Sensor and Simulation Notes

Note 412

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Transient Gain of Antennas Related to the Traditional Continuous-Wave (CW) Definition of Gain

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Abstract

There is some ambiguity in relating the usual CW (IEEE) definition of antenna gain to its time-domain response (transmitting and receiving). This involves issues of direction of incidence, polarization, and phase dispersion. In this paper we develop restrictions necessary to simplify this relationship.

1. Introduction

In [3, 10] the problem of appropriate definition of antenna gain and radiation pattern in time domain is considered in some detail. Using time-domain reciprocity the definitions are made to apply in both transmission and reception. Since the time-domain parameters are equivalent to the related frequency-domain parameters considered over all frequencies, the time-domain parameters are not simple numbers, but in general vector convolution operators. Specifically the gain is related to an appropriately normalized effective height convolution operator. This can in turn be reduced to a scalar number by application of appropriate mathematical norms. (A summary of these results, with an example, is included as Appendix A.)

Figure 1.1 indicates various parameters appropriate to an antenna in transmission and reception. Summarizing from [2] in transmission we have the radiated far electric field

$$\tilde{\vec{E}}_f(\vec{r},s) = \frac{e^{-\gamma_r}}{r}\tilde{\vec{F}}_t(\vec{1}_0,s)\tilde{V}_t(s)$$

$$\tilde{V}_t(s) = \frac{\tilde{Z}_{in}(s)}{\tilde{Z}_L(s)+\tilde{Z}_{in}(s)}\tilde{V}_s(s) \equiv \text{transmitted voltage}$$

$$\tilde{V}_s(s) \equiv \text{source voltage} \tag{1.1}$$

$$\tilde{Z}_L(s) \equiv \text{load impedance}$$

$$\tilde{Z}_{in}(s) = \frac{\tilde{V}_t(s)}{\tilde{I}_t(s)} \equiv \text{antenna input impedance}$$

$$= \tilde{Y}_{in}^{-1}(s)$$

$$\vec{1}_0 \equiv \text{direction to far-field observer}$$

$$r \equiv \text{distance to far-field observer}$$

$$\sim \; \equiv \text{Laplace transform (two-sided)}$$

$$s \equiv \Omega + j\omega \equiv \text{complex frequency or Laplace-transform variable}$$

$$\gamma = \frac{s}{c} \equiv \text{propagation constant}$$

$$c = [\mu_0\varepsilon_0]^{-\frac{1}{2}} \equiv \text{speed of light}$$

Note that the antenna may have a transmission line (e.g., a coaxial cable or strip line) of characteristic impedance Z_c, connecting the antenna to the source/receiver. This input impedance is in general not the same as this, except in cases where this is part of the specific design. In time domain, however, if one puts a pulse into the port then $V(t)/I(t)$ will be Z_c for a time given by the round-trip transit time.

