Sensor and Simulation Notes

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Three-Dimensional Coils for Low-Frequency Magnetic Illumination and Detection

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Abstract

This paper considers several types of three-dimensional coils for transmitting and receiving low-frequency magnetic fields to and from targets of interest "inside" the coil structures. Point symmetries are imposed on the coil designs to make the magnetic field uniform to various orders in the region of interest by elimination of unwanted terms in the vector spherical harmonic expansions. Various types of coils are considered, including bodies of revolution (circular loops) surrounding the target, arrays of point magnetic dipoles, and line magnetic dipoles.

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1. Introduction

There are various types of coils that one might use for producing low-frequency magnetic fields and measuring the scattered magnetic fields, such for detection and identification of metallic targets [7–9, 17]. The case of coils formed by long (ideally infinite) parallel wires giving two-dimensional coils has been considered in a previous paper [5]. The present paper considers various three-dimensional coil structures that can also be used for this application. Since multiple coils can be used for the various field components and for transmit and receive, various combinations of the two- and three-dimensional coils are possible.

A fundamental concept in designing such coils for producing approximately uniform magnetic fields (and for uniformity of detection involving both transmitter and receiver coils) is symmetry. For the three-dimensional coils it is convenient to expand the static magnetic fields in terms of vector spherical harmonics, the lowest order terms giving the uniform field of interest. Many of the higher order terms are suppressed by the imposition of appropriate point symmetries (rotation and reflection) on the coil designs. This leaves a minimum of terms (only one for third-order field uniformity) to be suppressed by appropriate choice of remaining geometric parameters.

There are additional considerations in the selection of coil geometry. One would like the receiver coils to be insensitive to approximately uniform externally incident magnetic fields (e.g., 60 Hz or 50 Hz). This can be achieved by making them as quadrupoles (in transmission, by reciprocity). One would also like each transmitter coil to have negligible mutual inductance (coupling) to other transmitter and receiver coils. This is also attained in some cases by symmetry (particularly if the two coils are associated with orthogonal field components: x, y, z). For the case of transmitter and receiver coils associated with the same field component, the receiver coil can be placed "inside" the transmitter coil to take advantage of the approximately uniform field of the transmitter. In addition, however, one may wish to adjust other parameters of the geometry (retaining the symmetry constraints) to minimize this coupling.

Scanning the contents, various types of coil designs are considered. These include various combinations of circular loops (bodies of revolution with a common axis), arrays of discrete (point) magnetic dipoles, and distributed magnetic dipoles (line dipoles) along paths parallel to the symmetry axis.

2. Symmetry Considerations

In [5], treating two-dimensional coils, translation symmetry (with respect to z) with ±z directed currents was assumed, giving rise to transverse (to z) magnetic fields which could be treated as analytic function of the complex coordinate $\zeta = x + jy$, thereby considerably simplifying the problem. Now, in considering coils for z-directed magnetic fields, with a finite extent of the coils (in the z and well as Ψ (cylindrical radius directions) the variation of the fields in all three coordinate directions (x, y, z) becomes important. For this analysis we have cylindrical (Ψ, ϕ, z) and spherical (r, θ, ϕ) coordinates as

$$\begin{aligned} x &= \Psi\cos(\phi) \quad , \quad y = \Psi\sin(\phi) \\ \Psi &= r\sin(\phi) \quad , \quad z = r\cos(\theta) \\ \vec{r} &= r\vec{1}_r = \Psi\vec{1}_\Psi + z\vec{1}_z = x\vec{1}_x + y\vec{1}_y + z\vec{1}_z \end{aligned} \tag{2.1}$$

In uniform isotropic media (e.g., air with permeability μ_0) and away from sources (the coil conductors) the magnetic field has both zero divergence and zero curl as

$$\begin{aligned} \nabla \cdot \vec{H}(\vec{r}) &= \frac{\partial H_x(\vec{r})}{\partial x} + \frac{\partial H_y(\vec{r})}{\partial y} + \frac{\partial H_z(\vec{r})}{\partial z} \\ &= \frac{1}{\Psi}\frac{\partial}{\partial\Psi}\left(\Psi H_\Psi(\vec{r})\right) + \frac{1}{\Psi}\frac{\partial H_\phi(\vec{r})}{\partial\phi} + \frac{\partial H_z(\vec{r})}{\partial z} \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 H_r(\vec{r})\right) + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)H_\theta(\vec{r})\right) + \frac{1}{r\sin(\theta)}\frac{\partial H_\phi(\vec{r})}{\partial\phi} \\ &= 0 \\ \nabla \times \vec{H}(\vec{r}) &= \left[\frac{\partial H_z(\vec{r})}{\partial y} - \frac{\partial H_y(\vec{r})}{\partial z}\right]\vec{1}_x + \left[\frac{\partial H_x(\vec{r})}{\partial z} - \frac{\partial H_z(\vec{r})}{\partial x}\right]\vec{1}_y + \left[\frac{\partial H_y(\vec{r})}{\partial x} - \frac{\partial H_x(\vec{r})}{\partial y}\right]\vec{1}_z \\ &= \left[\frac{1}{\Psi}\frac{\partial H_z(\vec{r})}{\partial\phi} - \frac{\partial H_\phi(\vec{r})}{\partial z}\right]\vec{1}_\Psi + \left[\frac{\partial H_\Psi(\vec{r})}{\partial z} - \frac{\partial H_z(\vec{r})}{\partial\Psi}\right]\vec{1}_\phi \\ &\quad + \frac{1}{\Psi}\left[\frac{\partial}{\partial\Psi}\left(\Psi H_\phi(\vec{r})\right) - \frac{\partial H_\Psi(\vec{r})}{\partial\phi}\right]\vec{1}_z \\ &= \frac{1}{r\sin(\theta)}\left[\frac{\partial}{\partial\theta}\left(\sin(\theta)H_\phi(\vec{r})\right) - \frac{\partial H_\theta(\vec{r})}{\partial\phi}\right]\vec{1}_r + \frac{1}{r}\left[\frac{1}{\sin(\theta)}\frac{2H_r(\vec{r})}{\partial\phi} - \frac{\partial}{\partial r}\left(rH_\phi(\vec{r})\right)\right]\vec{1}_\theta \end{aligned} \tag{2.2}$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}\left(rH_\theta(\vec{r})\right) - \frac{\partial H_r(\vec{r})}{\partial \theta}\right]\vec{1}_\phi$$

$$= \vec{0}$$

Noting that all three vector components of the curl are zero, this gives four equations relating the various derivatives of the field components in three convenient coordinate systems. In terms of a scalar magnetic potential we have

$$H(\vec{r}) = -\nabla\Phi_h(\vec{r}) \tag{2.3}$$

Away from sources this has a zero Laplacian as

$$\begin{aligned}
\nabla^2\Phi_h(\vec{r}) &= \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]\Phi_h(\vec{r}) \\
&= \frac{1}{\Psi}\frac{\partial}{\partial \Psi}\left[\Psi\frac{\partial \Phi_h(\vec{r})}{\partial z}\right] + \frac{1}{\Psi^2}\frac{\partial^2\Phi_h(\vec{r})}{\partial \Psi^2} + \frac{\partial^2\Phi_h(\vec{r})}{\partial z^2} \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial \Phi_h(\vec{r})}{\partial r}\right] + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial \theta}\left[\sin(\theta)\frac{\partial \Phi_h(\vec{r})}{\partial \theta}\right] \\
&\quad + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2\Phi_h(\vec{r})}{\partial \phi^2} \\
&= 0
\end{aligned} \tag{2.4}$$

As long as we do not consider paths which enclose currents (loop conductors) this potential is single valued. Our concern here is with the domain near the origin $\vec{r} = \vec{0}$ for field uniformity.

For present purposes, let the coils and currents (and hence potential and field) have $z = 0$ as a transverse symmetry plane. For this purpose we have the reflection symmetry group [16]

$$R_z = \{(1)\ ,\ (R_z)\}\ ,\ \ (R_z)^2 = (1) \equiv \text{identity} \tag{2.5}$$

with representation

$$\overleftrightarrow{R}_z = \overleftrightarrow{1} - 2\,\vec{1}_z\vec{1}_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{2.6}$$

$$\overleftrightarrow{R}_z^2 = \overleftrightarrow{1} = \vec{1}_x\vec{1}_x + \vec{1}_y\vec{1}_y + \vec{1}_z\vec{1}_z \equiv \text{identity dyadic}$$

The reflection dyadic gives a mirror coordinate $\vec{r}_m^{(z)}$ as

$$\vec{r} = (x,y,z) \ , \ \vec{r}_m^{(z)} = (x,y,-z) = \overleftrightarrow{R}_z \cdot \vec{r} \tag{2.7}$$

With respect to such a symmetry plane the currents, potential and field can be decomposed into two parts which separately satisfy the Maxwell equations, denoted as symmetric and antisymmetric. For present purposes we choose our source current density, and hence potential and field as *symmetric,* in which case we have

$$\begin{aligned} \overleftrightarrow{R}_z \cdot \vec{J}(\vec{r}) &= \vec{J}(\vec{r}_m^{(z)}) \\ \Phi_h(\vec{r}) &= -\Phi_h(\vec{r}_m^{(z)}) \\ \overleftrightarrow{R}_z \cdot \vec{H}(\vec{r}) &= -\vec{H}(\vec{r}_m^{(z)}) \end{aligned} \tag{2.8}$$

Coils for this case then have current components parallel to z = 0 reflected with the same sign, but current components perpendicular to z = 0 reflected with apposite sign. but current components perpendicular to z = 0 reflected with opposite sign. In particular on z = 0 we have

$$\begin{aligned} &\vec{J}(x,y,0) \cdot \vec{1}_z = 0 && \text{(parallel to } z = 0 \text{ plane)} \\ &\Phi_h(x,y,0) = 0 && \\ &\vec{H}(x,y,0) = H_z(x,y,0)\,\vec{1}_z && \text{(perpendicular to } z = 0 \text{ plane)} \end{aligned} \tag{2.9}$$

With $\vec{H}(0,0,0)$ taken as non zero (being in the center of the region of desired uniform magnetic field), then we can compare the magnetic field at general $\vec{r}$ to

$$\vec{H}_0 = H_0\,\vec{1}_z = \vec{H}(\vec{0}) \tag{2.10}$$

to estimate the field uniformity. Now with no currents in the region of interest, we have an even function of z for the z component as

$$H_z(x,y,-z) = H_z(x,y,z)$$
$$\left.\frac{\partial^\ell H_z(x,y,z)}{\partial z^\ell}\right|_{z=0} = 0 \quad \text{for } \ell \text{ odd} \tag{2.11}$$

showing some field uniformity near $\vec{r} = \vec{0}$ (second derivative with respect to z being, in general, non zero at $z = 0$), just due to the symmetry plane. Note that the other components are odd functions of z as

$$H_x(x,y,-z) = -H_x(x,y,z) \quad , \quad H_y(x,y,-z) = -H_y(x,y,z)$$
$$\left.\frac{\partial^\ell H_x(x,y,z)}{\partial z^\ell}\right|_{z=0} = 0 \quad \text{for } \ell \text{ even} \tag{2.12}$$
$$\left.\frac{\partial^\ell H_y(x,y,z)}{\partial z^\ell}\right|_{z=0} = 0 \quad \text{for } \ell \text{ even}$$

the first derivatives with respect to z being, in general, non zero. Similarly for the scalar potential we have an odd function of z with

$$\Phi_h(x,y,-z) = -\Phi_h(x,y,z) \quad , \quad \Phi_h(x,y,0) = 0$$
$$\left.\frac{\partial^\ell \Phi_h(x,y,z)}{\partial z^\ell}\right|_{z=0} = 0 \quad \text{for } \ell \text{ even} \tag{2.13}$$
$$\left.\frac{\partial \Phi_h(0,0,z)}{\partial z}\right|_{z=0} = -H_0$$

This all comes from just one symmetry plane!

Of course, one can go to other (higher order) point symmetries (rotations and reflections) [16], including the symmetry plane discussed above (in some cases as a subgroup). Axial symmetry planes (containing the z axis) with respect to which the fields are made to be antisymmetric can achieve greater uniformity when combined with the foregoing. Taking the $x = 0$ plane as such a plane we have

$$R_x = \{(1), (R_x)\} \quad , \quad (R_x)^2 = (1)$$
$$\overleftrightarrow{R}_x = \overleftrightarrow{1} - 2\,\vec{1}_z\vec{1}_z = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad , \quad \overleftrightarrow{R}_x^2 = \overleftrightarrow{1} \tag{2.14}$$

Changing our definition of $(\vec{r}_m^{(z)})$ in (2.7) to $(\vec{r}_m^{(x)})$ for this reflection we have *antisymmetric* currents, potentials, and fields (noting sign reversals from (2.8)) as

$$\begin{aligned} \overleftrightarrow{R}_x \cdot \vec{J}(\vec{r}) &= -\vec{J}(\vec{r}_m^{(x)}) \\ \Phi_h(\vec{r}) &= -\Phi_h(\vec{r}_m^{(x)}) \\ \overleftrightarrow{R}_x \cdot \vec{H}(\vec{r}) &= \vec{H}(\vec{r}_m^{(x)}) \end{aligned} \tag{2.15}$$

In this case on $x = 0$ we have

$$\begin{aligned} &\vec{J}(0,y,z) = J_x(0,y,z)\,\vec{1}_x && \text{(perpendicular to } x = 0 \text{ plane)} \\ &\vec{H}(0,y,z) \cdot \vec{1}_x = 0 && \text{(parallel to } x = 0 \text{ plane)} \end{aligned} \tag{2.16}$$

Now the x component of the magnetic field is an odd function of x as

$$\begin{aligned} &H_x(-x,y,z) = -H_x(x,y,z)\ , \quad H_x(0,y,z) = 0 \\ &\left.\frac{\partial^\ell H_x(x,y,z)}{\partial x^\ell}\right|_{x=0} = 0 \ \text{ for } \ell \text{ even} \end{aligned} \tag{2.17}$$

The other components are even functions of x as

$$\begin{aligned} &H_y(-x,y,z) = H_y(x,y,z)\ , \quad H_z(-x,y,z) = H_z(x,y,z) \\ &\left.\frac{\partial^\ell H_x(x,y,z)}{\partial x^\ell}\right|_{x=0} = 0 \ \text{ for } \ell \text{ odd} \\ &\left.\frac{\partial^\ell H_z(x,y,z)}{\partial x^\ell}\right|_{x=0} = 0 \ \text{ for } \ell \text{ odd} \end{aligned} \tag{2.18}$$

the first derivatives with respect to x, in particular being zero, giving the second derivatives as the first non-zero term. From the zero curl in (2.2) we also have

$$\begin{aligned} \frac{\partial H_x(0,y,z)}{\partial y} &= 0 \\ \frac{\partial H_x(0,y,z)}{\partial z} &= 0 \end{aligned} \tag{2.19}$$

One can similarly make $y = 0$ a symmetry plane and obtain the same results as in (2.17) through (2.19) for antisymmetric fields with the simple interchange of the roles of x and y. With all three symmetry planes (symmetric with respect to $z = 0$, antisymmetric with respect to $x = 0$ and $y = 0$) the large number of constraints on the field near the origin, $\vec{r} = \vec{0}$, give

$$\nabla \vec{H}(\vec{r}) = \overleftrightarrow{0} \tag{2.20}$$

i.e., all first derivatives of all field components are zero. With three orthogonal symmetry planes we can consider this a case of dihedral D_{2t} symmetry, at least in a geometric sense (ignoring the orientation of the current directions in the conductors). The $x = 0$ and $y = 0$ symmetry planes correspond to C_{2a} rotation symmetry with respect to the z axis. Reflection R_z occurs in a different sense (symmetric) involving a sign reversal when rotating by π about the x and y axes. As we shall see in the next section, such a high degree of symmetry is not required for the result in (2.20), but rather only $C_{2t} = C_2 \otimes R_z$, i.e., $x = 0$ and $y = 0$ need not be symmetry planes but the z axis need only be a 2-fold rotation axis.

Adjunction of rotation symmetry C_N about the z axis gives additional constraints. A 2-fold or higher order rotation axis ($N \geq 2$) gives special uniformity properties to the field, as we can see in terms of the spherical harmonic expansion of the fields (next section). A special case of interest is continuous rotation symmetry C_∞, which in adjunction with axial and transverse symmetry planes $C_{\infty at}$ is equivalent to dihedral symmetry $D_{\infty t}$ (a and t indicating axial and transverse symmetry planes, respectively [16]). In this case, we use cylindrical coordinates and have ϕ-directed ϕ-independent loop-current density as

$$\vec{J}(\vec{r}) = J_\phi(\vec{r})\,\vec{1}_\phi = J_\phi(\Psi, z)\,\vec{1}_\phi \tag{2.21}$$

giving a field with no ϕ component as

$$\vec{H}(\vec{r}) = H_\Psi(\Psi, z)\,\vec{1}_\Psi + H_z(\Psi, z)\,\vec{1}_z \tag{2.22}$$

the components in cylindrical coordinates being ϕ independent. Including the $z = 0$ symmetry plane we now have

$$
\begin{aligned}
H_z(\Psi,-z) &= H_z(\Psi,z) \\
H_\Psi(\Psi,-z) &= -H_\Psi(\Psi,z)\ ,\ \ H_\Psi(\Psi,0) = 0 \\
\vec{H}(\Psi,\phi,0) &= H_z(\Psi,0)\,\vec{1}_z \\
\Phi_h(\Psi,-z) &= -\Phi(\Psi,z)\ ,\ \ \Phi_h(\Psi,0) = 0
\end{aligned}
\tag{2.23}
$$

From this we have

$$
\begin{aligned}
\left.\frac{\partial^\ell H_z(\Psi,0)}{\partial z^\ell}\right|_{z=0} &= 0 \text{ for } \ell \text{ odd} \\
\frac{\partial^\ell H_\Psi(\Psi,z)}{\partial \Psi^\ell} &= 0 \text{ for all } \ell \geq 0
\end{aligned}
\tag{2.24}
$$

For zero divergence on the z axis we also have

$$
\begin{aligned}
\vec{H}(0,\phi,z) &= H_z(0,z)\ \vec{1}_z\ ,\ \ H_\Psi(0,z) = 0 \\
\frac{\partial^\ell H_\Psi(0,z)}{\partial z^\ell} &= 0 \text{ for all } \ell \geq 0
\end{aligned}
\tag{2.25}
$$

The curl equation with this gives

$$
0 = \frac{\partial H_\Psi(0,z)}{\partial z} = \left.\frac{\partial H_z(\Psi,z)}{\partial \Psi}\right|_{\Psi=0}
\tag{2.26}
$$

Thus first derivatives with respect to Ψ and z of both field components H_Ψ and H_z are zero at the origin. One can go on to higher orders of field uniformity as discussed in following sections.

