Sensor and Simulation Notes

Note 406

6 November 1996

Two-Dimensional Coils for Low-Frequency Magnetic Illumination and Detection

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Abstract

This paper considers several types of two-dimensional loop structures for transmitting and receiving low-frequency magnetic fields to and from nearby targets. Such targets may be "inside" the coils, or on the opposite side of a boundary (e.g., the ground surface) from the coils. Various design considerations include zero mutual inductance between transmitter and receiver coils, receiver-coil insensitivity to externally incident low-frequency magnetic fields, and uniformity of target detection over some test volume. Using complex-variable(conformal-transmission) techniques a large number of such designs are considered.

Contents

1. Introduction

There are various applications of coils (loops) operated in a quasi-magnetostatic regime, including detection and identification of metallic targets [7, 8, 13]. In this case the target is in the near field of the coils and an analysis of the static magnetic fields is appropriate. Such coils can take various shapes such as rectangular and circular. If a rectangular loop is sufficiently elongated it can be analyzed as a two-dimensional problem, much as a TEM transmission line, with parameters considered on a per-unit-length basis. Such loops consist of two or more wires parallel to the z axis with a zero sum for the currents (in the z direction).

This type of loop can be used to give a uniform magnetic field in some limited spatial domain. Combining two coils as transmitter and receiver one can measure components of a target's magnetic polarizability dyadic. In this case one would like that one coil not couple to the other so as to maximize the signal-to-noise ratio in the receiver coil. One might also desire some uniformity of detection over some spatial domain, this involving the combined properties of the two loops. Another design consideration for the receiver coil is to make it insensitive (zero equivalent area[10]) to an external incident magnetic field (e.g., from 50 Hz or 60 Hz power lines).

There are two general classes of coil geometry with respect to the target. In one case (such as for security applications) the target may pass through the coil set where a good region of uniform detection is readily attained. In another case (such as for detecting/identifying targets buried in soil) the coils are on the opposite side of an approximate planar boundary from the target, making uniform detection more difficult. Various examples of such coil systems are considered in this paper.

2. Coils for Uniform Magnetic Field

Consider now some cases of coils for producing a uniform magnetic field "inside" the coil ensemble. Around some appropriate point ζ_0, related to the symmetry of the problem, one can expand the magnetic field in a power (Taylor) series as

$$H(\zeta) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n H(\zeta_0)}{d\zeta_0^n} [\zeta - \zeta_0]^n \tag{2.1}$$

which has a radius of convergence given by the nearest singularity in the complex ζ plane.

With the assumption that

$$H(\zeta_0) \neq 0 \tag{2.2}$$

one can relate the coefficient of the next term to some effective distance as

$$\begin{aligned} D_1 &\equiv \left[\frac{dH(\zeta_0)}{d\zeta_0}\right]^{-1} H(\zeta_0) \\ H(\zeta) &= H(\zeta_0)\left[1 + \frac{\zeta - \zeta_0}{D_1} + \cdots\right] \end{aligned} \tag{2.3}$$

Larger values of D_1 (related to some characteristic cross-section dimension of the coil) imply a more uniform field. If, however, the first derivative is zero (a desirable situation) one can look at higher order derivatives. If the Nth derivative is the first non-zero derivative, then we have

$$\begin{aligned} D_N &\equiv \left[N!\left[\frac{d^N H(\zeta_0)}{d\zeta_0^N}\right]^{-1} H(\zeta_0)\right]^{\frac{1}{N}} \\ H(\zeta) &= H(\zeta_0)\left[1 + \left[\frac{\zeta - \zeta_0}{D_N}\right]^N + \cdots\right] \end{aligned} \tag{2.4}$$

Again, D_N represents some characteristic distance for field uniformity, with now the first "error" term given by the Nth-power term.

Often the symmetry of the loop currents and resulting magnetic field will make certain derivatives be zero on planes or axes of symmetry. For present purposes reflection symmetry is quite important.

This is an involution group (two elements) [11]. In the present context we have important reflection planes, the x = 0 plane with

$$\begin{aligned} R_x &= \{(1)\,,(R_x)\} \quad \text{(group)} \\ (1) &\equiv \text{identity} \\ (R_x) &\equiv \text{reflection} \\ (R_x)^2 &= (1) \end{aligned} \tag{2.5}$$

and dyadic representation

$$\begin{aligned} (1) &\rightarrow \overleftrightarrow{1} = \vec{1}_x \vec{1}_x + \vec{1}_y \vec{1}_y + \vec{1}_z \vec{1}_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ (R_x) &\rightarrow \overleftrightarrow{R}_x = -\vec{1}_x \vec{1}_x + \vec{1}_y \vec{1}_y + \vec{1}_z \vec{1}_z = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{2.6}$$

and the $y = 0$ plane with a similar form. These can, of course, be rotated about the z axis for additional reflection planes.

With respect to such symmetry planes the fields are symmetric or antisymmetric depending on whether $\vec{H}$ is perpendicular or parallel, respectively, to the symmetry plane (Appendix D). For present purposes we will sometimes have two symmetry planes and the fields will be symmetric with respect to the $x = 0$ plane and antisymmetric with respect to the $y = 0$ plane. This gives a two-fold rotation axis (the $z = 0$ axis) with axial symmetry planes with group designation

$$C_{2a} = R_x \otimes R_y \tag{2.7}$$

This will assure that all odd-order derivatives are zero at $\zeta = 0$, giving D_2 as the first "uniformity distance." Note that on reflection symmetric means currents preserving sign, and antisymmetric means currents reversing sign. So the geometry (location) of the current filaments is what has the C_{2a} symmetry; the signs of the currents still need to be considered.

Higher order symmetries are also possible, making yet more derivatives zero at the origin.

2.1 Two wires

Our first case has filamentary currents of $\pm I$ at $\zeta = \pm jb$ respectively, as in fig. 2.1A, giving an x-directed magnetic field at the origin. Summing terms from Appendix B for line currents we have

$$w(\zeta) = -\ell n(\zeta - jb) + \ell n(\zeta + jb) = \ell n\left(\frac{\zeta + jb}{\zeta - jb}\right)$$

$$H(\zeta) = -\frac{I}{2\pi} j\left\{[\zeta - jb]^{-1} - [\zeta + jb]^{-1}\right\} = \frac{I}{\pi} b\left[\zeta^2 + b^2\right]^{-1} \tag{2.8}$$

$$H(0) = H_x(0) = \frac{I}{\pi b}$$

The symmetry makes odd derivatives of $H(\zeta)$ zero at the origin. So looking at the second derivative we have

$$\frac{d^2 H(\zeta)}{d\zeta^2} = -\frac{I}{2\pi} j(2!)\left\{[\zeta - jb]^{-3} - [\zeta + jb]^{-3}\right\}$$

$$\left.\frac{d^2 H(\zeta)}{d\zeta^2}\right|_{\zeta=0} = -\frac{2I}{\pi b^3} \tag{2.9}$$

$$D_2^2 = -b^2 \, , D_2 = \pm jb$$

More details concerning this configuration are found in [1].

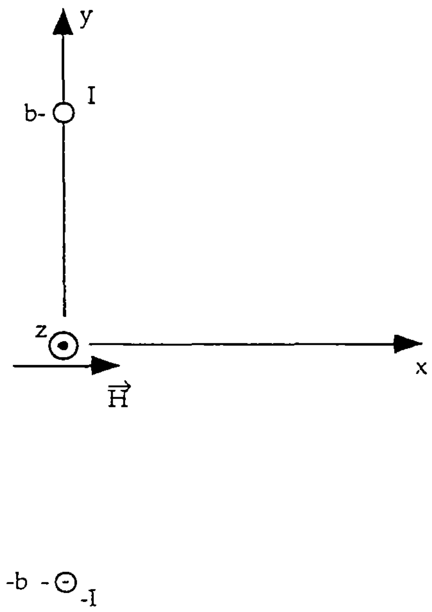
2.2 Four wires

As indicated in fig. 2.1B, let there be four wires symmetrically located with respect to the x = 0 and y = 0 symmetry planes with currents as indicated to give a uniform H_x near the origin. For this purpose it is convenient to define

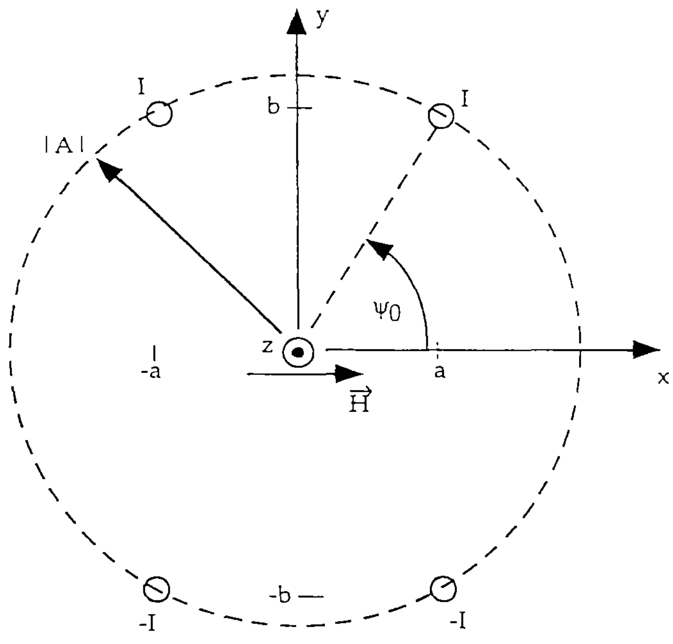
$$A \equiv a + jb = |A| e^{j\psi_0} \tag{2.10}$$

The wires are then located at A and $-A^*$, each with current I, and at $-A$ and A^*, each with current -I. This case is also discussed in [1].

The complex potential and magnetic field are now



A. Two wires



B. Four wires

Fig. 2.1. Filamentary Currents

$$w(\zeta) = -\ell n(\zeta - A) + \ell n(\zeta - A^*) - \ell n(\zeta + A^*) + \ell n(\zeta + A)$$

$$= \ell n\left(\frac{\zeta - A^*}{\zeta - A}\right) + \ell n\left(\frac{\zeta + A}{\zeta + A^*}\right)$$

$$H(\zeta) = -\frac{I}{2\pi} j\left\{[\zeta - a]^{-1} - [\zeta - A^*]^{-1} + [\zeta + A^*]^{-1} - [\zeta + A]^{-1}\right\} \tag{2.11}$$

$$H(0) = H_x(0) = \frac{2I}{\pi}\frac{\text{Im}[A]}{|A|^2} = \frac{2I}{\pi}\frac{\sin(\psi_0)}{|A|} = \frac{2I}{\pi}\frac{b}{a^2 + b^2}$$

Again odd derivatives of the magnetic field are zero at the origin. Looking at the second derivative of the field we have

$$\frac{d^2 H(\zeta)}{d\zeta^2} = -\frac{I}{2\pi} j(2!)\left\{[\zeta - A]^{-3} - [\zeta - A^*]^{-3} + [\zeta + A^*]^{-3} - [\zeta + A]^{-3}\right\}$$

$$\left.\frac{d^2 H(\zeta)}{d\zeta^2}\right|_{\zeta=0} = \frac{2I}{\pi} j\left[A^{-3} - A^{*-3}\right] = \frac{4I}{\pi}|A|^{-3}\sin(3\psi_0) \tag{2.12}$$

The second derivative can be set to zero at the origin by making $3\psi_0$ some integer multiple of π. For convenience we take

$$\psi_0 \equiv \frac{\pi}{3}$$

$$|A| = \left[a^2 + b^2\right]^{\frac{1}{2}} = 2a = \frac{2}{\sqrt{3}} b \simeq 1.155b \tag{2.13}$$

$$b = \sqrt{3}a \simeq 1.732a$$

Next consider the fourth derivative as

$$\frac{d^4 H(\zeta)}{d\zeta^4} = -\frac{12I}{\pi} j\left\{[\zeta - A]^{-5} - [\zeta - A^*]^{-5} + [\zeta + A^*]^{-5} - [\zeta + A]^{-5}\right\}$$

$$\left.\frac{d^4 H(\zeta)}{d\zeta^4}\right|_{\zeta=0} = \frac{24I}{\pi} j\left[A^{-5} - A^{*-5}\right] = \frac{48I}{\pi}|A|^{-5}\sin(5\psi_0)$$

$$= -\frac{48I}{\pi}|A|^{-5}\sin(\psi_0) = -\frac{24\sqrt{3}I}{\pi}|A|^{-5} \tag{2.14}$$

$$D_4^4 = -|A|^4 = -\left[a^2 + b^2\right]^2 \quad , \frac{D_4}{|A|} = [-1]^{\frac{1}{4}}$$

$$[-1]^{\frac{1}{4}} = e^{\pm j\frac{\pi}{4}}, e^{\pm j\frac{3\pi}{4}}$$

$$D_4^4 = -|A|^4 = -\left[a^2 + b^2\right]^2 \quad , \frac{D_4}{|A|} = [-1]^{\frac{1}{4}}$$

$$[-1]^{\frac{1}{4}} = e^{\pm j\frac{\pi}{4}}, e^{\pm j\frac{3\pi}{4}}$$

For further insight into this special four-wire case, consider the points in fig. 2.1B on the x axis at $\pm|A|$. Imagine that these are wires with zero current. This gives six wires uniformly spaced on a circle of radius $|A|$ with an angle of $\pi/3$ between adjacent wires. This geometry has C_{6a} symmetry including the 6 axial symmetry planes. Such a structure has a variety of field patterns corresponding to various combination of currents on the wires. Special cases of these correspond to eigenmodes of the bicirculant impedance matrix that describes such a structure [6, 11]. The case being considered corresponds to one such mode.