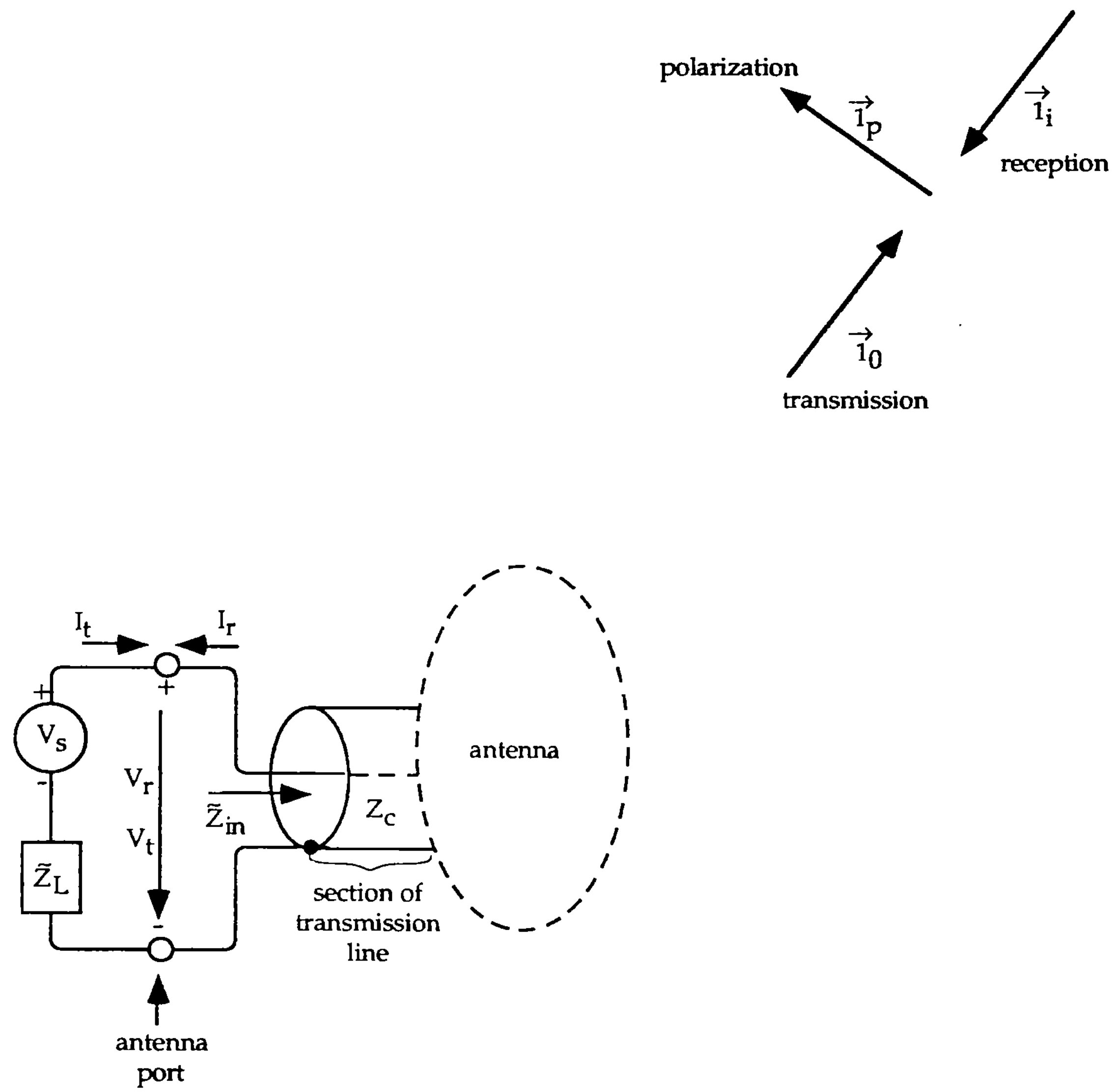

Fig. 1.1. Antenna in Transmission and Reception

In reception we have

$$\tilde{\vec{E}}^{(inc)}(\vec{r},s) = \tilde{\vec{E}}_0(s)e^{-\gamma \vec{1}_i \cdot \vec{r}} \equiv \text{incident plane wave}$$

$$\vec{1}_i \equiv \text{direction of incidence}$$

$$\tilde{\vec{E}}_0(s) \cdot \vec{1}_i = 0$$

$$\tilde{V}_r(s) = \tilde{\vec{h}}_t(\vec{1}_i,s) \cdot \tilde{\vec{E}}^{(inc)}(\vec{0},s) = \tilde{\vec{h}}_t(\vec{1}_i,s) \cdot \tilde{\vec{E}}_0(s) \tag{1.2}$$

$$\tilde{\vec{h}}_t(\vec{1}_i,s) \equiv \text{effective height}$$

$$\tilde{\vec{h}}_t(\vec{1}_i,s) \cdot \vec{1}_i = 0 \text{ (no } \vec{1}_i \text{ component of incident electric field)}$$

$$\tilde{V}_r(s) \equiv \text{voltage across load } \tilde{Z}_L(s)$$

$$\tilde{Z}_L(s) = \frac{\tilde{V}_r(s)}{\tilde{I}_r(s)}$$

Note the opposite direction conventions of I_t and I_r.

Now applying reciprocity [2] we have the fundamental result

$$\tilde{\vec{F}}_t(\vec{1}_0,s) = \frac{s\mu_0}{4\pi}\left[\tilde{Y}_{in}(s) + \tilde{Y}_L(s)\right]\tilde{\vec{h}}_t(-\vec{1}_0,s) \tag{1.3}$$

where we have here defined $\tilde{\vec{h}}_t$ such that it has only a transverse part with respect to $\vec{1}_i = -\vec{1}_0$. Otherwise we can use the transverse dyads

$$\overleftrightarrow{1}_0 = \overleftrightarrow{1} - \vec{1}_0\vec{1}_0 \, , \; \overleftrightarrow{1}_i = \overleftrightarrow{1} - \vec{1}_i\vec{1}_i \tag{1.4}$$

to dot multiply $\tilde{\vec{h}}_t$ to make it purely transverse. Note that for the special, but useful, case that

$$\tilde{Z}_{in}(s) = \tilde{Z}_L(s) = R \tag{1.5}$$

i.e., a frequency-independent input impedance with a matched load, we have

$$\tilde{\vec{F}}_t(\vec{1}_0,s) = \frac{s\mu_0}{2\pi R}\tilde{\vec{h}}_t(-\vec{1}_0,s) \tag{1.6}$$

2. Effective Area (CW) of Antenna in Reception

The effective area or effective aperture of an antenna [8,9] is traditionally defined via

$$\tilde{P}_r(j\omega) \equiv \tilde{A}_{eff}(j\omega)\tilde{S}^{(inc)}(j\omega) \tag{2.1}$$

where the received power $\tilde{P}_r(j\omega)$ is taken as the real power (or real part of the power) into the antenna load $\tilde{Z}_L(j\omega)$. Note that the above form has been used on the $j\omega$ axis of the s plane. Furthermore there is the polarization and angle-of-incidence dependence of these parameters to consider.

In analytic (s-plane) form the received power can be defined as

$$\tilde{P}_r^{(1)}(s) \equiv \tilde{V}_r(s)\tilde{I}_r(-s) = \tilde{V}_r(s)\tilde{V}_r(-s)\tilde{Y}_L(-s) \tag{2.2}$$

Alternately one could use

$$\tilde{P}_r^{(2)}(s) \equiv \tilde{V}_r(-s)\tilde{I}_r(s) = \tilde{V}_r(-s)\tilde{V}_r(s)\tilde{Y}_L(s) \tag{2.3}$$

Another interesting form then averages these as

$$\tilde{P}_r(s) \equiv \tilde{V}_r(s)\tilde{V}_r(-s)\frac{1}{2}\left[\tilde{Y}_L(s) + \tilde{Y}_L(-s)\right] \tag{2.4}$$

which gives an even function of s. Of course if the load is a frequency-independent resistance R, then the above all give the same result as

$$\tilde{P}_r(s) = \frac{1}{R}\tilde{V}_r(s)\tilde{V}_r(-s) \tag{2.5}$$

This is an important case for application with transient waveforms (pulses). Note that this power can also be defined with a factor of 1/2 by averaging over a cycle, but this is a common factor with the Poynting vector, not influencing the effective area.