3. Representation of Static Magnetic Fields in Spherical Coordinates with Implications of Symmetry

Beginning with the magnetic potential we have the usual expansion in spherical coordinates [3, 6, 12]

$$\begin{aligned}
\Phi_n(\vec{r}) &= -\sum_{n=1}^{\infty}\sum_{m=0}^{n}\sum_{e,o} a_{n,m,{}^e_o}\; r^n\; Y_{n,m,{}^e_o}(\theta,\phi) \\
Y_{n,m{}^e_o} &= P_n^{(m)}(\cos(\theta)) \begin{Bmatrix}\cos(m\phi)\\ \sin(m\phi)\end{Bmatrix} \\
P_n^{(m)}(\xi) &\equiv (-1)^m\left[1-\xi^2\right]^{\frac{m}{2}} \frac{d^m P_n(\xi)}{d\xi^m} \\
&\equiv \text{associated Legendre functions} \\
P_n(\xi) &\equiv P_n^{(0)}(\xi) \equiv \frac{1}{2^n n!}\frac{d^n}{d\xi^n}\left[\left[\xi^2-1\right]^n\right] \\
&\equiv \text{Legendre functions}
\end{aligned} \tag{3.1}$$

Here we have begun the summation with $n = 1$, since there should be no singularity at the origin, and a constant term has no gradient (no magnetic field). The magnetic field is found from

$$\begin{aligned}
\vec{H}(\vec{r}) &= -\nabla\Phi_n(\vec{r}) \\
&= \sum_{n=1}^{\infty}\sum_{m=0}^{n}\sum_{e,o} a_{n,m,{}^e_o}\left\{n\; r^{n-1} Y_{n,m,{}^e_o}(\theta,\phi)\,\vec{1}_r + r^{n-1}\vec{Q}_{n,m,{}^e_o}(\theta,\phi)\right\} \\
\vec{Q}_{n,m,{}^e_o}(\theta,\phi) &= \nabla_s Y_{n,m,{}^e_o}(\theta,\phi) \quad \text{(gradient on unit sphere)} \\
&= \vec{1}_\theta \frac{\partial}{\partial\theta} Y_{n,m,{}^e_o}(\theta,\phi) + \vec{1}_\theta \frac{1}{\sin(\theta)}\frac{\partial}{\partial\phi} Y_{n,m,{}^e_o}(\theta,\phi)
\end{aligned} \tag{3.2}$$

Note that for $m = 0$, only the even terms (index e) are non zero; these are ϕ independent.

Constrain the solution to have $z = 0$ as a symmetry plane (R_z symmetry) about which the currents, potential, and field are symmetric as in (2.8). This requires that the terms in (3.1) be odd functions of $\cos(\theta)$ $(=\xi)$ which in turn requires that n + m be odd so that

$$a_{n,m,{}^e_o} = 0 \text{ for } n+m \text{ even} \tag{3.3}$$

The leading terms ($n = 1$ with $m = 0$) are

$$\vec{H}_1(\vec{r}) = \vec{H}(\vec{0}) = a_{n,m,e}\left\{\cos(\theta)\,\vec{1}_r - \sin(\theta)\,\vec{1}_\theta\right\}$$
$$= a_{1,o,e}\,\vec{1}_z = H_0\,\vec{1}_z \tag{3.4}$$
$$\Phi_{h_1}(\vec{r}) = -a_{1,o,e}\; r\,\cos(\theta) = -a_{1,0,e}\,z$$
$$= -H_0\,z$$

So now we can write

$$\Phi_h(\vec{r}) = -H_0\,z - \sum_{n=2}^{\infty}\;\sum_{\substack{m=0\\ m+n\\ odd}}^{n}\;\sum_{e,o} a_{n,m,{e \atop o}}\; r^n\, Y_{n,m,{e \atop o}}(\theta,\phi)$$
$$\vec{H}(\vec{r}) = H_0\,\vec{1}_z + \sum_{n=2}^{\infty}\;\sum_{\substack{m=0\\ m+n\\ odd}}^{n}\;\sum_{e,o} a_{n,m,{e \atop o}}\; r^{n-1}\left\{n\, Y_{n,m,{e \atop o}}(\theta,\phi)\,\vec{1}_r + \vec{Q}_{n,m,{e \atop o}}(\theta,\phi)\right\} \tag{3.5}$$

where the sum over m is now in increments of 2, beginning with $m = 0$ for n odd and with $m = 1$ for n even. Thus R_z has eliminated roughly half the terms in the summation.

Next, consider C_N symmetry, an N-fold rotation with respect to the z axis. This group is given by [16]

$$C_N = \left\{(C_N)_\ell \,\middle|\, \ell = 1,2,\ldots,N\right\}$$
$$(C_N)_\ell \equiv \text{rotation by } \frac{2\pi\ell}{N} = (C_N)_1^\ell \tag{3.6}$$
$$(C_N)_1^N = (C_N)_N = (1)$$

This has a matrix representation

$$\left(C_{n,m}(\phi_\ell)\right) = \begin{pmatrix}\cos(\phi_\ell) & -\sin(\phi_\ell)\\ \sin(\phi_\ell) & \cos(\phi_\ell)\end{pmatrix} = \exp\left(\phi_\ell\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right)$$
$$\phi_\ell = \frac{2\pi\ell}{N}\;,\quad \ell = 1,2,\ldots,N \tag{3.7}$$
$$\left(C_{n,m}(0)\right) = \left(C_{n,m}(2\pi)\right) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \equiv \text{identity transverse to } z \text{ axis}$$

(There is also a scalar representation using $e^{j\ell\phi_\ell}$.) This is cast in two-dimensional form for operation on a plane of constant z. (One can add a 1 in the z, z position if desired for a three-dimensional form).

So now lwt our loops (and the resulting magnetic field) have C_N symmetry with respect to the z axis. This means that on successive rotation by $\phi_1 = 2\pi/N$ the currents (and hence potential and field) are invariant. Note that since C_N is a proper rotation (no reflections required) there are no sign changes on some of the field components (referred to (Ψ,ϕ,z) or (r,θ,ϕ) coordinates) in this symmetry operation. Then the only m index values in the $\cos(m\phi)$ and $\sin(m\phi)$ terms in the expansions (3.1) and (3.2) with non-zero $a_{n,m,{}^e_o}$ coefficients are those given by

$$m = \nu N\,,\ \nu = 0,1,2,\ldots = \text{integers} \geq 0 \tag{3.8}$$

The resuloting magnetic field in (3.2) then reduces to

$$\vec{H}(\vec{r}) = \sum_{n=1}^{\infty} \sum_{m=0}^{n,N} \sum_{e,o} a_{n,m,{}^e_o} r^{n-1} \left\{ n Y_{n,m,{}^e_o}(\theta,\phi) \vec{1}_r + \vec{Q}_{n,m,{}^e_o}(\theta,\phi) \right\}$$

$$\sum_{m=0}^{n,N} \equiv \text{ sum over } m \text{ (from zero here) in steps of N (i.e., } m = 0,N,2N,\ldots) \text{ for all } m \leq n \tag{3.9}$$

For non-trivial C_N symmetry, i.e., for $N \geq 2$ we have only $m = 0$ included in the $n = 1$ term, giving the same form for the field at the origin as in (3.4). In this case (3.9) can be rewritten as

$$\vec{H}(\vec{r}) = H_0 \vec{1}_z + \sum_{n=2}^{\infty} \sum_{m=0}^{n,N} \sum_{e,o} a_{n,m,{}^e_o} r^{n-1} \left\{ n Y_{n,m,{}^e_o}(\theta,\phi) \vec{1}_r + \vec{Q}_{n,m,{}^e_o}(\theta,\phi) \right\} \tag{3.10}$$

This shows that C_2 (or higher) symmetry is also sufficient to give a z-directed magnetic field at the origin. (No R_z symmetry has been assumed here.) By extension, since the location of the origin can slide up and down the axis without affecting the C_N rotation symmetry, this shows that everywhere on the z axis the magnetic field has only a z component with C_2 (or higher) symmetry.

Higher order C_N symmetry sets more and more coefficients to zero. From (3.8) we see that the $m = 0$ terms are retained for all N. However, we can set all $m \geq 1$ terms to zero for all n up through some n_0 by setting

$$N = n_0 + 1 \tag{3.11}$$

giving

$$\vec{H}(\vec{r}) = H_0 \vec{1}_z + \sum_{n=2}^{n_0} a_{n,0,e}\left\{n\, r^{n-1} Y_{n,o,e}(\theta,\phi)\, \vec{1}_r + r^{n-1} \vec{Q}_{n,0,e}(\theta,\phi)\right\}$$
$$+ O\left(r^{n_o}\right) \text{ as } r \to 0$$
$$= H_0 \vec{1}_z + \sum_{n=2}^{n_0} a_{n,0,e}\, r^{n-1}\left\{n\, P_n(\cos(\theta))\, \vec{1}_r + \left[\frac{\partial}{\partial\theta} P_n(\cos(\theta))\right]\vec{1}_\theta\right\}$$
$$+ O\left(r^{n_o}\right) \text{ as } r \to 0 \tag{3.12}$$

So terms up through r^{n_0-1} or r^{N-2} have no ϕ dependence. In designing coils for uniform magnetic field then one can choose what one desires for a first error term as order r^{n_0} and choose $N = n_0 + 1$ for the rotation symmetry to give the form in (3.12). This leaves single terms for each n from 2 through n_0 to consider for possible elimination to obtain uniformity order r^{n_0-1}.

Adjoining N-fold rotation symmetry with the transverse symmetry plane gives the symmetry group

$$C_{Nt} = C_N \otimes R_z \tag{3.13}$$

with $2N$ elements. This eliminates yet more terms in the expansion. Combining (3.5) with (3.10), observe that $m + n$ can only be odd. This reduces the terms to be considered in the summation as given in Table 3.1.

Using the results in Table 3.1, the magnetic field can be written for $N \geq 3$ as

$$\vec{H}(\vec{r}) = H_0 \vec{1}_z + \sum_{n=3}^{\infty,2} \sum_{m=0}^{n,N} \sum_{e,o} a_{n,m,{}^e_o}\, r^{n-1}\left\{nY_{n,m,{}^e_o}(\theta,\phi)\, \vec{1}_r + \vec{Q}_{n,m,{}^e_o}(\theta,\phi)\right\} \quad \text{for } N \text{ even}$$

$$\vec{H}(\vec{r}) = H_0 \vec{1}_z + \sum_{n=2}^{\infty,2} \sum_{m=N}^{n,2N} a_{n,m,{}^e_o}\, r^{n-1}\left\{nY_{n,m,{}^e_o}(\theta,\phi)\, \vec{1}_r + \vec{Q}_{n,m,{}^e_o}(\theta,\phi)\right\}$$
$$+ \sum_{n=3}^{\infty,2} \sum_{m=0}^{n,2N} a_{n,m,{}^e_o}\, r^{n-1}\left\{nY_{n,m,{}^e_o}(\theta,\phi)\, \vec{1}_r + \vec{Q}_{n,m,{}^e_o}(\theta,\phi)\right\} \quad \text{for } N \text{ odd} \tag{3.14}$$

Table 3.1. Terms in Summation for C_{Nt} Symmetry

N	n, m and form of summation
N = even $N \geq 2$	n = even, $n \geq 2$, m = odd $\sum_{m=0}^{n,N}$ has no terms (all m = even) $\Rightarrow 0$
	n = odd, $n \geq 3$, m = even $\sum_{m=0}^{n,N}$ has terms for m = 0, N, $2N$, ... $\Rightarrow \sum_{m=0}^{n,N}$
N = odd $N \geq 3$	n = even, $n \geq 2$, m = odd $\sum_{m=0}^{n,N}$ has terms for m = N, $3N$, ... $\Rightarrow \sum_{m=N}^{n,2N}$
	n = odd, $n \geq 3$, m = even $\sum_{m=0}^{n,N}$ has terms for m = 0, $2N$, $4N$, ... $\Rightarrow \sum_{m=0}^{n,2N}$

As discussed previously, higher order C_N symmetry sets more and more coefficients to zero. As we can see now in (3.12) the m = 0 terms are retained only in odd-n terms (even powers of r). As before we can set all $m \geq 1$ terms to zero for all n up through some n_0 by setting

$$N = n_0 + 1 \tag{3.15}$$

What remains in (3.14) is

$$\vec{H}(\vec{r}) = H_0 \vec{1}_z + \sum_{n=3}^{n_0,2} a_{n,0,e} r^{n-1} \left\{ n\, P_n(\cos(\theta))\, \vec{1}_r + \left[\frac{\partial}{\partial \theta} P_n(\cos(\theta)) \right] \vec{1}_\theta \right\} + O(r^{n_0+1}) \text{ as } r \to 0 \tag{3.16}$$

where the remaining terms, including the ϕ-dependent terms, have been included in the $O(r^{n_0+1})$ or $O(r^N)$ (or error in the truncation). This result applies for both even and odd N.

For field uniformity we would like as many powers or r (after the r^0, or constant term) to be zero. Define the number of such terms (highest r^{n-1} with zero coefficients from r^1 through r^{n-1}, as the *order of field uniformity*. The conditions for achieving various orders of field uniformity are presented in Table 3.2. Note that besides the symmetry conditions, there still remain the terms for generally non-zero $a_{n,0,e}$ for n odd and $n \geq 3$. These remaining coefficients can be made zero by appropriate choice of the spatial dependence of the currents (source for the field) within the symmetry constraints (basically the θ dependence).

Table 3.2 Conditions for Various Orders of Field Uniformity

Order of Field Uniformity	Symmetry Conditions	Additional Conditions
0 (z-directed field)	R_z or C_N with $N \geq 2$	None
1	C_{Nt} with $N \geq 2$	None
2	C_{Nt} with $N \geq 3$	$a_{3,0,e} = 0$
3	C_{Nt} with $N \geq 4$	$a_{3,0,e} = 0$

A special case of interest concerns C_∞ symmetry (continuous rotation symmetry). This implies that only $m = 0$ terms remain in the expansion. As discussed in the previous section, by restricting the currents to be ϕ-directed makes $H_\phi = 0$ and the remaining components are only functions of Ψ and z. This also implies axial symmetry planes ($C_{\infty a}$). Adjoining R_z symmetry gives dihedral symmetry $D_{\infty d} = D_{\infty t}$. Examples of this symmetry are discussed later.

4. Further Consideration of Field Uniformity

As discussed in the previous section the magnetic field can be expanded in powers of r with coefficients that are functions of θ and ϕ. The first non-zero coefficient of r^{n-1} (after the uniform $H_0 \vec{1}_z$ term from $n = 1$) determines the power $n - 1$ of r which can be considered as the order of the nonuniformity. Application of symmetry (C_{2t}) has been shown to achieve second order nonuniformity (near $\vec{r} = \vec{0}$). Since this second order nonuniformity means that the first spatial derivatives of the field are zero, we can refer to this case as first order uniformity. Of course, even higher order uniformity is in principle achievable by higher order symmetries and/or other special features of the loop design. In general then, the order of the uniformity is $n - 2$ if r^{n-1} is the first nonuniformity term.

As discussed in [5], besides uniformity of the field, there is the concept of uniformity of detection. In this case, there are two sets of coils, either of which may be considered as the transmitter and the other as the receiver by reciprocity. The receiver is detecting the scattered field from a target near $\vec{r} = \vec{0}$. This leads to the concept of the product of the fields near $\vec{r} = \vec{0}$ from the two coils (as transmitters) as the quantity one wishes to be uniform. In [5] the fields were two-dimensional, allowing a complex-variable formulation in which field multiplication was straight forward. In the present case of three-dimensional fields expressed as vectors, this multiplication is more complicated.

As a special case let us consider that the target magnetic polarizability $\tilde{\overleftrightarrow{M}}(s)$ is dominated by its z, z component $\tilde{M}_{z,z}(s)$, and the two magnetic fields are both aligned in the z direction. Then we form

$$U'(\vec{r}) = \vec{H}_2(\vec{r}) \cdot \tilde{\overleftrightarrow{M}}(s) \cdot \vec{H}_1(\vec{r}) = \tilde{M}_{z,z}(s)\, \vec{H}_2(\vec{r}) \cdot \vec{H}_1(\vec{r}) \tag{4.1}$$

so that our uniformity function can be defined in the form

$$U(\vec{r}) = \vec{H}_2(\vec{r}) \cdot \vec{H}_1(\vec{r}) \tag{4.2}$$

where

$$\vec{H}_2(\vec{0}) = \vec{H}_1(\vec{0}) = H_0 \, \vec{1}_z \tag{4.3}$$

Then assuming that the two fields have $n - 1$ equal to $n_1 - 1$ and $n_2 - 1$ for the first nonuniformity terms we have

$$
\begin{aligned}
U(\vec{r}) = H_0^2 \Bigg\{ 1 \\
&+ \frac{r^{n_1-1}}{H_0} \vec{1}_z \cdot \sum_{m=0}^{n_1} \sum_{e,o} a^{(1)}_{n_1,m,{}^e_o} \left[Y_{n_1,m,{}^e_o}(\theta,\phi)\, \vec{1}_r + \vec{Q}_{n_1,m,{}^e_o}(\theta,\phi) \right] \\
&+ \frac{r^{n_2-1}}{H_0} \vec{1}_z \cdot \sum_{m=0}^{n_2} \sum_{e,o} a^{(2)}_{n_2,m,{}^e_o} \left[Y_{n_2,m,{}^e_o}(\theta,\phi)\, \vec{1}_r + \vec{Q}_{n_2,m,{}^e_o}(\theta,\phi) \right] \\
&+ \cdots \Bigg\}
\end{aligned}
\tag{4.4}
$$

If $n_1 \neq n_2$ then the smallest of these dominates the nonuniformity, in which case the smaller term (say n_1) in (4.4) gives the nonuniformity term of order $n_1 - 1$. Then one can define

$$
A^{(1)}_{n_1} = \sup_{\theta,\phi} \left| \frac{1}{H_0} \vec{1}_z \cdot \sum_{m=0}^{n_1} \sum_{e,0} a^{(1)}_{n_1,m,{}^e_0} \left[Y_{n_1,m,{}^e_0}(\theta,\phi)\, \vec{1}_r + \vec{Q}_{n_1,m,{}^e_0}(\theta,\phi) \right] \right|
\tag{4.5}
$$

giving $A^{(1)}_{n_1}\ r^{n_1-1}$ as a bound on the "error" or nonuniformity in the field. Then similar to [5] we can define an effective distance D_{n_1-1} via

$$
\begin{aligned}
U(\vec{r}) &= H_0^2 \left\{ 1 + \left[\frac{r}{D_{n_1-1}} \right]^{n_1-1} + \cdots \right\} \\
D_{n_1-1} &= \left[A^{(1)}_{n_1} \right]^{-\frac{1}{n_1-1}}
\end{aligned}
\tag{4.6}
$$

If $n_1 = n_2$, then we need to add the two sets of coefficients before finding the least upper bound over θ,ϕ as in (4.5). Of course, if we can make our two coil designs such that

$$
a^{(1)}_{n_1,m,{}^e_0} + a^{(2)}_{n_2,m,{}^e_0} = 0
\tag{4.7}
$$

then the system is uniform to order greater than $n_1 - 2$ with first nonuniformity term of order greater than $n_1 - 1$.

