Sensor and Simulation Notes

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Optimization of Transient Radiation

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Abstract

This paper considers further application of norm concepts to the optimization of vector temporal waveforms radiated from transient antennas with associated pulsers. By representing the radiated far field as an integral over either the antenna currents, or the fields on an appropriate surface near the antenna, the usual time-derivative relationship for the far fields is exhibited, so that norms of the far-field become norms of the near field on a planar aperture. This leads to the focusing condition as part of the optimization conditions. Developing the norms of the temporal waveforms as appropriate norms over the frequency spectrum of the pulse, one can design a pulse-radiating system to give a pulse with desirable spectral properties over some wide frequency band of interest.

1. Introduction

In designing antennas with pulsed sources for radiating transient pulses, one is faced with the problem of deciding what it is that one wants. What is it about the far radiated fields that one wishes to optimize, and what does optimize mean (total energy, peak field, peak spectral content across some band of frequencies, particular polarization, etc.)? There is also the question of what constraints one wishes to impose on the antenna (e.g., physical dimensions) and pulser(s) (e.g., voltage, energy, peak and average power, etc.).

Much has been developed for pulse radiators, primarily in the context of simulators for the nuclear electromagnetic pulse (EMP); this is reviewed in [2, 15]. In recent years this has led to the class of pulse radiators known as impulse radiating antennas (IRAs) reviewed in [24, 25]. With this technology expanding into various types of pulse radiators, it is important to understand which is better for a given application. There may be various "optimum" solutions, depending on the problem at hand. Various example problems have been considered in [7] and the references therein. (See also [21].) In [7] norm concepts are introduced in a systematic way to extend the definitions of antenna gain and radiation pattern to apply to radiated temporal pulses. By considering reciprocity in the time domain the definitions are made to apply to both transmission and reception, allowing for the additional time derivative in transmission, or equivalently the additional time integral in reception.

The present paper considers further application of norms to characterize the radiated far-field temporal waveforms. First, the various integral representations of the fields in terms of currents on the antenna, and in terms of equivalent electric and magnetic surface current densities on surfaces surrounding the antenna are developed. For the case that the boundary surface is a plane the integral expressions simplify, particularly for the far field. Then, using results from the appendices, norms of vector temporal waveforms are considered in terms of norms over the frequency spectrum of the two-sided-Laplace/Fourier transform of such waveforms. This leads into concepts of comparing waveforms via norms in the frequency domain which include weighting functions which emphasize portions of the spectrum of the pulse which are deemed important for the problem at hand.

2. Fields From Currents

Begin with the Maxwell equations for free space

$$
\begin{aligned}
\nabla \times \vec{E}(\vec{r},t) &= -\mu_0 \frac{\partial}{\partial t}\vec{H}(\vec{r},t) - \vec{J}_m(\vec{r},t) \\
\nabla \times \vec{H}(\vec{r},t) &= \varepsilon_0 \frac{\partial}{\partial t}\vec{E}(\vec{r},t) + \vec{J}(\vec{r},t)
\end{aligned}
\tag{2.1}
$$

where both electric and magnetic current densities are included for generality and later use in the equivalence theorem. This is conveniently cast in terms of the combined field as [26]

$$
\begin{aligned}
&\left[\nabla\times - \frac{qj}{c}\right]\vec{E}_q(\vec{r},t) = qj\ Z_0\ \vec{J}_m(\vec{r},t) \\
&c \equiv [\mu_0\varepsilon_0]^{-\frac{1}{2}} \equiv \text{speed of light} \\
&Z_0 \equiv \left[\frac{\mu_0}{\varepsilon_0}\right]^{\frac{1}{2}} \equiv \text{wave impedance of free space} \\
&\vec{E}_q(\vec{r},t) \equiv \vec{E}(\vec{r},t) + qj\ Z_0\ \vec{H}(\vec{r},t) \equiv \text{combined field} \\
&\vec{J}_q(\vec{r},t) \equiv \vec{J}(\vec{r},t) + \frac{qj}{Z_0}\vec{J}_m(\vec{r},t) \equiv \text{combined current density} \\
&q = \pm 1 \equiv \text{separation index}
\end{aligned}
\tag{2.2}
$$

The associated charge densities are given by

$$
\begin{aligned}
\nabla\cdot\vec{J}(\vec{r},t) &= -\frac{\partial}{\partial t}\rho(\vec{r},t)\ ,\ \nabla\cdot\vec{J}_m(\vec{r},t) = -\frac{\partial}{\partial t}\rho_m(\vec{r},t) \\
\rho_q(\vec{r},t) &= \rho(\vec{r},t) + \frac{qj}{Z_0}\rho_m(\vec{r},t)\ ,\ \nabla\cdot\vec{J}_q(\vec{r},t) = -\frac{\partial}{\partial t}\rho_q(\vec{r},t)
\end{aligned}
\tag{2.3}
$$

While this is cast in terms of free-space parameters, it can be converted to other uniform isotropic media by a change of the constitutive parameters, although if these are frequency dependent, this is directly done in frequency domain with subsequent transformation to the time domain. The electric current density can also be organized to include polarization and conduction current density (say in some antenna or scatterer), and the magnetic current density can also include the magnetic polarization associated with permeability different from free space. For present purposes the medium will be assumed to have frequency independent ε_0 and μ_0 with conductivity $\sigma = 0$ (i.e., free-space like) so that the simple forms above are directly applicable to analyzing antenna radiation in time domain.

The fields are in turn derivable from the usual potentials as

$$\vec{E}(\vec{r},t) = -\nabla\Phi(\vec{r},t) - \frac{\partial}{\partial t}\vec{A}(\vec{r},t) - \frac{1}{\varepsilon_o}\nabla\times\vec{A}_m(\vec{r},t)$$

$$\vec{H}(\vec{r},t) = -\frac{1}{\mu_o}\nabla\times\vec{A}(\vec{r},t) - \nabla\Phi_m(\vec{r},t) - \frac{\partial}{\partial t}\vec{A}_m(\vec{r},t) \tag{2.4}$$

Defining combined potentials as

$$\vec{A}_q(\vec{r},t) = \vec{A}(\vec{r},t) + q\,j\,Z_o\,\vec{A}_m(\vec{r},t)$$

$$\Phi_q(\vec{r},t) = \Phi(\vec{r},t) + q\,j\,Z_o\,\Phi_m(\vec{r},t) \tag{2.5}$$

the combined field is given by

$$\vec{E}_q(\vec{r},t) = -\nabla\Phi_q(\vec{r},t) - \frac{\partial}{\partial t}\vec{A}_q(\vec{r},t) + j\,q\,c\,\nabla\times\vec{A}_p(\vec{r},t) \tag{2.6}$$