2.3 Two line magnetic dipoles on the x axis

Consider two line magnetic dipoles $\overrightarrow{m}'$ at $x = \pm a$ as illustrated in fig. 2.2A. Here we take

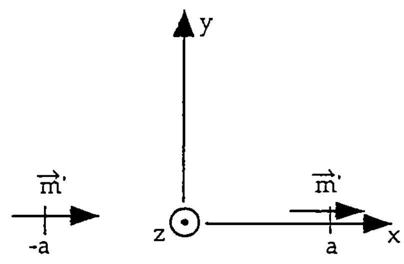
$$\overrightarrow{m}' = m'_x \overrightarrow{1}_x \quad , \quad m' = m'_x \text{ (real)} \tag{2.15}$$

This particular orientation is chosen to give an H_x near the origin with the two symmetry planes as discussed previously. Summing terms from Appendix C for line magnetic dipoles we have

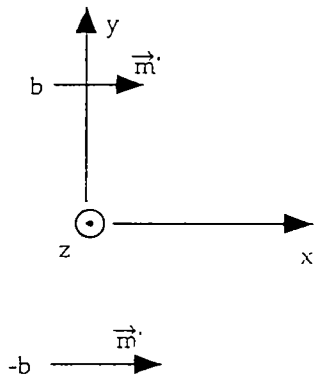
$$\begin{aligned} w_m(\zeta) &= j\left\{[\zeta - a]^{-1} + [\zeta + a]^{-1}\right\} = \frac{j2\zeta}{\zeta^2 + a^2} \\ H(\zeta) &= \frac{m'}{2\pi}\left\{[\zeta - a]^{-2} + [\zeta + a]^{-2}\right\} \\ H(0) &= H_x(0) = \frac{m'}{\pi a^2} \end{aligned} \tag{2.16}$$

Again, odd derivatives of $H(\zeta)$ are zero at the origin. The second derivative gives

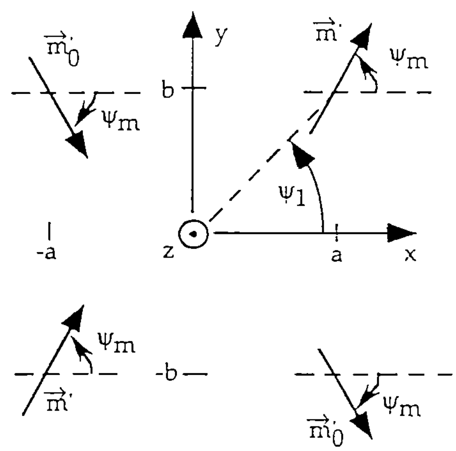
$$\begin{aligned} \frac{d^2H(\zeta)}{d\zeta^2} &= \frac{3m'}{\pi}\left\{[\zeta - a]^{-4} + [\zeta + a]^{-4}\right\} \\ \left.\frac{d^2H(\zeta)}{d\zeta^2}\right|_{\zeta=0} &= \frac{6m'}{\pi a^4} \\ D_2^2 = \frac{a^2}{3} \quad &, \quad \frac{D_2}{a} = \pm\frac{1}{\sqrt{3}} \simeq \pm.577 \end{aligned} \tag{2.17}$$



A. Two on x axis



B. Two on y axis



C. Four

Fig. 2.2. Line Magnetic Dipoles

2.4 Two line magnetic dipoles on the y axis

A similar result (to that in Section 2.3) is obtained if one positions the two line magnetic dipoles $\vec{m}'$ at $y = \pm b$ as illustrated in fig. 2.2B. Again we have

$$\vec{m}' = m'_x \vec{1}_x \quad , \quad m' = m'_x \text{ (real)} \tag{2.18}$$

This gives a negative H_x at the origin with the same two symmetry planes. In this case we have

$$\begin{aligned} w_m(\zeta) &= j\left\{[\zeta - jb]^{-1} + [\zeta + jb]^{-1}\right\} = \frac{j2\zeta}{\zeta^2 - b^2} \\ H(\zeta) &= \frac{m'}{2\pi}\left\{[\zeta - jb]^{-2} + [\zeta + jb]^{-2}\right\} \\ H(0) &= H_x(0) = -\frac{m'}{\pi b^2} \end{aligned} \tag{2.19}$$

The first derivative being zero at the origin, the second derivative gives

$$\begin{aligned} \frac{d^2 H(\zeta)}{d\zeta^2} &= \frac{3m'}{\pi}\left\{[\zeta - jb]^{-4} + [\zeta + jb]^{-4}\right\} \\ \left.\frac{d^2 H(\zeta)}{d\zeta^2}\right|_{\zeta=0} &= \frac{6m'}{\pi b^4} \\ D_2^2 = -\frac{b^2}{3} \quad &, \quad \frac{D_2}{b} = \pm j\frac{1}{\sqrt{3}} \simeq \pm .577j \end{aligned} \tag{2.20}$$

2.5 Four line magnetic dipoles

Consider four line magnetic dipoles as illustrated in fig. 2.2B. Located at $\pm A$ and $\pm A^*$ as defined in (2.10), the line dipoles are two each $\vec{m}'$ and $\vec{m}'_0$ with

$$\begin{aligned} \vec{m}'_0 = \overleftrightarrow{R}_y \cdot \vec{m}' \quad &, \quad \left|\vec{m}'\right| = \left|\vec{m}'_0\right| = \left|\vec{m}'\right| \\ m' &= \left|\vec{m}'\right| e^{j\Psi_m} \\ m'_0 = m'^* &= \left|\vec{m}'\right| e^{-j\Psi_m} \\ A = a + jb &= |A| e^{j\Psi_1} \end{aligned} \tag{2.21}$$

This preserves the fields and currents symmetric with respect to the x = 0 plane and antisymmetric with respect to the y = 0 plane, as with the other cases considered.

The complex potential and magnetic field are now

$$
\begin{aligned}
w_m(\zeta) &= j\left\{e^{j\psi_m}[\zeta - A]^{-1} + e^{-j\psi_m}[\zeta - A*]^{-1} + e^{-j\psi_m}[\zeta + A*]^{-1} + e^{j\psi_m}[\zeta + A]^{-1}\right\} \\
&= j2\left\{e^{j\psi_m}\zeta\left[\zeta^2 - A^2\right]^{-1} + e^{-j\psi_m}\left[\zeta^2 - A*^2\right]^{-1}\right\} \\
H(\zeta) &= \frac{|m'|}{2\pi}\left\{e^{j\psi_m}[\zeta - A]^{-2} + e^{-j\psi_m}[\zeta - A*]^{-2} + e^{-j\psi_m}[\zeta + A*]^{-2} + e^{j\psi_m}[\zeta + A]^{-2}\right\} \\
H(0) &= H_x(0)\frac{|m'|}{\pi}\left\{e^{j\psi_m}A^{-2} + e^{-j\psi_m}A*^{2}\right\} = \frac{2|m'|}{\pi|A|^2}\cos(\psi_m - 2\psi_0)
\end{aligned}
\tag{2.22}
$$

With odd derivatives of the magnetic field zero, look at the second derivative of the field as

$$
\begin{aligned}
\frac{d^2H(\zeta)}{d\zeta^2} &= \frac{3|m'|}{\pi}\left\{e^{j\psi_m}[\zeta - A]^{-4} + e^{-j\psi_m}[\zeta - A*]^{-4} + e^{-j\psi_m}[\zeta + A*]^{-4} + e^{j\psi_m}[\zeta + A]^{-4}\right\} \\
\left.\frac{d^2H(\zeta)}{d\zeta^2}\right|_{\zeta=0} &= \frac{6|m'|}{\pi}\left\{e^{j\psi_m}A^{-4} + e^{-j\psi_m}A*^{-4}\right\} = \frac{12|m'|}{\pi|A|^4}\cos(\psi_m - 4\psi_1)
\end{aligned}
\tag{2.23}
$$

Setting the second derivative to zero gives

$$
\psi_m - 4\psi_1 = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \cdots
\tag{2.24}
$$

Next consider the fourth derivative as

$$
\begin{aligned}
\frac{d^4H(\zeta)}{d\zeta^4} &= \frac{60|m'|}{\pi}\left\{e^{j\psi_m}[\zeta - A]^{-6} + e^{-j\psi_m}[\zeta - A*]^{-6} + e^{-j\psi_m}[\zeta + A*]^{-6} + e^{j\psi_m}[\zeta + A]^{-6}\right\} \\
\left.\frac{d^4H(\zeta)}{d\zeta^4}\right|_{\zeta=0} &= \frac{120|m'|}{\pi}\left\{e^{j\psi_m}A^{-6} + e^{-j\psi_m}A*^{-6}\right\} = \frac{240|m'|}{\pi|A|^6}\cos(\psi_m - 6\psi_1)
\end{aligned}
\tag{2.25}
$$

One can try to make this fourth derivative also zero, but with (2.24) this restricts ψ_1 to degenerate cases of 0 or $\pi/2$ with $\psi_m = \pi/2$ which, in turn, makes $H(0) = 0$ (not desirable here). So let the fourth derivative be nonzero giving (with (2.24))

$$
\begin{aligned}
D_4^4 &= \frac{1}{5}|A|^4 \frac{\cos(\psi_m - 2\psi_1)}{\cos(\psi_m - 6\psi_1)} \\
&= \frac{1}{5}|A|^4 \frac{\cos(\psi_m - 4\psi_1)\cos(2\psi_1) - \sin(\psi_m - 4\psi_1)\sin(2\psi_1)}{\cos(\psi_m - 4\psi_1)\cos(2\psi_1) + \sin(\psi_m - 4\psi_1)\sin(2\psi_1)} \\
&= -\frac{1}{5}|A|^4 \\
\frac{D_4}{|A|} &= [-1]^{\frac{1}{4}} 5^{-\frac{1}{4}} = .669[-1]^{\frac{1}{4}}
\end{aligned}
\tag{2.26}
$$

So with the constraint of (2.24) we have some flexibility of choosing ψ_m and ψ_1. A rather interesting choice has

$$
\begin{aligned}
&\psi_m \equiv \frac{\pi}{2}, \psi_1 \equiv \frac{\pi}{4}, a = b = \frac{|A|}{\sqrt{2}} \simeq .707|A| \\
&H(0) = H_x(0) = \frac{2|m'|}{\pi|A|^2}
\end{aligned}
\tag{2.27}
$$

because such a loop configuration does not couple to a uniform externally incident quasi-static magnetic field, regardless of orientation. This is then an interesting candidate for a receiver coil for the magnetic field scattered by a target near the z axis.

3. Combining Transmitter and Receiver Coils for Uniformity of Detection

The transmitter coil produces an incident magnetic field $\vec{H}_{inc}$, such as discussed in Section 2. This in turn produces a scattered magnetic field associated with an induced magnetic dipole moment

$$\begin{aligned} \tilde{\vec{m}}(s) &= \tilde{\overleftrightarrow{M}}(s) \cdot \tilde{\vec{H}}_{inc}(s) \\ \tilde{\overleftrightarrow{M}}(s) &= \tilde{\overleftrightarrow{M}}^T(s) \equiv \text{ magnetic polarizability dyadic (reciprocity)} \end{aligned} \tag{3.1}$$

Here we have included the frequency dependence of the target scattering through the two-sided Laplace transform and complex frequency $s = \Omega + j\omega$ as in [7, 8, 13], since even in the quasi-static regime for the loops the targets of interest have important frequency dependence which can be utilized for identification.

The induced magnetic dipole moment gives the leading term in the scattering, but this is a good approximation provided the dimensions of the scatterer (say largest d) have $d << |A|$ where $|A|$ is a characteristic cross section of both transmitter and receiver coils (e.g., as in Section 2). One can compute the scattered magnetic field from $\vec{m}$ in the usual way [7, 13]. This is then integrated over the receiver coil to give the intercepted magnetic flux, the time derivative of which is the open-circuit voltage in the receiver coil.

