{B}}$ from equations 12 and 13. In taking the required derivatives we have the intermediate results

$$\frac{\partial R}{\partial x} = \frac{x-x'}{R} , \quad \frac{\partial R}{\partial y} = \frac{y-y'}{R} , \quad \frac{\partial R}{\partial z} = \frac{z}{R} \tag{15}$$

and

$$\frac{\partial}{\partial R}\left[\frac{\gamma R+1}{R^3} e^{-\gamma R}\right] = -\frac{e^{-\gamma R}}{R^4}\left[(\gamma R)^2 + 3\gamma R + 3\right] \tag{16}$$

The components of $\vec{\tilde{B}}$ are then

$$\tilde{B}_x = \frac{1}{2\pi s}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{-\frac{(x-x')(y-y')}{R^2}\left[(\gamma R)^2 + 3\gamma R + 3\right]\frac{e^{-\gamma R}}{R^3}\tilde{E}'_x + \left[2\gamma R + 2 - \frac{(y-y')^2+z^2}{R^2}\left[(\gamma R)^2 + 3\gamma R + 3\right]\right]\frac{e^{-\gamma R}}{R^3}\tilde{E}'_y\right\}dx'dy' \tag{17}$$

$$\tilde{B}_y = \frac{1}{2\pi s}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\frac{(x-x')(y-y')}{R^2}\left[(\gamma R)^2 + 3\gamma R + 3\right]\frac{e^{-\gamma R}}{R^3}\tilde{E}'_y - \left[2\gamma R + 2 - \frac{(x-x')^2+z^2}{R^2}\left[(\gamma R)^2 + 3\gamma R + 3\right]\right]\frac{e^{-\gamma R}}{R^3}\tilde{E}'_x\right\}dx'dy' \tag{18}$$

$$\tilde{B}_z = \frac{1}{2\pi s}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{-\frac{(y-y')z}{R^2}\left[(\gamma R)^2 + 3\gamma R + 3\right]\frac{e^{-\gamma R}}{R^3}\tilde{E}'_x + \frac{(x-x')z}{R^2}\left[(\gamma R)^2 + 3\gamma R + 3\right]\frac{e^{-\gamma R}}{R^3}\tilde{E}'_y\right\}dx'dy' \tag{19}$$

Thus, we have the transform of the six field components for specified transforms, $\tilde{E}'_x$ and $\tilde{E}'_y$, of the tangential components of the electric field on the z=0 plane. Alternatively one can calculate the six field components in the time domain, if E'_x and E'_y are specified, by performing a convolution integral of each of these quantities with the inverse transforms of the appropriate Laplace-domain coefficients as they appear in the above equations. For the special case that the medium for $z>0$ corresponds to free space (with $\varepsilon = \varepsilon_o$, $\mu = \mu_o$, $\sigma = 0$) the forms of the inverse transforms simplify since $\gamma = s/c$ where we also have

$$c \equiv \frac{1}{\sqrt{\mu_o \varepsilon_o}} \quad , \; Z_o \equiv \sqrt{\frac{\mu_o}{\varepsilon_o}} \tag{20}$$

Then $e^{-\gamma R}$ introduces a time delay so that we have

Laplace domain	time domain
$e^{-\gamma R}\, \tilde{E}'_x(s)$	$E'_x\,(t - \frac{R}{c})$
$e^{-\gamma R}\, \tilde{E}'_y(s)$	$E'_y\,(t - \frac{R}{c})$

The polynomials in γR become derivatives in the time domain, operating on the retarded tangential fields in the following forms

Laplace domain	time domain
γR	$\frac{R}{c}\frac{\partial}{\partial t}$
$(\gamma R)^2$	$\left(\frac{R}{c}\right)^2 \frac{\partial^2}{\partial t^2}$

In the case that $\sigma \neq 0$ the forms of the inverse transforms of the integrands in equations 12, 13, 17, 18, and 19 are somewhat more complicated.

III. Fields Associated with a One-Dimensional Distribution of a Particular Single-Component Tangential Electric Field on a Plane

We now specialize the results of the previous section to a case of present interest to us. Specifically, we set $E'_y \equiv 0$ and constrain E'_x to be independent of y'. Then because of the symmetry thereby introduced three of the field components are zero, specifically

$$E_y = 0 \quad , \quad B_x = 0 \quad , \; B_z = 0 \tag{21}$$

The remaining field components then have the forms

$$\tilde{E}_x = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\gamma R + 1}{R^2} \; \frac{z}{R} \; e^{-\gamma R} \; \tilde{E}'_x \; dx' dy' \tag{22}$$

$$\tilde{E}_z = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\gamma R + 1}{R^2} \; \frac{x - x'}{R} \; e^{-\gamma R} \tilde{E}'_x \; dx' dy' \tag{23}$$

$$\tilde{B}_y = \frac{1}{2\pi s} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\frac{(x-x')^2+z^2}{R^2} \quad [(\gamma R)^2 + 3\gamma R + 3] - 2\gamma R - 2 \right] \frac{e^{-\gamma R}}{R^3} \tilde{E}_x' \, dx'dy' \tag{24}$$

Since $\tilde{E}_x'$ is independent of y' we can perform the integration over y' before further specifying $\tilde{E}_x'$. For our present considerations we introduce two cylindrical coordinate systems as illustrated in figure 3B. There is the $(\rho,\phi,-y)$ system where

$$\rho \equiv [x^2 + z^2]^{1/2} \quad , \quad \tan(\phi) \equiv \frac{z}{x} \tag{25}$$

and there is the $(\rho',\phi',-y)$ system where

$$\rho' \equiv [(x-x')^2 + z^2]^{1/2} \quad , \quad \tan(\phi') \equiv \frac{z}{x-x'} \tag{26}$$

Note that $-y$ is used in both cases to make the systems right handed. The first system is centered on the axis $(x,z) = (0,0)$ while the second is centered on $(x,z) = (x',0)$. Introducing a change of variable

$$\eta \equiv y-y' \tag{27}$$

we then have

$$R = [\rho'^2 + \eta^2]^{1/2} \tag{28}$$

Equations 22 through 24 can now be rewritten as

$$\tilde{E}_x = \int_{-\infty}^{\infty} \tilde{E}_x' \frac{\sin(\phi')}{\rho'} \left\{ \int_{-\infty}^{\infty} \frac{1}{2\pi} [\gamma R+1] \frac{\rho'^2}{R^3} e^{-\gamma R} \, d\eta \right\} dx' \tag{29}$$

$$\tilde{E}_z = -\int_{-\infty}^{\infty} \tilde{E}_x' \frac{\cos(\phi')}{\rho'} \left\{ \int_{-\infty}^{\infty} \frac{1}{2\pi} [\gamma R+1] \frac{\rho'^2}{R^3} e^{-\gamma R} \, d\eta \right\} dx' \tag{30}$$

$$\tilde{B}_y = \int_{-\infty}^{\infty} \frac{\tilde{E}_x'}{\rho'} \left\{ \int_{-\infty}^{\infty} \frac{\rho'}{2\pi s} \left[\frac{\rho'^2}{R^2} \quad [(\gamma R)^2 + 3\gamma R + 3] - 2\gamma R - 2 \right] \frac{e^{-\gamma R}}{R^3} \, d\eta \right\} dx' \tag{31}$$

In appendix A the first integral with respect to η is solved. Using equation A5 the electric field components are then

$$\tilde{E}_x = \int_{-\infty}^{\infty} \tilde{E}_x' \frac{\sin(\phi')}{\rho'} \frac{1}{\pi} \gamma\rho' K_1(\gamma\rho') \, dx' \tag{32}$$

$$\tilde{E}_z = -\int_{-\infty}^{\infty} \tilde{E}_x' \frac{\cos(\phi')}{\rho'} \frac{1}{\pi} \gamma\rho' K_1(\gamma\rho') \, dx' \tag{33}$$

In equation 31 the integral over η can also be evaluated, but a simpler approach is to use the relation between $\vec{B}$ and $\vec{E}$ in equation 14 which gives

$$\begin{aligned}
\tilde{B}_y &= \frac{1}{s}\int_{\infty}^{\infty} \tilde{E}_x' \frac{\gamma}{\pi}\left\{\frac{\partial}{\partial\rho'}\left(\frac{K_1(\gamma\rho')}{\rho'}\right)\left[\frac{\partial\rho'}{\partial x}[-(x-x')] - \frac{\partial\rho'}{\partial z} z\right]\right. \\
&\qquad \left. + \frac{K_1(\gamma\rho')}{\rho'}\left[\frac{\partial[-(x-x')]}{\partial x} - \frac{\partial z}{\partial z}\right]\right\} dx' \\
&= \frac{1}{s}\int_{-\infty}^{\infty} \tilde{E}_x' \frac{\gamma}{\pi}\left\{-\rho' \frac{\partial}{\partial\rho'}\left(\frac{K_1(\gamma\rho')}{\rho'}\right) - 2\frac{K_1(\gamma\rho')}{\rho'}\right\} dx' \\
&= \frac{1}{s}\int_{-\infty}^{\infty} \tilde{E}_x' \frac{\gamma}{\pi\rho'}\left\{\gamma\rho' K_2(\gamma\rho') - 2K_1(\gamma\rho')\right\} dx' \\
&= \frac{\gamma}{s}\int_{-\infty}^{\infty} \tilde{E}_x' \frac{1}{\pi\rho'} \gamma\rho' K_o(\gamma\rho') \, dx'
\end{aligned} \tag{34}$$

where we have used the derivative and recurrence relationships for the Bessel functions. For convenience in frequency domain analysis one can use $\gamma = jk$ and other forms of the Bessel functions as[4]

$$\begin{aligned}
\frac{1}{\pi}\gamma\rho' K_1(\gamma\rho') &= -j\frac{k\rho'}{2} H_1^{(2)}(k\rho') \\
\frac{1}{\pi}\gamma\rho' K_o(\gamma\rho') &= \frac{k\rho'}{2} H_o^{(2)}(k\rho')
\end{aligned} \tag{35}$$

4. See AMS 55, Handbook of Mathematical Functions, National Bureau of Standards, 1964 for the various Bessel function relationships.