With the scalar Green's function for free space (using complex frequency for the two-sided Laplace transform (Appendix A))

$$\tilde{G}_o(\vec{r},\vec{r};s) = \frac{e^{-\gamma|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|}\;,\; \gamma \equiv \frac{s}{c} \tag{2.7}$$

which satisfies the radiation condition for outgoing waves one can express the potentials in complex frequency domain as [1, 26]

$$\tilde{\vec{A}}(\vec{r},s) = \mu_o\int_V \tilde{G}_o(\vec{r},\vec{r};s)\;\tilde{\vec{J}}(\vec{r}',s)\;dV'\;,\; \Phi(\vec{r},s) = \frac{1}{\varepsilon_o}\int_V \tilde{G}_o(\vec{r},\vec{r}';s)\;\tilde{\rho}(\vec{r}',s)\;dV'$$

$$\tilde{\vec{A}}_m(\vec{r},s) = \varepsilon_o\int_V \tilde{G}_o(\vec{r},\vec{r}';s)\;\tilde{\vec{J}}_m(\vec{r}',s)\;dV'\;,\; \tilde{\Phi}_m(\vec{r},s) = \frac{1}{\mu_o}\int_V \tilde{G}_o(\vec{r},\vec{r}';s)\;\rho_m(\vec{r}',s)\;dV' \tag{2.8}$$

$$\tilde{\vec{A}}_q(\vec{r},s) = \mu_o\int_V \tilde{G}_o(\vec{r},\vec{r};s)\;\tilde{\vec{J}}_q(\vec{r}',s)\;dV'\;,\; \tilde{\Phi}_q(\vec{r},s) = \frac{1}{\varepsilon_o}\int_V \tilde{G}_o(\vec{r},\vec{r}';s)\;\tilde{\rho}_q(\vec{r}',s)\;dV'$$

In time domain we have

$$G_o(\vec{r},\vec{r};t) = \frac{1}{4\pi|\vec{r}-\vec{r}'|}\,\delta\left(t - \frac{|\vec{r}-\vec{r}'|}{c}\right) \tag{2.9}$$

which operates on currents and charges as a temporal convolution. The combined potentials then have the temporal form

$$\vec{A}_q(\vec{r},t) = \mu_o \int_V G_o(\vec{r},\vec{r}';t) \circ \vec{J}_q(\vec{r}',t)\,dV', = \mu_o \int_V \int_{-\infty}^{\infty} G_o(\vec{r},\vec{r}';t-t')\, \vec{J}_q(\vec{r}',t')\,dt'\,dV'$$

$$= \mu_o \int_V \frac{1}{4\pi|\vec{r}-\vec{r}'|} \vec{J}_q\left(\vec{r}',t-\frac{|\vec{r}-\vec{r}'|}{c}\right) dV'$$

$$\Phi_q(\vec{r},t) = \frac{1}{\varepsilon_o}\int_V G_o(\vec{r},\vec{r}';t) \circ \rho_q(\vec{r}',t)\,dV', = \frac{1}{\varepsilon_o}\int_V \int_{-\infty}^{\infty} G_o(\vec{r},\vec{r}';t-t')\, \rho_q(\vec{r}',t')\,dt'\,dV'$$

$$= \frac{1}{\varepsilon_o}\int_V \frac{1}{4\pi|\vec{r}-\vec{r}'|} \rho_q\left(\vec{r}',t-\frac{|\vec{r}-\vec{r}'|}{c}\right) dV' \tag{2.10}$$

$\circ \equiv$ convolution with respect to time

and similarly for the other (electric and magnetic) potentials. Note that the temporal convolution reduces to the evaluation of the currents and charges in retarded time with respect to the observer at $\vec{r}$.

One can go directly from the currents to fields via the free-space dyadic Green's function which can also be written in the form of an impedance kernel as [10]

$$\tilde{\overset{\leftrightarrow}{Z}}(\vec{r},\vec{r}';s) = s\mu_o\, \tilde{\overset{\leftrightarrow}{G}}_o(\vec{r},\vec{r}';s)$$

$$= \frac{Z_o\gamma^2}{4\pi}\left\{\left[-2\zeta^{-3}-2\zeta^{-2}\right]e^{-\zeta}\,\vec{1}_R\,\vec{1}_R + \left[\zeta^{-3}+\zeta^{-2}+\zeta^{-1}\right]e^{-\zeta}\,\overset{\leftrightarrow}{1}_R\right\} + \frac{Z_o}{3\gamma}\,\delta(\vec{r}-\vec{r}')\,\overset{\leftrightarrow}{1} \tag{2.11}$$

$$R \equiv |\vec{r}-\vec{r}'|\,,\quad \vec{1}_R \equiv \frac{\vec{r}-\vec{r}'}{R}\,,\quad \overset{\leftrightarrow}{1}_R \equiv \overset{\leftrightarrow}{1} - \vec{1}_R\,\vec{1}_R$$

$$\zeta = \gamma R\,,\quad \overset{\leftrightarrow}{1} \equiv \vec{1}_x\,\vec{1}_x + \vec{1}_y\,\vec{1}_y + \vec{1}_z\,\vec{1}_z \equiv \text{identity}$$

Note the inclusion of the delta function at $\vec{r} = \vec{r}'$. This is for use with a small spherical volume about the source point for defining a principal-value integral [19]. Similar formulas hold for other shapes of the volume around the source point (including for use in integration over surface current densities) [20]. This kernel has the form of an impedance (per meter^2) as it is used in surface integral equations when operating on the surface current density to give the tangential part of the incident electric field. For the scattered or radiated electric field a minus sign appears with this kernel.