Applying these concepts to the vector magnetic field one can form for $\vec{H}_1(\vec{r})$

$$A_{n_1}^{(1)} = \sup_{\theta,\phi} \left| \frac{1}{H_0} \cdot \sum_{m=0}^{n_1} \sum_{e,0} a_{n_1,m,{}_0^e}^{(1)} \left[Y_{n_1,m,{}_0^e} \vec{1}_r + \vec{Q}_{n_1,m,{}_0^e}(\theta,\phi) \right] \right| \tag{4.8}$$

where now we take the least upper bound of the magnetitude of a vector. In this case we have

$$\left| \frac{\vec{H}_1(\vec{r}) - \vec{H}_1(0)}{H_0} \right| \le A_{n_1}^{(1)} \, r^{n_1-1} + \cdots \tag{4.9}$$

with the first term as an approximate bound for r small compared to the distance of the coil conductors nearest $\vec{r} = \vec{0}$. The definition of effective distance in (4.6) can then also be applied to the magnetic field as well.

5. Body-of-Revolution Coils

The special casec of interest of C_∞ symmetry leaves only $m = 0$ terms in the expansion. Together with the transverse $z = 0$ symmetry plane this restricts the potential and field in (3.5) to the form ($C_{\infty at}$ or $D_{\infty d}$ or $D_{\infty t}$ symmetry)

$$\begin{aligned}\Phi_h(\vec{r}) &= -H_0 z - \sum_{n=2}^{\infty,2} a_{n,0,e}\, r^n\, P_n(\cos(\theta)) \\ \vec{H}(\vec{r}) &= H_0\, \vec{1}_z + \sum_{n=2}^{\infty,2} a_{n,0,e}\, r^{n-1} \left\{ n\, P_n(\cos(\theta))\, \vec{1}_r + \left[\frac{\partial}{\partial\theta} P_n(\cos(\theta))\right] \vec{1}_\theta \right\}\end{aligned} \tag{5.1}$$

where the notation indicates that only even values of n are included in the sum. For this case, the current density, being divergenceless, has only a ϕ-independent ϕ component. This leads to the attainment of higher order uniformities by making successive, even-n coefficients zero. This leads to the well-known Helmholtz two-coil and Maxwell three-coil arrangements, and coils of yet higher order uniformity [1, 2, 10, 11].

This class of coils consists of circular coils on a sphere of radius a with each coil lying on a constant θ. For this purpose we expand the potentials and fields both interior and exterior to the sphere of radius a as

$$\begin{aligned}\Phi_n^{(in)}(\vec{r}) &= -a \sum_{n=1}^{\infty} a_n \left[\frac{r}{a}\right]^n P_n(\cos(\theta)) \qquad 0 \le r < a \\ \Phi_n^{(ex)}(\vec{r}) &= a \sum_{n=1}^{\infty} b_n \left[\frac{r}{a}\right]^{-n-1} P_n(\cos(\theta)) \qquad a < r \\ \vec{H}^{(in)}(\vec{r}) &= \sum_{n=1}^{\infty} a_n \left[\frac{r}{a}\right]^{n-1} \left\{ n\, P_n(\cos(\theta))\, \vec{1}_r + \left[\frac{\partial}{\partial\theta} P_n(\cos(\theta))\right] \vec{1}_\theta \right\} \\ \vec{H}^{(ex)}(\vec{r}) &= \sum_{n=1}^{\infty} b_n \left[\frac{r}{a}\right]^{-n-2} \left\{ [n+1]\, P_n(\cos(\theta))\, \vec{1}_r - \left[\frac{\partial}{\partial\theta} P_n(\cos(\theta))\right] \vec{1}_\theta \right\}\end{aligned} \tag{5.2}$$

This is the general form for $C_{\infty a}$ symmetry. The adjunction of the $z = 0$ symmetry plane giving $C_{\infty at}$ symmetry simply means that the coefficients for even n are all zero. Note from (3.1) that

$$\frac{\partial}{\partial\theta} P_n(\cos(\theta)) = P_n^{(1)}(\cos(\theta)) \tag{5.3}$$

Assume that the surface current density on the sphere for a single coil at $\theta = \theta_\ell$ with current I_ℓ has the form

$$\vec{J}_{s_\ell}(\vec{r}_s) = J_{s_{\phi_\ell}}(\theta)\,\vec{1}_\phi = I_\ell\,\delta(a(\theta-\theta_\ell))\,\vec{1}_\phi \qquad (5.4)$$
$$\vec{r}_s = \vec{r} \text{ on spherical surface, } |\vec{r}_s| = a$$

The boundary condition at $r = a$ is in general

$$\vec{1}_r \cdot \left[\vec{H}^{(ex)}(\vec{r}_s) - \vec{H}^{(in)}(\vec{r}_s)\right] = 0 \qquad (5.5)$$
$$\vec{1}_r \times \left[\vec{H}^{(ex)}(\vec{r}_s) - \vec{H}^{(in)}(\vec{r}_s)\right] = \vec{J}_s(\vec{r}_s)$$

Expand the surface current density in (5.4) in the form

$$J_{s_{\phi_\ell}}(\theta) = I_\ell\ \delta(a(\theta-\theta_\ell)) = \frac{I_\ell}{a}\sum_{n=1}^{\infty} c_n P_n^{(1)}(\cos(\theta)) \qquad (5.6)$$

Multiply by $P_n^{(1)}(\cos(\theta))\sin(\theta)$ and integrate over θ from 0 to π (using the orthogonality of the Legendre functions [12]) to find

$$c_n = \frac{2n+1}{2n(n+1)} P_n^{(1)}(\cos(\theta_\ell))\sin(\theta_\ell) \qquad (5.7)$$

Enforcing the boundary conditions at $r = a$ gives

$$n\,a_n = [n+1]\,b_n \qquad \text{(continuity of } H_r)$$
$$a_n + b_n = \frac{I_\ell}{a}\,c_n \qquad \text{(discontinuity of } H_\theta) \qquad (5.8)$$
$$a_n = -\frac{n+1}{2n+1}\,\frac{I_\ell}{a}\,c_n = -\frac{I_\ell}{2na}\,P_n^{(1)}(\cos(\theta_\ell))(\theta_\ell)$$

For a set of L such coils we merely sum over ℓ to give [15]

$$\vec{H}(\vec{r}) = \frac{1}{2a}\sum_{n=1}^{\infty}\left[\frac{r}{a}\right]^{n-1}\left\{\left\{P_n(\cos(\theta))\,\vec{1}_r + \frac{1}{n}P_n^{(1)}(\cos(\theta))\,\vec{1}_\theta\right\}\right.$$

$$\sum_{\ell=1}^{L} -P_n^{(1)}(\cos(\theta_\ell))\sin(\theta_\ell)\,\vec{1}_{\theta_\ell}\Bigg\} \tag{5.9}$$

where now we drop the superscript (in) with the understanding that we are dealing with fields near the origin. Similar expansions can be found for the exterior field and for the potential. At $\vec{r} = \vec{0}$ the magnetic field is

$$\vec{H}(\vec{0}) = H_0\,\vec{1}_z \quad , \quad H_0 = \frac{1}{2a}\sum_{\ell=1}^{L}\sin^2(\theta_\ell)\,I_\ell \tag{5.10}$$

Now for various choices of I_ℓ and θ_ℓ we have the traditional coils on a sphere for uniform magnetic fields. If a is replaced by a_ℓ with different spherical radii, the above formulae (with a_ℓ moved into the summation over ℓ) give additional possibilities.

5.1 Single circular loop

The first case is the single circular loop of radius a with

$$I_1 = I \quad , \quad \theta_1 = \frac{\pi}{2} \tag{5.11}$$

for which we have

$$H_0 = \frac{I}{2a} \tag{5.12}$$

The first nonuniformity term comes from $n = 3$ giving for fields near the origin

$$\begin{aligned}\vec{H}(\vec{r}) = H_0\Bigg\{\vec{1}_z - \frac{3}{2}\Bigg\{\left[\frac{5}{2}\cos^3(\theta) - \frac{3}{2}\cos(\theta)\right]\vec{1}_r \\ + \left[-\frac{5}{2}\cos^2(\theta) + \frac{1}{2}\right]\sin(\theta)\,\vec{1}_\theta\Bigg\}\left[\frac{r}{a}\right]^2 \\ + \cdots\Bigg\}\end{aligned} \tag{5.13}$$

Thus the field is first-order uniform.

This special case of a circular loop also has a solution for the fields in terms of elliptic integrals [4, 15] as

$$\begin{aligned} H_z(\Psi,z) &= \frac{I}{2\pi}\left[[a+\Psi]^2+z^2\right]^{-\frac{1}{2}}\left[K(m_0)+\frac{a^2-\Psi^2-z^2}{[a-\Psi]^2+z^2}\,E(m_0)\right] \\ H_\Psi(\Psi,z) &= \frac{I}{2\pi}\,\frac{z}{\Psi}\left[[a+\Psi]^2+z^2\right]^{-\frac{1}{2}}\left[-K(m_0)+\frac{a^2+\Psi^2+z^2}{[a-\Psi]^2+z^2}\,E(m_0)\right] \\ m_0 &= 4\,a\,\Psi\left[[a+\Psi]^2+z^2\right]^{-1} \end{aligned} \tag{5.14}$$

where m_0 is called the parameter of the elliptic integrals [12]. On the z axis this reduces to

$$\begin{aligned} \vec{H}(z\,\vec{1}_z) &= H_z(0,z)\,\vec{1}_z \\ H_z(0,z) &= \frac{I}{2a}\left[1+\left[\frac{z}{a}\right]^2\right]^{-\frac{3}{2}} \end{aligned} \tag{5.15}$$

This solution can be applied to multiple circular loops by summing the fields of the individual loops, allowing for the various loop currents, radii, and shifted coordinates. For numerical computation of the fields for general $\vec{r}/a$ (not necessarily small) these closed-form expressions should be more efficient.

5.2 Two circular loops

Considering two coils with a $z = 0$ symmetry plane gives

$$\begin{aligned} I_1 = I_2 = I \quad &, \quad \theta_1 = \pi - \theta_2 \\ \cos(\theta_2) = -\cos(\theta_1) \quad &, \quad \sin(\theta_2) = \sin(\theta_1) \end{aligned} \tag{5.16}$$

Then set the $n = 3$ term in (5.9) to zero as

$$\begin{aligned} 0 &= \sum_{\ell=1}^{2} -P_3^{(1)}(\cos(\theta_\ell))\sin(\theta_\ell) \\ 0 &= P_3^{(1)}(\cos(\theta_1)) = -\frac{3}{2}\sin(\theta_1)\left[5\cos^2(\theta_1)-1\right] \\ \cos(\theta_1) &= \left[\frac{1}{5}\right]^{\frac{1}{2}}, \quad \sin(\theta_1) = \left[\frac{4}{5}\right]^{\frac{1}{2}}, \quad \tan(\theta_1) = 2, \quad \theta_1 \simeq 63.43^\circ \end{aligned} \tag{5.17}$$

In terms of the cylindrical coordinates the two loops are located at

$$\Psi_1 = \Psi_2 = \left[\frac{4}{5}\right]^{\frac{1}{2}} a \ , \quad z_1 = z_2 = \left[\frac{1}{5}\right]^{\frac{1}{2}} a \ , \quad \frac{\Psi_1}{z_1} = -\frac{\Psi_2}{z_2} = 2 \tag{5.18}$$

This gives the usual Helmholtz-coil configuration with the spacing equal to the radius.

With the above constraint we then have

$$H_0 = \frac{I}{2a} \sum_{\ell=1}^{2} -P_1^{(1)}\left(\cos(\theta_\ell)\right) \sin(\theta_\ell) = \frac{I}{a} \sin^2(\theta_1) = \frac{4}{5} \frac{I}{a} \tag{5.19}$$

The first nonuniformity term comes from $n = 5$ giving

$$\begin{aligned} \vec{H}\,(\vec{r}) = \Bigg\{ \vec{1}_z &- \frac{9}{5}\Bigg\{\left[\frac{63}{8}\cos^5(\theta) - \frac{35}{4}\cos^3(\theta) + \frac{15}{8}\cos(\theta)\right]\vec{1}_r \\ &- \frac{3}{8}\left[21\cos^4(\theta) - 14\cos^2(\theta) + 1\right]\sin(\theta)\,\vec{1}_\theta\Bigg\}\left[\frac{r}{a}\right]^4 + \cdots\Bigg\} \end{aligned} \tag{5.20}$$

Thus, the field is third-order uniform.

5.3 Combined coils for three-axis magnetic fields with third-order uniformity

At this juncture, let us consider how we might combine the Helmholtz coil with other coils to give a set of three coils, each of which can be driven to give separately three orthogonal fields near $\vec{r} = \vec{0}$. Appropriate care needs to be given that the presence of each coil does not significantly perturb the fields of the other two, such as by presenting a shorted loop orthogonal to the magnetic field of one of the other loops. In addition, it is desirable that each coil not induce an open circuit voltage in another coil, a result which can be achieved by symmetry.

One way to design such a three-axis system is with three Helmholtz coils with axes aligned with the Cartesian (x, y, z) axes. If the three loop pairs are each as discussed previously, with equal dimensions, except rotated to align the fields with the three orthogonal axes, then one needs to take care that, on the eight positions on the sphere of radius a where the loops would intersect, each conductor is slightly displaced from the others so that electrical contact is not made among these conductors. Considering just the three coils (conductors without currents) the geometry has octahedral (O) symmetry, adjoined by symmetry planes. Adding the conductors to drive the currents in the three coils, these can be positioned outisde the loops using appropriate symmetry planes so as to avoid disturbing the fields from all three coils.

One can also make a three orthogonal single-loop system using the results of Section 5.1 (with first-order field uniformity). This can also have octahedral symmetry, such as in the case of the octahedral three-axis loop (O3L) sensor [14]. Instead of bringing the connecting conductors for the drive currents to the center ($\vec{r} = \vec{0}$), however, these additional conductors will need to be outside as discussed above.

Another interesting way to make a three-axis system with third-order uniformity is illustrated in fig. 5.1. Here, the Helmholtz coil with conductors on $(\Psi, z) = (\Psi_1, \pm z_1)$ produces a z-directed magnetic field as discussed above. Combined with this are two two-wire loops as discussed in [5]. These have been analyzed as two-dimensional structures to give third-order uniformity by placing the wires parallel to the z axis on planes specified by planes of constant ϕ equal to certain integer multiples of $\pi/6$ as indicated. These wires are all located on a circular cylinder given by $\Psi = \Psi_0$. The currents on each of the coils are labelled with superscripts according to the orientation of the magnetic field at the origin produced by each of these coils. Positive orientation for the currents on the two-dimensional coils is the $+z$ direction.

The symmetry of this configuration of conductors is a 4-fold symmetry axis (z axis) with axial symmetry planes (C_{4a}) together with a transverse ($z = 0$) symmetry plane. This gives dihedral D_{4t} symmetry. Note that the coils for the x and y coils need to be closed at top and bottom (say $z = \pm z_0$). This can be accomplished while preserving the above symmetry, if desired. An interesting form of this, as indicated in fig. 5.1A, has these conductors remain on the same planes of constant ϕ as the wires to which they connect. Then the magnetic field from the $D_{\infty t}$ z-coil which has only Ψ and z components passes between these conductors with negligible distortion. Note, however, that as these conductors approach the z axis, they should connect across to the conductor closing their respective loop, and to no other conductors there (at least at either top or bottom) so that shorted turns are not formed in one coil that exclude magnetic field from another coil. There are also other configurations for closing the x- and y-loops at top and bottom while retaining the symmetry and avoiding coupling among the three coils. Adding conductors outside the coils to drive the currents in the three coils, these can be located on appropriate symmetry planes to avoid disturbing the various fields.

There is still the question of the relative sizes of the coils. One can choose Ψ_0 and Ψ_1 with a great degree of freedom. In the context of a walk-through metal detector, the vertical spacing (height) between the two z-loops is $2z_1 = \Psi_1$, and the horizontal spacing (width) between the vertical conductors for the x-loops is Ψ_0. One could then, for example, choose Ψ_1 / Ψ_0 as 2 or 3, like a typical door. Of course, there needs to be some space (clearance) allowed so that the target does not come too close to the loop conductors.