Equations 32, 33, and 35 express the results for the transforms of the three field components for the assumed form of E_x' independent of y' with $E_y' = 0$. These same results can be reached by considering a line magnetic dipole at a particular x' giving an $\tilde{E}_x'$ (independent of y') and associated fields in the form of an expanding cylindrical wave described by the Bessel functions as above. Integrating over x' then gives the above results.

Our interest now centers on the case that the medium for z>0 corresponds to free space for which we have $\gamma = s/c$. The three field components can then be written

$$\tilde{E}_x = \int_{-\infty}^{\infty} s\tilde{E}_x' \frac{\sin(\phi')}{\rho'} \left[\frac{\rho'}{\pi c} K_1 \left(\frac{s\rho'}{c}\right)\right] dx' \tag{36}$$

$$\tilde{E}_z = -\int_{-\infty}^{\infty} s\tilde{E}_x' \frac{\cos(\phi')}{\rho'} \left[\frac{\rho'}{\pi c} K_1 \left(\frac{s\rho'}{c}\right)\right] dx' \tag{37}$$

$$\tilde{B}_y = \frac{1}{c}\int_{-\infty}^{\infty} s\tilde{E}_x' \frac{1}{\rho'} \left[\frac{\rho'}{\pi c} K_o \left(\frac{s\rho'}{c}\right)\right] dx' \quad . \tag{38}$$

The functions in the square brackets have inverse transforms given by[5]

Laplace domain	time domain
$\frac{\rho'}{\pi c} K_o \left(\frac{s\rho'}{c}\right)$	$\frac{1}{\pi}\left[\left(\frac{ct}{\rho'}\right)^2 - 1\right]^{-1/2} u\left(t - \frac{\rho'}{c}\right)$
$\frac{\rho'}{\pi c} K_1 \left(\frac{s\rho'}{c}\right)$	$\frac{1}{\pi}\frac{ct}{\rho'}\left[\left(\frac{ct}{\rho'}\right)^2 - 1\right]^{-1/2} u\left(t - \frac{\rho'}{c}\right)$

For convenience we give names to these two functions as

$$e_{o_\phi}' \equiv -\frac{1}{\pi}\frac{ct}{\rho'}\left[\left(\frac{ct}{\rho'}\right)^2 - 1\right]^{-1/2} u\left(t - \frac{\rho'}{c}\right) \tag{39}$$

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5. See reference 3 and use equation 29.3.119 and use a time derivative and delay in equation 29.3.121.

$$h'_o \equiv \frac{1}{\pi}\left[\left(\frac{ct}{\rho'}\right)^2 - 1\right]^{-1/2} u\left(t - \frac{\rho'}{c}\right) \tag{40}$$

Then we can write the three field components in the time domain as

$$E_x = -\int_{-\infty}^{\infty} \frac{\partial E'_x}{\partial t} * e'_{o_\phi} \frac{\sin(\phi')}{\rho'} \, dx' \tag{41}$$

$$E_z = \int_{-\infty}^{\infty} \frac{\partial E'_x}{\partial t} * e'_{o_\phi} \frac{\cos(\phi')}{\rho'} \, dx' \tag{42}$$

$$B_y = \frac{1}{c} \int_{-\infty}^{\infty} \frac{\partial E'_x}{\partial t} * h'_o \frac{dx'}{\rho'} \tag{43}$$

where * is used to indicate the convolution integral with respect to time. With E'_x specified as a function of x' and t one can use these last equations to calculate the fields directly in the time domain.

IV. Fields Above the Distributed Source

With the results of the last section we now calculate the fields above the distributed source with the approximations as discussed in the introduction. As was illustrated in figure 2 we specify the tangential electric field on the z=0 plane for $0<x<d$. Outside this range the tangential electric field is constrained to be zero. For the calculations in this note we first specify the tangential electric field to be of the form

$$E'_x \equiv E_o \, u\left(t - \frac{x'}{c}\right) u(x') \tag{44}$$

Thus we consider the field at the source starting at x=0 and propagating to the right at the speed of light with a constant amplitude, E_o. Later we shall stop the source at x'=d. While a step-function time dependence is used here, other time dependences can be considered by using convolution integral techniques.

For convenience we define normalized waveforms for the three field components as

$$h_o \equiv \frac{Z_o H_y}{E_o} = \frac{cB_y}{E_o} \;, \quad e_{o_\phi} \equiv \frac{E_\phi}{E_o} \;, \quad e_{o_\rho} = \frac{E_\rho}{E_o} \tag{45}$$

Note that we are using the cylindrical coordinate system, (ρ, ϕ, y), which is centered on the beginning of the source. Considering first h_o from equation 43 we have

$$h_o = \frac{1}{\pi}\int_0^{\infty} \delta(t - \frac{x'}{c}) * \left\{ \left[\left(\frac{ct}{\rho'}\right)^2 - 1 \right]^{-1/2} u(t - \frac{\rho'}{c}) \right\} \frac{dx'}{\rho'}$$

$$= \frac{1}{\pi}\int_0^{\infty} [(ct-x')^2 - \rho'^2]^{-1/2} \, u(t - \frac{x'+\rho'}{c}) \, dx' \tag{46}$$

Introducing a change of variable

$$\rho^2 \eta = (ct-x')^2 - z^2 - (x-x')^2$$
$$\rho^2 d\eta = -2(ct-x) \, dx' \tag{47}$$

and defining a normalized time

$$\tau \equiv \frac{ct}{\rho} \tag{48}$$

we then have for $\tau > 1$

$$h_o = -\frac{\rho}{2\pi(ct-x)} \int_{\tau^2-1}^{0} \eta^{-1/2} \, d\eta \tag{49}$$

while $h_o = 0$ for $\tau < 1$. Then we have

$$h_o = \frac{u(\tau-1)}{2\pi} \, [\tau - \cos(\phi)]^{-1} \int_0^{\tau^2-1} \eta^{-1/2} d\eta$$

$$= \frac{u(\tau-1)}{\pi} \, \frac{[\tau^2-1]^{1/2}}{\tau - \cos(\phi)} \tag{50}$$

which is a rather simple closed-form result.

Next consider the normalized electric field components. Instead of evaluating equations 41 and 42 with the assumed source dependence we can take advantage of the fact that h_o is a function of ρ and t in the combination ct/ρ or τ. Starting from one of Maxwell's equations

$$\nabla \times \vec{H} = \varepsilon_o \frac{\partial \vec{E}}{\partial t} \tag{51}$$

we have in cylindrical coordinates $(\rho,\phi,-y)$

$$\varepsilon_o \frac{\partial E_\rho}{\partial t} = -\frac{1}{\rho}\frac{\partial H_y}{\partial \rho}$$

$$\varepsilon_o \frac{\partial E_\phi}{\partial t} = \frac{\partial H_y}{\partial \rho} \tag{52}$$

which in terms of the normalized field components are

$$\frac{\partial e_{o_\rho}}{\partial t} = -\frac{c}{\rho}\frac{\partial h_o}{\partial \phi}$$

$$\frac{\partial e_{o_\phi}}{\partial t} = c\ \frac{\partial h_o}{\partial \rho} \tag{53}$$

Now consider τ and ϕ as the only independent variables so that we use

$$\frac{\partial}{\partial t} = \frac{c}{\rho}\frac{\partial}{\partial \tau} \quad , \quad \frac{\partial}{\partial \rho} = \frac{\partial \tau}{\partial \rho}\frac{\partial}{\partial \tau} = -\ \frac{\tau}{\rho}\frac{\partial}{\partial \tau} \tag{54}$$

Substituting these operators into equations 53 gives

$$\frac{\partial e_{o_\phi}}{\partial \tau} = -\ \frac{\partial h_o}{\partial \phi} \tag{55}$$

$$\frac{\partial e_{o_\phi}}{\partial \tau} = -\ \tau\frac{\partial h_o}{\partial \tau} \tag{56}$$

We use these equations to calculate the electric field components from the result of equation 50.