The incident plane wave in (1.2) has a magnetic field

$$\begin{aligned} \vec{\tilde{H}}^{(inc)}(\vec{r},s) &= \frac{1}{Z_0}\vec{1}_i \times \vec{\tilde{E}}^{(inc)}(\vec{r},s) = \frac{1}{Z_0}\vec{1}_i \times \vec{\tilde{E}}_0(s)e^{-\gamma \vec{1}_i \cdot \vec{r}} \\ Z_0 &= \frac{1}{Y_0} = \left[\frac{\mu_0}{\varepsilon_0}\right]^{\frac{1}{2}} \equiv \text{wave impedance of free space} \end{aligned} \tag{2.6}$$

Now define an analytic form of the Poynting vector as

$$
\begin{aligned}
\tilde{\vec{S}}^{(inc)}(\vec{r},s) &= \tilde{\vec{E}}^{(inc)}(\vec{r},s)\times\tilde{\vec{H}}^{(inc)}(\vec{r},-s) \\
&= \frac{1}{Z_0}\vec{1}_i\,\tilde{\vec{E}}^{(inc)}(\vec{r},s)\cdot\tilde{\vec{E}}^{(inc)}(\vec{r},-s) \\
&= \frac{1}{Z_0}\vec{1}_i\,\vec{E}_0(s)\cdot\vec{E}_0(-s)
\end{aligned}
\tag{2.7}
$$

which is conveniently independent of r. The above form then allows us to define a scalar form of the Poynting vector as

$$
\begin{aligned}
\tilde{\vec{S}}^{(inc)}(s) &= \vec{1}_i\,\tilde{S}^{(inc)}(s) \\
\tilde{\vec{S}}^{(inc)}(s) &= \frac{1}{Z_0}\tilde{\vec{E}}_0(s)\cdot\tilde{\vec{E}}(-s)
\end{aligned}
\tag{2.8}
$$

which is also an even function of s.

This allows us to write

$$
\tilde{A}_{eff}(\vec{1}_i,s) = \frac{Z_0}{2}\left[\tilde{Y}_L(s)+\tilde{Y}_L(-s)\right]\frac{\tilde{V}_r(s)\tilde{V}_r(-s)}{\tilde{\vec{E}}_0(s)\cdot\tilde{\vec{E}}_0(-s)}
\tag{2.9}
$$

which is an even function of s. From (2.1) we then have

$$
\tilde{A}_{eff}(\vec{1}_i,s) = \frac{Z_0}{2}\left[\tilde{Y}_L(s)+\tilde{Y}_L(-s)\right]\frac{\tilde{\vec{E}}_0(s)\cdot\tilde{\vec{h}}_t(\vec{1}_i,s)\,\tilde{\vec{h}}_t(\vec{1}_i,-s)\cdot\tilde{\vec{E}}_0(-s)}{\tilde{\vec{E}}_0(s)\cdot\tilde{\vec{E}}_0(-s)}
\tag{2.10}
$$

Note that

$$
\tilde{\overleftrightarrow{H}}_t(\vec{1}_i,j\omega) \equiv \tilde{\vec{h}}_t(\vec{1}_i,j\omega)\,\tilde{\vec{h}}_t(\vec{1}_i,-j\omega) = \tilde{\vec{h}}_t(\vec{1}_i,j\omega)\,\tilde{\vec{h}}_t^{*}(\vec{1}_i,j\omega)
\tag{2.11}
$$

is obviously Hermitian. With $s = j\omega$ then (2.10) is maximized by choosing

$$
\begin{aligned}
\tilde{\vec{E}}_0(j\omega) &\equiv C_0^{*}(j\omega)\,\tilde{\vec{h}}_t^{*}(\vec{1}_i,j\omega) = C_0(-j\omega)\,\tilde{\vec{h}}_t(\vec{1}_i,-j\omega) \\
C_0 &= \text{scalar function} \neq 0
\end{aligned}
\tag{2.12}
$$

This C_0 can be a simple real constant if desired. Then we have a maximum

$$\tilde{A}_{eff}^{(0)}(\vec{1}_i, j\omega) = Z_0 \operatorname{Re}\left[\tilde{Y}_L(j\omega)\right]\vec{\tilde{h}}_t(j\omega)\cdot\vec{\tilde{h}}_t(-j\omega)$$

$$= Z_0 \operatorname{Re}\left[\tilde{Y}_L(j\omega)\right]\left|\vec{\tilde{h}}_t(j\omega)\right|^2 \tag{2.13}$$

$$0 \le \tilde{A}_{eff}(\vec{1}_i, j\omega) \le \tilde{A}_{eff}^{(0)}(\vec{1}_i, j\omega)$$

This suggests that we define an analytic form in the s plane as

$$\tilde{A}_{eff}^{(0)}(\vec{1}_i, s) \equiv \frac{Z_0}{2}\left[\tilde{Y}_L(s)+\tilde{Y}_L(-s)\right]\vec{\tilde{h}}_t(\vec{1}_i, s)\cdot\vec{\tilde{h}}_t(\vec{1}_i, -s) \tag{2.14}$$

Note that there also exists a polarization of the incident electric field such that

$$\vec{\tilde{A}}_{eff}(\vec{1}_i, s) = 0 \tag{2.15}$$

namely an electric field such that

$$\vec{\tilde{E}}_0(j\omega)\cdot\vec{\tilde{h}}_t(\vec{1}_i, j\omega) = 0 \tag{2.16}$$