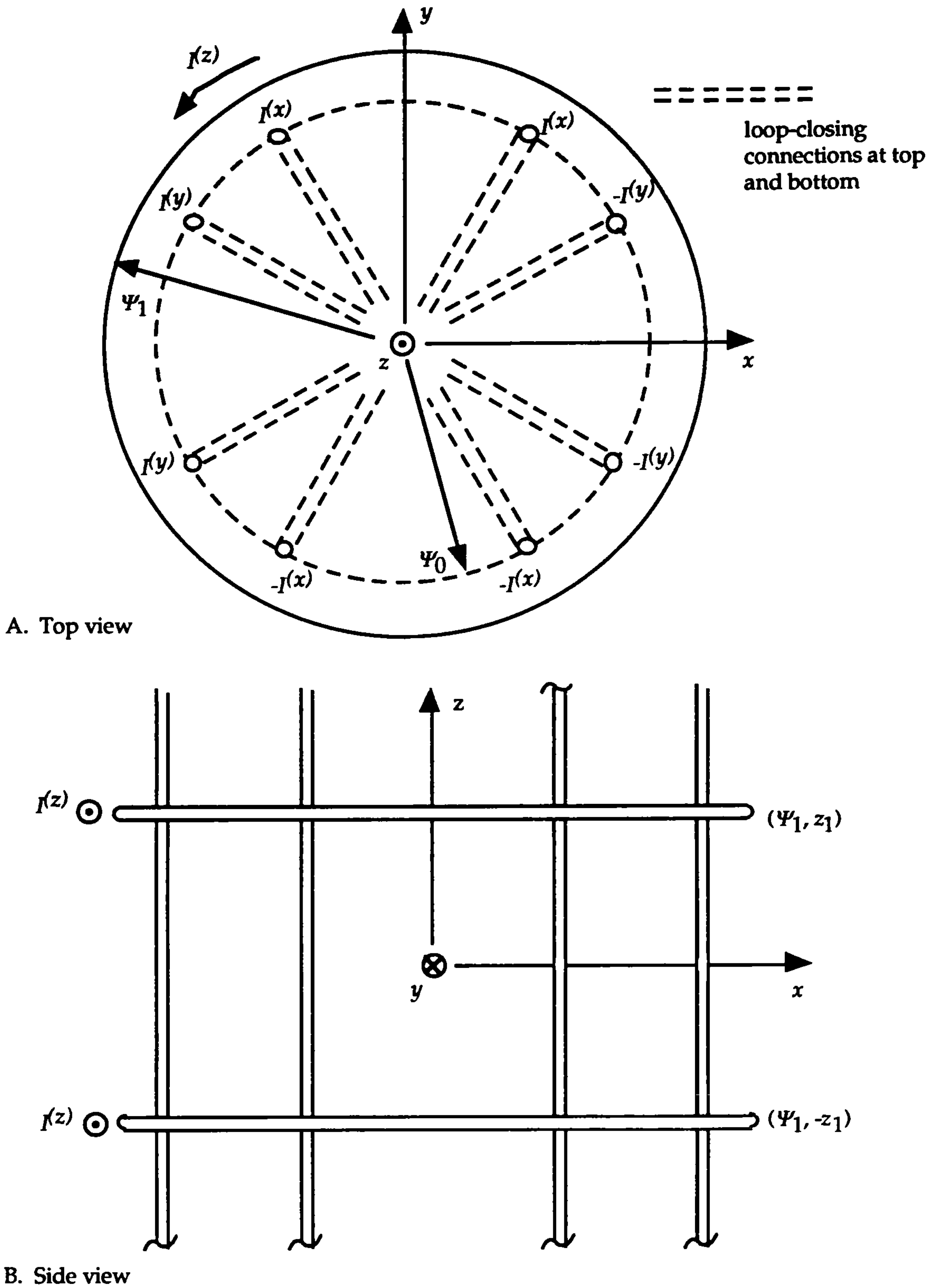


Fig. 5.1 Combined Coils for Three-Axis Magnetic Fields with Third-Order Uniformity

There are yet other design questions. How large should z_0 / Ψ_0 be so that end effects for the x- and y-coils are not significant (or allow some appropriate correction to the field)? The coils in fig. 5.1 are appropriate as transmitters, since in reception they are sensitive to externally incident low-frequency magnetic fields. These coils then need to be combined with a different set of 3-axis receiver coils.

6. Special Ring Coils

One of the design considerations for receiver coils is insensitivity to externally incident, approximately uniform, low-frequency magnetic fields (such as 60 Hz or 50 Hz) [5]. This requires (by reciprocity) that in transmission such coils have no net magnetic dipole moment. One way to achieve this is to place loops on a constant cylindrical radius Ψ, carrying equal but opposite currents so that the dipole moments cancel.

For this purpose we can rewrite (5.9) as

$$\vec{H}(\vec{r}) = \sum_{n=1}^{\infty} \left\{ \left[P_n(\cos(\theta)) \vec{1}_r + \frac{1}{n} P_n^{(1)}(\cos(\theta)) \vec{1}_\theta \right\} r^{n-1} \right. \\ \left. \sum_{n=1}^{\infty} -P_n^{(1)}(\cos(\theta_\ell)) \sin(\theta_\ell) \frac{I_\ell}{2a_\ell^n} \right\} \tag{6.1}$$

At $\vec{r} = \vec{0}$ the magnetic field is

$$\vec{H}(\vec{0}) = H_0 \, \vec{1}_z \quad , \quad H_0 = \sum_{\ell=1}^{L} \sin^2(\theta_\ell) \frac{I_\ell}{2a_\ell} \tag{6.2}$$

By various choices of the I_ℓ, a_ℓ, and θ_ℓ numerous coil designs can be explored.

6.1 Single ring coil

By a ring coil, let us mean something like that illustrated in fig. 6.1, consisting of two coaxial circular loops of the same cylindrical radius Ψ, carrying equal but opposite currents. This has zero magnetic dipole moment, but of course has a magnetic quadrupole moment. The two loop coordinates are described by

$$\Psi_\ell = a_\ell \sin(\theta_\ell) \quad , \quad z_\ell = a_\ell \cos(\theta_\ell) \quad , \quad \ell = 1, 2 \tag{6.3}$$

We will set

$$\Psi_1 = \Psi_2 = \Psi_0 \quad , \quad I = I_1 = -I_2 \tag{6.4}$$

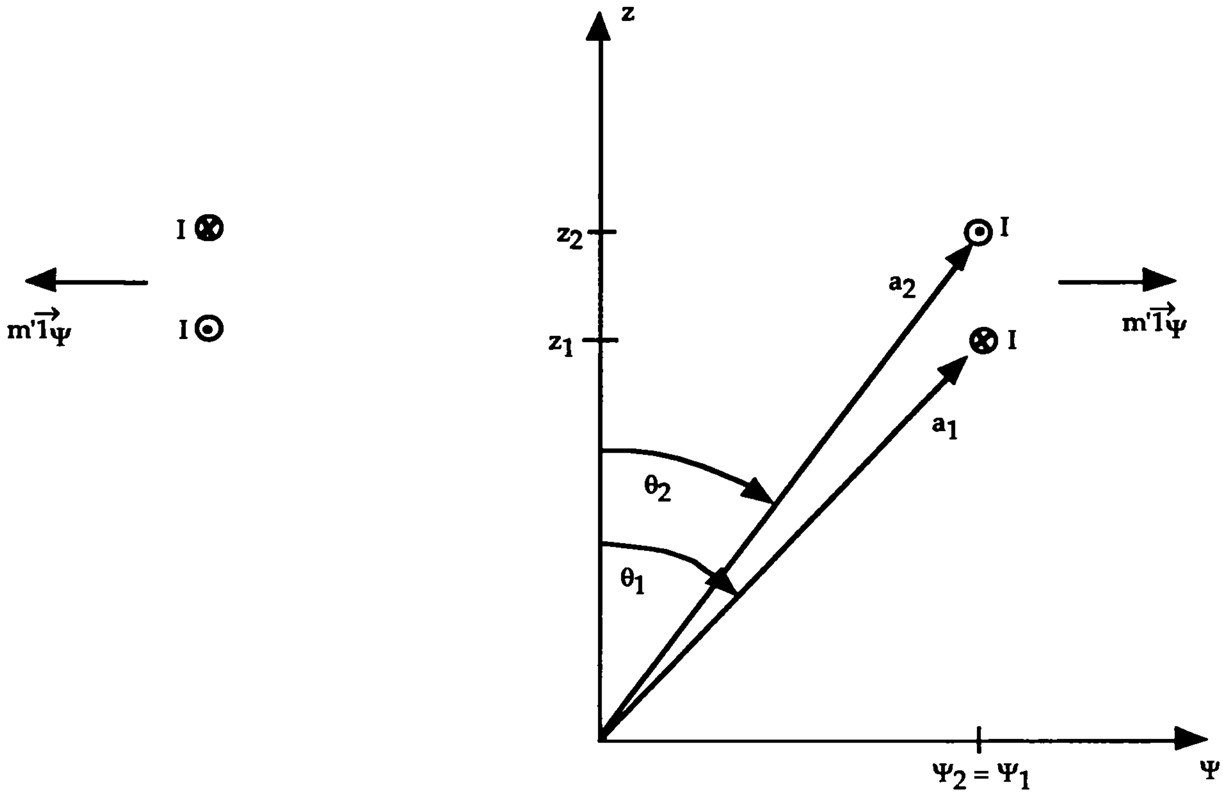


Fig. 6.1. Quadrupole Ring Coil

Then write

$$
\begin{aligned}
z_2 &= z_1 + \Delta z \;=\; z_0 + \Delta z \\
a_2 &= a_1 + \Delta a \;=\; a_0 + \Delta a \\
\theta_2 &= \theta_1 + \Delta\theta \;=\; \theta_0 + \Delta\theta
\end{aligned} \tag{6.5}
$$

As this indicates, we are concerned with two closely spaced circular loops of equal area. In taking the limit of close loops, let us maintain

$$
m' = I \Delta z \tag{6.6}
$$

constant as $\Delta z \to 0$. Here, m' takes the form of a line-magnetic-dipole moment. However, as a vector, it is pointing in the $\vec{1}_\Psi$ direction as illustrated in fig. 6.1. As a vector, when integrated around the ring, the net result is zero, so m' should be thought of as only characterizing an incremental length.

Now, from (6.1) we have the magnetic field from our ring coil as

$$
\begin{aligned}
\vec{H}^{(in)}(\vec{r}) = m' \sum_{n=1}^{\infty} &\left\{ \left\{ P_n(\cos(\theta)) \vec{1}_r + \frac{1}{n} P_n^{(1)}(\cos(\theta)) \vec{1}_\theta \right\} \left[\frac{r}{a_0} \right]^{n-1} \right. \\
&\left. \frac{1}{2a_0 \Delta z} \left\{ -P_n^{(1)}(\cos(\theta_0)) \sin(\theta_0) + P_n^{(1)}(\cos(\theta_0 + \Delta\theta)) \sin(\theta_0 + \Delta\theta) \right\} \left[\frac{a_0}{a_0 + \Delta a} \right] \right\} \\
H_0 = \frac{m'}{2a_0 \Delta z} &\left\{ \sin^2(\theta_0) - \sin^2(\theta_0 + \Delta\theta) \frac{a_0}{a_0 + \Delta a} \right\}
\end{aligned} \tag{6.7}
$$

As this form suggests, we can consider the limit as $\Delta z \to 0$ so as to obtain more convenient formulae.

Let us relate Δz and Δa to $\Delta\theta$ via the leading terms in a power (Taylor) series expansion for small $\Delta\theta$. For Δz we have

$$
\begin{aligned}
z_0 &= \Psi_0 \cot(\theta_0) \\
z_0 + \Delta z &= \Psi_0 \cot(\theta_0 + \Delta\theta) \\
&= \Psi_0 \left[\cot(\theta_0) - \csc^2(\theta_0) \Delta\theta + O\left((\Delta\theta)^2\right) \right] \\
\Delta z &= -\Psi_0 \csc^2(\theta_0) \Delta\theta + O\left((\Delta\theta)^2\right) \\
&= -a_0 \csc(\theta_0) \Delta\theta + O\left((\Delta\theta)^2\right)
\end{aligned} \tag{6.8}
$$

For Δa we have

$$
\begin{aligned}
a_0 &= \Psi_0 \csc(\theta_0) \\
a_0 + \Delta a &= \Psi_0 \csc(\theta_0 + \Delta\theta) \\
&= \Psi_0 \left[\csc(\theta_0) - \csc(\theta_0)\cot(\theta_0)\,\Delta\theta + O\left((\Delta\theta)^2\right)\right] \\
\Delta a &= -a_0 \cot(\theta_0)\,\Delta\theta \;+\; O\left((\Delta\theta)^2\right)
\end{aligned}
\tag{6.9}
$$

Then there are various functions to expand as

$$
\begin{aligned}
&\sin(\theta_0 + \Delta\theta) = \sin(\theta_0) + \cos(\theta_0)\Delta\theta + O\left((\Delta\theta)^2\right) \\
&\left[\frac{a_0}{a_0 + \Delta a}\right]^n = \left[1 - \cot(\theta_0)\Delta\theta + O\left((\Delta\theta)^2\right)\right]^{-n} \\
&= 1 + n\cot(\theta_0)\Delta\theta + O\left((\Delta\theta)^2\right) \\
&P_n^{(1)}(\cos(\theta_0 + \Delta\theta)) = P_n^{(1)}(\cos(\theta_0)) + \left[\frac{d}{d\theta_0} P_n^{(1)}(\cos(\theta_0))\right]\Delta\theta + O((\Delta\theta)^2)
\end{aligned}
\tag{6.10}
$$

Concerning the associated Legendre function we have [12, 13]

$$
\begin{aligned}
\frac{d}{d\theta} P_n^{(1)}(\cos(\theta)) &= -\sin(\theta)\frac{d\,P_n^{(1)}(\cos(\theta))}{d\cos(\theta)} \\
\frac{d\,P_n^{(1)}(\xi)}{d\,\xi} &= \xi\left[1-\xi^2\right]^{-1} P_n^{(1)}(\xi) \;+\; n\,[n+1]\left[1-\xi^2\right]^{-\frac{1}{2}} P_n(\xi) \\
\frac{d}{d\theta} P_n^{(1)}(\cos(\theta)) &= -\cot(\theta)\,P_n^{(1)}(\cos(\theta)) \;-\; n[n+1]\,P_n(\cos(\theta))
\end{aligned}
\tag{6.11}
$$

Collecting terms and taking the limit as $\Delta\theta \to 0$ we have

$$
\begin{aligned}
&\frac{1}{2a_0\Delta z}\left\{-P_n^{(1)}(\cos(\theta_0)) + P_n^{(1)}(\cos(\theta_0 + \Delta\theta))\sin(\theta_0 + \Delta\theta)\right\}\left[\frac{a_0}{a_0 + \Delta a}\right] \\
&= -\frac{\sin(\theta_0)}{2{a_0}^2}\,\frac{1}{\Delta\theta}\left\{\left[\frac{d}{d\theta_0} P_n^{(1)}(\cos(\theta_0))\right][\Delta\theta]\sin(\theta_0)\right. \\
&\quad \left. + P_n^{(1)}(\cos(\theta_0))\,\cos(\theta_0)\,\Delta\theta + P_n^{(1)}(\cos(\theta_0))\sin(\theta_0)\,n\cot(\theta_0)\Delta\theta\right\} \\
&= -\frac{\sin(\theta_0)}{2{a_0}^2}\left\{\sin(\theta_0)\frac{d}{d\theta_0} P_n^{(1)}(\cos(\theta_0)) + [n+1]\cos(\theta_0) P_n^{(1)}\,(\cos(\theta_0))\right\} \\
&= \frac{\sin(\theta_0)}{2{a_0}^2}\left\{\cos(\theta_0)\,P_n^{(1)}(\cos(\theta_0)) \;+\; n[n+1]\sin(\theta_0)\,P_n(\cos(\theta_0))\right. \\
&\qquad\qquad \left. -\,[n+1]\cos(\theta_0) P_n^{(1)}\cos(\theta_0)\right\}
\end{aligned}
\tag{6.12}
$$

$$= \frac{\sin(\theta_0)}{2a_0^2}\left\{n[n+1]\sin(\theta_0)\,P_n(\cos(\theta_0)) - n\cos(\theta_0)\,P_n^{(1)}(\cos(\theta_0))\right\}$$

So now we write for our ring coil at a_0 , θ_0

$$\begin{aligned}\overrightarrow{H}^{(in)}(\vec{r}) &= m' \sum_{n=1}^{\infty}\left\{\left\{P_n(\cos(\theta))\,\vec{1}_r + \frac{1}{n}\,P_n^{(1)}(\cos(\theta))\,\vec{1}_\theta\right\}\left[\frac{r}{a_0}\right]^{n-1}\right. \\ &\left.\frac{\sin(\theta_0)}{2a_0^2}\left\{n[n+1]\sin(\theta_0)\,P_n(\cos(\theta_0)) - n\cos(\theta_0)\,P_n^{(1)}(\cos(\theta_0))\right\}\right\} \\ H_0 &= \frac{3m'}{2a_0^2}\sin^2(\theta_0)\cos(\theta_0)\end{aligned} \tag{6.13}$$

Consider briefly the special case of $\theta_0 = \pi/2$ giving $H_0 = 0$, i.e., zero field at the origin. More generally, this case is an *antisymmetric* field, implying H_z is an odd function of z. As discussed in previous sections, it is desirable to have a *symmetric* field with respect to the $z = 0$ plane. This leads to the desirability of multiple ring coils at other values of θ_0.

6.2 Two ring coils

The result of (6.13) can be generalized to an arbitrary number L of ring coils as

$$\begin{aligned}\overrightarrow{H}^{(inc)}(\vec{r}) &= \sum_{n=1}^{\infty}\left\{\left\{P_n(\cos(\theta))\,\vec{1}_r + \frac{1}{n}\,P_n^{(1)}(\cos(\theta))\,\vec{1}_\theta\right\} r^{n-1}\right. \\ &\left.\sum_{\ell=1}^{L}\frac{m'_\ell \sin(\theta_\ell)}{2a_\ell^{n+1}}\left\{n[n+1]\sin(\theta_\ell)P_n(\cos(\theta_\ell)) - n\cos(\theta_\ell)\,P_n^{(1)}(\cos(\theta_\ell))\right\}\right\} \\ H_0 &= \sum_{\ell=1}^{L}\frac{3m'_\ell}{2a_\ell^2}\sin^2(\theta_\ell)\cos(\theta_\ell)\end{aligned} \tag{6.14}$$

where the ring coils can be at various a_ℓ , θ_ℓ with excitations m'_ℓ. As discussed previously, we would like to have a symmetric field distribution for field uniformity near the origin. This is accomplished by choosing L even and pairing rings as

$$\begin{aligned}\theta_\ell &= \pi - \theta_{L-\ell+1} \quad , \quad a_\ell = a_{L-\ell+1} \\ \Psi_\ell &= \Psi_{L-\ell+1} \quad , \quad z_\ell = -z_{L-\ell+1} \\ m'_\ell &= -m'_{L-\ell+1}\end{aligned} \tag{6.15}$$

This is sufficient to assure that only odd n appear in the sum.

A special case of interest for this symmetric field distribution is two ring coils as in fig. 6.2 with

$$\begin{aligned} &L = 2 \ , \quad \theta_2 = \pi - \theta_1 \ , \quad a_2 = a_1 = a \\ &\Psi_2 = \Psi_1 \ , \quad z_2 = -z_1 \ , \quad m'_1 = -m'_2 = m' \end{aligned} \tag{6.16}$$

Then (6.13) becomes

$$\begin{aligned} \vec{H}^{(in)}(\vec{r}) &= \frac{m'}{a^2} \sum_{n=1}^{\infty,2} \left\{ \left\{ P_n(\cos(\theta)) \vec{1}_r + \frac{1}{n} P_n^{(1)}(\cos(\theta)) \right\} \left[\frac{r}{a} \right]^{n-1} \right. \\ &\qquad \left. \sin(\theta_1) \left\{ n[n+1] \sin(\theta_1) P_n(\cos(\theta_1)) - n \cos(\theta_1) P_n^{(1)}(\cos(\theta_1)) \right\} \right\} \\ H_0 &= \frac{3m'}{a^2} \sin^2(\theta_1) \cos(\theta_1) \end{aligned} \tag{6.17}$$

Note in fig. 6.2 how the reversal of the currents in the second ring coil makes the currents (and hence the magnetic field) symmetric with respect to the z = 0 plane.