As discussed previously the coil length ℓ in the z direction is assumed large compared to the cross-section dimensions. This allows us to translate the target in the z direction, in which case $\tilde{\vec{H}}_{inc}$ is unchanged as are $\vec{m}$ and the signal in the receiving coil. Then let us as a gedankenexperiment subdivide the target induced dipole moment into many (proportionally smaller) magnetic dipoles, which in the limit can be thought of as a line dipole moment given by

$$\begin{aligned} \tilde{\vec{m}}'_{eff}(s) &\equiv \frac{\tilde{\vec{m}}(s)}{\ell} \equiv \text{ effective line magnetic dipole moment} \\ \tilde{\vec{m}}'_{eff}(s) &= \tilde{\overleftrightarrow{M}}'_{eff}(s) \cdot \tilde{\vec{H}}_{inc}(s) \\ \tilde{\overleftrightarrow{M}}'_{eff}(s) &= \ell\, \tilde{\overleftrightarrow{M}}(s) \equiv \text{effective line magnetic polarizability} \end{aligned} \tag{3.2}$$

Note that $\vec{H}_{inc}$ has no z component and the receiving coil is not sensitive to a z component of $\vec{m}$. So for present purposes we can regard $\overleftrightarrow{M}$ as having only x and y components (transverse dyadic), and similarly for $\overleftrightarrow{M}'_{eff}$.

Now line magnetic dipoles give fields as discussed in Appendix C. These fields in turn couple to the two-dimensional receiver coil via the linked magnetic flux, or change in u (the real part of the complex potential) between the wires of the receiver coil (as discussed in Appendix D). By reciprocity one can look at the magnetic field at the target when driving a current through the receiver coil to obtain the same result. The response of a line magnetic dipole to a (transverse) magnetic field is proportional to the dot product. So let us form

$$U' \equiv \vec{H}_2 \cdot \vec{m}'_{eff} = \vec{H}_2 \cdot \overleftrightarrow{M}'_{eff} \cdot \vec{H}_1$$

$$\vec{H}_1 \equiv \text{ magnetic field from transmitter coil} \qquad (3.3)$$

$$\vec{H}_2 \equiv \text{ magnetic field from receiver coil driven as transmitter}$$

where the frequency dependence has been suppressed, noting the quasi-static nature of the problem. Looking at the form of (3.3) we can see that it is basically the product of $\vec{H}_1$ and $\vec{H}_2$ at some (x,y) coordinates which governs the sensitivity of our detection system. How uniform this is over the cross section is a measure of the uniformity of detection, and it is this that we would like to quantify.

Consider the case first that $\overleftrightarrow{M}'_{eff}$ has only an x,x component. Then we reduce (3.3) to the consideration of

$$U_{x,x} = H_{1x} H_{2x} \qquad (3.4)$$

As in Section 2 let there be two symmetry planes (x = 0 and y = 0). Let the two coils produce only x components of the field at the origin (z axis). Then we can look at derivatives of the above product to find the uniformity of detection. So we let

$$U(\zeta) = H_1(\zeta) H_2(\zeta) \qquad (3.5)$$

and consider the series expansion as in (2.1) through (2.4) for this product, replacing $H(\zeta)$ by $U(\zeta)$.

For the cross-polarized (x,y) component of $\overleftrightarrow{M}'_{eff}$ let the transmit coil produce an x-directed field at the origin while the receive coil produces a y-directed field there. The symmetry conditions in Section 2 (symmetric/antisymmetric with respect to the two symmetry planes) then have the roles of the two symmetry planes interchanged when considering the receiver coil (y directed) as compared to the transmitter coil (x directed). Then we consider

$$U_{x,y} = H_{1x}H_{2y} \tag{3.6}$$

To consider the uniformity of this we again form the product in (3.5) as an analytic function of ζ. Note, however, we have

$$U(0) = \begin{cases} \text{real for } H_1(0) \text{ and } H_2(0) \text{ x - directed} \\ \text{imaginary for } H_1(0) \text{ and } H_2(0) \text{ respectively x - and y - directed} \end{cases} \tag{3.7}$$

As discussed in Section 2 there are various ways to produce a rather uniform magnetic field for which various derivatives are zero at the origin. These can be combined in transmitter and receiver coil to achieve uniformity in the field product. Form

$$\frac{dU(\zeta)}{d\zeta} = \frac{dH_1(\zeta)}{d\zeta}H_2(\zeta) + H_1(\zeta)\frac{dH_2(\zeta)}{d\zeta} = 0 \tag{3.8}$$

if first derivatives of both fields are zero and note furthermore in general that

$$\frac{d^n U(\zeta)}{d\zeta^n} = 0 \tag{3.9}$$

if all derivatives $m = 1, 2, \cdots, n$ of both fields are zero.

This can be extended by using (2.4) with origin now at ζ_0 and letting

$$\begin{aligned} N_1 &\equiv N \text{ for "error" term in } H_1(\zeta) \\ N_2 &\equiv N \text{ for "error" term in } H_2(\zeta) \end{aligned} \tag{3.10}$$

Then let

$$N_0 \equiv \text{ lesser of } N_1, N_2 \tag{3.11}$$

giving

$$U(\zeta) = \begin{cases} U(\zeta_0)\left[1 + \left[\frac{\zeta - \zeta_0}{D_{N_0}}\right]^{N_0} + \cdots\right] \text{for } N_1 \neq N_2 \\ U(\zeta_0)\left[1 + \left[\frac{\zeta - \zeta_0}{D_{N_1}}\right]^{N_1} + \left[\frac{\zeta - \zeta_0}{D_{N_2}}\right]^{N_2} + \cdots\right] \text{ for } N_1 = N_2 \end{cases} \tag{3.12}$$

In the first case it is the lowest order "error" term in either field; in the second case both contribute. The characteristic distance in (2.4) can then be applied to the two-field problem as

$$D_N = D_{N_0} = \begin{cases} D_{N_0} & \text{for } N_1 \neq N_2 \\ \left[D_{N_1}^{-N_0} + D_{N_2}^{-N_0} \right]^{-N_0} & \text{for } N_1 = N_2 \end{cases} \tag{3.13}$$

For $N_1 \neq N_2$ the characteristic distance D_N is then less than that for each of the two fields (D_{N_1} and D_{N_2}).

Some cases of interest have the first derivatives of the fields non-zero. The derivative of the product can still be zero provided

$$\begin{aligned} 0 &= \left[\frac{1}{H_1(\zeta)} \frac{dH_1(\zeta)}{d\zeta} + \frac{1}{H_2(\zeta)} \frac{dH_2(\zeta)}{d\zeta} \right] \Bigg|_{\zeta=\zeta_0} \\ &= \left[\frac{d\, \ell n\left(H_1(\zeta)\right)}{d\zeta} + \frac{d\, \ell n\left(H_2(\zeta)\right)}{d\zeta} \right] \Bigg|_{\zeta=\zeta_0} \end{aligned} \tag{3.14}$$

Another way to view this is to require that the D_1 coefficients in (2.4) for the two fields sum to zero. If this requirement is met then one can go on the look at higher derivatives which generalize (3.14) to more complicated forms.

4. Two Coplanar Loops

Section 2.1 has considered the two-wire loop as in fig. 2.1A. Let us now take two such loops, one for transmission and another for reception. As in fig. 4.1 let the two loops be coplanar on the x = 0 plane. Note the symmetrical positioning of the two loops with respect to the y = 0 plane. Relating to Section 2.1, note the shift with

$$y_2 + y_1 = 2b \tag{4.1}$$

for the wire spacing in each of the two loops.

The transmitter coil with current I_1 at $y = y_2$ and $-I_1$ at $y = -y_1$ produces a complex potential

$$\begin{aligned} w(\zeta) &= -\ell n(\zeta - jy_2) + \ell n(\zeta + jy_1) = \ell n\left(\frac{\zeta + jy_1}{\zeta - jy_2}\right) \\ &= u(\zeta) + jv(\zeta) \end{aligned} \tag{4.2}$$

For zero coupling between the two loops we have from Appendix D that the change in u (proportional to magnetic flux) linking the second loop (with I_2 at $y = y_1$ and $-I_2$ at $y = -y_2$) be zero. This is

$$\begin{aligned} \Delta u &= \mathrm{Re}\left[\ell n\left(\frac{jy_1 + jy_1}{jy_1 - jy_2}\right) - \ell n\left(\frac{-jy_2 + jy_1}{-jy_2 - jy_2}\right)\right] \\ &= \mathrm{Re}\left[\ell n\left(\frac{-4y_1y_2}{(y_1 - y_2)^2}\right)\right] \\ &= \ell n\left(\left|\frac{4y_1y_2}{(y_2 - y_1)^2}\right|\right) \end{aligned} \tag{4.3}$$

Setting this to zero gives

$$\begin{aligned} (y_2 - y_1)^2 &= 4y_1y_2 \\ y_2^2 - 6y_1y_2 + y_1^2 &= 0 \\ \frac{y_2}{y_1} &= 3 + \sqrt{8} \simeq 5.83 \end{aligned} \tag{4.4}$$

where the plus sign is chosen to make $y_2 > y_1$. In terms of the half-spacing b of the two wires in each coil we have

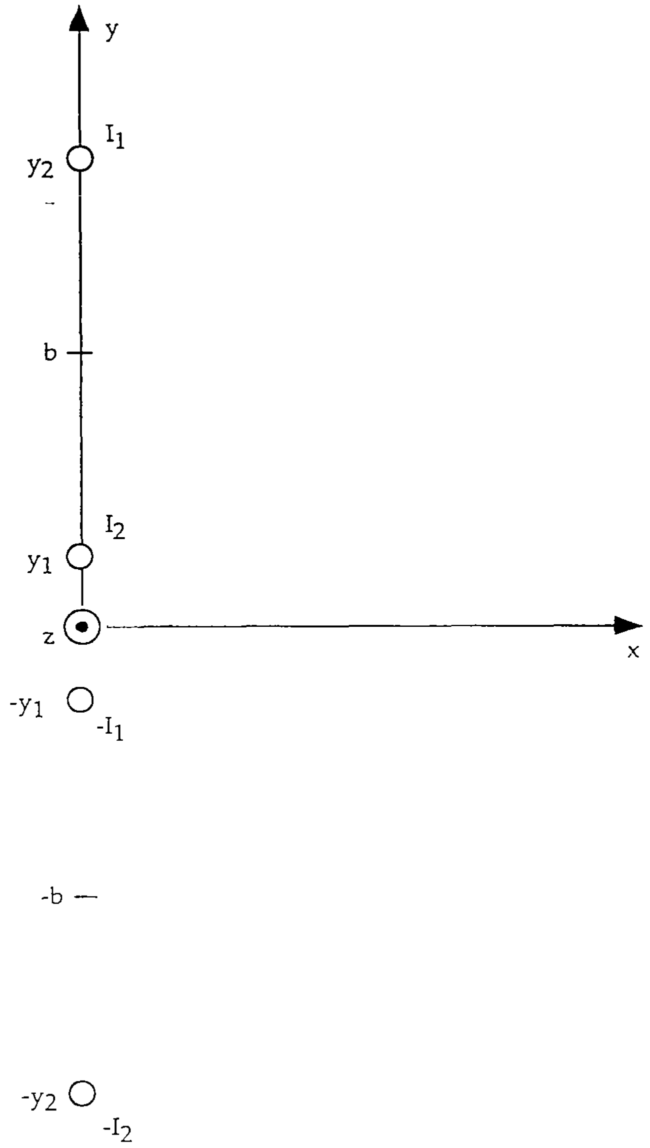


Fig. 4.1. Two Coplanar Loops

$$\frac{y_1}{b} = 1 - 2^{-\frac{1}{2}} = .293\,,\ \frac{y_2}{b} = 1 + 2^{-\frac{1}{2}} = 1.707 \tag{4.5}$$

fixing the wire positions in fig. 4.1.

The two magnetic fields are

$$\begin{aligned} H_1(\zeta) &= -\frac{I_1}{2\pi} j\left\{[\zeta - jy_2]^{-1} - [\zeta + jy_1]^{-1}\right\} \\ H_2(\zeta) &= -\frac{I_2}{2\pi} j\left\{[\zeta - jy_1]^{-1} - [\zeta + jy_2]^{-1}\right\} \end{aligned} \tag{4.6}$$

giving at the origin

$$\begin{aligned} H_1(0) = H_{1x}(0) &= \frac{I_1}{2\pi}\left[y_1^{-1} + y_2^{-1}\right] = \frac{2I_1}{\pi b} \\ H_2(0) = H_{2x}(0) &= \frac{I_2}{2\pi}\left[y_1^{-1} + y_2^{-1}\right] = \frac{2I_2}{\pi b} \\ U(0) = H_1(0)H_2(0) &= \frac{4}{\pi^2 b^2} I_1 I_2 \end{aligned} \tag{4.7}$$

showing the symmetry in this two-coil configuration. The first derivative of U is zero there by symmetry. The second derivative is

$$\begin{aligned} \frac{d^2U(\zeta)}{d\zeta^2} &= \frac{d^2H_1(\zeta)}{d\zeta^2} H_2(\zeta) + \frac{2dH_1(\zeta)}{d\zeta}\frac{dH_2(\zeta)}{d\zeta} + H_1(\zeta)\frac{d^2H_2(\zeta)}{d\zeta^2} \\ &= \frac{I_1 I_2}{(2\pi)^2}\Big\{-2\Big[[\zeta - jy_2]^{-3} - [\zeta + jy_1]^{-3}\Big]\Big[[\zeta - jy_1]^{-1} - [\zeta + jy_2]^{-1}\Big] \\ &\quad -2\Big[[\zeta - jy_2]^{-2} - [\zeta + jy_1]^{-2}\Big]\Big[[\zeta - jy_1]^{-2} - [\zeta r jy_2]^{-2}\Big] \\ &\quad -2\Big[[\zeta - jy_2]^{-1} - [\zeta + jy_1]^{-1}\Big]\Big[[\zeta - jy_1]^{-3} - [\zeta + jy_2]^{-3}\Big]\Big\} \\ \left.\frac{d^2U(\zeta)}{d\zeta^2}\right|_{\zeta=0} &= -\frac{I_1 I_2}{(2\pi)^2}\left\{2\left[y_1^{-3} + y_2^{-3}\right]\left[y_1^{-1} + y_2^{-1}\right] - \left[y_1^{-2} - y_2^{-2}\right]^2\right\} \\ &= -\frac{96 I_1 I_2}{\pi^2 b^4} \end{aligned} \tag{4.8}$$

Then we can write

$$\begin{aligned} U(\zeta) &= U(0)\left[1 + \left(\frac{\zeta}{D_2}\right)^2 + O(\zeta^4)\right] \text{ as } \zeta \to 0 \\ D_2^2 &= -\frac{b^2}{12}\ ,\quad \frac{D_2}{b} = \pm\frac{j}{2\sqrt{3}} \simeq \pm .289j \end{aligned} \tag{4.9}$$

as a measure of the uniformity of the field near the origin (symmetry axis). The third derivative is again zero by symmetry.