So the effective area can be considered as a function of the polarization $\vec{1}_p$ with

$$\tilde{A}_{eff}^{(0)}(\vec{1}_i, j\omega) = \sup_{\vec{1}_p} \tilde{A}_{eff}(\vec{1}_i, \vec{1}_p, j\omega) \tag{2.17}$$

where the polarization can even be complex as

$$\vec{1}_p = \frac{\vec{\tilde{E}}_0(j\omega)}{\left|\vec{\tilde{E}}_0(j\omega)\right|} \tag{2.18}$$

Here we have maximized over $\vec{1}_p$. We can also maximize over $\vec{1}_i$ if we desire as

$$\tilde{A}_{eff}^{(1)}(j\omega) = \sup_{\vec{1}_i \text{ real}} \tilde{A}_{eff}^{(0)}(\vec{1}_i, j\omega) \tag{2.19}$$

The corresponding $\vec{1}_i$ is opposite to the $\vec{1}_0$ for maximum gain in transmission.

3. CW Antenna Gain

Antenna gain is usually defined as

$$\tilde{G}(\vec{1}_0,j\omega)=\frac{4\pi r^2}{Z_0\,\mathrm{Re}[P_{in}(j\omega)]}\,\tilde{\vec{E}}_f(\vec{1}_0,j\omega)\cdot\tilde{\vec{E}}_f^{*}(\vec{1}_0,j\omega) \tag{3.1}$$

where the usual far-field behavior is assumed. The power into the antenna in analytic form can be defined as

$$\tilde{P}_{in}^{(1)}(s)\equiv\tilde{V}_t(s)\tilde{I}_t(-s)=\tilde{V}_t(s)\tilde{V}_t(-s)\tilde{Y}_{in}(-s) \tag{3.2}$$

analogous to the received power in (2.2). Similarly we have alternate forms as

$$\begin{aligned}\tilde{P}_{in}^{(2)}(s)&\equiv\tilde{V}_t(-s)\tilde{I}_t(s)=\tilde{V}_t(s)\tilde{V}_t(-s)\tilde{Y}_{in}(s)\\ \tilde{P}_{in}(s)&\equiv\tilde{V}_t(s)\tilde{V}_t(-s)=\tilde{V}_t(s)\tilde{V}_t(-s)\frac{1}{2}\left[\tilde{Y}_{in}(s)+\tilde{Y}_{in}(-s)\right]\end{aligned} \tag{3.3}$$

If the antenna has a frequency-independent input resistance R (which is the case for special antennas of interest), then the above all reduce to

$$\tilde{P}_{in}(s)=\frac{1}{R}\tilde{V}_t(s)\tilde{V}_t(-s) \tag{3.4}$$

The gain now takes the analytic form

$$\tilde{G}(\vec{1}_0,s)=\frac{8\pi r^2}{Z_0\left[\tilde{Y}_{in}(s)+\tilde{Y}_{in}(-s)\right]}\,\frac{\tilde{\vec{E}}_f(\vec{1}_0,s)\cdot\tilde{\vec{E}}_f(\vec{1}_0,-s)}{\tilde{V}_t(s)\tilde{V}_t(-s)} \tag{3.5}$$

which is an even function of s, as well as real and positive for $s=j\omega$. From (1.1) and (1.3) we have

$$\begin{aligned}\tilde{G}(\vec{1}_0,s)&=\frac{8\pi}{Z_0\left[\tilde{Y}_{in}(s)+\tilde{Y}_{in}(s)\right]}\,\tilde{\vec{F}}_t(\vec{1}_0,s)\cdot\tilde{\vec{F}}_t(\vec{1}_0,-s)\\ &=-\frac{[s\mu_0]^2}{2\pi Z_0}\,\frac{\left[\tilde{Y}_{in}(s)+\tilde{Y}_L(s)\right]\left[\tilde{Y}_{in}(-s)+\tilde{Y}_L(-s)\right]}{\tilde{Y}_{in}(s)+\tilde{Y}_{in}(-s)}\,\tilde{\vec{h}}_t(-\vec{1}_0,s)\cdot\tilde{\vec{h}}_t(-\vec{1}_0,-s)\end{aligned} \tag{3.6}$$

One can now define

$$\tilde{\vec{g}}(\vec{1}_0,s)\equiv\frac{s\mu_0}{\sqrt{2\pi Z_0}}\,\frac{\tilde{Y}_{in}(s)+\tilde{Y}_L(s)}{\left[\tilde{Y}_{in}(s)+\tilde{Y}_{in}(s)\right]^{\frac{1}{2}}}\,\tilde{\vec{h}}_t(-\vec{1}_0,s) \tag{3.7}$$

so that

$$\tilde{G}(\vec{1}_0,s)=\tilde{\vec{g}}(\vec{1}_0,s)\cdot\tilde{\vec{g}}(\vec{1}_0,-s) \tag{3.8}$$

In this form $\tilde{\vec{g}}$ represents a relationship between voltage or current and field (linear) while $\tilde{G}$ is in terms of power.