A related kernel is

$$\tilde{\overset{\leftrightarrow}{Z}}_m(\vec{r},\vec{r}';s) = -Z_o\nabla\times\tilde{\overset{\leftrightarrow}{G}}_o(\vec{r},\vec{r}';s) = -\,Z_o\,\nabla\tilde{G}_o(\vec{r},\vec{r}';s)\times\overset{\leftrightarrow}{1}$$

$$= \frac{Z_o\gamma^2}{4\pi}\left[\zeta^{-2}+\zeta^{-1}\right]e^{-\zeta}\,\vec{1}_R\times\overset{\leftrightarrow}{1} \tag{2.12}$$

which is used for the magnetic field (from the electric current density). The two kernels can be combined as [16]

$$\overleftrightarrow{Z}_q(\vec{r},\vec{r}\,';s) = \overleftrightarrow{Z}(\vec{r},\vec{r}\,';s) + qj\,\overleftrightarrow{Z}_m(\vec{r},\vec{r}\,';s) \tag{2.13}$$

which can be used to find the combined field as

$$\begin{aligned}\vec{E}_q(\vec{r},s) &= -\int_V \tilde{\overleftrightarrow{Z}}_q(\vec{r},\vec{r}\,';s)\cdot\tilde{\vec{J}}_q(\vec{r}\,',s)\,dV' \\ \vec{E}_q(\vec{r},t) &= -\int_V \overleftrightarrow{Z}_q(\vec{r},\vec{r}\,';t)\circ\vec{J}_q(\vec{r}\,',t)\,dV' \\ &= -\int_V \int_{-\infty}^{\infty} \overleftrightarrow{Z}_q(\vec{r},\vec{r}\,';t-t')\cdot\vec{J}_q(\vec{r}\,',t')\,dt'\,dV'\end{aligned} \tag{2.14}$$

This formula can be readily decomposed into separate formulas for the electric and magnetic fields in terms of the electric and magnetic current densities. In a form similar to (2.10) the integral over t' evaluates the current density in retarded time $t - R/c$. However, there are additional powers of s (in γ) in (2.11) and (2.12). These become temporal derivatives and integrals [4, 8], allowing explicit representation in time domain as an integral over $\vec{r}\,'$.

For present purposes, our primary interest is in the far fields for which we take the leading $1/r$ term with [4]

$$\begin{aligned}& r \equiv |\vec{r}\,|\,,\quad \vec{1}_R \equiv \frac{\vec{r}}{r}\,,\quad \overleftrightarrow{1}_r \equiv \overleftrightarrow{1} - \vec{1}_r\,\vec{1}_r \\ & R = r - \vec{1}_r\cdot\vec{r}\,' + O\left(r^{-1}\right) \text{ as } r\to\infty \\ & \vec{1}_R = \vec{1}_r + O\left(r^{-1}\right) \text{ as } t\to\infty\end{aligned} \tag{2.15}$$

where $\vec{r}\,' = \vec{0}$ is assumed in the region (of finite linear dimensions) with the currents. The kernels then become

$$\begin{aligned}\tilde{\overleftrightarrow{Z}}^{(f)}(\vec{r},\vec{r}\,';s) &= \frac{s\mu_0}{4\pi r}e^{-\gamma[r-\vec{1}_r\cdot\vec{r}\,']}\,\overleftrightarrow{1}_r \\ \tilde{\overleftrightarrow{Z}}_m^{(f)}(\vec{r},\vec{r}\,';s) &= \frac{s\mu_0}{4\pi r}e^{-\gamma[r-\vec{1}_r\cdot\vec{r}\,']}\,\vec{1}_r\times\overleftrightarrow{1} \\ \tilde{\overleftrightarrow{Z}}_q^{(f)}(\vec{r},\vec{r}\,';s) &= \frac{s\mu_0}{4\pi r}e^{-\gamma[r-\vec{1}_r\cdot\vec{r}\,']}[\overleftrightarrow{1}_r + qj\,\vec{1}_r\times\overleftrightarrow{1}]\end{aligned} \tag{2.16}$$

so that the electric and magnetic parts are the same except for the factor $q\,j$ with the rotation by $\pi/2$ about $\vec{1}_r$. For large r then replace (for fixed s)

$$\tilde{\overleftrightarrow{Z}}_q(\vec{r},\vec{r}\,';s) = \tilde{\overleftrightarrow{Z}}_q^{(f)}(\vec{r},\vec{r}\,';s) + O\left(\frac{e^{-\gamma r}}{r^2}\right) \quad \text{as } r \to \infty \tag{2.17}$$

This gives the fields for large r

$$\tilde{\vec{E}}_q(\vec{r},s) = \tilde{\vec{E}}_q^{(f)}(\vec{r},s) + O\left(\frac{e^{-\gamma r}}{r^2}\right) \text{as } r \to \infty \tag{2.18}$$

where the far fields are given by

$$\begin{aligned}\tilde{\overleftrightarrow{E}}_q^{(f)}(\vec{r},s) &= -\int_V \tilde{\overleftrightarrow{Z}}_q^{(f)}(\vec{r},\vec{r}\,';s)\cdot\tilde{\vec{J}}_q(\vec{r}\,',s)\,dV' \\ &= -\frac{s\mu_o}{4\pi r}e^{-\gamma r}\left[\overleftrightarrow{1}_r + q\,j\,\vec{1}_r\times\overleftrightarrow{1}\right]\cdot\int_V e^{\vec{1}_r\cdot\vec{r}\,'}\tilde{\vec{J}}_q(\vec{r}\,',s)\,dV'\end{aligned} \tag{2.19}$$

In time domain it is convenient to use retarded time

$$t_r \equiv t - \frac{r}{c} \tag{2.20}$$

giving

$$\vec{E}_q^{(f)}(\vec{r},t) = -\frac{\mu_o}{4\pi r}\left[\overleftrightarrow{1}_r + q\,j\,\vec{1}_r\times\overleftrightarrow{1}\right]\cdot\frac{\partial}{\partial t}\int_V \tilde{\vec{J}}_q\left(\vec{r}\,',t_r+\frac{\vec{1}_r\cdot\vec{r}\,'}{c}\right)dV' \tag{2.21}$$

As discussed in [6] one needs to be careful in the interpretation of the far field in time domain since it includes frequencies extending to infinity, for which the far field is not strictly valid as the $1/r$ term in the asymptotic expansion as $r \to \infty$. In time domain this implies that the above formula applies for temporal changes that are not too fast, how fast depending on the magnitude of r.

A convenient feature of the far field is the relation between the electric and magnetic fields as

$$\begin{aligned}\vec{E}^{(f)}(\vec{r},t) &= -\frac{\mu_o}{4\pi r}\left\{\overleftrightarrow{1}_r\cdot\frac{\partial}{\partial t}\int_V \vec{J}\left(\vec{r}\,',t_r+\frac{\vec{1}_r\cdot\vec{r}\,'}{c}\right)dV' - \frac{1}{Z_o}\vec{1}_r\times\int_V \vec{J}_m\left(\vec{r}\,',t_r+\frac{\vec{1}_r\cdot\vec{r}\,'}{c}\right)dV'\right\} \\ &= -\,Z_o\,\vec{1}_r\times\vec{H}^{(f)}(\vec{r},t)\end{aligned} \tag{2.22}$$

Thus, for the far field, it is convenient to consider only the electric field since the magnetic field is so simply related to it. If one uses (2.22) for integrating over the electric current density on an antenna then only one term is used. Later use, however, will involve integration over equivalent sources on surfaces away from the antenna, for which both electric and magnetic surface current densities can be present.