Consider the dependence of the $n = 3$ coefficient on θ_1 as

$$\begin{aligned} &\sin(\theta_1) \left\{ n[n+1] \sin(\theta_1) P_n(\cos(\theta_1)) - n \cos(\theta_1) P_n^{(1)}(\cos(\theta_1)) \right\} \\ &= 3 \sin(\theta_1) \left\{ 4 \sin(\theta_1) \left[\frac{5}{2} \cos^3(\theta_1) - \frac{3}{2} \cos(\theta_1) \right] + \frac{3}{2} \cos(\theta_1) \sin(\theta_1) \left[5 \cos^2(\theta_1) - 1 \right] \right\} \\ &= \frac{15}{2} \sin^2(\theta_1) \cos(\theta_1) \left\{ 7 \cos^2(\theta_1) - 3 \right\} \end{aligned} \tag{6.18}$$

This coefficient is zero (for $\theta_1 \neq 0 \, , \, \pi/2$) at

$$\cos(\theta_1) = \left[\frac{3}{7}\right]^{\frac{1}{2}} , \ \sin(\theta_1) = \left[\frac{4}{7}\right]^{\frac{1}{2}} , \ \tan(\theta_1) = \frac{2}{\sqrt{3}} , \ \theta_1 \approx 49.89^\circ \tag{6.19}$$

Note that this result differs from that for the Helmholtz coil in (5.17) and (5.18).

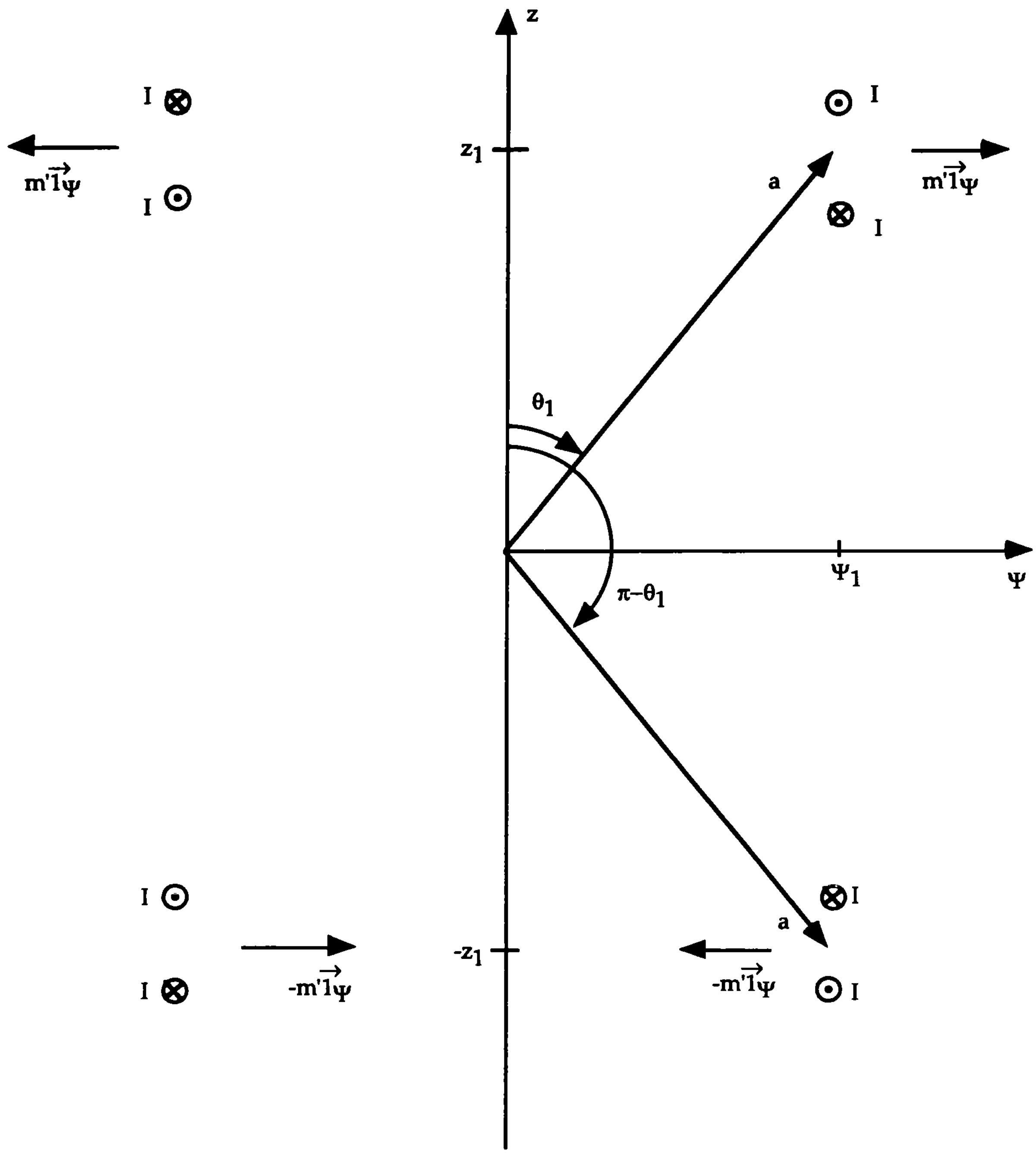


Fig. 6.2. Two Ring Coils for Symmetric Field Distribution.

With the above constraint we then have

$$H_0 = \frac{12}{7}\left[\frac{3}{7}\right]^{\frac{1}{2}} \frac{m'}{a^2} \tag{6.20}$$

The first nonuniformity term comes from $n = 5$ giving

$$\begin{aligned}\vec{H}(\vec{r}) = H_0 \Bigg\{ \vec{1}_z - \frac{55}{7}\Bigg\{ &\left[\frac{63}{8}\cos^5(\theta) - \frac{35}{4}\cos^3(\theta) + \frac{15}{8}\cos(\theta)\right]\vec{1}_r \\ &- \frac{3}{8}\left[21\cos^4(\theta) - 14\cos^2(\theta) + 1\right]\vec{1}_\theta \Bigg\}\left[\frac{r}{a}\right]^4 \\ &+ \cdots \Bigg\}\end{aligned} \tag{6.21}$$

Comparing this result to (5.20) for the Helmholtz coil we see that the fourth derivative in the present example is somewhat larger, but the result is still third-order uniform.

This two-ring coil has the advantage in reception of being insensitive to uniform externally incident magnetic fields. If a Helmholtz coil is used as the transmitter, there will be some coupling to the two-ring coil because the magnetic field of the Helmholtz coil is not perfectly uniform. One can adjust the relative sizes of the two coils, keeping them coaxial with common origin. By making the two-ring coil smaller than (contained within) the Helmholtz coil one can try to minimize the mutual coupling. Using the results of (5.14). For a single loop, one can calculate the magnetic field throughout space for the Helmholtz coil. (See plots in [1, 2, 10, 11].) Ideally, the Helmholtz field should be perpendicular to $\vec{1}_\Psi$ at the ring coils (fig. 6.2). One can, of course, make each ring coil so that m' is moved in a direction slightly off from $\vec{1}_\Psi$ (retaining antisymmetric field) so as to be exactly perpendicular to the Helmholtz field. The resulting small sensitivity to externally incident fields can be cancelled by other means. Note that the formulae for the ring coil are only an approximation, since in practice Δz is not zero. So one can consider the placement of each of the coils in the ring for best results.

7. Arrays of Small Numbers of Elementary Magnetic Dipoles

As discussed in Section 3, one need not have complete rotation symmetry ($C_{\infty a}$ as discussed in Sections 5 and 6) to still have various degrees of field uniformity. Table 3.2 shows that C_{Nt} symmetry with $N = 1, 2, 3$ can give interesting field-uniformity conditions. Besides the R_z part of the symmetry (giving a symmetric magnetic field with respect to the $z = 0$ plane), there is the N-fold rotation symmetry (about the z axis) C_N. Let us now consider combinations of magnetic dipoles, appropriately positioned and oriented to achieve such field uniformities.

Consider an elementary magnetic dipole $\vec{m}_0$ at some position $\vec{r}_0$ as indicated in fig. 7.1. Then we have

$$
\begin{aligned}
\vec{H}_0(\vec{r}) &= \frac{1}{4\pi R^3}\left[3\,\vec{1}_R\,\vec{1}_R - \overleftrightarrow{1}\right]\cdot\vec{m}_0 \\
\vec{r}_0 &= \Psi_0\,\vec{1}_\Psi + z_0\,\vec{1}_z \\
\vec{m}_0 &= m_0\,\vec{1}_0 \\
\vec{R} &= \vec{r} - \vec{r}_0 \quad , \quad R = |\vec{R}| \quad , r_0 = |\vec{r}_0| \\
\vec{1}_R &= \frac{\vec{r} - \vec{r}_0}{R}
\end{aligned}
\tag{7.1}
$$

Considering the orientation of $\vec{m}_0$, if it is oriented in the ϕ direction then the magnetic field at the origin has no z component, as can be seen from (7.1) or from symmetry considerations. So let us restrict $\vec{m}_0$ to have only Ψ and z components. The vector then lies in (is parallel to) a plane of constant ϕ and its orientation can be described by an angle ϕ_m with respect to the z axis as indicated in fig. 7.1. This constant-ϕ plane (including its extension through the axis by addition of π to ϕ) is a symmetry plane about which the magnetic field is antisymmetric.

The z component of the magnetic field on the z axis (i.e., $H_z(z\,\vec{1}_z)$) is independent of which constant-ϕ plane one chooses for placing $\vec{m}_0$. Then one can superimpose the fields from various such magnetic dipoles on various constant-ϕ planes and obtain the same $H_z(z\,\vec{1}_z)$ as if these magnetic dipoles were all on the same constant-ϕ plane. Later, the choices of the constant-ϕ planes will be used to obtain the desired C_N symmetry. For the moment let us consider the R_z part of the symmetry.

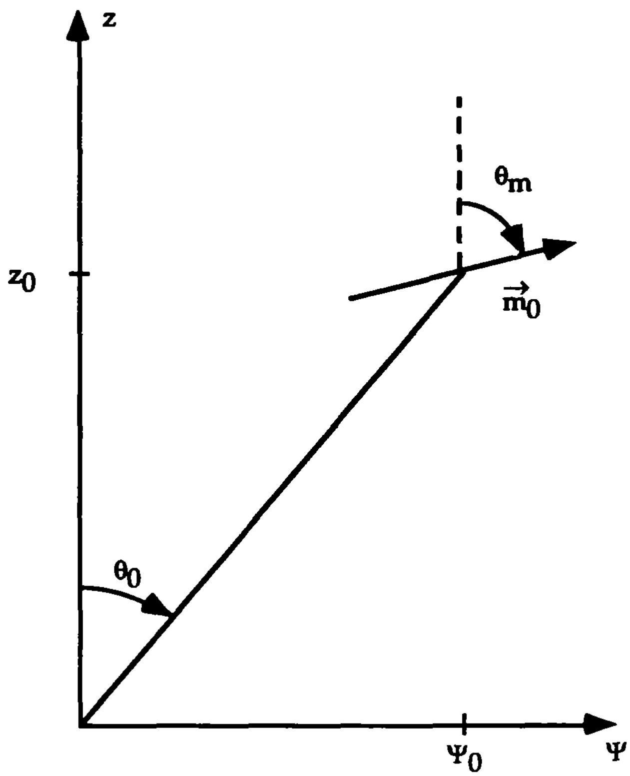


Fig. 7.1 Elementary Magnetic Dipole on Plane of Constant ϕ

7.1 Two magnetic dipoles with R_z symmetry

As illustrated in fig. 7.2, let there be a second magnetic dipole at $\overleftrightarrow{R}_z \cdot \vec{r}_0$ (or $(\Psi, z) = (\Psi_0 - z_0)$) of moment $-\overleftrightarrow{R}_z \cdot \vec{m}_0$ to give a symmetric magnetic field with respect to the $z = 0$ plane. By replicating this special pair on planes of constant ϕ on successive rotations by $2\pi / N$, the desired C_{Nt}, symmetry as in Table 3.2, can be achieved. So let us consider the properties of such a pair for the magnetic field near the origin.

For a given orientation angle θ_m for the dipoles, we can regard the resulting field as the superposition of the fields for two cases given by $\theta_m = 0, \pi/2$. Each of these needs to be weighted by $\cos(\theta_m)$ and $\sin(\theta_m)$ to give the result for an arbitrary θ_m, since we have

$$
\begin{aligned}
\vec{m}_0 &= m_0 \sin(\theta_m)\, \vec{1}_\Psi + m_0 \cos(\theta_m)\, \vec{1}_z \\
-\overleftrightarrow{R}_z \cdot \vec{m}_0 &= -m_0 \sin(\theta_m)\, \vec{1}_\Psi + m_0 \cos(\theta_m)\, \vec{1}_z
\end{aligned}
\tag{7.2}
$$

So let us consider each of these cases separately.

7.2 $\theta_m = \pi/2$, $\vec{m}_0 = m_0 \vec{1}_\Psi$

Consider the magnetic field from the first dipole at $\vec{r}_0$. Along the z axis the z component is

$$
\begin{aligned}
H_{0_z}(z \vec{1}_z) &= \frac{m_0}{4\pi R^3} 3\left[\vec{1}_R \cdot \vec{1}_z\right]\left[\vec{1}_R \cdot \vec{1}_\Psi\right] \\
&= -\frac{m_0 \Psi_0}{4\pi R^5}\left[z - z_0\right] \\
&= -\frac{m_0 \Psi_0}{4\pi}\left[z - z_0\right]\left[\left[z - z_0\right]^2 + \Psi_0^2\right]^{-\frac{5}{2}}
\end{aligned}
\tag{7.3}
$$

The first two z derivatives along the z axis are

$$
\begin{aligned}
\frac{\partial}{\partial z} H_{0_z}(z \vec{1}_z) &= -\frac{m_0 \Psi_0}{4\pi}\left\{\left[\left[z - z_0\right]^2 + \Psi_0^2\right]^{-\frac{5}{2}} - 5\left[z - z_0\right]^2\left[\left[z - z_0\right]^2 + \Psi_0^2\right]^{-\frac{7}{2}}\right\} \\
\frac{\partial}{\partial z^2} H_{0_z}(z \vec{1}_z) &= -\frac{m_0 \Psi_0}{4\pi}\left\{-15\left[z - z_0\right]\left[\left[z - z_0\right]^2 + \Psi_0^2\right]^{-\frac{7}{2}} \right. \\
&\qquad \left. - 35\left[z - z_0\right]^3\left[\left[z - z_0\right]^2 + \Psi_0^2\right]^{-\frac{9}{2}}\right\}
\end{aligned}
\tag{7.4}
$$

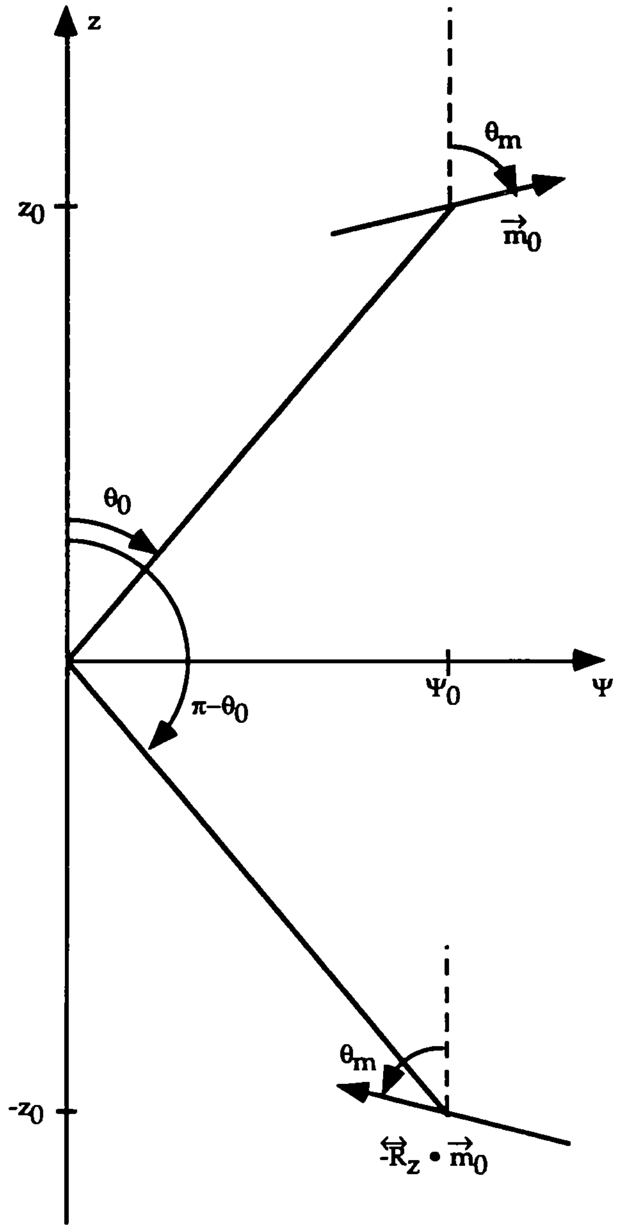


Fig. 7.2 Two Elementary Magnetic Dipoles on Plane of Constant ϕ with R_z Symmetry

The total magnetic field includes a cimilar term from the second dipole at $\vec{R}_z \cdot \vec{r}_0$. At z = 0, the R_z smmetry makes there only be a z component of the field with odd z derivatives zero. Including a factor of 2 for the two sources gives at the origin

$$\vec{H}(\vec{0}) = \frac{m_0}{2\pi} \Psi_0 z_0 r_0^{-5} \vec{1}_z = \frac{m_0}{2\pi} r_0^{-3} \cos(\theta_0) \sin(\theta_0) \vec{1}_z$$
$$\left. \frac{\partial^2}{\partial z^2} H_z(\vec{r}) \right|_{\vec{r}=\vec{0}} = \frac{5m_0}{2\pi} \Psi_0 z_0 r_0^{-7} \left[-3 r_0^2 + 7 z_0^2\right] \tag{7.5}$$

The second derivative can be set to zero by choosing

$$\cos(\theta_0) = \frac{z_0}{r_0} = \left[\frac{3}{7}\right]^{\frac{1}{2}} \quad , \quad \sin(\theta_0) = \frac{\Psi_0}{r_0} = \left[\frac{4}{7}\right]^{\frac{1}{2}}$$
$$\tan(\theta_0) = \frac{\Psi_0}{z_0} = \frac{2}{\sqrt{3}} \quad , \quad \theta_0 \simeq 49.89^\circ \tag{7.6}$$

in exact agreement with (6.19) for the ring coils in the previous section. This should not be surprising since the two dipoles form a quadrupole (as do the ring coils), and rotating the two dipoles around the z axis (by varying ϕ describing the dipole locations) and adding (integrating) the sources and fields over 2π gives the ring-coil results. The present case of two dipoles giving a quadrupole also makes this configuration insensitive to an externally incident uniform magnetic field, a good property for receiving coils.