In terms of the effective area we have

$$\begin{aligned}\tilde{G}(\vec{1}_0,s) &= -\frac{[s\mu_0]^2}{\pi Z_0^2}\frac{\left[\tilde{Y}_{in}(s)+\tilde{Y}_L(s)\right]\left[\tilde{Y}_{in}(-s)+\tilde{Y}_L(-s)\right]}{\left[\tilde{Y}_{in}(s)+\tilde{Y}_{in}(-s)\right]\left[\tilde{Y}_L(s)+\tilde{Y}_L(-s)\right]}\tilde{A}_{eff}^{(0)}(-\vec{1}_0,s) \\ &= -\frac{\gamma^2}{\pi}\frac{\left[\tilde{Y}_{in}(s)+\tilde{Y}_L(s)\right]\left[\tilde{Y}_{in}(-s)+\tilde{Y}_L(-s)\right]}{\left[\tilde{Y}_{in}(s)+\tilde{Y}_{in}(-s)\right]\left[\tilde{Y}_L(s)+\tilde{Y}_L(-s)\right]}\tilde{A}_{eff}^{(0)}(-\vec{1}_0,s)\end{aligned} \tag{3.9}$$

For $s = j\omega$ this reduces to

$$\begin{aligned}\tilde{G}(\vec{1}_0,j\omega) &= \frac{1}{4\pi}\left[\frac{\omega}{c}\right]^2\frac{\left|\tilde{Y}_{in}(j\omega)+\tilde{Y}_L(j\omega)\right|^2}{\mathrm{Re}\left[\tilde{Y}_{in}(j\omega)\right]\mathrm{Re}\left[\tilde{Y}_L(j\omega)\right]}\tilde{A}_{eff}^{(0)}(-\vec{1}_0,j\omega) \\ &= \frac{\pi}{\lambda^2}\frac{\left|\tilde{Y}_{in}(j\omega)+\tilde{Y}_L(j\omega)\right|^2}{\mathrm{Re}\left[\tilde{Y}_{in}(j\omega)\right]\mathrm{Re}\left[\tilde{Y}_L(j\omega)\right]}\tilde{A}_{eff}^{(0)}(-\vec{1}_0,j\omega)\end{aligned} \tag{3.10}$$

$$\lambda = \frac{f}{c} = \frac{2\pi\omega}{c} \equiv \text{wavelength}$$

4. Special Case of Resistive Matched Load

The foregoing results simplify considerably for a resistive matched load. Returning to fig. 1.1 we note that we can now restrict

$$\tilde{Z}_{in}(s) = \tilde{Z}_L(s) = Z_c \equiv f_g Z_0 \tag{4.1}$$

so that the resistive impedance is the same as the characteristic impedance of a transmission-line antenna feed. This of course also requires more of the antenna design so that the antenna presents a resistive "termination" to the feed line of resistance Z_c. Fortunately some kinds of antennas have this property, at least to a reasonable approximation. (See, e.g., the TDR measurements in [4].)

From (2.10) the effective area becomes

$$\tilde{A}_{eff}(\vec{1}_i, s) = \frac{1}{f_g} \frac{\tilde{\vec{E}}_0(s) \cdot \tilde{\vec{h}}_t(\vec{1}_i, s)\, \tilde{\vec{h}}_t(\vec{1}_i, -s) \cdot \tilde{\vec{E}}_0(-s)}{\tilde{\vec{E}}_0(s) \cdot \tilde{\vec{E}}_0(-s)} \tag{4.2}$$

with the frequency-dependent admittances now removed. The maximum over polarization as in (2.14) now becomes

$$\tilde{A}^{(0)}_{eff}(\vec{1}_i, s) = \frac{1}{f_g} \tilde{\vec{h}}_t(\vec{1}_i, s) \cdot \tilde{\vec{h}}_t(\vec{1}_i, -s) \tag{4.3}$$

The gain in (3.9) and (3.10) is now

$$\begin{aligned} \tilde{G}(\vec{1}_0, s) &= -\frac{\gamma^2}{\pi} \tilde{A}^{(0)}_{eff}(-\vec{1}_0, s) = -\frac{\gamma^2}{\pi f_g} \tilde{\vec{h}}_t(-\vec{1}_0, s) \cdot \tilde{\vec{h}}_t(-\vec{1}_0, -s) \\ \tilde{G}(\vec{1}_0, j\omega) &= \frac{4\pi}{\lambda^2} \tilde{A}^{(0)}_{eff}(-\vec{1}_0, j\omega) = \frac{4\pi}{\lambda^2 f_g} \tilde{\vec{h}}_t(-\vec{1}_0, j\omega) \cdot \tilde{\vec{h}}_t(-\vec{1}_0, -j\omega) \end{aligned} \tag{4.4}$$

So this case of resistive matched load considerably simplifies the formulae by the removal of some of the frequency-dependent factors.

5. Relation of Effective Height to Effective Area and Gain Under Restrictive Assumptions

It is the effective height which gives the transmission and reception properties of an antenna in a linear sense, including phase. In time domain it is the effective height as a convolution operator that is directly related to the pulse response. To what extent can the effective height be inferred from the effective area or gain?

Note first that effective area and gain are scalars, whereas effective height is a vector (related to the polarization). On the $j\omega$ axis we have

$$\tilde{A}_{eff}^{(0)}(\vec{1}_i, j\omega) = \frac{1}{f_g} \tilde{\vec{h}}_t(\vec{1}_i, j\omega) \cdot \tilde{\vec{h}}_t^*(\vec{1}_i, j\omega) = \frac{1}{f_g} \left| \tilde{\vec{h}}_t(\vec{1}_i, j\omega) \right|^2$$

$$\tilde{G}(\vec{1}_0, j\omega) = \frac{4\pi}{\lambda^2 f_g} \tilde{\vec{h}}_t(-\vec{1}_0, j\omega) \cdot \tilde{\vec{h}}_t^*(-\vec{1}_0, j\omega) = \frac{4\pi}{\lambda^2 f_g} \left| \tilde{\vec{h}}_t(-\vec{1}_0, j\omega) \right|^2 \tag{5.1}$$

Thus the best that can be inferred from $\tilde{A}_{eff}^{(0)}$ or $\tilde{G}$ is the magnitude of the effective height. If one has polarization dependent information as in (4.2) (including complex polarizations perpendicular to $\vec{1}_i$) then the orientation of $\tilde{\vec{h}}_t$ can also be inferred.