3. Fields From Fields on Boundary Surfaces

Let all the sources be contained in a volume bounded by surface S with outward pointing normal $\vec{1}_S$ as indicated in fig. 3.1. Then for computing the external fields (outside $V \cup S$) we have the equivalence principle [3, 5, 26]

$$
\begin{aligned}
\vec{J}_{s_q}^{(eq)}(\vec{r}_s,t) &= -\frac{qj}{Z_0}\vec{1}_S(\vec{r}_s)\times\vec{E}_q(\vec{r}_s,t)\ , \quad \vec{r}_s \in S \\
\tilde{\vec{E}}_q(\vec{r},s) &= -\int_S \tilde{\overleftrightarrow{Z}}_q(\vec{r},\vec{r}\,';s)\cdot\vec{J}_{s_q}^{(eq)}(\vec{r}\,'_s,s)\, dS' \\
&= \frac{qj}{Z_0}\int_S \tilde{\overleftrightarrow{Z}}_q(\vec{r},\vec{r}\,';s)\cdot\left[\vec{1}_S(\vec{r}\,'_s)\times\tilde{\vec{E}}_q(\vec{r}\,'_s,s)\right]dS' \\
&\text{for } \vec{r} \notin [V \cup S]
\end{aligned}
\tag{3.1}
$$

Note that $\vec{1}_S$ is here pointing into the region where the fields are to be computed. While in Section 2 the magnetic current density on antennas may be neglected in the case of non permeable materials, here on *S* both kinds (electric and magnetic) for equivalent surface current density are required.

As a special case, fig. 3.2 shows S replaced by a plane P of infinite extent. As a simple convention one can take P as the $z = 0$ plane with

$$
\vec{1}_P = \vec{1}_z \quad \text{(unit surface normal)}
\tag{3.2}
$$

and the fields $\vec{E}_q$ are calculated for $z > 0$ from equivalent surface currents on P. The equivalent currents on P (or more generally on S) can, of course, be calculated in turn from the currents (source) for $z < 0$ using the formulas in Section 2. Note that the choice of $z = 0$ is somewhat arbitrary as long as it is to the right (in fig. 3.2) of the sources. Thus, one can in principle compute equivalent sources on a plane of constant z, compute fields to the right, compute new equivalent sources on a plane of larger constant z, etc., continuing on indefinitely to the right.

For this special case of an equivalent source plane P, there are alternate forms the equivalent sources can take, based on constructing fields in a symmetric or antisymmetric sense with respect to P [9, 26]. Suppose that $\vec{E}_q$ is given on P. Let us construct $\vec{E}_q$ such that it is symmetric with respect to P, i.e.,

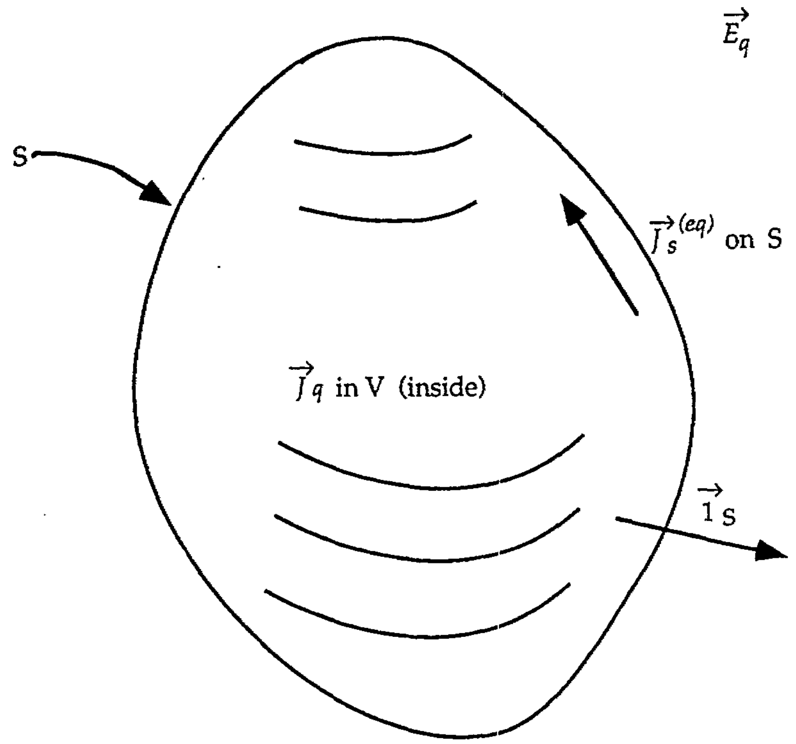

Fig. 3.1 Equivalence Principle for Representing Fields Outside a Volume Containing Sources

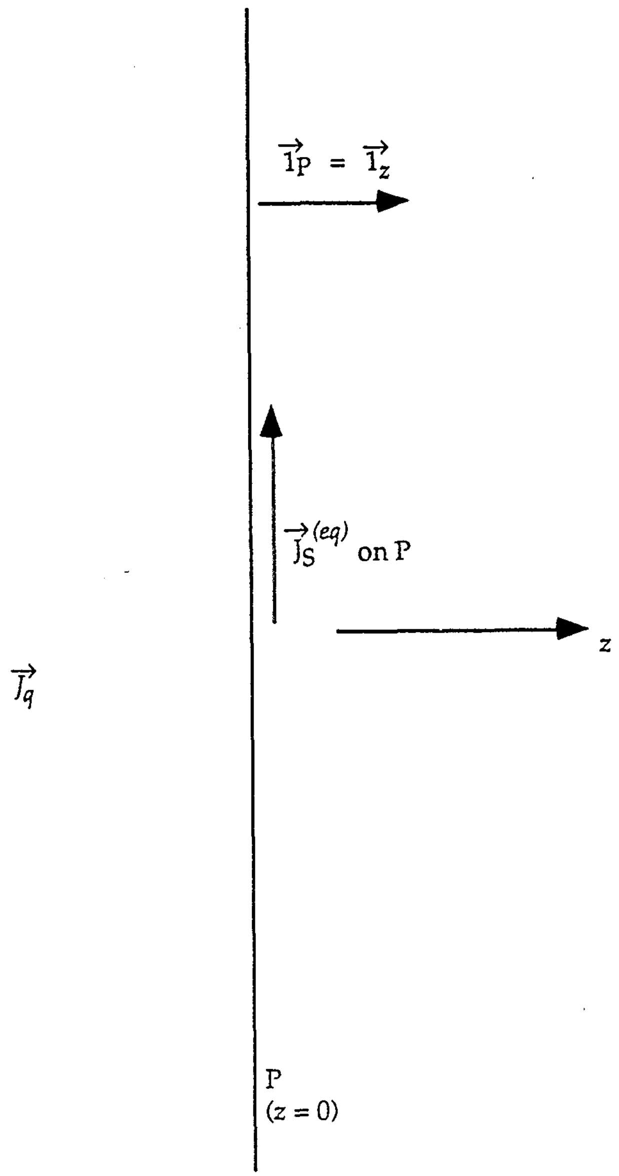

Fig. 3.2 Equivalence Principle for Representing Fields on One Side of a Plane from Sources on the Other Side