One can consider a target located near $\zeta = 0$ as "inside" the coil system, and in this case a certain detection uniformity is achieved. Another case of interest concerns when the target is away from the coils (away from the origin, in particular) as in the case of a buried target.

A first case for an external target is that of a target located on the +x axis away from the origin. (For a buried target the reader can rotate his head 90° down to the left when looking at fig. 4.1.) The ground surface is at some $x = x_g > 0$ and the target is at $x > x_g$, $y = 0$. For this case the incident magnetic field $H_1(x)$ has both x and y components, while $H_2(x)$ has the same x but opposite y components (i.e., conjugate). For an isotropic target (e.g., a sphere) one may consider $H_1(x)H_2(x)$ for sensitivity of detection, while for a target with a dominant $M_{x,x}$ component it is just the x components (vertical in this case) or real parts in the form $\mathrm{Re}[H_1(x)]\mathrm{Re}[H_2(s)]$ that one may consider for sensitivity of detection. For large x these reduce to the same thing as both coils are approximated as line magnetic dipoles (Appendix C) in this limit giving

$$\begin{aligned} H_1(x) &= \frac{m_1'}{2\pi} x^{-2}\left[1 + O(x^{-1})\right] \text{ as } |x| \to \infty \\ H_2(x) &= \frac{m_2'}{2\pi} x^{-2}\left[1 + O(x^{-1})\right] \text{ as } |x| \to \infty \\ m_1' &= 2bI_1 \quad , \quad m_2' = 2bI_2 \end{aligned} \tag{4.10}$$

While the fields fall off like x^{-2}, the product falls off like x^{-4}.

By placing the ground surface on a plane of constant $y = y_g < -y_2$ with the target at $y < y_g$ with x = 0, the x axis in fig. 4.1 is parallel to the ground surface and the x-component of the field is horizontal in the ground near the target. So by rotating the coil pair above the ground surface one can rotate the polarization in the ground to obtain a second component of the target magnetic polarizability. (Furthermore, rotating this about the y axis gives a third component). By symmetry the y fields on the x = 0 plane have only x components. For large negative y these are

$$\begin{aligned} H_1(jy) &= -\frac{m_1'}{2\pi} y^{-2}\left[1 + O(y^{-1})\right] \text{ as } |y| \to \infty \\ H_2(jy) &= -\frac{m_2'}{2\pi} y^{-2}\left[1 + O(y^{-1})\right] \text{ as } |y| \to \infty \end{aligned} \tag{4.11}$$

Note that the fields are pointed in the -x direction (for positive currents). Otherwise the fall off for large $|y|$ in this case is the same as for large $|x|$ in the previous case.

5. Three Coplanar Loops

Impose more symmetry with the three-coplanar-loop configuration in fig. 5.1. In this case the transmitter loop has both x = 0 and y = 0 as symmetry planes (wires at $\zeta = \pm jb$). Likewise the receiver loops (two) have the same symmetry planes (wires at $\zeta = \pm jy_3, \pm jy_4$). Note that the two receiver loops can be operated in two different modes depending on the upper or lower signs on I_2 in one of the coils (the one on $\zeta = -jy_3, -jy_4$). If the receiver coils operate in differential mode (lower signs), then the reception is insensitive to a uniform externally incident magnetic field (quasi static). Unfortunately, we do not have such luck in the case of the common mode (upper signs).

The transmitter coil produces a complex potential (as in (2.8))

$$w(\zeta) = \ell n\left(\frac{\zeta - jb}{\zeta + jb}\right) \tag{5.1}$$

for zero coupling from the transmitter coil to each of the receiver coils separately (and hence to both common and differential modes) we need zero magnetic flux to link each of the loops. For the first receiver coil (at $\zeta = jy_3, jy_4$) this is

$$\begin{aligned} \Delta u &= \mathrm{Re}\left[\ell n\left(\frac{jy_4 - jb}{jy_4 + jb}\right) - \ell n\left(\frac{jy_3 - jb}{jy_3 + jb}\right)\right] \\ &= \ell n\left(\frac{(y_4 - b)((b + y_3)}{(y_4 + b)(b - y_3)}\right) \end{aligned} \tag{5.2}$$

Setting this to zero gives

$$y_3 y_4 = b^2 \;,\; y_4 > b \;, 0 < y_3 < b \tag{5.3}$$

By symmetry this applies to the second receiver coil as well. Note that (5.3) allows the receiver coils to be line magnetic dipoles in the limit of $y_3, y_4 \to b$, or rather large coils in the limit of $y_3 \to 0$, $y_4 \to \infty$.

5.1 Behavior near the origin in common mode

The field of the transmitter coil is treated in Section 2.1. That of the receiver coils is

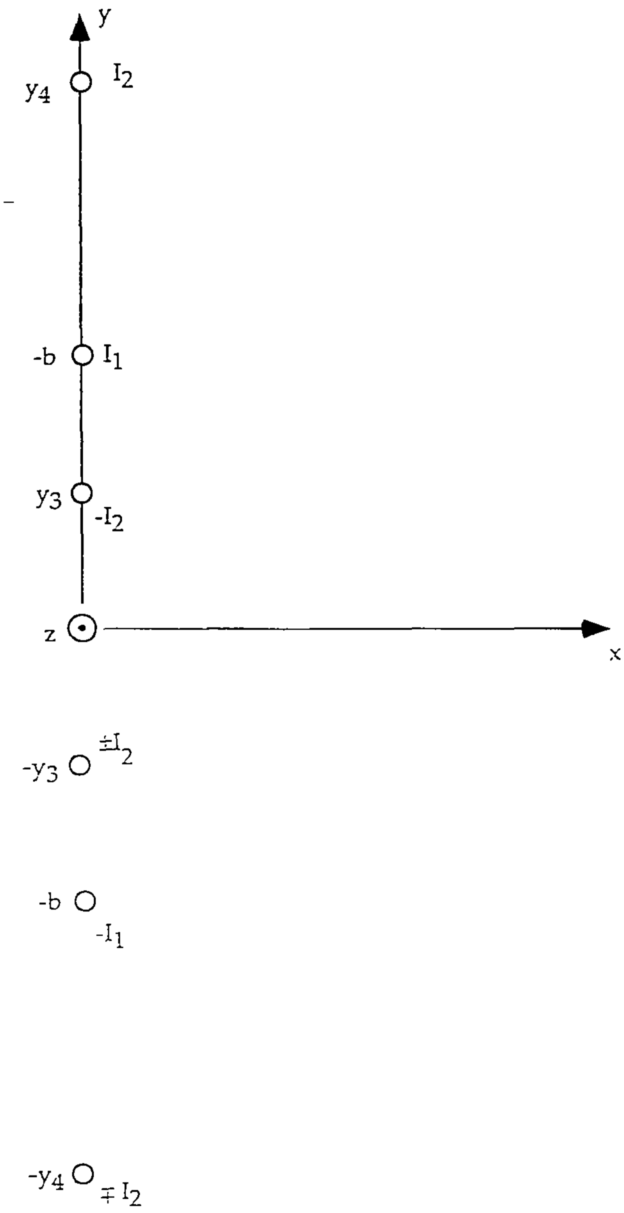


Fig. 5.1. Three Coplanar Loops

$$H_2(\zeta) = -\frac{I_2}{2\pi} j\left\{[\zeta - jy_4]^{-1} - [\zeta - jy_3]^{-1} + [\zeta + jy_3]^{-1} - [\zeta + jy_4]^{-1}\right\}$$

$$= \frac{I_2}{\pi}\left\{y_4\left[\zeta^2 + y_4^2\right]^{-1} - y_3\left[\zeta^2 + y_3^2\right]^{-1}\right\} \qquad (5.4)$$

$$H_2(0) = H_{2x}(0) = -\frac{I_2}{\pi}\left[y_3^{-1} - y_4^{-1}\right]$$

This field is -x directed (for positive I_2) at the origin. By symmetry the odd derivatives are zero at the origin.

Our uniformity function is

$$U(\zeta) = H_1(\zeta)H_2(\zeta) \ , \quad H_1(0) = H_{1x}(0) = \frac{I_1}{2b}$$

$$U(0) = H_1(0)H_2(0) = \frac{I_1 I_2}{\pi^2 b}\left[y_4^{-1} - y_3^{-1}\right] \qquad (5.5)$$

Odd derivatives at the origin are zero. Looking at the second derivatives we have

$$\frac{d^2 H_2(\zeta)}{d\zeta^2} = -\frac{I_2}{\pi} j\left\{[\zeta - jy_4]^{-3} - [\zeta - jy_3]^{-3} + [\zeta + jy_3]^{-3} - [\zeta + jy_4]^{-3}\right\}$$

$$\left.\frac{d^2 H_2(\zeta)}{d\zeta^2}\right|_{\zeta=0} = \frac{2I_2}{\pi}\left[y_3^{-3} - y_4^{-3}\right]$$

$$H_2(\zeta) = H_2(0)\left[1 + \left(\frac{\zeta}{D_2^{(2)}}\right)^2 + \cdots\right] \qquad (5.6)$$

$$D_2^{(2)^2} = -\frac{y_3^{-1} - y_4^{-1}}{y_3^{-3} - y_4^{-3}} = -b^4 \frac{y_4 - y_3}{y_4^3 - y_3^3} = -b^4\left[y_4^2 + b^2 + y_3^2\right]^{-1} < 0$$

$$D_2^{(1)^2} = -b^2 < 0$$

Our uniformity function is then

$$U(\zeta) = U(0)\left[1 + \left(\frac{\zeta}{D_2}\right)^2 + \cdots\right]$$

$$D_2^{-2} = D_2^{(1)^{-2}} + D_2^{(2)^{-2}} < 0 \qquad (5.7)$$

For a given b, D_2 is maximized in magnitude by maximizing $\left|D_2^{(2)}\right|$ which occurs for

$y_3 = y_4 = b$ (line magnetic dipole)

$$D_2^{(2)^2} = -\frac{b^2}{3} \tag{5.8}$$

$$D_2^2 = -\frac{b^2}{4}, \quad D_2 = \pm j\frac{b}{2}$$

So the second derivative (normalized) of the uniformity function cannot be made zero in this configuration, but can be minimized in magnitude by making the receiver coils line magnetic dipoles.

5.2 Behavior near the origin in differential mode

The receiver coil now has a magnetic field

$$\begin{aligned} H_2(\zeta) &= -\frac{I_2}{2\pi} j\left\{[\zeta - jy_4]^{-1} - [\zeta - jy_3]^{-1} - [\zeta + jy_3]^{-1} + [\zeta + jy_4]^{-1}\right\} \\ &= -\frac{I_2}{\pi} j\zeta\left\{\left[\zeta^2 + y_4^2\right]^{-1} - \left[\zeta^2 + y_3^2\right]^{-1}\right\} \end{aligned} \tag{5.9}$$

$$H_2(0) = 0$$

So the differential mode is not suitable for targets near the origin.