At this point one can impose additional assumptions. Suppose that the orientation of $\tilde{\vec{h}}_t$ is known for $\vec{1}_i$ of interest (say from symmetry in the antenna) and this orientation is real so that we can write

$$\tilde{\vec{h}}_t(j\omega) = \vec{1}_t \tilde{h}_t(j\omega) \quad , \quad \tilde{A}_{eff}(\vec{1}_i, j\omega) = \frac{1}{f_g} \left| \tilde{h}_t(j\omega) \right|^2$$

$$\tilde{G}(\vec{1}_0, j\omega) = \frac{4\pi}{\lambda^2 f_g} \left| \tilde{h}_t(j\omega) \right|^2 \tag{5.2}$$

Then $\tilde{G}$ and $\tilde{A}_{eff}^{(0)}$ can infer $\left|\tilde{h}_t(j\omega)\right|$ which does not include phase. If $\vec{1}_t$ is frequency-independent (which it is in some cases) then the problem is reduced to determining $\tilde{h}(j\omega)$ (and hence $\tilde{h}(s)$) as our transfer function of interest.

Even with the foregoing assumptions there still is the problem of phase. From (5.2) we have

$$\left| \tilde{h}_t(\vec{1}_i, j\omega) \right| = \left[f_g \tilde{A}_{eff}^{(0)}(\vec{1}_i, j\omega) \right]^{\frac{1}{2}} = \frac{\lambda}{2} \left[\frac{f_g}{\pi} \tilde{G}(\vec{1}_0, j\omega) \right]^{\frac{1}{2}} \tag{5.3}$$

which seems a simple way to relate the effective height to gain. However, in the way gain is defined (Section 3), it has no phase information (on the $j\omega$ axis). So various possible effective heights can correspond to the same gain function. The simplest case has

$$\tilde{h}_t(\vec{1}_i, s) = \tilde{h}_t^{(0)}(\vec{1}_i, s) e^{-st_0} \tag{5.4}$$

which is just an added delay of a time t_0 to some other effective height. Such a delay (such as by a length of transmission line) is not important here. However, for transfer functions (effective height being such a function) one can have more generally

$$\begin{aligned} &\tilde{h}_t(\vec{1}_i, s) = \tilde{h}_t^{(0)}(\vec{1}_i, s)\tilde{a}(s) \\ &\left|\tilde{a}(j\omega)\right| = 1 \end{aligned} \tag{5.5}$$

Such a function $\tilde{a}(s)$ is called an all-pass function [6,7] which can be realized various ways in an idealized circuit synthesis sense (but perhaps with practical difficulties involving real circuit elements over a very large band of frequencies).

The effect of this phase ambiguity is to give various transient responses to the various antennas with the same gain function of frequency. Some antennas are less dispersive than others. This can be interpreted loosely as keeping a pulse more together (not spreading out in time). This also tends to increase the amplitude of the radiated transient field. An interesting question concerns the construction of some optimal $\tilde{h}_t$ from a given $\tilde{G}$. This is something like a minimum -phase problem. From (3.8) for a fixed polarization $\vec{1}_t$ as in (5.2) we have

$$\begin{aligned} &\tilde{G}(\vec{1}_0, j\omega) = \left|\tilde{g}(\vec{1}_0, j\omega)\right|^2 \\ &\left|\tilde{g}(\vec{1}_0, j\omega)\right| = \tilde{G}^{\frac{1}{2}}(\vec{1}_0, j\omega) = \frac{2}{\lambda}\left[\frac{\pi}{f_g}\right]^{\frac{1}{2}}\left|\tilde{h}_t(j\omega)\right| \\ &\qquad = \frac{\omega}{c}\left[\pi f_g\right]^{-\frac{1}{2}}\left|\tilde{h}_t(j\omega)\right| \end{aligned} \tag{5.6}$$

The function $\ell n\left(\left|\tilde{g}(\vec{1}_0, j\omega)\right|\right)$ or $\ell n\left(\left|\tilde{h}_t(j\omega)\right|\right)$ can then be Hilbert transformed to obtain a phase function $\arg\left(\left|\tilde{g}(\vec{1}_0, j\omega)\right|\right)$ with $\ell n\left(\left|\tilde{h}_t(j\omega)\right|\right)$ which is a minimum phase function, thereby giving at least one form of the complex function(s). This may give some kind of "best" effective-height function for a given gain function. Realizing this in a practical antenna is another matter.