$$
\begin{aligned}
\vec{E}^{(sy)}(\vec{r}_m,t) &= \overleftrightarrow{R}\cdot\vec{E}(\vec{r},t)\\
\vec{H}^{(sy)}(\vec{r}_m,t) &= \overleftrightarrow{R}\cdot\vec{E}(\vec{r},t)\\
\vec{r}_m &= \overleftrightarrow{R}\cdot\vec{r}\quad\text{(mirror position or coordinates)}\\
\vec{r} &= (x,y,z),\ \vec{r}_m = (x,y,-z)\ \text{with}\ z>0\\
\overleftrightarrow{R} &= \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}\quad\text{(reflection dyadic)}
\end{aligned}
\tag{3.3}
$$

Then tangential $\vec{E}$ is continuous through P, but tangential $\vec{H}$ is discontinuous through P, giving a surface current density on P as

$$
\begin{aligned}
\vec{J}_s^{(sy)}(\vec{r}_s,t) &= 2\,\vec{1}_z\times\vec{H}(\vec{r}_{s+},t)\\
\vec{r}_{s+} &\equiv \text{coordinate just to the right of P}
\end{aligned}
\tag{3.4}
$$

Note that in (3.3) the fields for z > 0 are the original fields resulting from sources to the left of P. The mirror fields at $\vec{r}_m$ are artificially constructed to give fields symmetric with respect to P. Then substituting from (3.4) in (3.1) gives

$$
\begin{aligned}
\tilde{\vec{E}}_q(\vec{r},s) &= -\int_P \tilde{\overleftrightarrow{Z}}_q(\vec{r},\vec{r}'_s;s)\cdot\tilde{\vec{J}}_s^{(sy)}(\vec{r}'_s,s)\,dS'\\
&= -2\int_P \tilde{\overleftrightarrow{Z}}_q(\vec{r},\vec{r}'_s;s)\cdot\left[\vec{1}_z\times\tilde{\vec{H}}(\vec{r}_{s+},s)\right]dS'\\
&\text{for } z>0
\end{aligned}
\tag{3.5}
$$

Thus we have the fields to the right in terms of only the magnetic field on P, but with a factor of 2 appearing.

Another form the fields can take is found by replacing (3.3) by antisymmetric fields as

$$
\begin{aligned}
\vec{E}^{(as)}(\vec{r}_m,t) &= -\overleftrightarrow{R}\cdot\vec{E}(\vec{r},t)\\
\vec{H}^{(as)}(\vec{r}_m,t) &= \overleftrightarrow{R}\cdot\vec{H}(\vec{r},t)\\
z &> 0
\end{aligned}
\tag{3.6}
$$

Then tangential $\vec{H}$ is continuous through P, but tangential $\vec{E}$ is discontinuous through P, giving a magnetic surface current density on P as [16]

$$\vec{J}_{s_m}^{(as)}(\vec{r}_s, t) = -2\,\vec{1}_z \times \vec{E}(\vec{r}_{s+}, t) \tag{3.7}$$

Again the fields to the right are the original fields, and the fields to the left are constructed to give antisymmetric fields with respect to P. Then substituting from (3.7) in (3.1) gives

$$\begin{aligned}\tilde{\vec{E}}_q(\vec{r},s) &= -\frac{qj}{Z_o}\int_P \tilde{\overleftrightarrow{Z}}_q(\vec{r},\vec{r}';s)\cdot\tilde{\vec{J}}_{s_m}^{(as)}(\vec{r}'_s, s)\,dS' \\ &= 2\frac{qj}{Z_o}\int_P \overleftrightarrow{Z}_q(\vec{r},\vec{r}';s)\cdot\left[\vec{1}_z\times\tilde{\vec{E}}(\vec{r}'_{s+}, s)\right]dS' \\ &\text{for } z > 0\end{aligned} \tag{3.8}$$

Thus we have the fields to the right in terms of only the electric field on P, but with a factor of 2 again appearing. This is the form in [4, 6, 23].

Comparing (3.5) and (3.8) for sources on P in terms of electric and magnetic fields, note that one can take a linear combination of these integrals for $\tilde{\vec{E}}_q$, with coefficients summing to one, to obtain yet other valid formulas. In particular with both coefficients equal to $1/2$, then the formula with combined sources (as in (3.1) with S taken as P) is reproduced. In this special form, however, the formula is valid for all shapes of S that enclose the sources (and not the observer).

In Section 2, the kernel is evaluated for large r to give far fields. Here the same applies to integration over equivalent sources. Thus (3.1) becomes

$$\begin{aligned}\tilde{\vec{E}}_q^{(f)}(\vec{r},s) &= -\int_S \tilde{\overleftrightarrow{Z}}^{(f)}(\vec{r},\vec{r}'_s;s)\cdot\tilde{\vec{J}}_{s_q}^{(eq)}(\vec{r}'_s, s)\,dS' \\ &= \frac{qj}{Z_o}\int_S \tilde{\overleftrightarrow{Z}}_q(\vec{r},\vec{r}';s)\cdot\left[\vec{1}_S(\vec{r}'_s)\times\tilde{\vec{E}}_q(\vec{r}'_s, s)\right]dS' \\ &= \frac{qj\gamma}{Z_o}e^{-\gamma r}[\vec{1}_r + qj\,\vec{1}_r\times\overleftrightarrow{1}]\cdot\int_S e^{\gamma\vec{1}_r\cdot\vec{r}'_s}\,\vec{1}_S(\vec{r}'_s)\times\tilde{\vec{E}}_q(\vec{r}'_s, s)\,dS' \\ \vec{E}_q^{(f)}(\vec{r},t) &= \frac{qj}{4\pi r}[\vec{1}_r + qj\,\vec{1}_r\times\overleftrightarrow{1}]\cdot\frac{\partial}{\partial t}\int_S \vec{1}_S(\vec{r}'_s)\times\vec{E}_q\left(\vec{r}'_s, t_r + \frac{\vec{1}_r\cdot\vec{r}'_s}{c}\right)dS'\end{aligned} \tag{3.9}$$

As mentioned previously, one must use such expressions in time domain with care due to the inclusion of frequencies for $s\to\infty$.