For e_{o_ρ} we then have

$$e_{o_\rho} = \frac{u(\tau-1)}{\pi}\sin(\phi) \int_1^\tau \frac{[\nu^2-1]^{1/2}}{[\nu-\cos(\phi)]^2}\ d\nu \tag{57}$$

This integral is considered in appendix A; from equation A10 we have

$$e_{o_\rho} = \frac{u(\tau-1)}{\pi} \left\{ -\sin(\phi) \left\{ \frac{[\tau^2-1]^{1/2}}{\tau-\cos(\phi)} - \operatorname{arccosh}(\tau) \right\} + \cos(\phi) \arccos \left[\frac{1-\tau\cos(\phi)}{\tau-\cos(\phi)} \right] \right\} \tag{58}$$

For e_{o_ϕ} we have, by integrating equation 56 by parts,

$$e_{o_\phi} = - \nu h_o(\nu) \Big|_1^\tau + \int_1^\tau h_o(\nu)\, d\nu$$

$$= \frac{u(\tau-1)}{\pi} \left\{ - \frac{\tau[\tau^2-1]^{1/2}}{\tau-\cos(\phi)} + \int_1^\tau \frac{[\nu^2-1]^{1/2}}{\nu-\cos(\phi)}\, d\nu \right\} \tag{59}$$

The remaining integral is considered in appendix A; from equation A13 we have, combining terms,

$$e_{o_\phi} = \frac{u(\tau-1)}{\pi} \left\{ -\cos(\phi) \left\{ \frac{[\tau^2-1]^{1/2}}{\tau-\cos(\phi)} - \operatorname{arccosh}(\tau) \right\} - \sin(\phi) \arccos \left[\frac{1-\tau\cos(\phi)}{\tau-\cos(\phi)} \right] \right\} \tag{60}$$

Equations 50, 58, and 60 give the results for the three field components produced by the assumed step function source as in equation 44. These three field components have simpler forms at selected values of ϕ, e.g., we have at $\phi=\pi$

$$h_o = \frac{u(\tau-1)}{\pi} \left[\frac{\tau-1}{\tau+1}\right]^{1/2} , \quad e_{o_\rho} = 0$$

$$e_{o_\phi} = \frac{u(\tau-1)}{\pi} \left\{ \left[\frac{\tau-1}{\tau+1}\right]^{1/2} - \operatorname{arccosh}(\tau) \right\} \tag{61}$$

at $\phi = \pi/2$

$$h_o = \frac{u(\tau-1)}{\pi} \left[1 - \frac{1}{\tau^2}\right]^{1/2} , \quad e_{o_\rho} = - \frac{u(\tau-1)}{\pi} \left\{ \left[1 - \frac{1}{\tau^2}\right]^{1/2} - \operatorname{arccosh}(\tau) \right\}$$

$$e_{o_\phi} = - \frac{u(\tau-1)}{\pi} \arccos \left[\frac{1}{\tau}\right] \tag{62}$$

and finally at $\phi = 0$

$$h_o = \frac{u(\tau-1)}{\pi} \left[\frac{\tau+1}{\tau-1}\right]^{1/2} , \quad e_{o_\rho} = u(\tau-1)$$

$$e_{o_\phi} = - \frac{u(\tau-1)}{\pi} \left\{ \left[\frac{\tau+1}{\tau-1}\right]^{1/2} - \operatorname{arccosh}(\tau) \right\} \tag{63}$$

The three field components are plotted as functions of τ for various values of ϕ in figures 4 through 6. Note that h_o and e_{o_ϕ} have large magnitudes just after $\tau = 1$ for ϕ near zero and that their behaviors are singular at $\tau = 1$ for $\phi = 0$. This behavior near $\phi = 0$ can be thought of as being due to the fact that the source is propagating to the right at c, which is the same propagation speed as the wave above the distributed source. Since the distributed source is "phased" in this way there is a strong constructive interference near $\phi = 0$ of the waves produced by each small element of the distributed source. As the wave propagates to the right it picks up more and more energy. For another view of this wave look at the contour plots in figures 7 through 10 where we have defined

$$e_o \equiv \left[e_{o_\rho}^{\ 2} + e_{o_\phi}^{\ 2} \right]^{1/2} \tag{64}$$

as the normalized magnitude of the electric field. Note that for the contour plots the radial coordinate is $1/\tau$ so that the cartesian axes are $\frac{\cos(\phi)}{\tau}$ and $\frac{\sin(\phi)}{\tau}$, or equivalently x/ct and y/ct, respectively. Since $\tau = ct/\rho$ the contour plots of figures 7 through 10 can be considered as displaying the dependence of the various quantities on ρ and ϕ for a particular t. On the other hand figures 4 through 6 can be considered as displaying the dependence of these quantities on t for particular values of ρ and ϕ.

Associated with the magnetic field at $\phi = 0$ there is an equal surface current density in the source in the +x direction. For our assumed step-function source this surface current density then has a singularity at $\tau = 1$ as in the first of equations 63. This singularity can be avoided by having the source rise slower and in a smooth manner while still maintaining the propagation to the right at speed c. Another way to look at this problem of the current which the source has to supply because of the fields above the source is to consider the charge which the source must supply out to any given time. If the source is, for example, an array of capacitors this charge can be compared to the charge stored on the capacitors which provides the current associated with the magnetic field below the distributed source. Some allowance can then be made for this additional charge associated with the fields above the source.

The charge associated with the fields above the source is given by

$$Q_w = \int_0^t H_y(t')\ dt' \tag{65}$$

This is a charge per unit width (coulombs/meter) for the distributed source. Defining a corresponding normalized charge per unit width as

$$q_o \equiv \frac{Z_o}{E_o} \frac{c}{\rho} Q_w \tag{66}$$

we then have

$$q_o = \frac{c}{\rho}\int_0^t h_o\left(\frac{ct'}{\rho}\right)\Bigg|_{\phi=0} dt' = \int_0^\tau h_o(\nu)\Bigg|_{\phi=0} d\nu$$

$$= \frac{u(\tau-1)}{\pi}\int_1^\tau \left[\frac{\nu+1}{\nu-1}\right]^{1/2} d\nu$$

$$= \frac{u(\tau-1)}{\pi}\left\{\left[\tau^2-1\right]^{1/2} + \text{arccosh}(\tau)\right\} \quad (67)$$

This normalized charge per unit width is plotted in figure 11. Note that even though h_o for $\phi = 0$ is singular at $\tau = 1$, the singularity is integrable, giving a continuous q_o. Also note for a given τ that Q_w is proportional to ρ as is seen from the normalization defined by equation 66. However τ in equation 67 also contains ρ so that for a given retarded time, $t - \rho/c$, the dependence of Q_w on ρ is somewhat more complicated.

Up till now we have considered a form for the distributed source given by equation 44 for which E_x' is a step function in retarded time starting at $x'=0$ and propagating to the right indefinitely. This can be modified with a new definition for E_x' as

$$E_x' \equiv E_o u(t - \frac{x'}{c})\ [u(x') - u(x'-d)] \quad (68)$$

so that the source is now stopped at $x' = d$. Note, however, that the additional term, $-E_o u(t - \frac{x'}{c})\ u(x'-d)$, is just like the original term except for a change in sign and a shift in the starting point (and corresponding time) to $x' = d$. By a similar change in sign and shift of origin we can use the previous results for the field components to obtain the fields associated with this term. Then adding the fields associated with the two terms we have the fields associated with the source as given in equation 68. The results of equations 50, 58, and 60 for the fields then apply for times before the stopping of the source at $x' = d$ can propagate to the observer.

Using the source of equation 68 the charge per unit width associated with the fields above the distributed source can be calculated from equation 65. The new magnetic field, which for $0<x<d$ and $z = 0$ (just above the plane), is

$$H_y = \frac{E_o}{Z_o}\ \frac{1}{\pi}\left\{\frac{\left[\left(\frac{ct}{x}\right)^2-1\right]^{1/2}}{\frac{ct}{x} - 1}\ u(\frac{ct}{x} - 1) - \frac{\left[\left(\frac{ct-d}{d-x}\right)^2-1\right]^{1/2}}{\frac{ct-d}{d-x} + 1}\ u\left(\frac{ct-d}{d-x} - 1\right)\right\} \quad (69)$$

where the first term uses h_o for $\phi = 0$ and the second term uses $-h_o$ for $\phi = \pi$ with a shift of origin and a corresponding time delay. For the present form of source we define a normalized retarded time as

$$\tau_d = \frac{ct-x}{d} \tag{70}$$

from which the magnetic field can be written in the form

$$\frac{Z_o}{E_o} H_y = \frac{1}{\pi}\left\{\left[\frac{\tau_d+2\frac{x}{d}}{\tau_d}\right]^{1/2} u(\tau_d) - \left[\frac{\tau_d+2\frac{x}{d}-2}{\tau_d}\right]^{1/2} u(\tau_d+2\frac{x}{d}-2)\right\} \tag{71}$$