One type of ideal effective-height function is

$$\vec{h}_t(\vec{1}_i, s) = h_t e^{-st_0} \vec{1}_t \tag{5.7}$$

for a particular choice of $\vec{1}_i = -\vec{1}_0$ (boresight). This approximately characterizes an impulse radiating antenna (IRA) [1] in reception [2]. (This holds over a large band of frequencies, limited by antenna size at low frequencies and construction details and conductor losses at high frequencies.) Note the inclusion of a simple delay t_0, such as occurs on ideal transmission lines (cables). By appropriate choice of reference for time or phase, this can be made zero. In this case we have

$$\begin{aligned}
\tilde{A}_{eff}^{(0)}(\vec{1}_i, s) &= \frac{1}{f_g} h_t^2 \\
\tilde{\vec{g}}(\vec{1}_0, s) &= \frac{s}{c}\left[\pi f_g\right]^{-\frac{1}{2}} h_t e^{-st_0} \vec{1}_t \\
\tilde{G}(\vec{1}_0, s) &= -\frac{s^2}{c^2}\frac{1}{\pi f_g} h_t^2 \\
\tilde{G}(\vec{1}_0, j\omega) &= -\frac{\omega^2}{c^2}\frac{1}{\pi f_g} h_t^2
\end{aligned} \tag{5.8}$$

As we can see the effective height gives the effective area and gain, but these latter eliminate the phase information and make the inference of the effective height have an unknown delay which could be a function of frequency.

6. Concluding Remarks

As we have seen, it is possible to relate the usual CW antenna gain (IEEE) to the effective height operator which more adequately expresses the time-domain response in both transmission and reception. This relationship results in some loss of information. There is a requirement first of a constant-resistance load at the antenna port, matched to the same constant-resistance input impedance of the antenna, unless one is willing to carry these as extra factors in the relationship. Second, one needs a frequency-independent polarization for the given angle of incidence (e.g., boresight), and a corresponding constant (frequency independent) orientation of the effective height. Third, the loss of phase information results in an unknown dispersion of a pulse. If one is willing to place additional constraints (assumptions, as in the case of an IRA of various types) then this phase ambiguity can be resolved.

Appendix A: Summary of the Gain Equations with an Example

We consider here how to convert our measured antenna parameter, $h(t)$, for single polarization on boresight to something close to the IEEE definition of frequency domain gain. We consider, by example, how to implement this as well as can be done.

We begin with the standard expressions in the frequency domain. Thus, the received power is

$$\tilde{P}_r = \tilde{A}_{eff}\tilde{S}^{(inc)} \tag{A.1}$$

where $\tilde{S}^{(inc)}$ is the incident power density in Watts/m^2 and $\tilde{A}_{eff}$ is the effective aperture. Gain is related to effective aperture by

$$\tilde{A}_{eff} = \frac{\lambda^2}{4\pi}\tilde{G} \tag{A.2}$$

Combining the above two equations, we have

$$\tilde{P}_r = \frac{\lambda^2\tilde{G}}{4\pi}\tilde{S}^{(inc)} \tag{A.3}$$

Take the square root, and recast into voltages

$$\frac{\tilde{V}_r}{\sqrt{\tilde{Z}_c}} = \frac{\lambda\sqrt{\tilde{G}}}{2\sqrt{\pi}}\frac{\tilde{E}^{(inc)}}{\sqrt{Z_0}} \tag{A.4}$$

where Z_0 is the impedance of free space and Z_c, the feed or input impedance, is assumed a positive constant. Thus, the final result in the frequency domain is

$$\tilde{V}_r = \frac{\lambda\sqrt{\tilde{G}f_g}}{2\sqrt{\pi}}\tilde{E}^{(inc)} \tag{A.5}$$

where $f_g = \tilde{Z}_c / Z_0$.

Let us now compare the above equation to one that we have been using in the time domain, i.e.,

$$V_r(t) = h_t(t) \circ E^{(inc)}(t) \tag{A.6}$$

where the "$\circ$" symbol indicates convolution. We routinely have already measured $h(t)$, so we just have to rescale to get gain. Converting this to the frequency domain, we have

$$\tilde{V}_r(j\omega) = \tilde{h}(j\omega)\tilde{E}^{(inc)}(j\omega) \tag{A.7}$$

Now compare equations (5) and (7), to get

$$\boxed{\tilde{G}(j\omega) = \frac{4\pi}{\lambda^2}\frac{\left|\tilde{h}_t(j\omega)\right|^2}{f_g} = \frac{4\pi f^2}{c^2}\frac{\left|h_t(j\omega)\right|^2}{f_g}} \tag{A.8}$$

We can use this to scale our $h(t)$ waveforms to get a frequency domain gain. Note the similarity between equations (A.2) and (A.8). The implication is that the effective aperture as a function of frequency is $h_t^2(j\omega)/f_g$, which is a rather simple and pleasing result.

There is a drawback with the definition in that it does not take into account dispersion, or time delay. If different frequencies have different time delays (as happens on more conventional antennas), the received pulse will not be clean. But the above definition of gain does not take this into account. Thus, using this definition of gain, two antennas with the same gain can have very different peak radiation E-fields.

Using this theory, we can calculate the gain of the 9-inch diameter reflector and lens IRAs, which were originally measured in [5]. The gain of the 9-inch diameter reflector IRA is shown in fig. A.1 and the gain of the 9-inch diameter lens IRA is shown in fig. A.2. When comparing the two gains, the most striking difference is apparent at the low frequencies, between 1 and 3 GHz. The dip in $h(j\omega)$ in this region becomes quite apparent when it is multiplied by frequency squared. We presume this dip occurs because the feed arms are not terminated in the reflector design. Future designs will include terminations of the feed arms. Even without the dip there is a 3 dB advantage for the lens design.