5.3 Behavior on the x-axis in common mode

For large ζ the field of the receiver coils is

$$H_2(\zeta) = \frac{I_2}{\pi}\frac{y_4 - y_3}{\zeta^2}\left[1 + O(\zeta^{-2})\right] \text{ as } \zeta \to \infty \tag{5.10}$$

which is the field of a line magnetic dipole with

$$m' = 2(y_4 - y_3)I_2 \tag{5.11}$$

Noting that $H(x)$ is real, is negative at the origin, and is positive for large x, this implies that there is at least one zero on the +x axis. From (5.4) this occurs at x = b, and by symmetry

$$H_2(\pm b) = 0 \tag{5.12}$$

For x > b there is a peak in the $H_2(x)$ where its derivative is zero. However, with (from (2.8))

$$H_1(\zeta) = \frac{I_1}{\pi} b\left[\zeta^2 + b^2\right]^{-1} \tag{5.13}$$

$H_1(x)$ has no peak for x > 0. So let us look for a peak in the uniformity function. Corresponding to (3.13) we form

$$H_1^{-1}(\zeta)\frac{dH_1(\zeta)}{d\zeta} = -2\zeta\left[\zeta^2+b^2\right]^{-1}$$

$$H_2^{-1}(\zeta)\frac{dH_2(\zeta)}{d\zeta} = -2\zeta\left\{y_4\left[\zeta^2+y_4^2\right]^{-1} - y_3\left[\zeta^2+y_3^2\right]^{-1}\right\}^{-1} \tag{5.14}$$

$$\left\{y_4\left[\zeta^2+y_4^2\right]^{-2} - y_3\left[\zeta^2+y_3^2\right]^{-2}\right\}$$

Setting the sum to zero gives the location of the maximum. Noting the constraint in (5.3) one can then find the location of the peak with y_3/b or y_4/b as a parameter.

Concentrating on the case of line-dipole receivers we have

$$H_2(\zeta) = \frac{m_2'}{2\pi}\left\{\left[\zeta - jb\right]^{-2} + \left[\zeta + jb\right]^{-2}\right\}$$

$$= \frac{m_2'}{\pi}\left\{\left[\zeta^2-b^2\right]\left[\zeta^2+b^2\right]^{-2}\right\}$$

$$m_2' = (y_4 - y_3)I_2 \quad \text{(constant as } y_3, y_4 \to b) \tag{5.15}$$

$$\frac{dH_2(\zeta)}{d\zeta} = \frac{2m_2'}{\pi}\zeta\left[-\zeta^2+3b^2\right]\left[\zeta^2+b^2\right]^{-3}$$

$$H_2^{-1}(\zeta)\frac{dH_2(\zeta)}{d\zeta} = 2\zeta\left[-\zeta^2+3b^2\right]\left[\zeta^2-b^2\right]^{-1}\left[\zeta^2+b^2\right]^{-1}$$

Summing this last expression with the first in (5.14) gives the location of the maximum on the x axis as

$$\frac{\zeta_0}{b} = \frac{x_0}{b} = \pm\sqrt{2} = \pm 1.414 \tag{5.16}$$

Continuing the development we have

$$U(\zeta) = H_1(\zeta)H_2(\zeta) = \frac{I_1 b m_2}{\pi^2}\left[\zeta^2-b^2\right]\left[\zeta^2+b^2\right]^{-3}$$

$$U(\zeta_0) = \frac{I_1 b m_2'}{\pi^2}(27b^4)^{-1} \tag{5.17}$$

$$\left.\frac{dU}{d\zeta}\right|_{\zeta=\zeta_0} = 0$$

$$\left.\frac{d^2U(\zeta)}{d\zeta^2}\right|_{\zeta=\zeta_0} = U(\zeta_0)\left[1+\left(\frac{\zeta-x_0}{D_2}\right)^2+\cdots\right]$$

$$D_2^2 = -\frac{3}{4}b^2\ ,\ \frac{D_2}{b} = \pm j\frac{\sqrt{3}}{2} \approx \pm .866j$$

So one can envision a region near $x_0 = \sqrt{2}b$ for measuring target magnetic polarizability (x,x component). This can be a region below the ground surface (surface of constant x with $0 < x < x_0$).

5.4 Behavior on the x-axis in differential mode

For large ζ the field of the receiver coils is

$$H_2(\zeta) = \frac{I_2}{\pi}j\zeta^{-3}\left[y_4^2 - y_3^2\right]\left[1+O(\zeta^{-2})\right] \text{ as } \zeta \to \infty \tag{5.18}$$

which is the field of a line magnetic quadrupole. Noting that $H_2(x)$ is imaginary we have

$$H_2(x) = -jH_{2y}(x)$$

$$H_{2y}(x) = -\frac{I_2}{\pi}x\left\{\left[x^2+y_3^2\right]^{-1} - \left[x^2+y_4^2\right]^{-1}\right\} \tag{5.19}$$

$$= -\frac{I_2}{\pi}x^{-3}\left[y_4^2 - y_3^2\right]^{-1} - \left[1+O(x^{-2})\right] \text{ as } x \to \pm\infty$$

so that on the $\pm x$ axis the magnetic field is the $\mp y$ direction.

There is a peak in $H_{2y}(x)$ where its derivative is zero. From (5.9) we have

$$\frac{dH_2(\zeta)}{d\zeta} = \frac{I}{2\pi}j\left\{[\zeta-jy_4]^{-2} - [\zeta-jy_3]^{-2} - [\zeta+jy_3]^{-2} + [\zeta+jy_4]^{-2}\right\}$$

$$= \frac{I}{\pi}j\left\{\frac{\zeta^2-y_4^4}{\left[\zeta^2+y_4^2\right]^2} - \frac{\zeta^2-y_3^2}{\left[\zeta^2+y_3^2\right]^2}\right\} \tag{5.20}$$

This has a zero on the x axis at

$$\zeta^2 = -\frac{y_3^2+y_4^2}{6} + \frac{1}{6}\left[y_3^4+14y_3^2+y_4^4\right]^{\frac{1}{2}} \tag{5.21}$$

which simplifies in the case of line dipoles ($y_3, y_4 \to b$) to

$$\frac{\zeta}{b} = \pm\frac{1}{\sqrt{3}} \approx \pm .577 \tag{5.22}$$

However, $H_1(x)$ does not have a zero here.

Concentrating then on the case of line-dipole receivers we have

$$H_2(\zeta) = \frac{m_2'}{2\pi}\left\{[\zeta - jb]^{-2} - [\zeta + jb]^{-2}\right\}$$

$$= \frac{2}{\pi} m_2' jb\zeta\left[\zeta^2 + b^2\right]^{-2}$$

$$m_2' = (y_4 - y_3)I_2 \quad \text{(constant as } y_3, y_4 \to \text{b)} \tag{5.23}$$

$$\frac{dH_2(\zeta)}{d\zeta} = \frac{2}{\pi} m_2' jb\left[-3\zeta^2 + b^2\right]\left[\zeta^2 + b^2\right]^{-3}$$

$$H_2^{-1}(\zeta)\frac{dH_2(\zeta)}{d\zeta} = \zeta^{-1}\left[-3\zeta^2 + b^2\right]\left[\zeta^2 + b^2\right]^{-1}$$

Summing this with the first in (5.14) gives the location of the maximum on the x axis as

$$\frac{\zeta_0}{b} = \frac{x_0}{b} = \pm\frac{1}{\sqrt{5}} \simeq \pm.447 \tag{5.24}$$

Continuing the development we have

$$U(\zeta) = H_1(\zeta)H_2(\zeta) = j2\frac{I_1 b^2 m_2'}{\pi^2}\zeta\left[\zeta^2 + b^2\right]^{-3}$$

$$U(\zeta_0) = j\frac{I_1 b m_2'}{\pi^2}\frac{2}{\sqrt{5}}\left(\frac{5}{6}\right)^3 b^{-4}$$

$$\left.\frac{dU}{d\zeta}\right|_{\zeta=\zeta_0} = 0 \tag{5.25}$$

$$\frac{d^2u(\zeta)}{d\zeta^2} = U(\zeta_0)\left[1 + \left(\frac{\zeta - x_0}{D_2}\right)^2 + \cdots\right]$$

$$D_2^2 = -\frac{6}{25}b^2, \quad \frac{D_2}{b} = \pm j\frac{\sqrt{6}}{5} \simeq \pm.490$$

There is then a region near $x_0 = b/\sqrt{5}$ for measuring the y,x component (cross pol) of the magnetic polarizability. This can also be a region below the ground surface (surface of constant x with $0 < x < x_0$).

6. Two Pairs of Collocated Line Magnetic Dipoles

As we progress to more complex loop structures, the analytic expressions for the fields and related optimization become more elaborate. For coils of small cross section (two-dimensional sense) these can be approximated as line magnetic dipoles. As we have seen in previous sections this simplifies the expressions somewhat. So now let both transmitter and receiver coils (two each) be line magnetic dipoles. The receiver coils are aligned antiparallel for insensitivity to externally incident low-frequency magnetic fields. Three cases are considered as indicated in fig. 6.1. Transmitter coils and fields are denoted by subscript 1; receiver coils and fields are denoted by subscript 2. As before the x = 0 and y = 0 planes are symmetry planes, about which the transmitter and receiver fields are symmetric or antisymmetric, to determine the appropriate magnetic-polarizability components. As indicated in fig. 6.1, the line dipoles are located at $(x, y) = (0, \pm b)$, and the transmitter coils do not couple to the receiver coils due to the symmetry. Our interest is in the magnetic fields on the +y axis away from the origin.

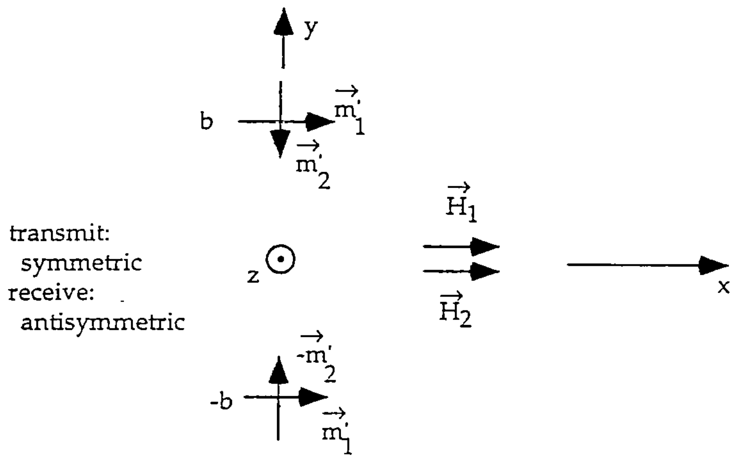
6.1 Transmit x, receive x: $M_{x,x}$

As our first example, consider the case depicted in fig. 6.1A. The transmitter coils and fields are the same as the case in Section 2.4. The fields and their derivatives are

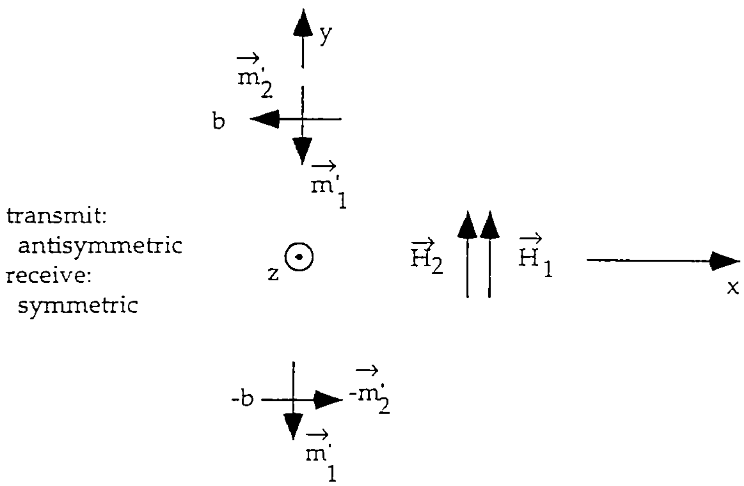
$$\begin{aligned} H_1(\zeta) &= \frac{m_1'}{2\pi}\left\{[\zeta - jb]^{-2} + [\zeta + jb]^{-2}\right\} \\ \frac{dH_1(\zeta)}{d\zeta} &= \frac{m_1'}{\pi}\left\{-[\zeta - jb]^{-3} - [\zeta + jb]^{-3}\right\} \\ \frac{d^2H_1(\zeta)}{d\zeta^2} &= \frac{3m_1'}{\pi}\left\{[\zeta - jb]^{-4} + [\zeta + jb]^{-4}\right\} \\ m_1' &= |m_1'| = \left|\vec{m}'_1\right| \end{aligned} \tag{6.1}$$

Similarly for the fields of the receiver coils we have

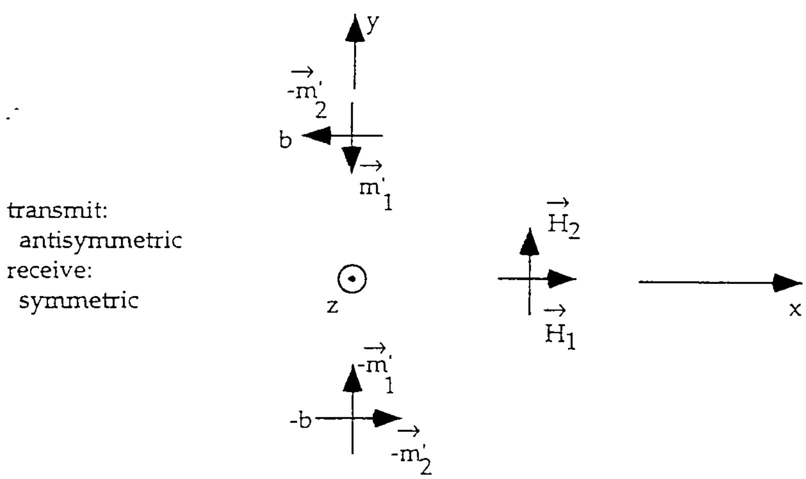
$$\begin{aligned} H_2(\zeta) &= \frac{m_2'}{2\pi}\left\{[\zeta - jb]^{-2} - [\zeta + jb]^{-2}\right\} \\ \frac{dH_2(\zeta)}{d\zeta} &= \frac{m_2'}{\pi}\left\{-[\zeta - jb]^{-3} + [\zeta + jb]^{-3}\right\} \end{aligned} \tag{6.2}$$