For the present case we define a normalized charge per unit width as

$$q_d \equiv \frac{Z_o}{E_o}\frac{c}{d} Q_w = \frac{Z_o}{E_o}\frac{c}{d}\int_0^t H_y(t')dt' \tag{72}$$

Then from equation 69, making appropriate substitutions for ct/x in the first term and (ct-d)/(d-x) in the second term we have

$$q_d = \frac{c}{d}\frac{1}{\pi}\left\{\frac{x}{c} u(ct-x)\int_1^{\frac{ct}{x}}\left[\frac{\nu+1}{\nu-1}\right]^{1/2} d\nu - \frac{d-x}{c} u(ct+x-2d)\int_1^{\frac{ct-d}{d-x}}\left[\frac{\nu-1}{\nu+1}\right]^{1/2} d\nu\right\}$$

$$= \frac{1}{\pi}\left\{\frac{x}{d} u(\tau_d)\left[[\nu^2-1]^{1/2}+\text{arccosh}(\nu)\right]_1^{\frac{\tau_d}{x/d}+1}\right\}$$

$$- \left(1-\frac{x}{d}\right) u(\tau_d+2\frac{x}{d}-2)\left[[\nu^2-1]^{1/2} - \text{arccosh}(\nu)\right]_1^{\frac{\tau_d}{1-\frac{x}{d}}-1}$$

$$= \frac{u(\tau_d)}{\pi}\left\{[\tau_d(\tau_d+2\frac{x}{d})]^{1/2} + \frac{x}{d}\ \text{arccosh}\left(\frac{\tau_d}{x/d}+1\right)\right\}$$

$$- \frac{u(\tau_d+2\frac{x}{d}-2)}{\pi}\left\{[\tau_d(\tau_d+2\frac{x}{d}-2)]^{1/2} - \left(1-\frac{x}{d}\right)\text{arccosh}\left(\frac{\tau_d}{1-\frac{x}{d}}-1\right)\right\} \tag{73}$$

In figure 12 q_d is plotted as a function of τ_d for various values of x/d. Note that including the end of the distributed source at x = d levels off q_d somewhat at large τ_d. Actually q_d is still increasing at large τ_d but only logarithmically.

V. Summary

With the results of this note we have estimates of the fields radiated above the distributed source and the associated currents in the distributed source. With these currents the distributed source delivers charge and energy which should be allowed for in the design of this type of a simulator. One should note that the initial surface current density (associated with the fields above the source) can be fairly large due to the fact that the propagation speed of the source is set equal to the propagation speed of the fields above the source. For the present calculations we have given the source a step-function time dependence; these results illustrate the behavior of the fields, current, and charge. For some particular design of a distributed-source simulator the time dependence of the source will differ somewhat, e.g., by having a nonzero rise time and a final decay to zero; the present results can still be applied using convolution-integral techniques. One feature not included in the present calculations is amplitude variation of the source along the z' coordinate; including such a variation of the source would alter the results to some extent, but not greatly if the relative amplitude variation is small.

One should note some of the limitations imposed on the results by the initial assumptions. The ground is rather finitely conducting beyond the ends of the distributed source (unless one purposely places conductors there). The distributed source also has only finite width. In the time domain, however, the presence of the finite width is not initially noticeable on the source (away from the sides) due to the propagation time required for the presence of the side to be noticed at the position of interest. Eventually, however, the presence of the sides will affect the currents in the source.

We would like to thank A1C Henry J. McDermott, Jr., and Amn Richard T. Clark for the numerical calculations and resulting graphs in this note.