Similarly (3.5) becomes

$$\tilde{\vec{E}}_q^{(f)}(\vec{r},s) = -\int_P \tilde{\overleftrightarrow{Z}}_q^{(f)}(\vec{r},\vec{r}'_s;s)\cdot \tilde{\vec{J}}_s^{(sy)}(\vec{r}'_s,s)\, dS'$$

$$= -2\int_P \tilde{\overleftrightarrow{Z}}_q^{(f)}(\vec{r},\vec{r}'_s;s)\cdot \left[\vec{1}_z \times \tilde{\vec{H}}(\vec{r}_{s+},s)\right] dS'$$

$$= -\frac{s\mu_0}{2\pi r} e^{-\gamma r}[\vec{1}_r + qj\,\vec{1}_r \times \overleftrightarrow{1}]\cdot \left\{\vec{1}_z \times \int_P e^{\gamma \vec{1}_r\cdot\vec{r}'_s}\, \tilde{\vec{H}}(\vec{r}_{s+},s)\, dS'\right\} \tag{3.10}$$

$$\tilde{\vec{E}}_q^{(f)}(\vec{r},t) = -\frac{\mu_0}{2\pi r}[\vec{1}_r + qj\,\vec{1}_r \times \overleftrightarrow{1}]\cdot \frac{\partial}{\partial t}\left\{\vec{1}_z \times \int_P \vec{H}\left(\vec{r}_{s+}, t_r + \frac{\vec{1}_r\cdot\vec{r}'}{c}\right) dS'\right\}$$

and (3.8) becomes

$$\tilde{\vec{E}}_q^{(f)}(\vec{r},s) = -\frac{qj}{Z_0}\int_P \tilde{\overleftrightarrow{Z}}_q^{(f)}(\vec{r}_s,\vec{r}'_s;s)\cdot \tilde{\vec{J}}_{s_m}^{(as)}(\vec{r}'_s,s)\, dS'$$

$$= \frac{2qj}{Z_0}\int_P \tilde{\overleftrightarrow{Z}}_q^{(f)}(\vec{r},\vec{r}'_s;s)\cdot \left[\vec{1}_z \times \tilde{\vec{E}}(\vec{r}'_s,s)\right] dS'$$

$$= -\frac{\gamma}{2\pi r}[\vec{1}_r \times \overleftrightarrow{1} - qj\,\vec{1}_r]\cdot \left\{\vec{1}_z \times \int_P e^{\gamma \vec{1}_r\cdot\vec{r}'_s}\, \tilde{\vec{E}}(\vec{r}'_s,s)\, dS'\right\} \tag{3.11}$$

$$\tilde{\vec{E}}_q^{(f)}(\vec{r},t) = -\frac{1}{2\pi c r}[\vec{1}_r \times \overleftrightarrow{1} - qj\,\vec{1}_r]\cdot \frac{\partial}{\partial t}\left\{\vec{1}_z \times \int_P \vec{E}\left(\vec{r}_s, t_r + \frac{\vec{1}_r\cdot\vec{r}'}{c}\right) dS'\right\}$$

Note that the plane P extends to $\pm\infty$ in both x and y directions which violates the condition of finite $\vec{r}'_s$ in deriving the far fields as in (2.15). However, if the fields of interest on P extend over a finite-dimensioned portion ("aperture") of P, this difficulty is avoided. This is a common approximation in antenna theory.

For the special case of the observer on the z axis (boresight) we have

$$\begin{aligned} \vec{1}_r &= \vec{1}_z \\ \overleftrightarrow{1}_r &= \overleftrightarrow{1}_z \equiv \overleftrightarrow{1} - \vec{1}_z\vec{1}_z = \vec{1}_x\vec{1}_x \times \vec{1}_y\vec{1}_y \end{aligned} \tag{3.12}$$

The domain of integration on P is assumed centered on $\vec{r} = \vec{0}$ and is assumed of finite extent as discussed above. Then (3.9) becomes

$$
\begin{aligned}
\vec{E}_q^{(f)}(\vec{r},s) &= \frac{\gamma}{4\pi r}\, e^{-\gamma r}\left[\overleftrightarrow{1}_z + qj\ \vec{1}_z \times \overleftrightarrow{1}_z\right] \cdot \int_P \tilde{\vec{E}}_q(\vec{r}\,'_s,s)\, dS' \\
\vec{E}_q^{(f)}(\vec{r},s) &= \frac{1}{4\pi c r}\left[\overleftrightarrow{1}_z + qj\ \vec{1}_z \times \overleftrightarrow{1}_z\right] \cdot \frac{\partial}{\partial t}\int_P \vec{E}_q(\vec{r}\,'_s,t_r)\, dS'
\end{aligned}
\tag{3.13}
$$

Similarly (3.10) becomes

$$
\begin{aligned}
\vec{E}_q^{(f)}(\vec{r},s) &= -\frac{s\mu_0}{2\pi r}\, e^{-\gamma r}\left[\vec{1}_z \times \overleftrightarrow{1}_z - qj\ \overleftrightarrow{1}_z\right] \cdot \int_P \tilde{\vec{H}}(\vec{r}_{s+},s)\, dS' \\
\vec{E}_q^{(f)}(\vec{r},s) &= -\frac{\mu_0}{2\pi r}\left[\vec{1}_z \times \overleftrightarrow{1}_z - qj\ \overleftrightarrow{1}_z\right] \cdot \frac{\partial}{\partial t}\int_P \vec{H}_q(\vec{r}_{s+},t_r)\, dS'
\end{aligned}
\tag{3.14}
$$

and (3.11) becomes

$$
\begin{aligned}
\vec{E}_q^{(f)}(\vec{r},s) &= \frac{\gamma}{2\pi r}\left[\overleftrightarrow{1}_z + qj\ \vec{1}_z \times \overleftrightarrow{1}_z\right] \cdot \int_P \tilde{\vec{E}}(\vec{r}_{s+},s)\, dS' \\
\vec{E}_q^{(f)}(\vec{r},t) &= \frac{1}{2\pi c r}\left[\overleftrightarrow{1}_z + qj\ \vec{1}_z \times \overleftrightarrow{1}_z\right] \cdot \frac{\partial}{\partial t}\int_P \vec{E}(\vec{r}_{s+},t_r)\, dS'
\end{aligned}
\tag{3.15}
$$