A. Transmit x, receive y: $M_{x,x}$



B. Transmit y, receive y: $M_{y,y}$



C. Transmit x, receive y: $M_{y,x}$

Fig. 6.1. Two Pairs of Colocated Line Magnetic Dipoles

$$\frac{d^2 H_2(\zeta)}{d\zeta^2} = \frac{3m_2'}{\pi}\left\{[\zeta - jb]^{-4} - [\zeta + jb]^{-4}\right\}$$

$$m_2' = |m_2'|e^{-j\frac{\pi}{2}} = -j|m_2'| = -j\left|\vec{m}'_2\right|$$

Form the field product and derivatives

$$U(\zeta) = H_1(\zeta)H_2(\zeta) = \frac{m_1' m_2'}{4\pi^2}\left\{[\zeta - jb]^{-4} - [\zeta + jb]^{-4}\right\}$$

$$= \frac{2|m_1'||m_2'|}{\pi^2 b^4}\frac{\zeta}{b}\left[\left(\frac{\zeta}{b}\right)^2 - 1\right]\left[\left(\frac{\zeta}{b}\right)^2 + 1\right]^{-4}$$

$$\frac{dU(\zeta)}{d\zeta} = \frac{m_1' m_2'}{\pi^2}\left\{-[\zeta - jb]^{-5} + [\zeta + jb]^{-5}\right\}$$

$$= \frac{-2|m_1'||m_2'|}{\pi^2 b^5}\frac{\zeta}{b}\left[5\left(\frac{\zeta}{b}\right)^4 - 10\left(\frac{\zeta}{b}\right)^2 + 1\right]\left[\left(\frac{\zeta}{b}\right)^2 + 1\right]^{-5} \qquad (6.3)$$

$$\frac{d^2 U(\zeta)}{d\zeta^2} = \frac{5m_1' m_2'}{\pi^2}\left\{-[\zeta - jb]^{-6} - [\zeta + jb]^{-6}\right\}$$

$$= \frac{20|m_1'||m_2'|}{\pi^2 b^6}\frac{\zeta}{b}\left[3\left(\frac{\zeta}{b}\right)^4 - 10\left(\frac{\zeta}{b}\right)^2 + 3\right]\left[\left(\frac{\zeta}{b}\right)^2 + 3\right]^{-6}$$

The derivative has zeros on the x axis at

$$\left(\frac{\zeta_0}{b}\right)^2 = -\pm\left[\frac{4}{5}\right]^{\frac{1}{2}} \qquad (6.4)$$

Choose for convenience, the zeros farther from the coil at

$$\frac{\zeta_0}{b} = \left[1 + \left[\frac{4}{5}\right]^{\frac{1}{2}}\right]^{\frac{1}{2}} \simeq 1.376 \qquad (6.5)$$

at which we have an effective uniformity distance of

$$D_2^2 = 2\left[\frac{d^2U(\zeta_0)}{d\zeta^2}\right]^{-1} U(\zeta_0)$$

$$= \frac{b^2}{5}\left[\left(\frac{\zeta_0}{b}\right)^2 - 1\right]\left[\left(\frac{\zeta_0}{b}\right)^2 + 1\right]^2\left[3\left(\frac{\zeta_0}{b}\right)^4 - 10\left(\frac{\zeta_0}{b}\right)^2 + 3\right]^{-1} \tag{6.6}$$

$$\frac{D_2}{b} = \pm j\frac{1}{\sqrt{5}}\left\{\left[\frac{4}{5}\right]^{\frac{1}{2}}\left[\frac{6}{5}+\left[\frac{4}{5}\right]^{\frac{1}{2}}\right]\left[\frac{2}{5}+\left[\frac{4}{5}\right]^{\frac{1}{2}}\right]^{-1}\right\}^{\frac{1}{2}}$$

$$\simeq \pm .538j$$

6.2 Transmit y, receive y: $M_{y,y}$

The second case is depicted in fig. 6.1B. The fields and the uniformity function are

$$H_1(\zeta) = \frac{m_1'}{2\pi}\left\{[\zeta - jb]^{-2} + [\zeta + jb]^{-2}\right\}$$

$$H_2(\zeta) = \frac{m_2'}{2\pi}\left\{[\zeta - jb]^{-2} - [\zeta + jb]^{-2}\right\}$$

$$U(\zeta) = H_1(\zeta)H_2(\zeta) = \frac{m_1'm_2'}{4\pi^2}\left\{[\zeta - jb]^{-4} - [\zeta + jb]^{-4}\right\} \tag{6.7}$$

$$m_1' = |m_1'|e^{-j\frac{\pi}{2}} = -j|m_1'| = -j\left|\vec{m'}_1\right|$$

$$m_2' = |m_2'|e^{j\pi} = -|m_2'| = -\left|\vec{m'}_2\right|$$

Except for a coefficient due to the rotation of the line magnetic dipoles by $-\pi/2$ from the previous case, the formulas are the same as in Section 6.1. The solutions are the same and need not be repeated. As indicated in fig. 6.1B, along with the rotation of the m's there is a rotation of the Hs (opposite sense).

6.3 Transmit x, receive y: $M_{y,x}$

The third case (cross polarization) is depicted in fig. 6.1C. The fields and their derivatives for the transmitter coils (H_1) are the same as for H_2 in (6.7), or as for H_2 in (6.2) with the line magnetic dipoles rotated. Summarizing we have

$$H_1(\zeta) = \frac{m_1'}{2\pi}\left\{[\zeta - jb]^{-2} - [\zeta + jb]^{-2}\right\}$$

$$H_2(\zeta) = \frac{m_2'}{2\pi}\left\{[\zeta - jb]^{-2} - [\zeta + jb]^{-2}\right\}$$

$$m_1' = |m_1'|e^{-j\frac{\pi}{2}} = -j|m_1'| = -j\left|\vec{m}'_1\right| \tag{6.8}$$

$$m_2' = |m_2'|e^{j\pi} = -|m_2'| = -\left|\vec{m}'_2\right|$$

Note that, except for the coefficients, the two fields have the same form. This is due to the fact that both sets of coils are in differential configurations.

While (6.8) is cast in the form of transmit x and receive y, the role of the coils can be interchanged (by interchanging 1 and 2 subscripts) to obtain the same result. This merely results from reciprocity ($M_{y,x} = M_{x,y}$). Each pair of coils having antiparallel line-magnetic-dipole moments, both coil pairs are insensitive to externally incident low-frequency magnetic fields. Said another way each coil pair is a line magnetic quadrupole.

Forming the uniformity function we have

$$U(\zeta) = H_1(\zeta)H_2(\zeta) = \frac{m_1'm_2'}{4\pi^2}\left\{[\zeta - jb]^{-2} - [\zeta + jb]^{-2}\right\}^2$$

$$= -\frac{4m_1'm_2'}{\pi^2 b^4}\left(\frac{\zeta}{b}\right)^2\left[\left(\frac{\zeta}{b}\right)^2 + 1\right]^{-4}$$

$$\frac{dU(\zeta)}{d\zeta} = \frac{8m_1'm_2'}{\pi^2 b^5}\frac{\zeta}{b}\left[3\left(\frac{\zeta}{b}\right)^2 - 1\right]\left[\left(\frac{\zeta}{b}\right)^2 + 1\right]^{-5} \tag{6.9}$$

$$\frac{d^2U(\zeta)}{d\zeta^2} = \frac{8m_1'm_2'}{\pi^2 b^6}\left[30\left(\frac{\zeta}{b}\right)^4 - 19\left(\frac{\zeta}{b}\right)^2 + 1\right]\left[\left(\frac{\zeta}{b}\right)^2 + 1\right]^{-6}$$

The derivative has zeros on the x axis at

$$\frac{\zeta_0}{b} = \frac{x_0}{b} = \pm\frac{1}{\sqrt{3}} = \pm .577 \tag{6.10}$$

the effective uniformity distance is

$$D_2^2 = 2\left[\frac{d^2U(\zeta_0)}{d\zeta_0^2}\right]^{-1} U(\zeta_0) = b^2\left(\frac{\zeta_0}{b}\right)^2\left[\left(\frac{\zeta_0}{b}\right)^2+1\right]^2\left[30\left(\frac{\zeta_0}{b}\right)^4-19\left(\frac{\zeta_0}{b}\right)^2+1\right]^{-1}$$

$$= -b^2\frac{8}{27} \tag{6.11}$$

$$\frac{D_2}{b} = \pm j\left[\frac{8}{27}\right]^{\frac{1}{2}} = \pm .544j$$

6.4 Extension to loops of larger cross section

While the foregoing discussion has been for line magnetic dipoles, the important features of the designs are extendible to loops of non-zero cross section dimensions as indicated in fig. 6.2. While the configuration has reflection symmetry with respect to both x = 0 and y = 0 planes, it is the x = 0 plane of concern here.

Designate the two loops at $y = y_5, y_6$ and $y = -y_5, -y_6$ on the x = 0 plane as the x loops due to the fact that their line-magnetic-dipole moments have only ±x components. These loops product a symmetric magnetic-field distribution [5, 11] with respect to the x = 0 plane, regardless of whether the moments are parallel or antiparallel. Designate the two loops at $(x,y) = (\pm a, b)$ and $(x,y) = (\pm a, -b)$ as the y-loops due to the fact that their line-magnetic-dipole moments have only ±y components. These loops produce an antisymmetric magnetic-field distribution [5, 11] with respect to the x = 0 plane, regardless of whether the moments are parallel or antiparallel. This fundamental symmetry property assures that no x-loop couples to any y-loop, and conversely. Enlarging the loops from zero-cross-section line magnetic dipoles to larger dimensions as in fig. 6.2 then preserves this zero-coupling property.

With the greater sensitivity of larger loop cross sections, the designs discussed previously in this section become more practical. Of course, the previous formulas then have some error when applied to such a configuration as in fig. 6.2. If desired, the formulas can be corrected by use of the formulas for line currents (eight of them here), thereby giving more exact but more complicated results.

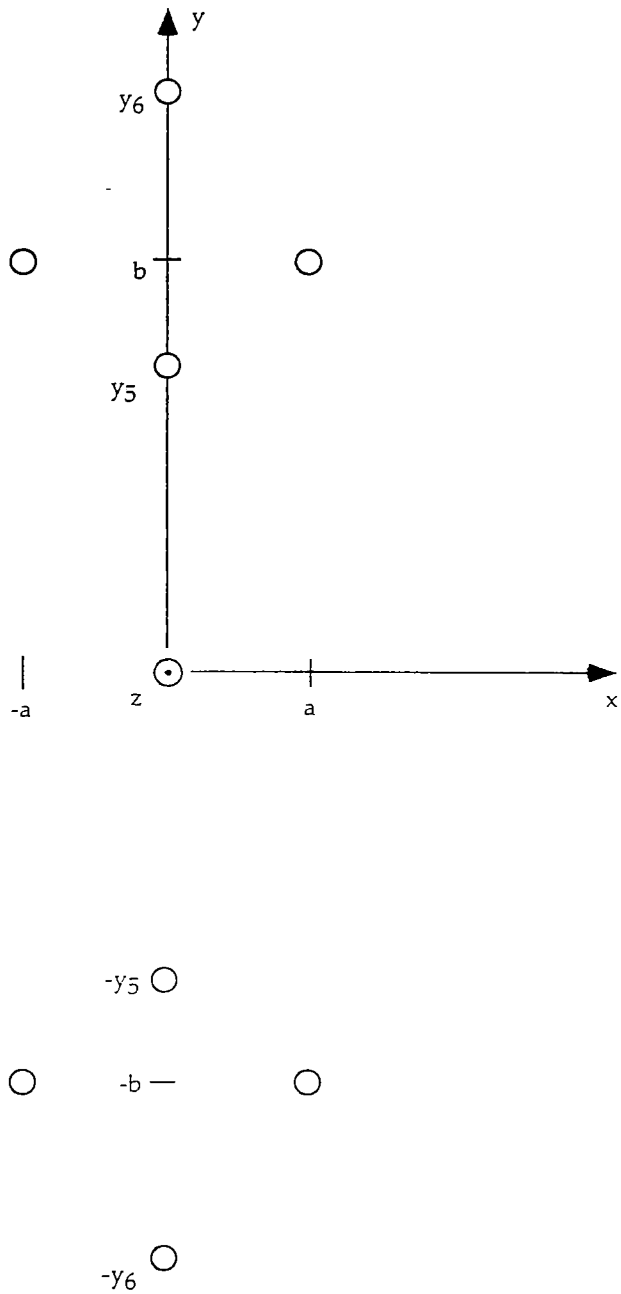


Fig. 6.2. Larger Coils Still Retaining The Requisite Symmetry

7. Crossed Loops

Returning to the case in which the target of interest is "inside" the loop structure consider the two-wire and four-wire loops discussed in Section 2. Now let there be two such loops in a cross-polarized configuration as illustrated in fig. 7.1. As in the previous sections the transmitter loop and associated fields are denoted by subscript 1, and for the receiver by 2.