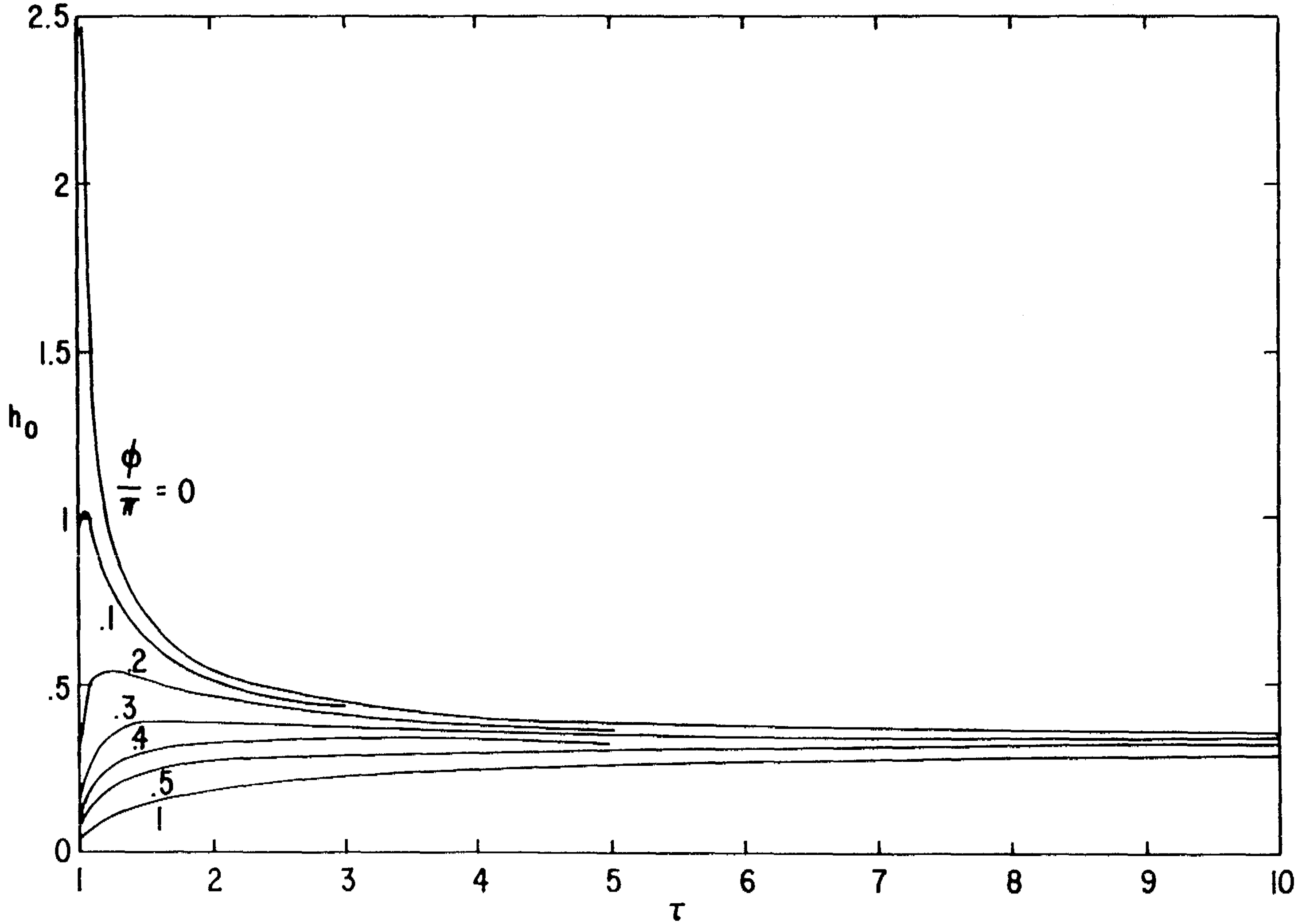

FIGURE 4. h_0 VS. τ WITH ϕ AS A PARAMETER

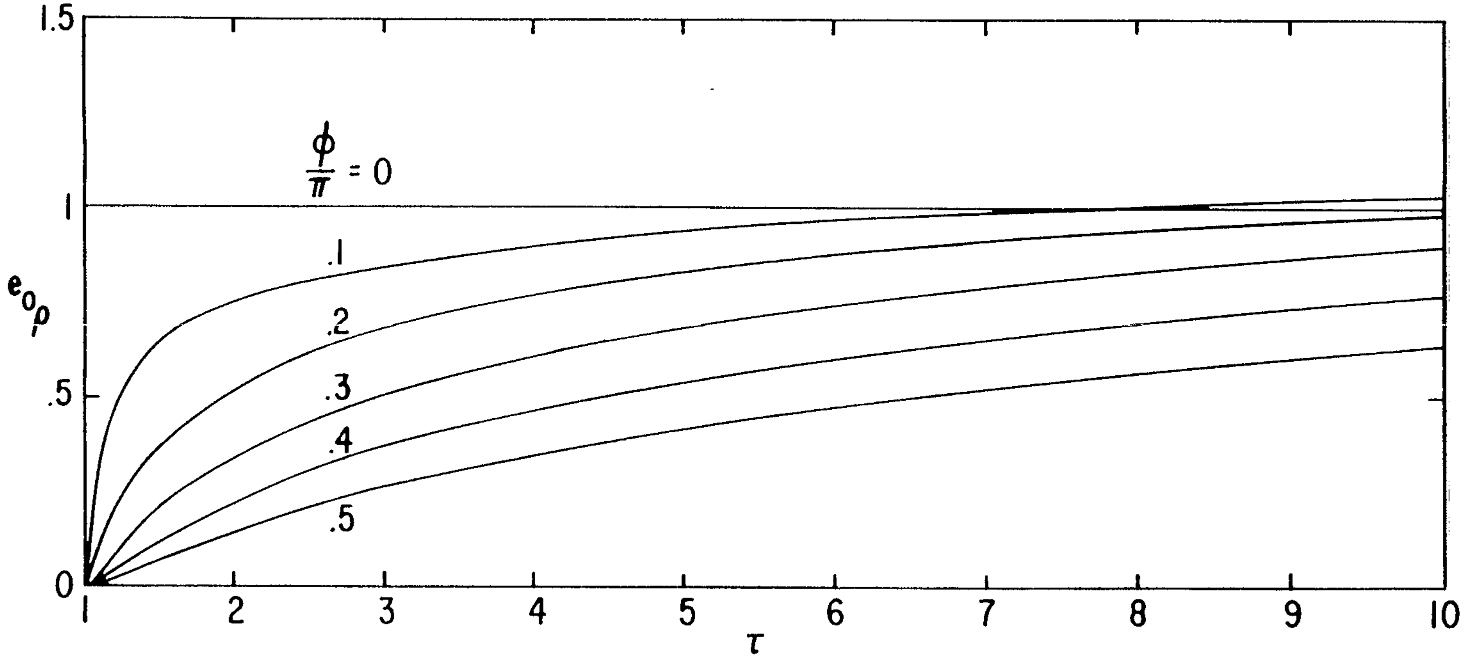

FIGURE 5. e_{0_ρ} VS. τ WITH ϕ AS A PARAMETER

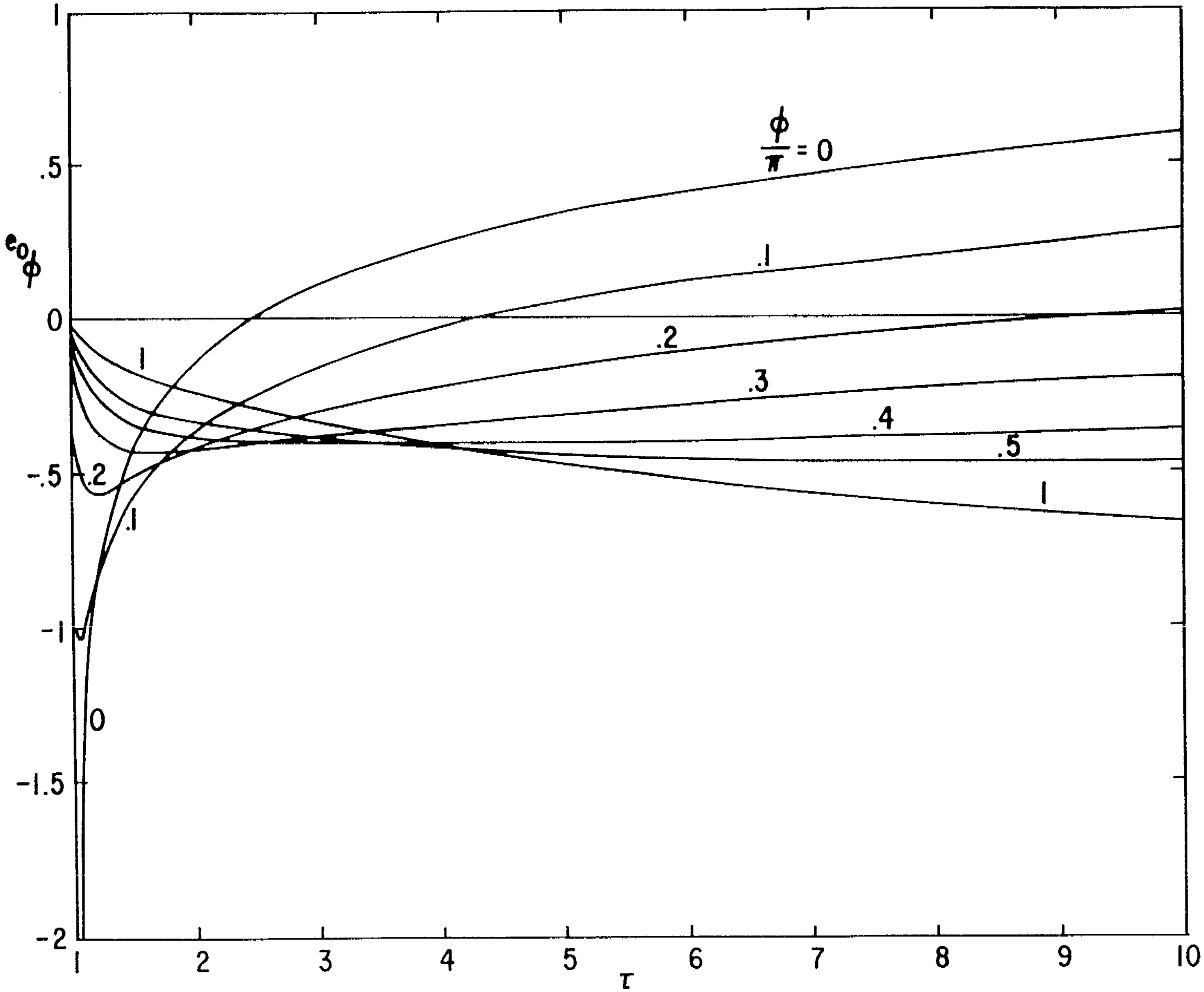

FIGURE 6. e_{o_ϕ} VS. τ WITH ϕ AS A PARAMETER

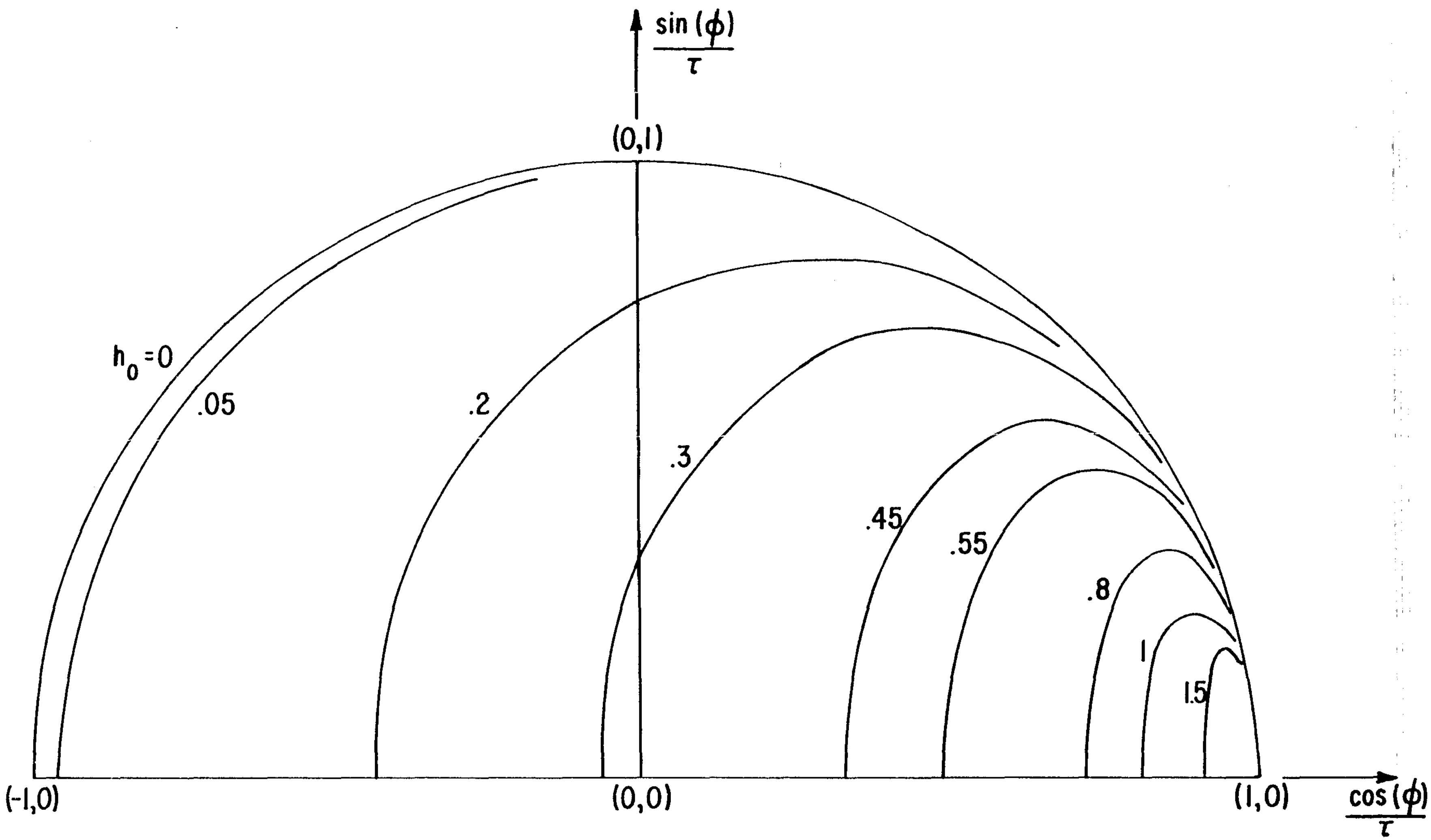

FIGURE 7. CONTOUR PLOT OF h_0

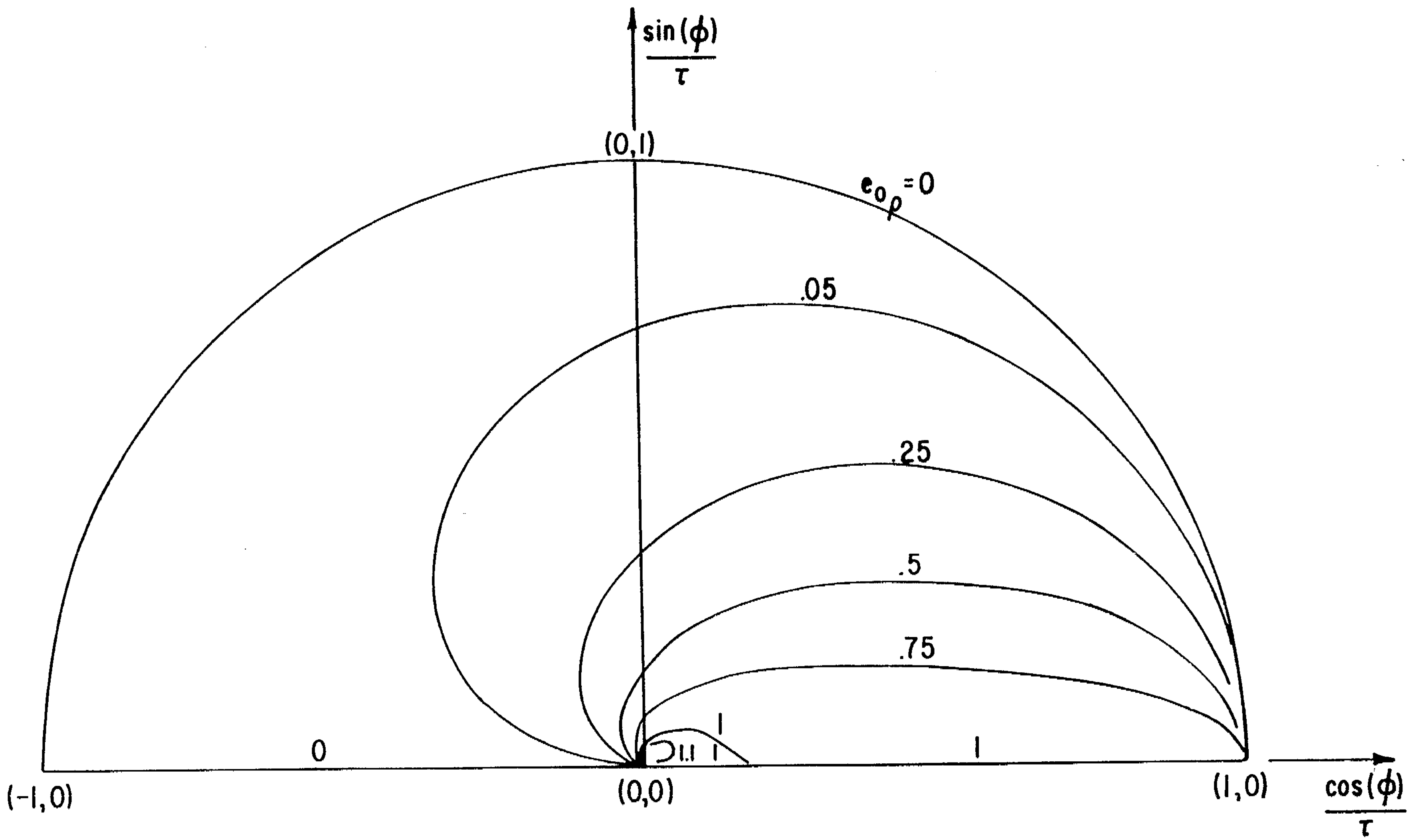

FIGURE 8. CONTOUR PLOT OF e_{0_ρ}

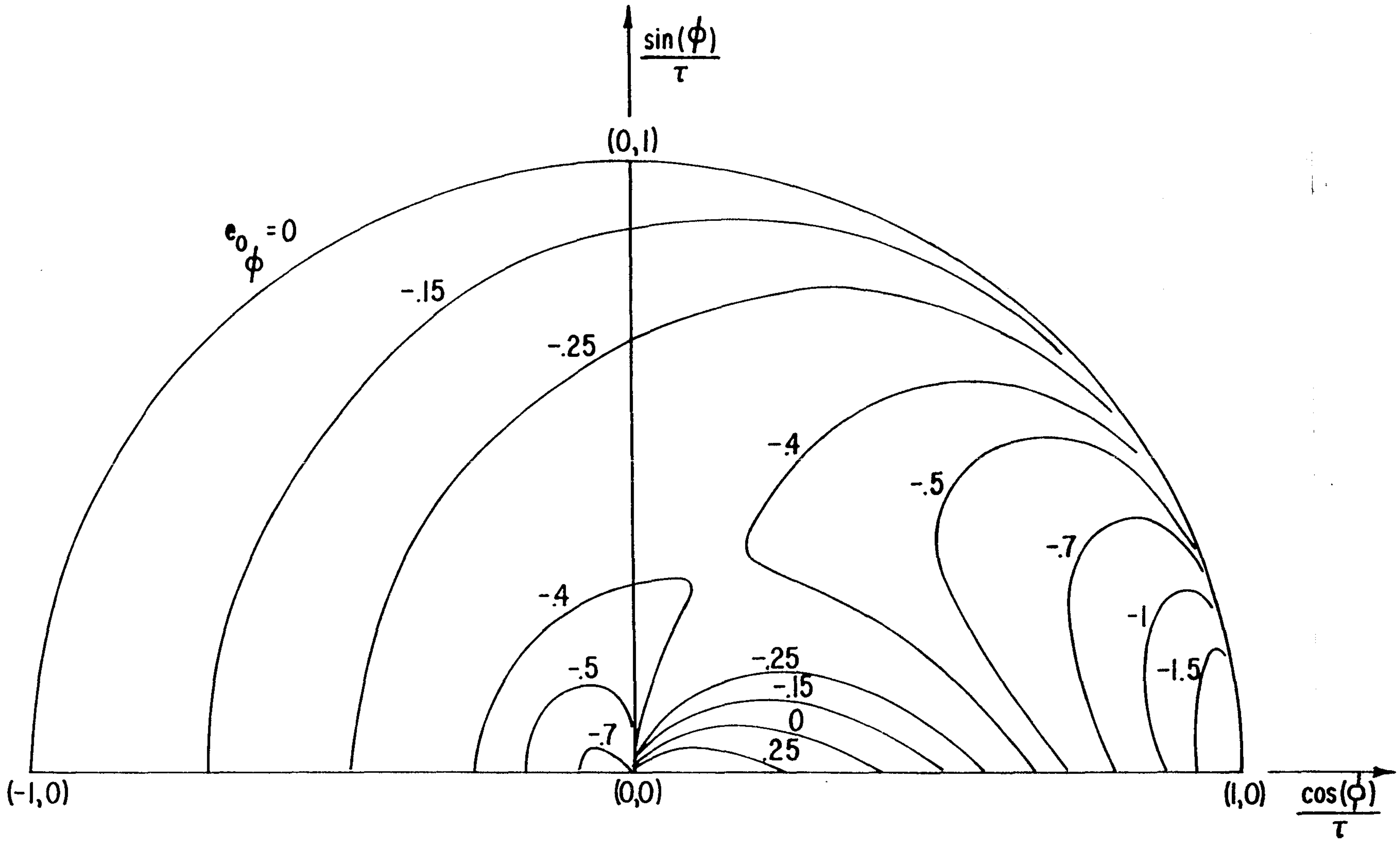

FIGURE 9. CONTOUR PLOT OF e_{o_ϕ}

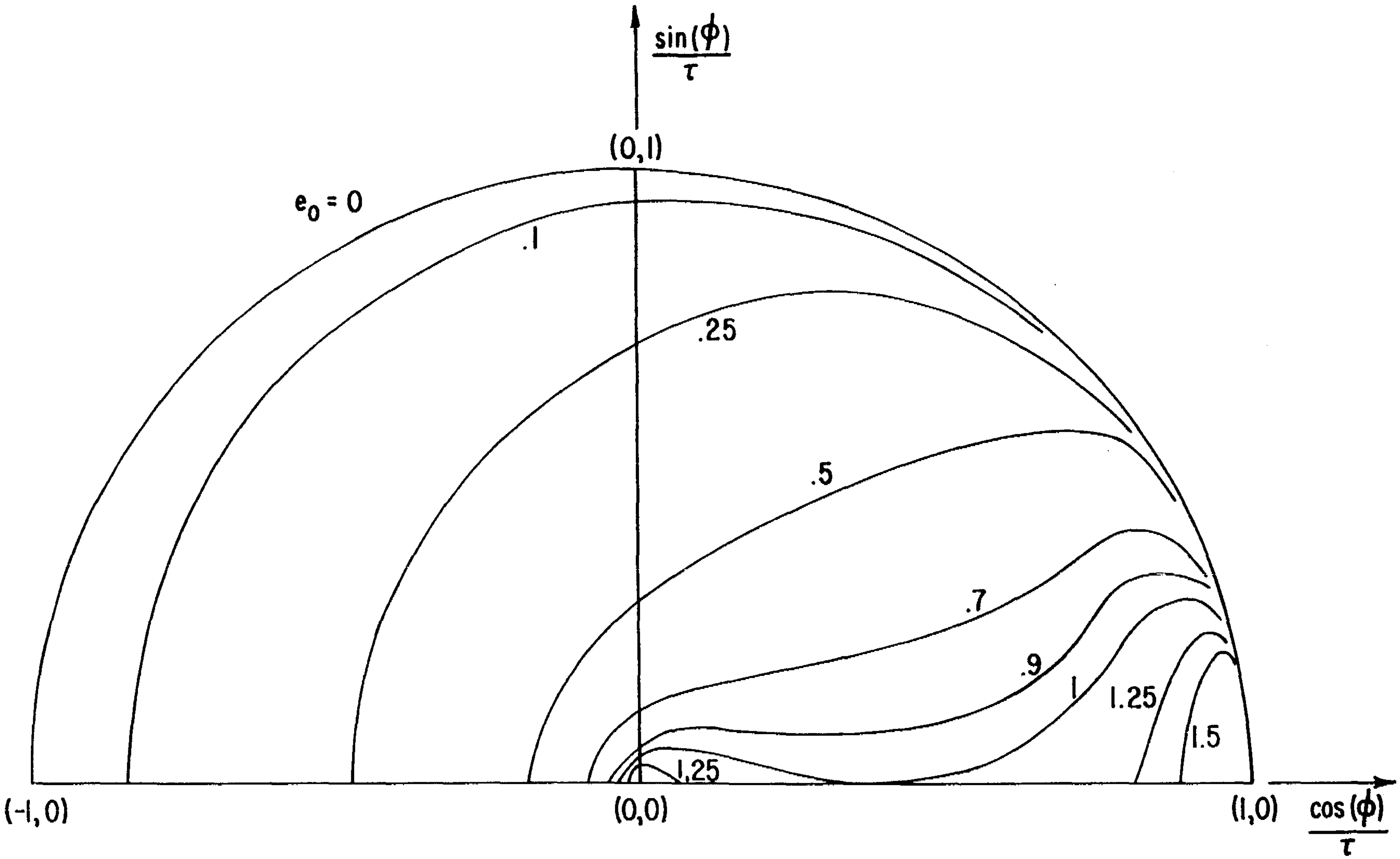

FIGURE 10. CONTOUR PLOT OF e_0

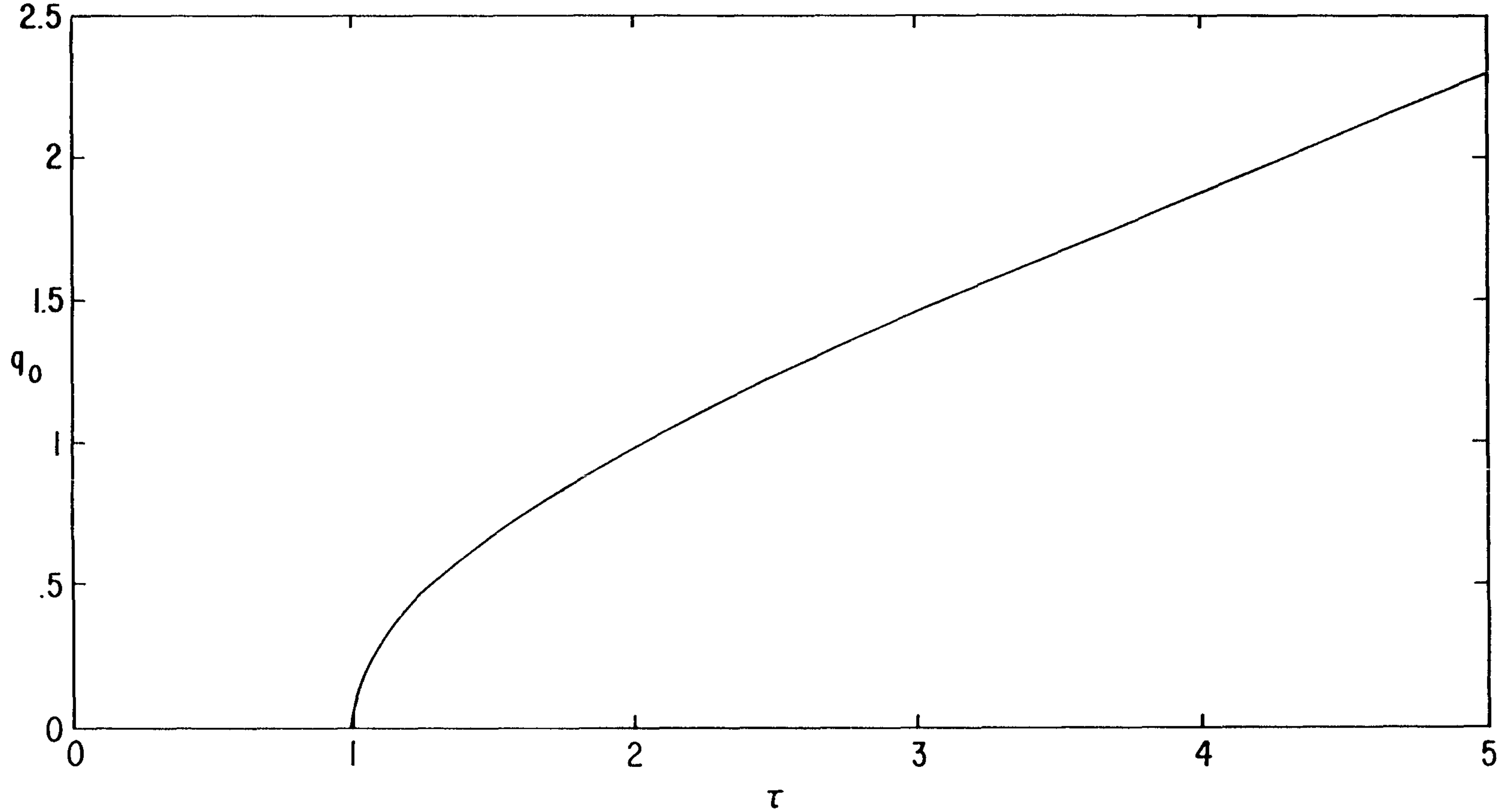

FIGURE 11. q_0 VS. τ

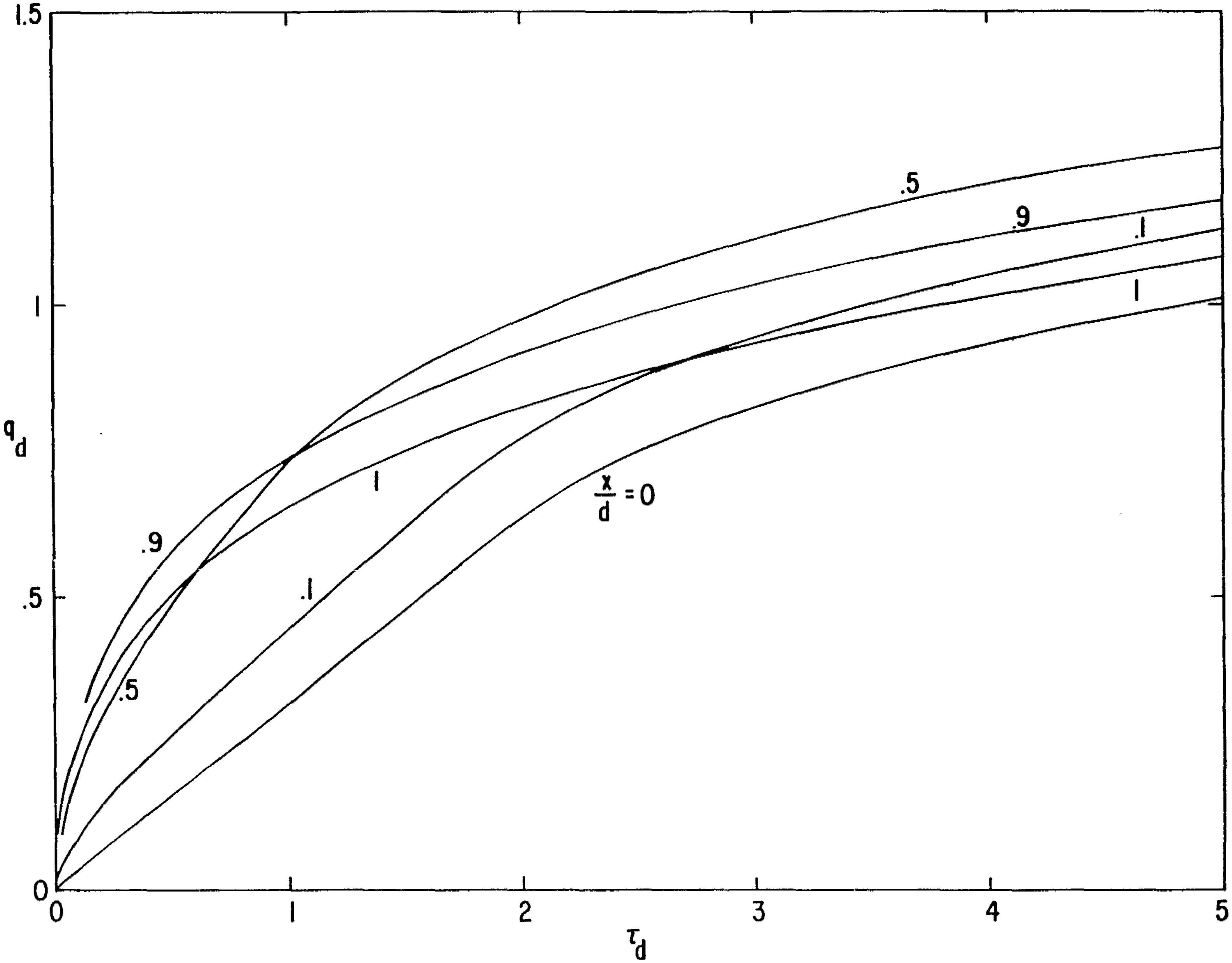

FIGURE 12. q_d VS. τ_d WITH x/d AS A PARAMETER

Appendix A: Evaluation of Some Integrals

In this appendix we evaluate some of the integrals which appear in previous sections of the note.

From equations 29 and 30 we have the integral

$$A_1 \equiv \int_{-\infty}^{\infty} \frac{1}{2\pi} [\gamma R+1] \frac{\rho'^2}{R^3} e^{-\gamma R} d\eta = \frac{1}{\pi}\int_0^{\infty} [\gamma R+1] \frac{\rho'^2}{R^3} e^{-\gamma R} d\eta \tag{A1}$$

Substitute

$$\xi = \frac{R}{\rho'} = \sqrt{\left(\frac{\eta}{\rho'}\right)^2 + 1} \ , \quad \eta = \rho'\sqrt{\xi^2 - 1} \ , \quad d\eta = \frac{\rho'\xi d\xi}{\sqrt{\xi^2 - 1}} \tag{A2}$$