Defining, for the far field,

$$
\begin{aligned}
\vec{V}_q^{(f)}(\vec{1}_r,t_r) &\equiv r\,\vec{E}_q^{(f)}(\vec{r},t) \equiv \vec{V}^{(f)}(\vec{1}_r,t_r) + \vec{V}_m^{(f)}(\vec{1}_r,t_r) \\
e^{-\gamma r}\,\tilde{\vec{V}}_q^{(f)}(\vec{1}_r,s) &\equiv r\,\tilde{\vec{E}}_q^{(f)}(\vec{r},s)
\end{aligned}
\tag{3.16}
$$

we have

$$
\begin{aligned}
\vec{V}_m^{(f)}(\vec{1}_r,t_r) &= \vec{1}_r \times \vec{V}^{(f)}(\vec{1}_r,t_r) \\
\vec{V}_q^{(f)}(\vec{1}_r,t_r) &= \left[\overleftrightarrow{1}_r + qj\ \vec{1}_r \times \overleftrightarrow{1}\right] \cdot \vec{V}^{(f)}(\vec{1}_r,t_r)
\end{aligned}
\tag{3.17}
$$

Thus, we have the common factor

$$
\begin{aligned}
\tilde{\vec{V}}^{(f)}(\vec{1}_r,s) &= -\frac{\gamma}{2\pi}\,\vec{1}_r \times \left\{\vec{1}_z \times \int_P e^{\gamma \vec{1}_r\cdot\vec{r}\,'_s}\,\tilde{\vec{E}}(\vec{r}\,'_s,s)\, dS'\right\} \\
&= -\frac{\gamma}{2\pi}\,\overleftrightarrow{1}_r \cdot \left\{\vec{1}_z \times \int_P e^{\gamma \vec{1}_r\cdot\vec{r}\,'_s}\,Z_0\,\tilde{\vec{H}}(\vec{r}\,'_s,s)\, dS'\right\}
\end{aligned}
$$

$$
\begin{aligned}
\vec{V}^{(f)}(\vec{1}_r,t) &= -\frac{1}{2\pi}\frac{\partial}{c\partial t}\left\{\vec{1}_r \times \left\{\vec{1}_z \times \int_P \vec{E}\left(\vec{r}'_s, t+\frac{\vec{1}_r\cdot\vec{r}'_s}{c}\right)dS'\right\}\right\} \\
&= -\frac{1}{2\pi}\frac{\partial}{c\partial t}\left\{\overleftrightarrow{1}_r \cdot \left\{\vec{1}_z \times \int_P Z_0\vec{H}\left(\vec{r}'_s, t+\frac{\vec{1}_r\cdot\vec{r}'_s}{c}\right)dS'\right\}\right\}
\end{aligned}
\tag{3.18}
$$

which is expressible in terms of the tangential electric or magnetic field on P (or a linear combination of the two). For the special case of boresight we have

$$
\begin{aligned}
&\vec{1}_r = \vec{1}_z \\
&\tilde{\vec{V}}^{(f)}(\vec{r},s) = \frac{\gamma}{2\pi}\overleftrightarrow{1}_z \cdot \int_P \tilde{\vec{E}}(\vec{r}'_s,s)\,dS' = -\frac{\gamma}{2\pi}\vec{1}_z \times \int_P Z_0\tilde{\vec{H}}(\vec{r}'_s,s)\,dS' \\
&\vec{V}^{(f)}(\vec{r},t) = \frac{1}{2\pi}\frac{\partial}{c\partial t}\left\{\overleftrightarrow{1}_z \cdot \int_P \vec{E}(\vec{r}'_s,t)\,dS'\right\} = -\frac{1}{2\pi}\frac{\partial}{c\partial t}\left\{\vec{1}_z \times \int_P Z_0\vec{H}(\vec{r}'_s,t)\,dS'\right\}
\end{aligned}
\tag{3.19}
$$

which is rather simple in form.

This factor has dimension volts and directly gives

$$
\begin{aligned}
\vec{E}^{(f)}(\vec{r},t) &= \frac{1}{r}\vec{V}^{(f)}(\vec{r},t_r) \\
\vec{H}^{(f)}(\vec{r},t) &= \frac{1}{r}\vec{1}_r \times \vec{V}^{(f)}(\vec{r},t_r) = \vec{1}_r \times \vec{E}^{(f)}(\vec{r},t)
\end{aligned}
\tag{3.20}
$$

so that we only need consider the far electric field, the far magnetic field being simply related to it.

4. Maximizing the Far Field

Define

$$\vec{\tilde{U}}(\vec{1}_r,s) \equiv \frac{\gamma}{2\pi}\overset{\leftrightarrow}{1}_z \cdot \int_P e^{-\gamma \vec{1}_r \cdot \vec{r}'_s} \vec{\tilde{E}}(\vec{r}_s,s)\, dS'$$

$$\vec{U}(\vec{1}_r,t) = \frac{1}{2\pi}\frac{\partial}{c\partial t}\left\{\overset{\leftrightarrow}{1}_z \cdot \int_P \vec{E}\left(\vec{r}'_s, t + \frac{\vec{1}_r \cdot \vec{r}'_s}{c}\right) dS'\right\} \tag{4.1}$$

so that

$$\vec{\tilde{V}}^{(f)}(\vec{1}_r,s) = -\vec{1}_r \times \left\{\vec{1}_z \times \vec{\tilde{U}}(\vec{1}_r,s)\right\}$$

$$\vec{V}^{(f)}(\vec{1}_r,t) = -\vec{1}_r \times \left\{\vec{1}_z \times \vec{U}(\vec{1}_r,s)\right\} \tag{4.2}$$

and note that only the tangential components of the electric field on P contribute. The above equations are written in terms of the electric field on P, but the similar ones in terms of the magnetic field on P could be used as well with the same results.