The symmetry of these configurations has the transmitter fields symmetric with respect to the x = 0 plane and antisymmetric with respect to the y = 0 plane. For the receiver fields the roles of these two planes are interchanged. As such there is no mutual inductance coupling the transmitter and receiver coils, except via the target. However, the receiver coil is sensitive to externally incident low-frequency magnetic fields. The target is assumed to be near the z axis so that this configuration is suitable for measuring $M_{x,y}$ $(= M_{y,x})$. For added symmetry the transmitter and receiver coils are identical except for a rotation of $\pi / 2$

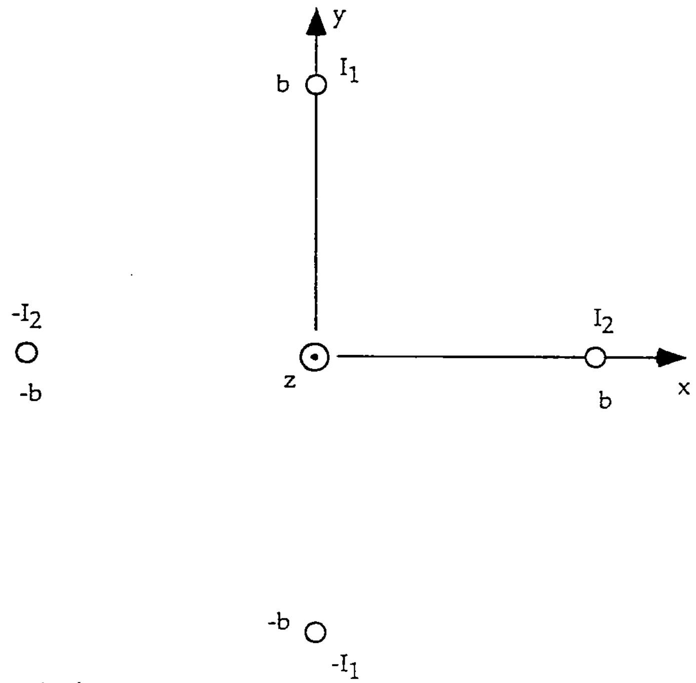
7.1 Two-wire loops

For the configuration in fig. 7.1A we have, related to Section 2.1, the fields and uniformity function

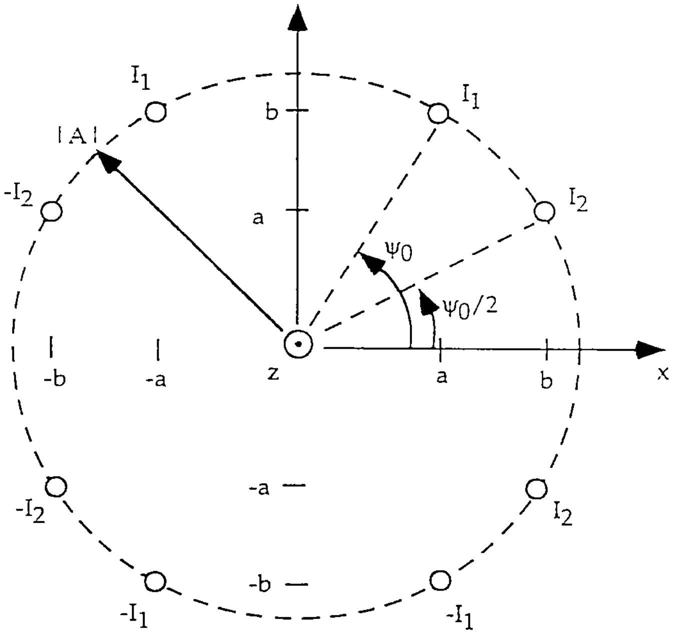
$$
\begin{aligned}
H_1(\zeta) &= -\frac{I_1}{2\pi} j\left\{[\zeta - jb]^{-1} - [\zeta + jb]^{-1}\right\} = \frac{I_1}{\pi} b\left[\zeta^2 + b^2\right]^{-1} \\
H_2(\zeta) &= -\frac{I_2}{2\pi} j\left\{[\zeta - b]^{-1} - [\zeta + b]^{-1}\right\} = -\frac{I_2}{\pi} jb\left[\zeta^2 - b^2\right]^{-1} \\
U(\zeta) = H_1(\zeta) H_2 &= -\frac{I_1 I_2}{\pi^2} jb^2\left[\zeta^4 - b^4\right]^{-1} \\
&= \frac{I_1 I_2}{\pi^2 b^2} j\left[1 - \left(\frac{\zeta}{b}\right)^4\right]^{-1} \\
&= \frac{I_1 I_2}{\pi^2 b^2} j\left[1 + \left(\frac{\zeta}{b}\right)^4 + O\left(\left(\frac{\zeta}{b}\right)^8\right)\right]^{-1} \quad \text{as} \quad \frac{\zeta}{b} \to 0
\end{aligned}
\tag{7.1}
$$

From this we conclude that the first "error" term is of fourth order and we have

$$
\begin{aligned}
\zeta_0 &= x_0 = 0 \\
U(0) &= \frac{I_1 I_2}{\pi^2 b^2} j
\end{aligned}
\tag{7.2}
$$



A. Two-wire loops



B. Four-wire loops

Fig. 7.1. Cross-Polarized Loops

$$D_4^4 = b^4, \frac{D_4}{b} = 1^{\frac{1}{4}}$$

$$1^{\frac{1}{4}} = \pm 1, \pm j$$

As we can see this is a quite uniform configuration for targets near the coordinate origin. The direction of field increase for one coil is the direction of field decrease for the second coil, the two second derivatives canceling each other and leaving the fourth derivative as the first "error" term.

7.2 Four-wire loops

For the configuration in fig. 7.1B we have, related to Section 2.2, the wires for loop 1 are located at

$$\begin{aligned} &\zeta = \pm A, \pm A^* \\ &A = a + jb = |A| e^{j\frac{\pi}{3}}, |A| = \left[a^2 + b^2\right]^{\frac{1}{2}} \\ &a = |A| \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}|A| \\ &b = |A| \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}|A| \approx .866|A| \end{aligned} \tag{7.3}$$

For loop 2 the configuration is rotated by $-\pi/2$ giving locations

$$\zeta = \pm jA, \pm jA^* \tag{7.4}$$

The fields and uniformity function are then

$$\begin{aligned} H_1(\zeta) &= -\frac{I_1}{2\pi} j \left\{ [\zeta - A]^{-1} - [\zeta - A^*]^{-1} + [\zeta + A^*]^{-1} - [\zeta + A]^{-1} \right\} \\ &= -\frac{I_1}{\pi} j \left\{ \frac{A}{\zeta^2 - A^2} - \frac{A^*}{\zeta^2 - A^{*2}} \right\} \\ &= \frac{2I_1}{\pi} b \frac{\zeta^2 + |A|^2}{\left[\zeta^2 - A^2\right]\left[\zeta^2 - A^{*2}\right]} \end{aligned}$$

$$H_2(\zeta) = -\frac{I_2}{2\pi} j\left\{[\zeta + jA]^{-1} - [\zeta + jA*]^{-1} + [\zeta - jA*]^{-1} - [\zeta - jA]^{-1}\right\}$$

$$= -\frac{I_2}{\pi} j\left\{\frac{A}{\zeta^2 + A^2} - \frac{A*}{\zeta^2 + A*^2}\right\} \tag{7.5}$$

$$= \frac{2I_2}{\pi} jb \frac{\zeta^2 - |A|^2}{[\zeta^2 + A^2][\zeta^2 + A*^2]}$$

$$U(\zeta) = H_1(\zeta)H_2(\zeta) = \frac{4I_1I_2}{\pi^2} jb^2 \frac{\zeta^4 - |A|^4}{[\zeta^4 - A^4][\zeta^4 - A*^4]}$$

$$= -\frac{4I_1I_2}{\pi^2} j \frac{b^2}{|A|^4} \frac{1 - \left(\frac{\zeta}{|A|}\right)^4}{\left[1 - \left(\frac{\zeta}{A}\right)^4\right]\left[1 - \left(\frac{\zeta}{A*}\right)^4\right]}$$

$$= -\frac{3I_1I_2}{\pi^2|A|^2} j\left[1 - 2\left(\frac{\zeta}{|A|}\right)^4 + O\left(\left(\frac{\zeta}{|A|}\right)^8\right)\right] \quad \text{as } \frac{\zeta}{|A|} \to 0$$

Again the first "error" term is of fourth order and we have

$$U(0) = -\frac{3I_1I_2}{\pi^2|A|^2} J = -\frac{9I_1I_2}{4\pi^2b^2}$$

$$D_{\frac{1}{4}}^{\frac{1}{4}} = -\frac{1}{2}|A|^4 = -\frac{8}{9}b^4, \ \frac{D_{\frac{1}{4}}}{b} = [-1]^{\frac{1}{4}}\left[\frac{8}{9}\right]^{\frac{1}{4}} \simeq .943[-1]^{\frac{1}{4}} \tag{7.6}$$

8. Concluding Remarks

This paper has considered a number of analytically tractable, two-dimensional coil designs. These designs are a progression of cases from simple to more complex geometries. For certain regions of space (inside or away from the coils) these can produce a uniform magnetic field or uniformity of detection appropriate for determining components of target magnetic-polarizability dyadics.

Besides the relation of the coils to the target there are other conditions to consider. Various examples here have no mutual coupling (ideally, with perfect geometry) between transmitter and receiver coils. Furthermore, various receiver-coil designs are insensitive to externally-incident uniform low-frequency magnetic fields. The basic unifying concept in these considerations is symmetry.

Note that the idea of two-dimensional coils is only an approximation, the coils being of finite length in the z direction compared to cross-section dimensions. So end effects will need to be considered. In addition, one would also like to transmit and receive z-directed magnetic fields, but this requires a different type of coil, one that will need to be combined with the foregoing designs.

Appendix A. Complex Potentials and Fields

For two-dimensional electromagnetic-field problems where the cross-section fields are governed by the Laplace equation (quasi-static problems and TEM modes) it is convenient to use

$$\begin{aligned} \zeta &= x + jy = \Psi e^{j\phi} \equiv \text{ complex coordinates} \\ w(\zeta) &= u(\zeta) + jv(\zeta) \equiv \text{ complex potential} \end{aligned} \tag{A.1}$$

where $w(\zeta)$ is a complex analytic function of ζ and is also referred to as a conformal transformation. For electric problems we have

$$\begin{aligned} \Phi_e(x,y) &= \frac{V}{\Delta u} u(\zeta) \equiv \text{electric potential} \\ V &\equiv \text{ voltage between appropriate conductors} \\ \Delta u &\equiv \text{ change in } u \text{ between these conductors} \\ \vec{E}(x,y) &= -\nabla\Phi(x,y) \equiv \text{ electric field} \end{aligned} \tag{A.2}$$

For magnetic problems we similarly have

$$\begin{aligned} \Phi_h(x,y) &= \frac{I}{\Delta v} v(\zeta) \equiv \text{ magnetic potential} \\ I &\equiv \text{ current on appropriate conductor (positive in } +z \text{ direction)} \\ \Delta v &\equiv \text{ change in } v \text{ in going around this conductor} \\ \vec{H}(x,y) &= -\nabla\Phi_h(x,y) \equiv \text{ magnetic field} \end{aligned} \tag{A.3}$$

where there is a discontinuity in v (but not the magnetic field) in traversing the branch cut as one goes around the conductor.

The fields also can be cast in a convenient complex form [2, 4, 9, 12] as

$$\begin{aligned} \vec{E}(x,y) &= -\frac{V}{\Delta u}\vec{e}_0(x,y) \\ \vec{e}_0(x,y) &= \nabla u(x,y) = e_{0_x}\vec{1}_x + e_{0_y}\vec{1}_y \\ e_0(\zeta) &\equiv e_{0_x}(\zeta) - je_{0_y}(\zeta) = \frac{dw(\zeta)}{d\zeta} \\ E(\zeta) &= E_x - jE_y = -\frac{V}{\Delta u}e_0(\zeta) = -\frac{V}{\Delta u}\frac{dw(\zeta)}{d\zeta} \end{aligned} \tag{A.4}$$

$$\vec{H}(x,y) = -\frac{I}{\Delta v}\vec{h}_0(x,y)$$

$$\vec{h}_0(x,y) = \nabla v(x,y) = h_{0_x}\vec{1}_x + h_{0_y}\vec{1}_y$$

$$\therefore \quad h_0(\zeta) \equiv h_{0_x}(\zeta) - jh_{0_y}(\zeta) = -j\frac{dw(\zeta)}{d\zeta} = -je_0(\zeta)$$

$$H(\zeta) = H_x - jH_y = -\frac{I}{\Delta v}h_0(\zeta) = \frac{I}{\Delta v}j\frac{dw(\zeta)}{d\zeta}$$

So fields can be conveniently and simply expressed via $dw/d\zeta$. For present purposes it is the magnetic fields that are of interest. The field components are directly obtained from the real and imaginary parts of $dw/d\zeta$.