giving

$$A_1 = \frac{1}{\pi}\int_1^{\infty} \frac{1}{\xi^2} [\gamma\rho'\xi+1] \frac{e^{-\gamma\rho'\xi}}{\sqrt{\xi^2 - 1}} d\xi$$

$$= \frac{\gamma\rho'}{\pi} \int_1^{\infty} \frac{e^{-\gamma\rho'\xi} d\xi}{\xi\sqrt{\xi^2 - 1}} + \frac{1}{\pi} \frac{\sqrt{\xi^2 - 1}}{\xi} e^{-\gamma\rho'\xi}\Bigg|_1^{\infty} + \frac{\gamma\rho'}{\pi}\int_1^{\infty} \frac{\sqrt{\xi^2 - 1}}{\xi} e^{-\gamma\rho'\xi} d\xi \tag{A3}$$

where the second part of the integral has been integrated by parts. The center term is zero by letting the real part of γ have a small positive part. Combining the two remaining integrals we have

$$A_1 = \frac{\gamma\rho'}{\pi} \int_1^{\infty} \frac{\xi e^{-\gamma\rho'\xi}}{\sqrt{\xi^2 - 1}} d\xi$$

$$= \frac{\gamma\rho'}{\pi} \sqrt{\xi^2 - 1}\, e^{-\gamma\rho'\xi}\Bigg|_1^{\infty} + \frac{(\gamma\rho')^2}{\pi} \int_1^{\infty} \sqrt{\xi^2 - 1}\, e^{-\gamma\rho'\xi} d\xi \tag{A4}$$

where we have again integrated by parts and the first term goes to zero. The remaining integral is tabulated[1a] giving

$$A_1 = \frac{1}{\pi}\gamma\rho' K_1(\gamma\rho') \tag{A5}$$

or in terms of $k = -j\gamma$ we have[2a]

1a. AMS 55, Handbook of Mathematical Functions, National Bureau of Standards, 1964, equation 9.6.23.

2a. Reference 1a, equation 9.6.4.