For this special case, the magnetic field at the origin is

$$\vec{H}(\vec{0}) = \frac{\sqrt{3}\, m_0}{7\pi} r_0^{-3} \tag{7.7}$$

and the first three z derivatives of the z component are zero there. This is a case in which the $a_{3,0,e}$ coefficient is set to zero as in Table 3.2. While the present results are based on the z derivatives of the z component on the z axis, this is extended to all derivatives at the origin via Table 3.2. For example, third-order uniformity is achieved by (7.6) together with four pairs of such magnetic dipoles placed on planes of $\phi = 0, \pi/2, \pi, 3\pi/2$ which are just the $x = 0$ and $y = 0$ planes to adjoin the four-fold rotation symmetry C_4 to the reflection symmetry R_z for each pair.

7.3 $\theta_m = 0$, $\vec{m}_0 = m_0 \vec{1}_z$

For the complementary orientation of $\vec{m}_0$ parallel to $\vec{1}_z$, the field from the first dipole has a z component along the z axis as

$$
\begin{aligned}
H_{0_z}(z\,\vec{1}_z) &= \frac{m_0}{4\pi R^3}\left[3\left[\vec{1}_R\cdot\vec{1}_z\right]^2 - 1\right] \\
&= \frac{m_0}{4\pi}\left\{3[z-z_0]^2\left[[z-z_0]^2+\Psi_0^2\right]^{-\frac{5}{2}} - \left[[z-z_0]^2+\Psi_0^2\right]^{-\frac{3}{2}}\right\}
\end{aligned}
\tag{7.8}
$$

The first two derivatives along the z axis are

$$
\begin{aligned}
\frac{\partial}{\partial z}H_{0_z}(z\,\vec{1}_z) &= \frac{m_0}{4\pi}\left\{9[z-z_0]\left[[z-z_0]^2+\Psi_0^2\right]^{-\frac{5}{2}}\right. \\
&\qquad \left. -15[z-z_0]^2\left[[z-z_0]^2+\Psi_0^2\right]^{-\frac{7}{2}}\right\} \\
\frac{\partial^2}{\partial z^2}H_{0_z}(z\,\vec{1}_z) &= \frac{m_0}{4\pi}\left\{9\left[[z-z_0]^2+\Psi_0^2\right]^{-\frac{5}{2}}\right. \\
&\qquad \left. -90[z-z_0]^2\left[[z-z_0]^2+\Psi_0^2\right]^{-\frac{7}{2}} + 105[z-z_0]^4\left[[z-z_0]^2+\Psi_0^2\right]^{-\frac{9}{2}}\right\}
\end{aligned}
\tag{7.9}
$$

As in the previous case, inclusion of the term from the second dipole makes $z = 0$ a symmetry plane with only a z component of the field there with odd z derivatives zero. Including a factor of 2 for the two sources gives at the origin.

$$
\begin{aligned}
\vec{H}(\vec{0}) &= \frac{m_0}{2\pi} r_0^{-5}\left[3\,z_0^2 - r_0^2\right]\vec{1}_z = \frac{m_0}{2\pi} r_0^{-3}\left[3\cos^2(\theta_0) - 1\right]\vec{1}_z \\
\frac{\partial^2}{\partial z^2}H_z(\vec{r})\Big|_{\vec{r}=\vec{0}} &= \frac{3\,m_0}{2\pi} r_0^{-9}\left[3\,r_0^4 - 30\,r_0^2\,z_0^2 + 35\,z_0^4\right]
\end{aligned}
\tag{7.10}
$$

The second derivative can be set to zero by choosing

$$\cos(\theta_0) = \frac{z_0}{r_0} = \left[\frac{3}{7} \pm \frac{2}{7}\left[\frac{6}{5}\right]^{\frac{1}{2}}\right]^{\frac{1}{2}} \approx .861,\ .340$$

$$\sin(\theta_0) = \frac{\Psi_0}{r_0} = \left[\frac{4}{7} \mp \frac{2}{7}\left[\frac{6}{5}\right]^{\frac{1}{2}}\right]^{\frac{1}{2}} \approx .508,\ .940 \qquad (7.11)$$

$$\theta_0 \approx 30.56^\circ\ ,\ 70.12^\circ$$

This gives two choices for dipole location for our consideration. In both cases, the dipole pair has a net dipole moment, making it sensitive to an externally incident uniform magnetic field. This type of dipole pair, including its extension to additional constant-ϕ planes, is then more suitable for transmitting coils.

For this special case, the magnetic field at the origin is

$$\vec{H}(0) = \frac{m_0}{2\pi} r_0^{-3} \left[\frac{2}{7} \pm \frac{6}{7}\left[\frac{6}{5}\right]^{\frac{1}{2}}\right] \approx \frac{m_0}{2\pi} r_0^{-3} [1.225,\ -.653] \qquad (7.12)$$

As before, adjunction of C_4 symmetry makes all first three derivatives at the origin zero as in Table 3.2. Note that the two cases of θ_0 in (7.11) give quite different results in (7.12), including opposite signs of the fields at the origin.

7.4 General θ_m

Combining the results of Sections 7.2 and 7.3 using (7.2) for general θ_m between 0 and $\pi/2$ we have the magnetic field at the origin as

$$\vec{H}(\vec{0}) = \frac{m_0}{2\pi} r_0^{-3} \Big[\cos(\theta_0)\ \sin(\theta_0)\ \sin(\theta_m) + \left[3\cos^2(\theta_0) - 1\right] \cos(\theta_0)\Big] \vec{1}_z \qquad (7.13)$$

The second z-derivative of the z component at the origin is

$$\frac{\partial^2}{\partial z^2} H_z(\vec{r})\Big|_{\vec{r}=\vec{0}} = \frac{m_0}{2\pi} r_0^{-9}\Big[\left[-15\, r_0^4 + 35\, r_0^2 z_0^2\right]\sin(\theta_m) + \left[9\, r_0^4 - 90\, r_0^2 z_0^2 + 105\, z_0^4\right]\cos(\theta_m)\Big] \tag{7.14}$$

This second derivative is made zero by requiring

$$\begin{aligned} \tan(\theta_m) &= -\frac{105\left[\frac{z_0}{r_0}\right]^4 - 90\left[\frac{z_0}{r_0}\right]^2 + 9}{35\left[\frac{z_0}{r_0}\right]^2 - 15} \\ &= -\frac{105\cos^4(\theta_0) - 90\cos^2(\theta_0) + 9}{35\cos^2(\theta_0) - 15} \end{aligned} \tag{7.15}$$

So, for a given θ_0 one can readily calculate a θ_m to meet this condition. Alternately for a given θ_m we can solve the quadratic equation for $[z_0 / r_0]^2$ as

$$\begin{aligned} 0 &= 105\cos(\theta_m)\left[\frac{z_0}{r_0}\right]^4 + \left[35\sin(\theta_m) - 90\cos(\theta_m)\right]\left[\frac{z_0}{r_0}\right]^2 + \left[9\cos(\theta_m) - 15\sin(\theta_m)\right] \\ \left[\frac{z_0}{r_0}\right]^2 &= \cos^2(\theta_0) \\ &= \frac{1}{210}\left\{90\cos(\theta_m) - 35\sin(\theta_m) \pm \left[\left[90\cos(\theta_m) - 35\sin(\theta_m)\right]^2 - 420\left[9\cos(\theta_m) - 15\sin(\theta_m)\right]\right]^{\frac{1}{2}}\right\} \end{aligned} \tag{7.16}$$

This gives a variety of possibilities for combinations of θ_0 and θ_m to obtain third-order field uniformity.

7.5 Combining transmitter and receiver arrays

For the receiver array the cost of $\theta_m = \pi / 2$ in Section 7.2 is appropriate due to its insensitivity to externally incident uniform magnetic fields. Denoting this case by subscript r we have

$$
\begin{aligned}
\theta_{m_r} &= 90^\circ \\
\theta_r &= \arccos\left(\left[\frac{3}{7}\right]^{\frac{1}{2}}\right) = 49.89^\circ \\
\frac{z_r}{r_r} &= \left[\frac{3}{7}\right]^{\frac{1}{2}} \simeq .655 \\
\frac{\Psi_r}{r_r} &= \left[\frac{4}{7}\right]^{\frac{1}{2}} \simeq .756 \\
\frac{z_r}{\Psi_r} &= \left[\frac{3}{4}\right]^{\frac{1}{2}} \simeq .866
\end{aligned}
\tag{7.17}
$$

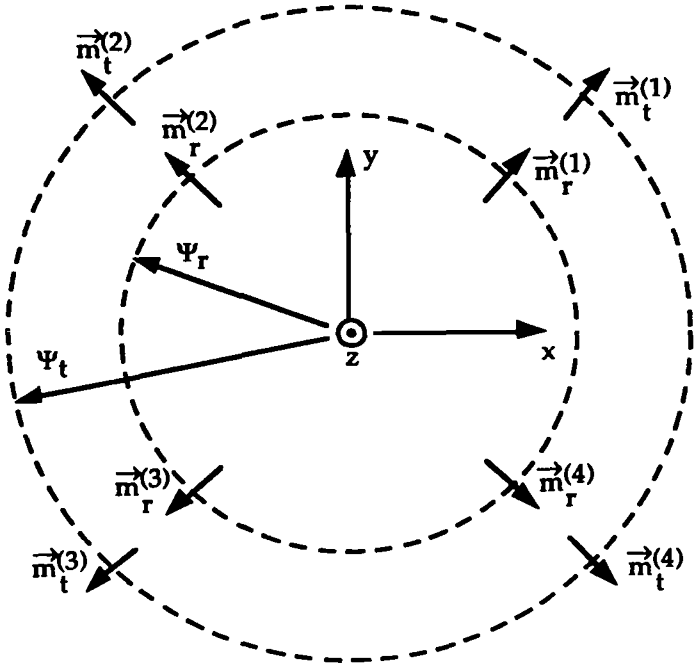
For the transmitter array, the case of general θ_m in Section 7.4 is appropriate. Denoting this case by subscript t, there are various possible combinations of θ_{m_t} and θ_t that one might use, together with the choice of r_t.

Noting that the receiver coils are oriented to have zero coupling to a uniform magnetic field, then one might place these "inside" the transmitter coils which are producing an approximately uniform magnetic field near the origin. So one might choose

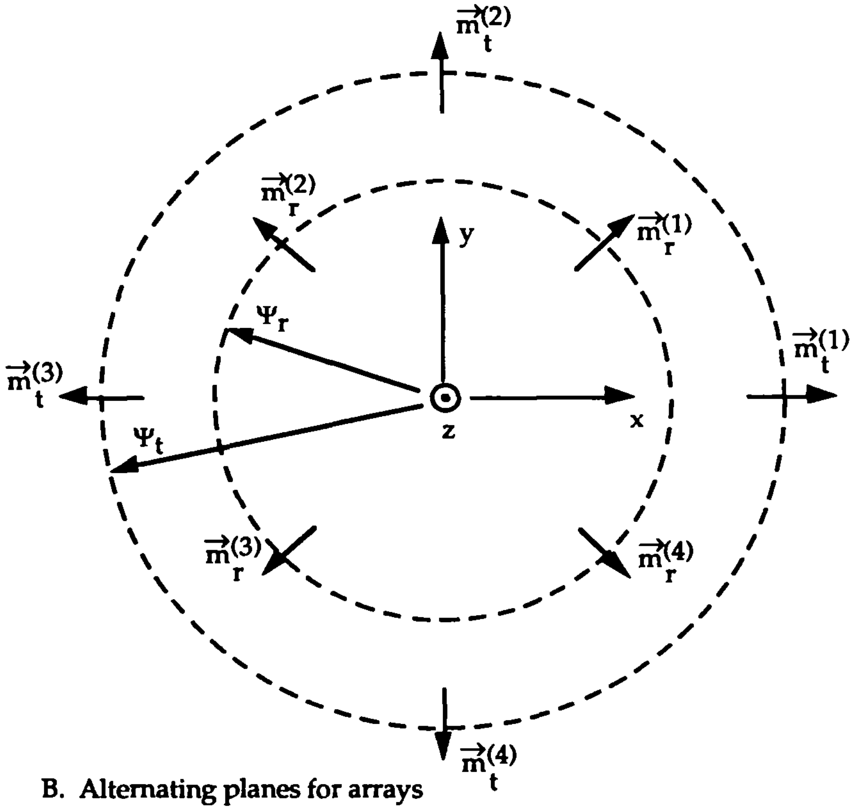
$$
r_r < r_t \tag{7.18}
$$

How small one should choose r_r / r_t depends on how little coupling between the two arrays one desires. Of course, there is also the choice of the θ_{m_t} , θ_t combination (Section 7.4) available for also minimizing this mutual coupling. As a practical matter, one would not like r_r to be too much less than r_t since some fraction of r_r (the smaller) is what one has available for placing targets (scatterers). Furhtermore r_t should not be too large so as to avoid excessive physical size of the array.

Combining the two magnetic-dipole arrays, the symmetries of the combined arrays also need to be appreciated. As illustrated in fig. 7.3, there are two cases of interest. Choosing $N = 4$ for each array for third-order field uniformity, each array has four axial symmetry planes, two containing the dipoles and two interspaced half-way between (angles of $\pi/4$ between successive symmetry planes). Note that the axial symmetry places increase C_4 to C_{4a} symmetry (eight elements). This allows the dipole-containing planes of one array to be coincident of the dipole-containing planes of the other (fig. 7.3A) or to be spaced between them with bisected angles (fig. 7.3B). Fig. 7.3 gives a top view (in the $-\vec{1}_z$ direction),



A. Coincident planes for arrays



B. Alternating planes for arrays

Fig 7.3 Combination of Transmitter and Receiver Magnetic-Dipole Arrays: Top Views showing only the dipoles in the upper half space. Those in the lower half space are related to these as in fig. 7.2 to give R_z symmetry so that each array has C_{4at} symmetry (16 elements) which can also be regarded as dihedral D_{4t} symmetry with the qualification that rotation about the four 2-fold axes has a sign change on the magnetic dipoles. Note also in fig. 7.3 that the transmitter dipoles (with superscripts to distinguish the different constant-ϕ planes) have only horizontal components (i.e., parallel to the $z = 0$ plane) while the transmitter dipoles have generally both horizontal and vertical components.

Comparing the two cases, note that the coincident planes in fig. 7.3A allows a clear path through the arrays (along either the $x = 0$ plane or $y = 0$ plane) for a walk-through metal detector. The case in fig. 7.3B is more crowded in this regard. In both cases, the optimum choice of θ_{m_t} (implying θ_t) is a function of r_r / r_t. Assuming r_r / r_t is not small (compared to 1), the expansion of the transmiter fields at the receive dipoles in powers of r as in Section 3 is not appropriate. More detailed calculations are needed to minimize the coupling between the arrays.

7.6 Combining dipole arrays with other kinds of transmitter coils

In Section 5, body-of-revolution coils such as the Helmholtz coil were considered for producing a uniform z-directed magnetic field. As has been noted in Section 7.2, the special receiver array has some properties similar to the special ring-coil pair discussed in Section 6. In particular, the coupling of the receiver array to the Helmholtz coil has the same positioning problem for minimum coupling due to the solution for θ_0 for the dipoles with $\theta_m = 90°$ being the same as for the ring coils in (6.19).

Referring to fig. 5.1, we see how a Helmholtz coil may be combined with (approximate) two-dimensional coils for H_x and H_y without mutual coupling (achieved by symmetry). Both transmitter and receiver dipole arrays discussed here also have the property of not coupling with such two-dimensional coils due to symmetry. In [5] there are discussed not only the H_x and H_y coils in fig. 5.1, but also line magnetic dipoles (two dimensional) for both transmitting and receiving H_x and H_y. Again symmetry makes the line dipoles not couple with the dipole arrays discussed here.

7.7 Other kinds of dipole arrays

Here we have addressed only the most basic types of dipole arrays, starting with two to obtain the R_z symmetry, and then adjoining rotation symmetry C_N. One need not be limited to two magnetic dipoles to achieve R_z symmetry for a symmetric magnetic field with respect to the $z = 0$ plane.

The case of three magnetic dipoles is illustrated in fig. 7.4. Here the case of two dipoles in fig. 7.2 is extended by the addition of

$$\vec{m}_1 = m_1 \vec{1}_z \tag{7.19}$$

at $(\Psi, z) = (\Psi_1, 0)$. This, by itself, has the appropriate symmetry, consistent with that for the other two dipoles. A special case of this has

$$\theta_m = 0 \quad , \quad \vec{m}_0 = m_0 \vec{1}_z \tag{7.20}$$

as in Section 7.3. If, in addition to this, we require

$$m_1 = -2m_0 \tag{7.21}$$

then we have the property that the net magnetic dipole moment is zero, making the array insensitive to uniform externally incident magnetic fields. This is an alternative to the design in Section 7.2 which also has this property. Note that as we increase the number of dipoles we have additional parameters (such as Ψ_1 here) available for optimization.