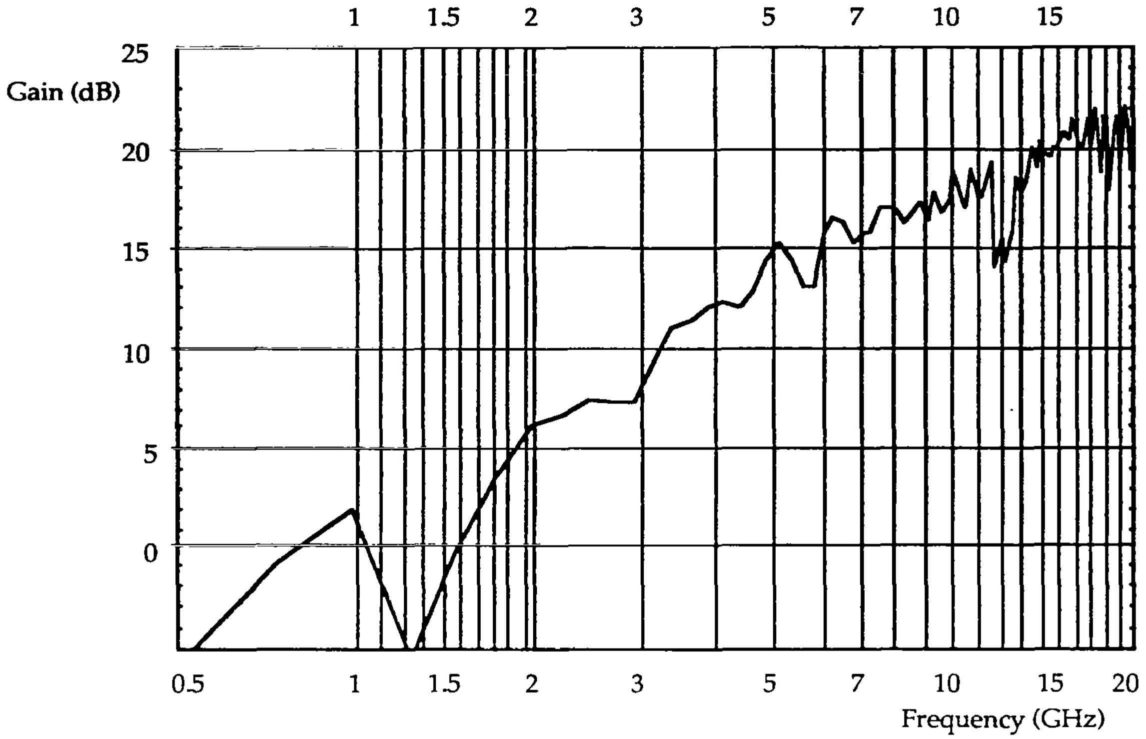

Figure A.1. Gain of the 9-inch reflector IRA.

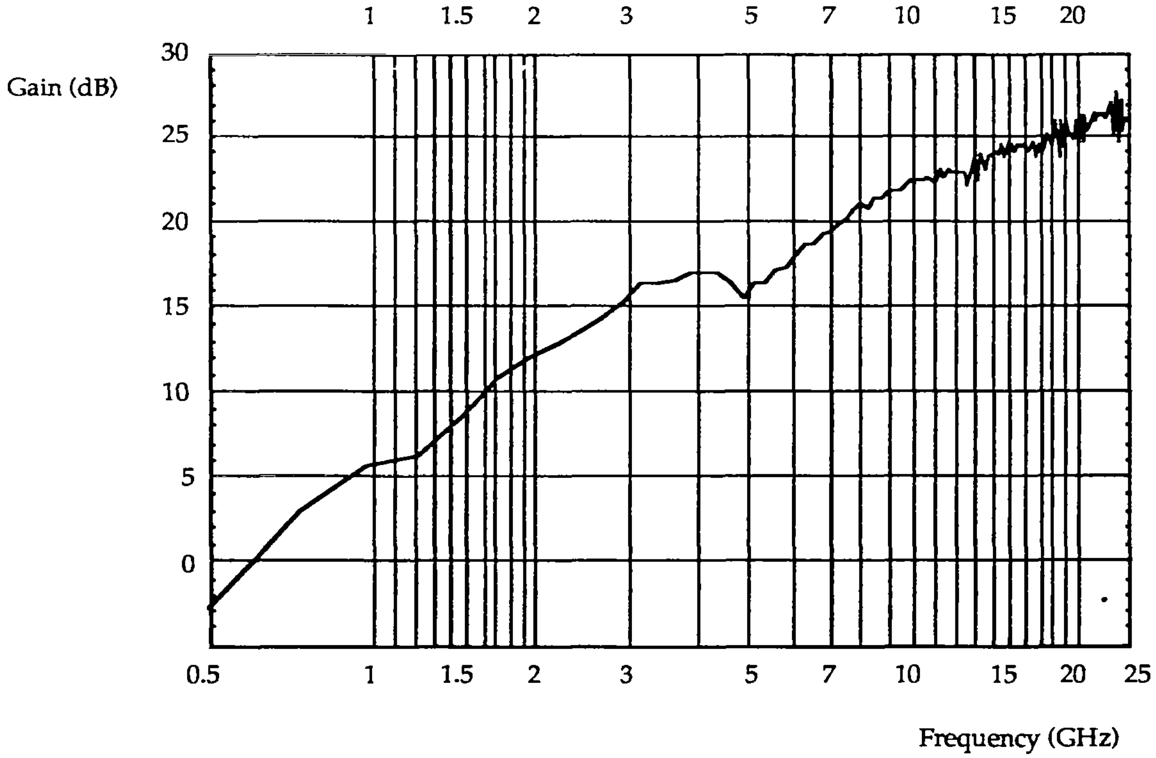

Figure A.2. Gain of the 9-inch lens IRA.

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