$$A_1 = -\frac{jk\rho'}{2} H_1^{(2)}(k\rho') \tag{A6}$$

where K_1 and $H_1^{(2)}$ are well-known types of Bessel functions.

From equation 57 we have the integral (for $\tau > 1$)

$$A_2 = \int_1^\tau \frac{[\nu^2-1]^{1/2}}{[\nu-\cos(\phi)]^2} d\nu \tag{A7}$$

Substitute

$$\eta = \nu - \cos(\phi) \tag{A8}$$

giving

$$A_2 = \int_{1-\cos(\phi)}^{\tau-\cos(\phi)} \eta^{-2}[\eta^2+2\cos(\phi)\eta - \sin^2(\phi)]^{1/2} d\eta$$

$$= -\left.\frac{[\eta^2+2\cos(\phi)\eta - \sin^2(\phi)]^{1/2}}{\eta}\right|_{1-\cos(\phi)}^{\tau-\cos(\phi)} + \int_{1-\cos(\phi)}^{\tau-\cos(\phi)} [\eta^2+2\cos(\phi)-\sin^2(\phi)]^{-1/2} d\eta$$

$$+ \cos(\phi)\int_{1-\cos(\phi)}^{\tau-\cos(\phi)} \eta^{-1}[\eta^2+2\cos(\phi)-\sin^2(\phi)]^{-1/2} d\eta \tag{A9}$$

where we have used Dwight[3a], equation 380.321. This further reduces to

$$A_2 = -\frac{[\tau^2-1]^{1/2}}{\tau-\cos(\phi)} + \ln\left|2[\eta^2+2\cos(\phi)\eta - \sin^2(\phi)]^{1/2}+2\eta+2\cos(\phi)\right|\Bigg|_{1-\cos(\phi)}^{\tau-\cos(\phi)}$$

$$+ \frac{\cos(\phi)}{\sin(\phi)} \arcsin\left[\frac{2\cos(\phi)\eta - 2\sin^2(\phi)}{2\eta}\right]\Bigg|_{1-\cos(\phi)}^{\tau-\cos(\phi)}$$

$$= -\frac{[\tau^2-1]^{1/2}}{\tau-\cos(\phi)} + \ln[\tau+(\tau^2-1)^{1/2}] + \frac{\cos(\phi)}{\sin(\phi)}\left\{-\arcsin\left[\frac{1-\tau\cos(\phi)}{\tau-\cos(\phi)}\right] + \frac{\pi}{2}\right\}$$