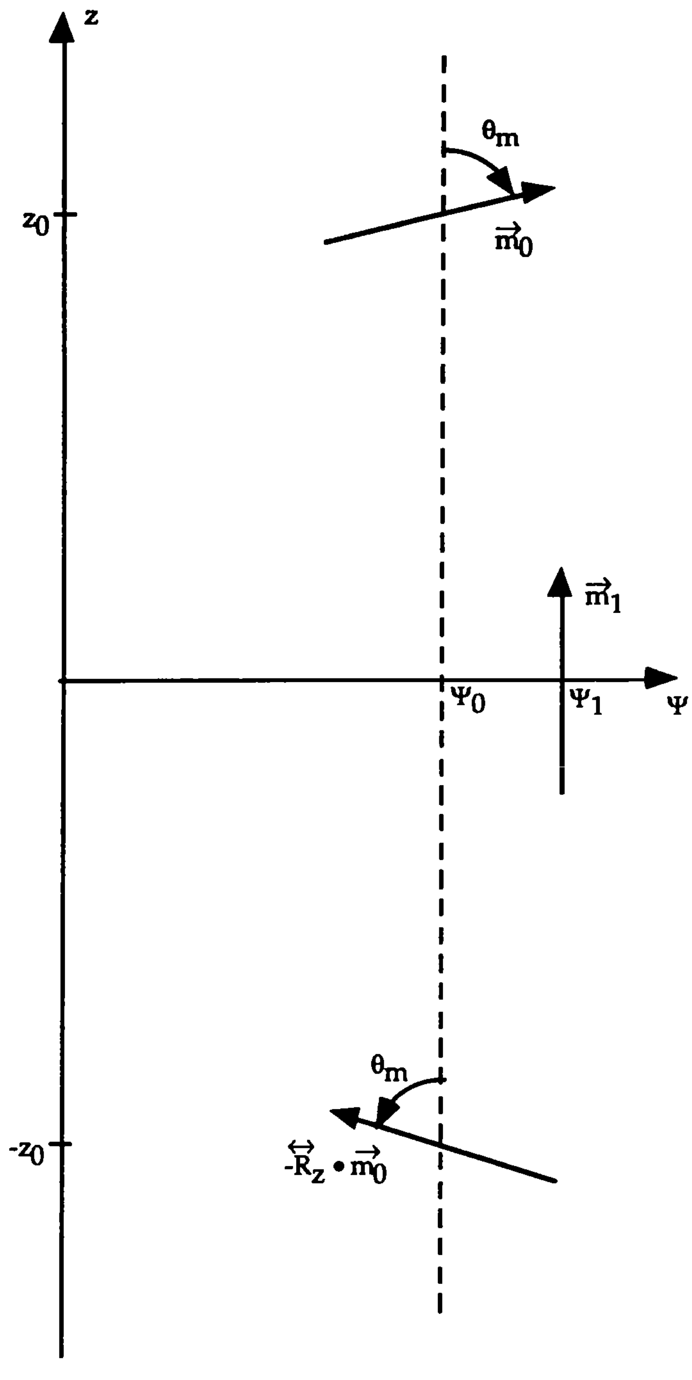


Fig. 7.4 Three Elementary Magnetic Dipoles on Plane of Constant ϕ with R_z Symmetry

8. Magnetic Dipoles Distributed along Paths Parallel to the z Axis

In the previous section, we have considered distinct magnetic dipoles, idealized as located at points. Now, consider distributions of such dipoles along paths in space described by a magnetic dipole moment per unit length. For present purposes, such paths shall be taken as parallel to the z axis.

8.1 Dipole moments parallel to z axis

As a first case, let the magnetic dipoles by oriented parallel to the z axis or a path in a constant-ϕ plane given by $\Psi = \Psi_0$ as illustrated in fig. 8.1. Instead of the discrete dipoles as in Section 7.3 we now have a distributed magnetic dipole moment per unit length described by

$$\vec{m}'(z) = m'(z)\ \vec{1}_z \tag{8.1}$$

Consistent with the requirement of R_z symmetry we now have an even function of z as

$$m'(-z) = m'(z) \tag{8.2}$$

giving a magnetic field symmetric with respect to the $z = 0$ plane.

Taking the result for a single magnetic dipole in (7.8) gives the z component of the magnetic field on the z axis as

$$H_z(z\ \vec{1}_z) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\left\{3[z-z']^2\left[[z-z']^2+\Psi_0^2\right]^{-\frac{5}{2}} - \left[[z-z']^2+\Psi_0^2\right]^{-\frac{3}{2}}\right\} m'(z')\,dz' \tag{8.3}$$

Changing variables gives

$$\begin{gathered}\frac{z'-z}{\Psi_0} \equiv \tan(\nu)\ ,\quad dz' = \Psi_0 \sec^2(\nu)\,d\nu \\ H_z(z\ \vec{1}_z) = \frac{1}{4\pi\Psi_0^2}\int_{-\pi/2}^{\pi/2}\left[3\sin^2(\nu) - 1\right]\cos(\nu)\ m'(z')d\nu\end{gathered} \tag{8.4}$$

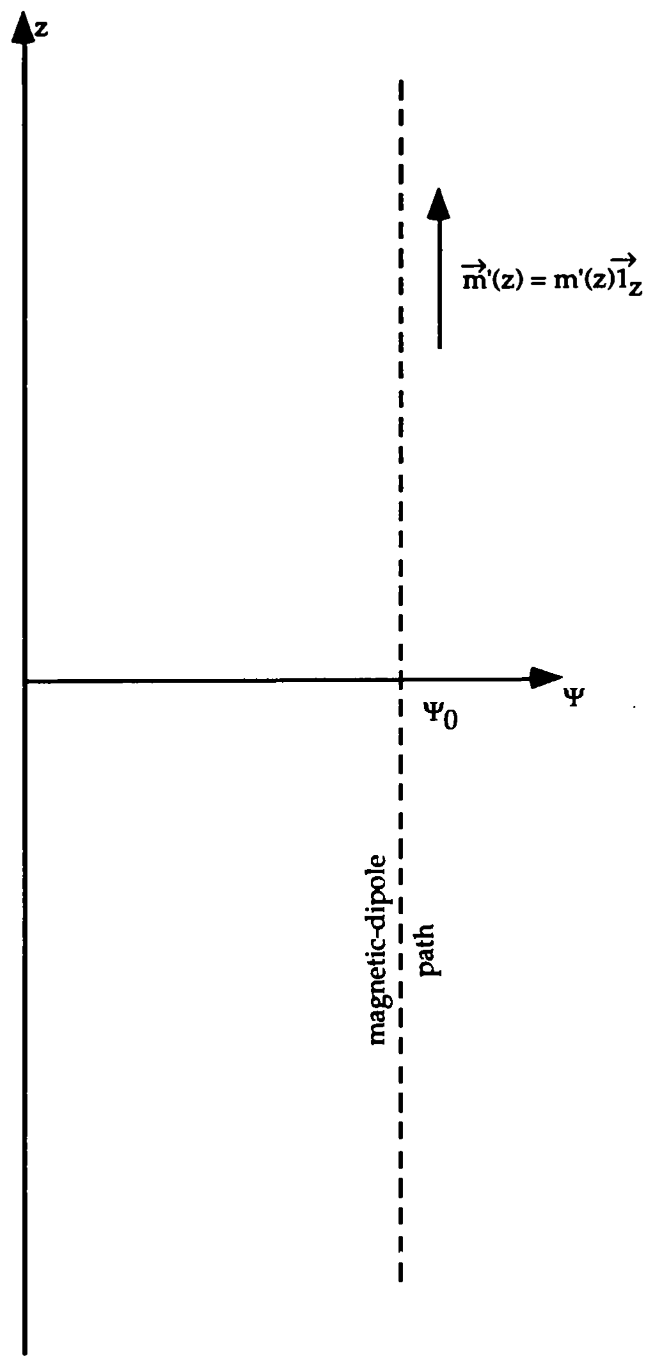


Fig. 8.1 Distributed Magnetic Dipoles Oriented Parallel to z Axis

Another change of variables gives

$$u \equiv \sin(v) \quad , \quad du = \cos(v)\,dv$$

$$\frac{z'-z}{\Psi_0} = \tan(\arcsin(u)) = u\left[1-u^2\right]^{-\frac{1}{2}}$$

$$u = \sin\left(\arctan\left(\frac{z'-z}{\Psi_0}\right)\right) = \frac{z'-z}{\Psi_0}\left[1 + \left[\frac{z'-z}{\Psi_0}\right]^2\right]^{-1} \tag{8.5}$$

$$H_z(z\,\vec{1}_z) = \frac{1}{4\pi\Psi_0^2}\int_{-1}^{1}\left[3u^2-1\right]m'(z')\,dz'$$

Now consider choices of $m'(z')$. Start with a constant magnetic dipole moment per unit length giving

$$m'(z) = m'$$

$$H_z(z\,\vec{1}_z) = \frac{m'}{4\pi\Psi_0^2}\left[u^3-u\right]\Bigg|_{-1}^{1} = 0 \tag{8.6}$$

More generally, it can be shown that

$$\vec{H}(\vec{r}) = \vec{0} \tag{8.7}$$

except, of course, "inside" the infinite solenoid formed by the distributed magnetic dipoles. So an infinite (uniform) solenoid has zero magnetic field outside it. As the length tends to infinity the field is like that of two magnetic monopoles (equal magnitude and opposite sign) being moved ever farther away.

This result can also be understood from reciprocity. Consider a dipole (as in (7.1)) oriented in the z direction and located at the origin. On the dipole path at $\Psi = \Psi_0$ we have

$$H_z\begin{cases} <0 \text{ for } |z| < \frac{\Psi_0}{\sqrt{2}} \\ >0 \text{ for } |z| > \frac{\Psi_0}{\sqrt{2}} \end{cases} \tag{8.8}$$

The dipole at the origin couples to the solenoid (dipole path) with different signs as one considers different positions along the solenoid. One could use this property by reversing the sense of the solenoid (change the sign of m') for certain regions of z consistent with (8.8) to make the signals add instead of cancel. Referring to Section 7.3, one can also see this phenomenon in the two choices of optimum location (via angle θ_0) for z-oriented dipoles giving opposite signs to the magnetic field near the origin as in (7.12).

A disadvantage of reversing dipole orientations according to (8.8) concerns target location. As the magnetic dipole representing the scatterer moves up and down along the z axis one is limited in the extent one can move away from the origin and retain this property of optimum polarity of the solenoid elements.

8.2 Dipole moments perpendicular to z axis

The complementary case has the magnetic dipoles oriented perpendicular to the z axis, but on the same path in a constant-ϕ plane given by $\Psi = \Psi_0$ as illustrated in fig. 8.2. Now the distributed magnetic dipole per unit length takes the form

$$\vec{m}'(z) = m'(z)\,\vec{1}_\Psi \tag{8.9}$$

Furthermore, consistent with the requirement of R_z symmetry we now have an odd function of z as

$$m'(-z) = -m'(z) \tag{8.10}$$

giving a magnetic field symmetric with respect to the $z = 0$ plane.

The result for a single such dipole in (7.3) gives the z component of the magnetic field on the z axis as

$$H_z(z\,\vec{1}_z) = \frac{\Psi_0}{4\pi}\int_{-\infty}^{\infty}[z'-z]\left[[z'-z]^2 + \Psi_0^2\right]^{-\frac{5}{2}} m'(z')\,dz' \tag{8.11}$$

Changing variables gives

$$\begin{aligned} &\frac{z'-z}{\Psi_0} \equiv \tan(v) \quad , \quad dz' \equiv \Psi\sec^2(v)\,dv \\ &H_z(z\,\vec{1}_z) = \frac{1}{4\pi\Psi_0^2}\int_{-\pi/2}^{\pi/2}\sin(v)\cos^2(v)\,m'(z')\,dv \end{aligned} \tag{8.12}$$

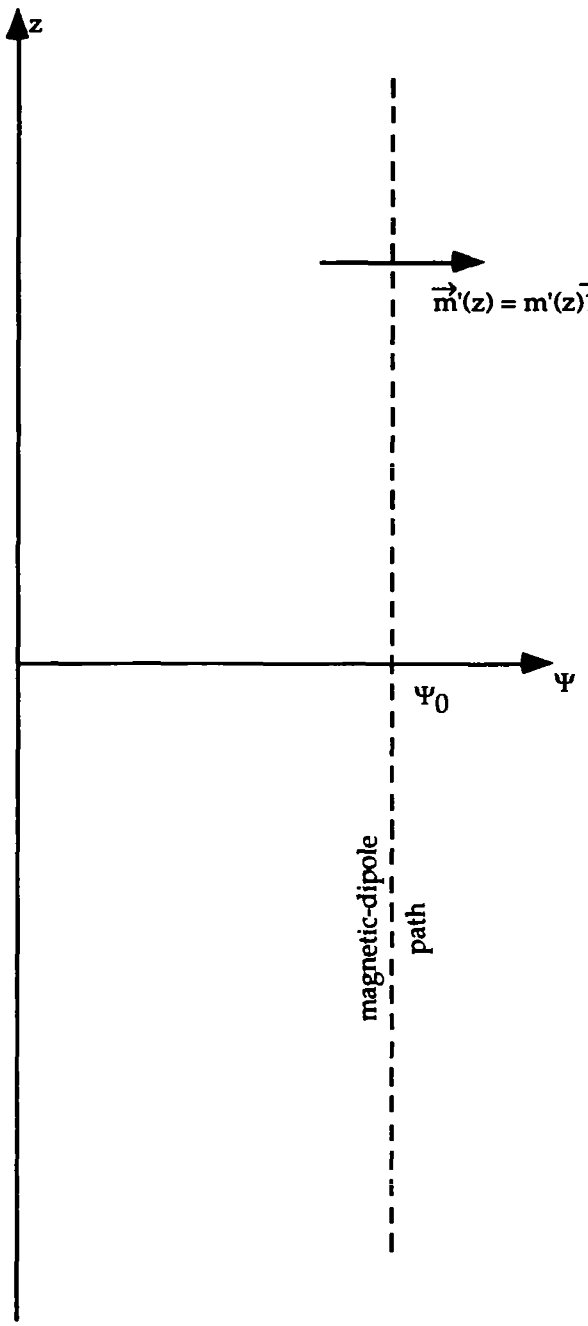


Fig. 8.2 Distributed Magnetic Dipoles Oriented Perpendicular to z Axis

At this point, choose the simplest form of $m'(z)$ consistent with (8.10), specifically as first power in z (linear variation), specifically

$$\begin{aligned} m'(z) &= \frac{m_0'}{z_1} z \\ z_1 &\equiv \text{convenient scaling distance} \\ m_0' &\equiv \text{reference dipole moment per unit length} \end{aligned} \tag{8.13}$$

Converting to the v variable

$$m'(z') = \frac{m_0'}{z_1} z' = \frac{m_0'}{z_1}\left[\Psi_0 \tan(v) + z\right] \tag{8.14}$$

Inserting this into (8.12) note first that the constant term $m_0' \Psi_0 z / z_1$ gives zero since the integrand is an odd function of v. The remaining term gives

$$H_z(z \vec{1}_z) = \frac{m_0'}{4\pi\Psi_0 z_1} \int_{-\pi/2}^{\pi/2} \sin^2(v) \cos(v)\, dv \tag{8.15}$$

Changing variables gives

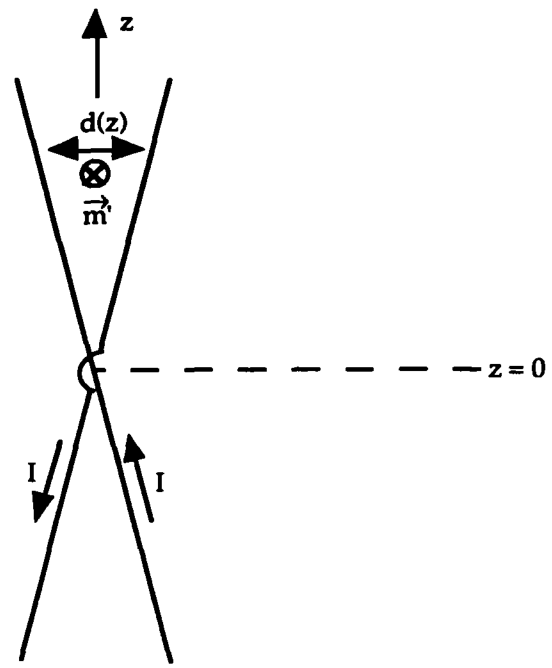
$$u \equiv \sin(v) \quad , \quad du = \cos(v)\, dv$$

$$\begin{aligned} H_z(z \vec{1}_z) &= \frac{m_0'}{4\pi\Psi_0 z_1} \int_{-1}^{1} u^2 du = \frac{m_0'}{4\pi\Psi_0 z_1} \left. \frac{u^3}{3} \right|_{-1}^{1} \\ &= \frac{m_0'}{6\pi\Psi_0 z_1} \end{aligned} \tag{8.16}$$

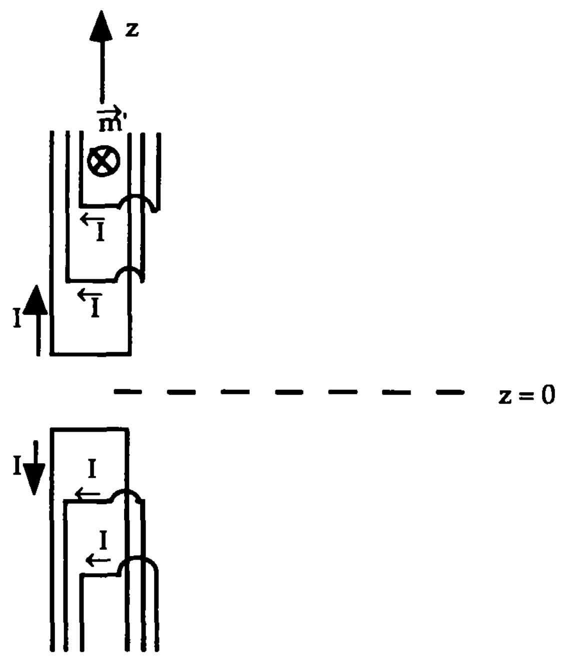
which is conveniently independent of z.

Consider for a moment how one might realize such a dipole distribution as in (8.13). One simple realization is the crossed wires (connected in series to have equal but "opposite" currents) as illustrated in fig. 8.3A. Here the X shape makes the wire spacing $d(z)$ a constant times z so that

$$\begin{aligned} m'(z) &= \mathrm{I}\, d(z) = m_0' \frac{z}{z_1} \\ d(z) &= \frac{m_0'}{\mathrm{I}} \frac{z}{z_1} \end{aligned} \tag{8.17}$$



A. Crossed wires



B. Multiple coils

Fig. 8.3 Coil Designs for Linear Variation of Magnetic Dipole Moment per Unit Length: View in Ψ Direction

Note that the view is from the z axis looking in the Ψ direction. The angle between the wires should not be large so that $d(z) << \Psi_0$, and we can think of the coil as being approximately located on a plane of constant ϕ. (Later the coil will be truncated at $z = \pm z_0$ so that $d(z)$ is bounded.) Where the wires cross at the $z = 0$ plane they need to be slightly displaced so that they do not touch.

Another realization of this type of dipole distribution is illustrated in fig. 8.3B. In this case multiple loops are used, each of the same wire spacing, but beginning at different distances from the $z = 0$ plane. As an approximation, the number of coils (each with equal currents I) at a given z is proportional to z. The beginnings of successive coils need to be spaced a distance small compared to Ψ_0 to approximate a continuous variation. Note the sign reversal as one goes to negative z.