Appendix B. Potentials and Fields of Line Currents

For a single filamentary current I (typically on a wire of small radius) located at $\zeta = \zeta' = x' + jy'$, we have a complex potential

$$w = -\ell n(\zeta - \zeta') \tag{B.1}$$

consistent with the usage in [1]. One could add an arbitrary constant to this. However, our interest in this type of potential is as a sum over two or more such terms weighted by the currents (including signs) on the wires at the various choices of ζ'. Including a constraint that the sum of the currents be zero for any of our two-dimensional loops, an additive constant cancels. Also for large ζ the above form has $|w| \to \infty$ as $|\zeta| \to \infty$, but with the constraint of zero current sum the resulting potential goes to zero for large $|\zeta|$. There is a branch cut in the logarithm function which can be placed at our convenience, considering the ensemble of wires at the various ζ'. In going around a wire we have

$$\Delta v = 2\pi \tag{B.2}$$

For computing the fields we need

$$\frac{dw}{d\zeta} = -[\zeta - \zeta']^{-1} = e_0(\zeta) = jh_0(\zeta) \tag{B.3}$$

and for later use we need

$$\frac{d^n w}{d\zeta^n} = (-1)_n (n-1)! [\zeta - \zeta']^{-n} \;, n \ge 1 \tag{B.4}$$

when considering the uniformity of the magnetic field produced in the vicinity of selected points. Specializing to the magnetic field we have

$$H(\zeta) = H_x - jH_y = -\frac{I}{\Delta v} h_0(\zeta) = \frac{I}{\Delta v} j \frac{dw}{d\zeta} = -\frac{I}{\Delta v} j[\zeta - \zeta']^{-1}$$

$$\frac{d^n H}{d\zeta^n} = \frac{I}{\Delta v} j(-1)^{n+1} n! [\zeta - \zeta']^{-n-1} \;, n \ge 0$$

Again note that when constructing the fields as well as the potentials for an ensemble of wires (two-dimensional loop), one sums over terms as above with the constraint that the sum of the currents be zero.

Appendix C. Potentials and Fields of Line Magnetic Dipoles

Start from two line currents near the origin as indicated in fig. C.1. With current I at complex coordinate

$$\zeta_m = de^{j\left(\frac{\pi}{2}+\psi_m\right)} = dje^{j\psi_m} = d[-\sin(\psi_m) + j\cos(\psi_m)] \tag{C.1}$$

and $-I$ at $-\zeta'$ we have a line magnetic dipole moment

$$\vec{m}' = 2dI\left[\cos(\psi_m)\,\vec{1}_x + \sin(\psi_m)\,\vec{1}_y\right]$$
$$\left|\vec{m}'\right| = 2dI \tag{C.2}$$

The potential is formed from (B.1) as

$$\begin{aligned} w(\zeta) &= -\ell n(\zeta - \zeta_m) + \ell n(\zeta + \zeta_m) \\ &= \ell n\left(\frac{\zeta + \zeta_m}{\zeta - \zeta_m}\right) = \ell n\left(\frac{1 + \frac{\zeta_m}{\zeta}}{1 - \frac{\zeta_m}{\zeta}}\right) \\ &= 2\frac{\zeta_m}{\zeta} + O\left(\left(\frac{\zeta_m}{\zeta}\right)^3\right) \text{ as } \frac{\zeta'}{\zeta} \to 0 \end{aligned} \tag{C.3}$$

Let $d \to 0$ with $\vec{m}'$ fixed

$$\begin{aligned} w'(\zeta) &\equiv \lim_{d\to 0} \frac{w(\zeta)}{2d} = je^{j\psi_m}\zeta^{-1} = u'(\zeta) + jv'(\zeta) \\ \Phi_h &= \lim_{d\to 0} -\frac{\left|\vec{m}'\right|}{2d}\frac{v(\zeta)}{\Delta v} = \frac{\left|\vec{m}'\right|}{2\pi}v' \\ H(\zeta) &= \lim_{d\to 0} -\frac{\left|\vec{m}'\right|}{2d}\frac{h_0(\zeta)}{\Delta v} = \lim_{d\to 0} -\frac{\left|\vec{m}'\right|}{2d}\frac{j}{\Delta v}\frac{dw(\zeta)}{d\zeta} \\ &= \left|\vec{m}'\right|\frac{j}{\Delta v}\frac{dw_m(\zeta)}{d\zeta} = \left|\vec{m}'\right|\frac{e^{j\psi_m}}{2\pi}\zeta^{-2} \end{aligned} \tag{C.4}$$

This allows us to write a complex form for the line magnetic dipole as

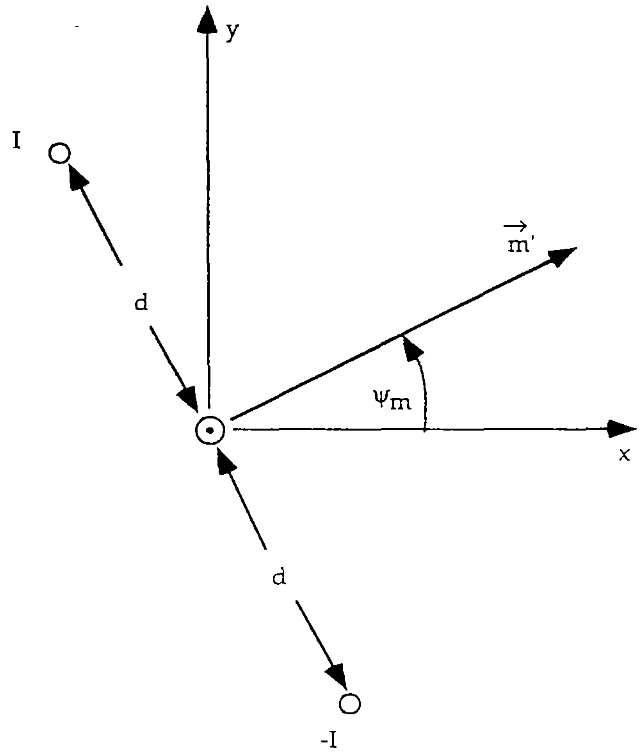


Fig. C.1. Line Magnetic Dipole

$$m' = 2dIe^{j\psi_m} = m'_x + jm'_y$$
$$|m'| = |\vec{m'}| \tag{C.5}$$

giving the complex potential and magnetic field

$$w_m(\zeta) = j\frac{m'}{|m'|}\zeta^{-1}$$
$$H(\zeta) = \frac{m'}{2\pi}\zeta^{-2} \tag{C.6}$$

In this complex form m' has orientation (argument) ψ_m.

This line magnetic dipole can be shifted to a complex coordinate ζ' giving

$$w_m(\zeta) = j\frac{m'}{|m'|}[\zeta - \zeta']^{-1}$$
$$H(\zeta) = \frac{m'}{2\pi}[\zeta - \zeta']^{-2} \tag{C.7}$$

We also have the various spatial derivatives

$$\frac{d^n w_m(\zeta)}{d\zeta^n} = j\frac{m'}{|m'|}(-1)^n n![\zeta - \zeta']^{-n-1} \quad \text{for } n \geq 0$$
$$\frac{d^n H(\zeta)}{d\zeta^n} = \frac{m'}{2\pi}(-1)^n (n+1)![\zeta - \zeta']^{-n-2} \quad \text{for } n \geq -1 \tag{C.8}$$

By summing over such terms corresponding to various line-magnetic-dipole positions with various ζ', the resulting potentials and fields apply to the ensemble. These can also be added to some set of corresponding terms for line currents from Appendix B.

Appendix D. Conditions for Non-Coupling of Two-Dimensional Magnetic Fields and Loop Structures (Zero Mutual Inductance

In measuring the magnetic field scattered by a target into a receiver coil it is desirable to minimize the direct coupling of the transmitter coil into the receiver coil to maximize the signal-to-noise ratio. This is accomplished by making the magnetic flux from one coil linking the second zero. The flux per unit length linking a coil characterized by two complex coordinates ζ_1 and ζ_2 is just

$$\begin{aligned}\Phi' &= \int_{\zeta_1}^{\zeta_2} \vec{B} \cdot \left[\left[\vec{1}_x\, dx + \vec{1}_y\, dy\right] \times \vec{1}_z = \mu_0 \int_{\zeta_1}^{\zeta_2} \vec{H} \cdot \left[-\vec{1}_y\, dx + \vec{1}_x\, dy\right]\right] \\ &= \mu_0 \int_{\zeta_1}^{\zeta_2} \left[-\vec{H}_y\, dx + \vec{H}_x\, dy\right] = \mu_0 I_m \left[\int_{\zeta_1}^{\zeta_2} H(\zeta) d\zeta\right] \\ &= \mu_0 \frac{I}{\Delta v} \mathrm{Re}\left[\int_{\zeta_1}^{\zeta_2} \frac{dw(\zeta)}{d\zeta} d\zeta\right] \\ &= \mu_0 \frac{I}{\Delta v}\left[u(\zeta_2) - u(\zeta_1)\right]\end{aligned} \tag{D.1}$$

using the general form for the magnetic field from a two-dimensional primary coil in Appendix A. Thus we obtain $\Phi' = 0$ by placing the receiver coil conductors on contours of constant $u(\zeta)$ from the transmitter coil.

A special case of interest for zero coupling involves a common symmetry plane S for transmitter and receiver coils. For magnetic fields we have

$$\begin{aligned}&\text{symmetric } (sy) \text{ field} \Rightarrow \vec{H} \perp \mathrm{S} \text{ on S} \\ &\text{antisymmetric } (as) \text{ field} \Rightarrow \vec{H} /\!/ \mathrm{S} \text{ on S}\end{aligned} \tag{D.2}$$

for appropriate configurations of currents in the coils (driving separately each coil as a transmitter). If under these conditions one coil produces a symmetric field and the other an antisymmetric field these coils do not couple to each other. They are in effect "cross-polarized." For a more detailed treatment of this type of symmetry see [5, 11].

If the receiver coil takes the form of a line magnetic dipole (when driven) as in Appendix C, it is merely necessary that the orientation ψ_m of the receiver be orthogonal to the magnetic field of the transmitter. This is expressed as

$$\vec{H} \cdot \vec{m}' = 0 = H_x m'_x + H_y m'_y = \mathrm{Re}\left[H(\zeta) m'\right] \tag{D.3}$$

$$\mathrm{Re}\left[H(\zeta)e^{j\psi_m}\right]=0$$

$$\mathrm{Im}\left[\frac{dw(\zeta)}{d\zeta}e^{j\psi_m}\right]=0$$

This gives a convenient complex form for determining line-dipole orientation ψ_m.

Applying (D.3) to the case of two line dipoles, let the first be situated at the origin as in fig. C.1. Designate parameters associated with this case with a subscript 1 so that its magnetic field is

$$H_1(\zeta)=\frac{m_1'}{2\pi}\zeta^{-2}=\frac{|m_1'|}{2\pi}\frac{e^{j(\psi_{m1}-2\phi)}}{|\zeta|^2} \tag{D.4}$$

The second line magnetic dipole is at some coordinate ζ, with ϕ as in (D.4) being the important part. Using a subscript 2 for this with orientation ψ_{m2} gives from (D.3)

$$\begin{aligned}
&\mathrm{Re}\left[H_1(\zeta)e^{j\psi_{m2}}\right]=0\\
&\mathrm{Re}\left[e^{j(\psi_{m1}+\psi_{m2}-2\phi)}\right]=\cos(\psi_{m1}+\psi_{m2}-2\phi)=0\\
&\psi_{m1}+\psi_{m2}-2\phi=\pm\frac{\pi}{2},\pm\frac{3\pi}{2},\cdots
\end{aligned} \tag{D.5}$$

as the condition for zero coupling. Without loss of generality we can take ψ_{m1} as defining $\phi=0$. Then for parallel $\vec{m}_1$ and $\vec{m}_2$

$$\begin{aligned}
&\psi_{m1}=\psi_{m2}=0\\
&\phi=\pm\frac{\pi}{4},\pm\frac{3\pi}{4}
\end{aligned} \tag{D.6}$$

and for perpendicular $\vec{m}_1$ and $\vec{m}_2$

$$\begin{aligned}
&\psi_{m1}=0,\quad \psi_{m2}=\pm\frac{\pi}{2}\\
&\phi=0,\quad \pm\frac{\pi}{2},\quad \pi
\end{aligned} \tag{D.7}$$

as special cases of interest. There are analogous cases for zero coupling between circular loops in the limit of point magnetic dipoles [3].

Note that for an ensemble of coils forming the receiver the conditions in (D.1) and (D.3) are replaced by summations of such terms, appropriately weighted.

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