$$= -\frac{[\tau^2-1]^{1/2}}{\tau-\cos(\phi)} + \operatorname{arccosh}(\tau) + \cot(\phi)\arccos\left[\frac{1-\tau\cos(\phi)}{\tau-\cos(\phi)}\right] \tag{A10}$$

for which we have used Dwight, equations 380.001 and 380.111.

3a. H. B. Dwight, Tables of Integrals and Other Mathematical Data, 4th ed., MacMillan, 1961.

From equation 59 we have the integral (for $\tau > 1$)

$$A_3 = \int_1^{\tau} \frac{[\nu^2-1]^{1/2}}{\nu-\cos(\phi)} d\nu = \int_{1-\cos(\phi)}^{\tau-\cos(\phi)} \eta^{-1}[\eta^2+2\cos(\phi)\eta - \sin^2(\phi)]^{1/2} d\eta \qquad \text{(A11)}$$

where we have used the substitution given by equation A8. Then using Dwight, equation 380.311, we have

$$A_3 = [\eta^2+2\cos(\phi)\eta - \sin^2(\phi)]^{1/2} \Big|_{1-\cos(\phi)}^{\tau-\cos(\phi)} + \cos(\phi) \int_{1-\cos(\phi)}^{\tau-\cos(\phi)} [\eta^2+2\cos(\phi)\eta - \sin^2(\phi)]^{-1/2} d\eta$$

$$-\sin^2(\phi) \int_{1-\cos(\phi)}^{\tau-\cos(\phi)} \eta^{-1}[\eta^2+2\cos(\phi)\eta - \sin^2(\phi)]^{-1/2} d\eta \qquad \text{(A12)}$$

These latter two integrals were evaluated in equation A10. Thus we have

$$A_3 = [\tau^2-1]^{1/2} + \cos(\phi)\operatorname{arccosh}(\tau) - \sin(\phi) \arccos\left[\frac{1-\tau\cos(\phi)}{\tau-\cos(\phi)}\right] \qquad \text{(A13)}$$