As a practical matter, the structure needs to be of finite extent in the z direction, and this will have an effect on the magnetic field produced. So now let (8.13) be modified to give zero for $|z| > z_0$, i.e.,

$$m'(z) = \begin{cases} m_0' \dfrac{z}{z_1} & \text{for } |z| < z_0 \\ 0 & \text{for } |z| > z_0 \end{cases} \tag{8.18}$$

At $z = \pm z_0$, there are appropriate connections to close the loops in fig. 8.3 to maintain current continuity and connect to sources.

The integral in (8.12) is then changed to

$$\begin{aligned} H_z(z \vec{1}_z) &= \frac{1}{4\pi\Psi_0^2} \int_{v_-}^{v_+} \sin(v) \cos^2(v)\, m(z')\, dv \\ v_- &\equiv \arctan\left(\frac{z_0 - z}{\Psi_0}\right) , \quad v_+ \equiv -\arctan\left(\frac{z_0 + z}{\Psi_0}\right) \end{aligned} \tag{8.19}$$

The two terms in m' in (8.14) lead to two integrals. Similar to (8.15), but with different limits we have

$$u_- \equiv \sin(\nu_-) = -\frac{z_0 + z}{\Psi_0}\left[1 + \left[\frac{z_0 + z}{\Psi_0}\right]^2\right]^{-\frac{1}{2}}$$

$$u_+ \equiv \sin(\nu_+) = \frac{z_0 + z}{\Psi_0}\left[1 + \left[\frac{z_0 - z}{\Psi_0}\right]^2\right]^{-\frac{1}{2}} \tag{8.20}$$

$$\int_{\nu_-}^{\nu_+} \sin^2(\nu)\cos(\nu)\, d\nu = \int_{u_-}^{u_+} u^2 du + \frac{1}{3}\left[u_+^3 - u_-^3\right]$$

We also need a second integral with a different change of variable as

$$\mathrm{v} \equiv \cos(\nu) \quad , \quad d\mathrm{v} = -\sin(\nu)d\nu$$

$$\frac{z' - z}{\Psi_0} = \tan(\arccos(\mathrm{v})) = \mathrm{v}^{-1}\left[1 - \mathrm{v}^2\right]^{\frac{1}{2}}$$

$$\mathrm{v} = \cos\left(\arctan\left(\frac{z' - z}{\Psi_0}\right)\right) = \left[1 + \left[\frac{z' - z}{\Psi_0}\right]^2\right]^{-1}$$

$$\mathrm{v}_- = \cos(\nu_-) = \left[1 + \left[\frac{z_0' + z}{\Psi_0}\right]^2\right]^{-1}$$

$$\mathrm{v}_+ = \cos(\nu_+) = \left[1 + \left[\frac{z_0' - z}{\Psi_0}\right]^2\right]^{-1} \tag{8.21}$$

$$\int_{\nu_-}^{\nu_+} \sin(\nu)\cos^2(\nu)d\nu = -\int_{\mathrm{v}_-}^{\mathrm{v}_+} \mathrm{v}^2 d\mathrm{v} = \frac{1}{3}\left[\mathrm{v}_-^3 - \mathrm{v}_+^3\right]$$

Combining these results we have

$$H_z(z\,\vec{1}_z) = \frac{m_0'}{4\pi\Psi_0^2\, z_1}\left\{\frac{\Psi_0}{3}\left[u_+^3 - u_-^3\right] + \frac{z}{3}\left[\mathrm{v}_-^3 - \mathrm{v}_+^3\right]\right\}$$

$$= \frac{m_0'}{12\pi\Psi_0 z_1}\left\{\left[\frac{z_0 - z}{\Psi_0}\right]^3\left[1 + \left[\frac{z_0 - z}{\Psi_0}\right]\right]^{-\frac{3}{2}} + \left[\frac{z_0 + z}{\Psi_0}\right]^3\left[1 + \left[\frac{z_0 + z}{\Psi_0}\right]^2\right]^{-\frac{3}{2}} - \frac{z}{\Psi_0}\left[\left[1 + \left[\frac{z_0 - z}{\Psi_0}\right]^2\right]^{-\frac{3}{2}} - \left[1 + \left[\frac{z_0 + z}{\Psi_0}\right]^2\right]^{-\frac{3}{2}}\right]\right\} \tag{8.22}$$

At $z = 0$, this simplifies to

$$H_z(\vec{0}) = \frac{m_0'}{6\pi\Psi_0 z_1}\left[\frac{z_0}{\Psi_0}\right]^3\left[1+\left[\frac{z_0}{\Psi_0}\right]^2\right]^{-\frac{3}{2}} = \frac{m_0'}{6\pi\Psi_0 z_1}\cos^3(\theta_0) \tag{8.23}$$

where θ_0 is the same as used previously (Section 7). Note the consistency with the previous results in (8.16) as $\theta_0 \rightarrow 0$.

Evaluating the z derivatives along the z axis we have

$$\begin{aligned}\frac{\partial}{\partial z}H_z(z\vec{1}_z) = \frac{m_0'}{12\pi\Psi_0^2 z_1}\Bigg\{3\Bigg\{-\left[\frac{z_0-z}{\Psi_0}\right]^2\left[1+\left[\frac{z_0-z}{\Psi_0}\right]^2\right]^{-\frac{5}{2}}+\left[\frac{z_0+z}{\Psi_0}\right]^2\left[1+\left[\frac{z_0+z}{\Psi_0}\right]^2\right]^{-\frac{5}{2}}\Bigg\}\\ +\Bigg\{-\left[1+\left[\frac{z_0-z}{\Psi_0}\right]^2\right]^{-\frac{3}{2}}+\left[1+\left[\frac{z_0-z}{\Psi_0}\right]^2\right]^{-\frac{3}{2}}\Bigg\}\\ +\frac{3z}{\Psi_0}\Bigg\{-\left[\frac{z_0-z}{\Psi_0}\right]\left[1+\left[\frac{z_0-z}{\Psi_0}\right]^2\right]^{-\frac{5}{2}}-\left[\frac{z_0+z}{\Psi_0}\right]\left[1+\left[\frac{z_0+z}{\Psi_0}\right]^2\right]^{-\frac{5}{2}}\Bigg\}\Bigg\}\end{aligned} \tag{8.24}$$

which is zero at the origin (as are all odd derivatives). The second derivative is

$$\begin{aligned}\frac{\partial^2}{\partial z^2}H_z(z\vec{1}_z) = \frac{m_0'}{4\pi\Psi_0^3 z_1}\Bigg\{-5\Bigg\{-\left[\frac{z_0-z}{\Psi_0}\right]^3\left[1+\left[\frac{z_0-z}{\Psi_0}\right]^2\right]^{-\frac{7}{2}}+\left[\frac{z_0+z}{\Psi_0}\right]^3\left[1+\left[\frac{z_0+z}{\Psi_0}\right]^2\right]^{-\frac{7}{2}}\Bigg\}\\ +\frac{z}{\Psi_0}\Bigg\{\left[1+\left[\frac{z_0-z}{\Psi_0}\right]^2\right]^{-\frac{5}{2}}-5\left[\frac{z_0-z}{\Psi_0}\right]^2\left[1+\left[\frac{z_0-z}{\Psi_0}\right]^2\right]^{-\frac{7}{2}}\\ -\left[1+\left[\frac{z_0-z}{\Psi_0}\right]^2\right]^{-\frac{5}{2}}+5\left[\frac{z_0+z}{\Psi_0}\right]^2\left[1+\left[\frac{z_0-z}{\Psi_0}\right]^2\right]^{-\frac{7}{2}}\Bigg\}\end{aligned} \tag{8.25}$$

At the origin this is

$$\frac{\partial^2}{\partial z^2} H_z(z\,\vec{1}_z)\bigg|_{z=0} = -\frac{5}{2\pi}\,\frac{m_0'}{\Psi_0^3\, z_1}\left[\frac{z_0}{\Psi_0}\right]^3\left[1+\left[\frac{z_0}{\Psi_0}\right]^2\right]^{-\frac{7}{2}} \tag{8.26}$$

Since this second derivative is not zero (except in the limit of $z_0 \to \infty$ with non-zero field at the origin), let us modify the design as indicated schematically in fig. 8.4. Where we have truncated the linear distribution of magnetic dipoles as in (8.18), let us add lumped (discrete) magnetic dipoles at the truncation positions ($z = \pm z_0$) of the form

$$\vec{m} = \begin{cases} m_0\,\vec{1}_\Psi & \text{at } z = z_0 \\ -m_0\,\vec{1}_\Psi & \text{at } z = -z_0 \end{cases} \tag{8.27}$$

This is precisely the configuration treated in Section 7.2, and the results there can be added to the present results.

The two contributions to the second derivative are added and set to zero to give

$$\begin{aligned} 0 = &-\frac{5}{2\pi}\,\frac{m_0'}{\Psi_0^3\, z_1}\left[\frac{z_0}{\Psi_0}\right]^3\left[1+\left[\frac{z_0}{\Psi_0}\right]^2\right]^{-\frac{7}{2}} \\ &-\frac{5}{2\pi}\,\frac{m_0}{\Psi_0^5}\left\{3\,\frac{z_0}{\Psi_0}\left[1+\left[\frac{z_0}{\Psi_0}\right]^2\right]^{-\frac{7}{2}} - 7\left[\frac{z_0}{\Psi_0}\right]^3\left[1+\left[\frac{z_0}{\Psi_0}\right]^2\right]^{-\frac{9}{2}}\right\} \end{aligned} \tag{8.28}$$

Defining a parameter

$$\xi \equiv \frac{m_0'}{z_1}\,\frac{\Psi_0^2}{m_0} \tag{8.29}$$

to normalize the two magnetic-dipole contributions, then (8.28) gives

$$\xi = \left[4\left[\frac{z_0}{\Psi_0}\right]^2 - 3\right]\left[\frac{z_0}{\Psi_0}\right]^{-2}\left[1+\left[\frac{z_0}{\Psi_0}\right]^2\right]^{-1} \tag{8.30}$$

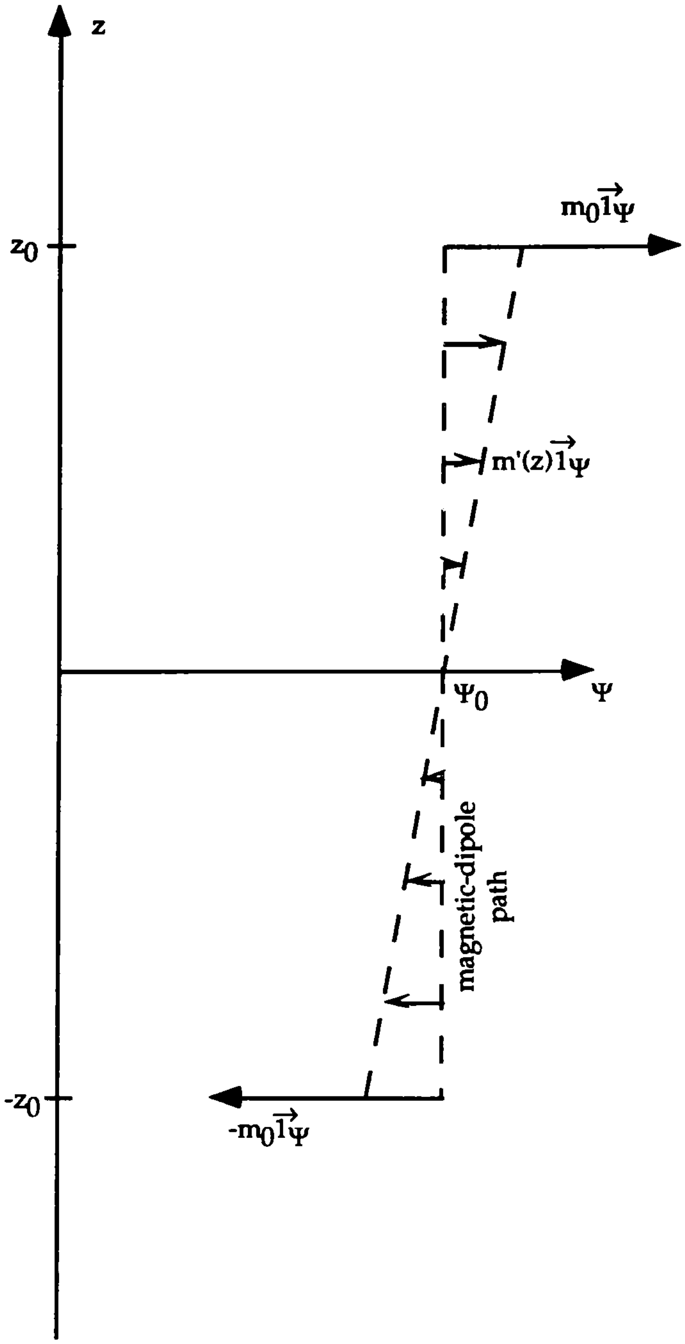


Fig. 8.4 Distributed Plus Two Lumped Magnetic Dipoles Oriented Perpendicular to z Axis

as the criterion for a zero second derivative. For m_0 to have the same sign as m_0', and the lumpted and distributed dipole moments to effectively add we need

$$\frac{z_0}{\Psi_0} > \frac{\sqrt{3}}{2} \tag{8.31}$$

Note that equality in this equation implies that $m_0' = 0$ and the situation is the same as that in Section 7.2 with only lumped dipoles. As z_0 / Ψ_0 becomes large we have

$$\xi = 4\left[\frac{z_0}{\Psi_0}\right]^{-2}\left[1+O\left(\left[\frac{z_0}{\Psi_0}\right]^{-2}\right)\right] \text{ as } \frac{z_0}{\Psi_0} \to \infty \tag{8.32}$$

so that $m_0 \to 0$ and the ideal uniform-field case on the z axis in (8.16) is approached.

Now the magnetic field at the origin can be found from the two contributions as

$$\begin{aligned} H_z(\vec{0}) &= \frac{m_0'}{6\pi\Psi_0\, z_1}\left[\frac{z_0}{\Psi_0}\right]^3\left[1+\left[\frac{z_0}{\Psi_0}\right]^2\right]^{-\frac{3}{2}} \\ &+ \frac{m_0}{2\pi}\frac{z_0}{\Psi_0^4}\left[1+\left[\frac{z_0}{\Psi_0}\right]^2\right]^{-\frac{5}{2}} \\ &= \left[1+\left[\frac{z_0}{\Psi_0}\right]^2\right]^{-\frac{5}{2}}\frac{z_0}{6\pi\Psi_0}\left\{\frac{m_0'}{\Psi_0\, z_1}\left[\frac{z_0}{\Psi_0}\right]^2\left[1+\left[\frac{z_0}{\Psi_0}\right]^2\right]+\frac{3\,m_0}{\Psi_0^3}\right\} \end{aligned} \tag{8.33}$$

Applying the constraint in (8.30) gives

$$\begin{aligned} H_z(\vec{0}) &= \left[1+\left[\frac{z_0}{\Psi_0}\right]^2\right]^{-\frac{5}{2}}\frac{z_0}{6\pi\Psi_0}\left\{\frac{m_0'}{\xi\Psi_0\, z_1}\left[4\left[\frac{z_0}{\Psi_0}\right]^2-3\right]+\frac{3\,m_0}{\Psi_0^3}\right\} \\ &= \frac{2\,m_0}{3\pi}\frac{z_0^3}{\Psi_0^6}\left[1+\left[\frac{z_0}{\Psi_0}\right]^2\right]^{-\frac{5}{2}} \end{aligned} \tag{8.34}$$

This can also be expressed in terms of m_0' as

$$H_z(\vec{0}) = \frac{2\,m_0'}{3\pi\,\Psi_0 z_1 \xi}\left[\frac{z_0}{\Psi_0}\right]^3\left[1+\left[\frac{z_0}{\Psi_0}\right]^2\right]^{-\frac{5}{2}}$$

$$= \frac{2\,m_0'}{3\pi\,\Psi_0 z_1}\left[\frac{z_0}{\Psi_0}\right]^5\left[1+\left[\frac{z_0}{\Psi_0}\right]^2\right]^{-\frac{3}{2}}\left[4\left[\frac{z_0}{\Psi_0}\right]^2 - 3\right]^{-1} \tag{8.35}$$

$$= \frac{m_0'}{6\pi\,\Psi_0 z_1}\left[1+\left[\frac{\Psi_0}{z_0}\right]^2\right]^{-\frac{3}{2}}\left[1-\frac{3}{4}\left[\frac{\Psi_0}{z_0}\right]^2\right]^{-1}$$

For large z_0 / Ψ_0, this gives

$$H_z(\vec{0}) = \frac{m_0'}{6\pi\,\Psi_0 z_1}\left[1-\frac{3}{4}\left[\frac{z_0}{\Psi_0}\right]^{-2} + O\left(\left(\frac{z_0}{\Psi_0}\right)^{-4}\right)\right] \text{ as } \frac{z_0}{\Psi_0}\to\infty \tag{8.36}$$

showin the asymptotic behavior toward (8.16). For the special choice of

$$\frac{z_0}{\Psi_0} = 1.144 \tag{8.37}$$

The result in (8.35) is the same as in (8.16), thereby giving the same result as the infinite array at the origin. From (8.29) and (8.30) the requisite value of m_0 can be obtained to correspond to the selected m_0' and z_0 / Ψ_0 in (8.35).

So, now we have the conditions for the first three derivatives of H_z to be zero at the origin. This is extended to all field components at the origin by the adjunction of other symmetry as discussed in Section 3. Speccifically C_4 symmetry is adequate for this purpose. So one need only replicate the dipole distributions discussed here as four such arrays spaced by successive rotations of the constant-ϕ plane (e.g., as in fig. 8.4) by $\pi/2$ (90°).

9. Concluding Remarks

This paper has explored the use of symmetry to aid in the design of three-dimensional coils for producing uniform magnetic fields (or receiving magnetic-dipole fields uniformly). Third-order field uniformity (as in the well-known example of a Helmholtz coil) can be achieved by $C_{4t} = C_4 \otimes R_z$ symmetry together with adjustment of the coil geometry to set one remaining expansion coefficient (the second derivative term) to zero (Table 3.2). As a practical matter axial symmetry planes are easily included, making the symmetry of a little higher order. Another aspect of symmetry is to minimize the direct coupling of one coil (or set of coils) to another. Such may be for transmit and receive, or for different field components.

Besides geometric symmetry, there are other techniques that can be used to minimize unwanted signals coupled into the various coils. One can cancel such signals by sampling the source of such signals and adding an appropriate (perhaps filtered) form of this into the response. For example, one can measure dI / dt in one coil (the transmitter) and add a constant times this to the voltage from the receive coil (proportional to the time derivative of the coupling magnetic flux) to minimize the net signal from the receive coil. Then there are various data processing issues. Geometric symmetry is only one (but an important one) of the design considerations.

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