If one wishes to maximize the far field in some sense, then this can be done in terms of the representations in (4.1) and (4.2). For a given direction to the observer $\vec{1}_r$ let us rearrange (4.2) as

$$\vec{1}_r \times \vec{V}^{(f)}(\vec{1}_r,t) = \overset{\leftrightarrow}{1}_r \cdot \left\{\vec{1}_z \times \vec{U}(\vec{1}_r,t)\right\} \tag{4.3}$$

Since $\vec{V}^{(f)}$ has only components transverse to $\vec{1}_r$ then maximizing $\vec{1}_r \times \vec{V}^{(f)}$ is equivalent to maximizing $\vec{V}^{(f)}$, this applying in both frequency and time domains. For a given $\vec{1}_z \times \vec{U}$ one can rotate the field distribution on P such that when dotted with $\vec{1}_r$ a maximum is achieved (say in the sense of vector magnitude). In time domain $\vec{1}_z \times \vec{U}$ is a real-valued vector parallel to P, so it can be rotated such that

$$\overset{\leftrightarrow}{1}_r \cdot \vec{1}_z \times \vec{U}(\vec{1}_r,t) = \vec{1}_z \times \vec{U}(\vec{1}_r,t)$$

$$\vec{1}_r \cdot \left[\vec{1}_z \times \vec{U}(\vec{1}_r,t)\right] = 0 \tag{4.4}$$

i.e., such that the full magnitude of $\vec{1}_z \times \vec{U}$ is achieved. Now $\vec{U}(\vec{1}_r, t)$ can rotate (parallel to P) as a function of time. So one may wish to constrain the polarization as in (4.4), thereby giving a linear polarization provided $\vec{1}_r \neq \vec{1}_z$. In frequency domain $\vec{1}_z \times \vec{\tilde{U}}$ is a complex-valued vector parallel to P, so the form in (4.4) is not immediately applicable. However, if $\vec{\tilde{U}}$ is expressible as a complex scalar times a real vector for frequencies of interest, then (4.4) applies in frequency domain as well. Such is the case, for example, if $\vec{U}$ has a constant linear polarization for all time, so that $\tilde{U}$ is also linearly polarized for all frequencies. If we consider multiple observer directions $\vec{1}_r$, then maximization conditions vary as a function of $\vec{1}_r$ as is seen in the polarization condition (4.4). If one considers $\vec{1}_r$ near $\vec{1}_z$ (boresight) then this polarization effect is not significant.

More important is the phasing (in frequency domain) or timing (in time domain) of the fields on P. The traditional concept (with good reason) is that the aperture fields should be focused on the observer (at ∞ in the $\vec{1}_r$ direction). To see this, consider now the tangential electric field on P as contained in $\vec{U}$ in (4.1). Let there be a region of area A of finite linear dimensions (centered on the z axis) where the tangential electric field is non zero. Then we can write, first in frequency domain,

$$\vec{\tilde{U}}(\vec{1}_r, s) = \frac{\gamma}{2\pi} \vec{1}_x \int_P e^{\gamma \vec{1}_r \cdot \vec{r}'_s} \tilde{E}_x(\vec{r}'_s, s) dS' + \frac{\gamma}{2\pi} \vec{1}_y \int_P e^{\gamma \vec{1}_r \cdot \vec{r}'_s} \tilde{E}_y(\vec{r}'_s, s) dS' \tag{4.5}$$

Consider the two orthogonal vector components separately. Considering the x component for frequencies on the $j\omega$ axis we have

$$\begin{aligned} s &= j\omega \,,\ \gamma = jk = j\frac{\omega}{c} \\ \tilde{U}_x(\vec{1}_r, j\omega) &= \frac{j\omega}{2\pi} \int_P e^{jk \vec{1}_r \cdot \vec{r}'_s} \tilde{E}_x(\vec{r}'_s, j\omega)\, dS' \end{aligned} \tag{4.6}$$

This can be bounded as

$$\begin{aligned} |\tilde{U}_x(\vec{1}_r, j\omega)| &= \frac{|\omega|}{2\pi} \left| \int_P e^{jk \vec{1}_r \cdot \vec{r}'_s} \tilde{E}_x(\vec{r}'_s, j\omega) dS' \right| \\ &\leq \frac{|\omega|}{2\pi} \int_P \left| e^{jk \vec{1}_r \cdot \vec{r}'_s} \tilde{E}_x(\vec{r}'_s, j\omega) \right| dS' \\ &= \frac{|\omega|}{2\pi} \int_P \left| \tilde{E}_x(\vec{r}'_s, j\omega) \right| dS' \end{aligned} \tag{4.7}$$

with equality provided

$$\arg\left(e^{jk\vec{1}_r\cdot\vec{r}'_s}\tilde{E}_x(\vec{r}'_s,\,j\omega)\right) = \text{constant over all of } A \qquad (4.8)$$

$$\neq \text{function of } \vec{r}'_s$$

i.e., with each little portion of the aperture having the same phase relative to the observer in the $\vec{1}_r$ direction. Note the local field magnitudes $|\tilde{E}_x(\vec{r}_s,\,j\omega)|$ are assumed constant while one is varying the phase to maximize $|\tilde{U}_x|$. The same conditions apply to the y component. Taking the magnitude of $\tilde{\vec{U}}$, this is then maximized by setting

$$\arg\left(e^{jk\vec{1}_r\cdot\vec{r}'_s}\tilde{E}_y(\vec{r}'_s,\,j\omega)\right) - \arg\left(e^{jk\vec{1}_r\cdot\vec{r}'_s}\tilde{E}_x(\vec{r}'_s,\,j\omega)\right)$$

$$= \begin{cases} 0 \text{ over all of } A \\ \text{or} \\ \pi \text{ over all of } A \end{cases} \qquad (4.9)$$

Of course, one can also vary the orientation of the electric field parallel to P (i.e., relative magnitudes of $\tilde{E}_x$ and $\tilde{E}_y$) for a given $|\overleftrightarrow{1}_z\cdot\tilde{\vec{E}}|$ at each $\vec{r}'_s$, resulting in a uniform polarization over the array to maximize $|\tilde{\overleftrightarrow{U}}|$. While the above derivation is for $s = j\omega$, more general s throughout the complex plane can be used in (4.5) with the same conclusions.

In time domain one can consider a tangential electric field on P of the form (as in TEM plane waves on the aperture [6])

$$\overleftrightarrow{1}_z\cdot\vec{E}(\vec{r}'_s,t) \equiv \vec{E}_t(\vec{r}'_s)\,f\left(t - \frac{\vec{1}_r\cdot\vec{r}'_s}{c}\right)$$

$$\vec{E}_t(\vec{r}'_s) \equiv \text{spatial distribution with } x \text{ and } y \text{ components} \qquad (4.10)$$

$$f(t) \equiv \text{waveform}$$

Then (4.1) gives

$$\vec{U}(\vec{1}_r,t) = \frac{1}{2\pi}\frac{\partial}{c\partial t}\int_P \vec{E}_t(\vec{r}'_s)\,f(t)\,dS'$$
$$= \frac{1}{2\pi}\left[\int_P \vec{E}(\vec{r}'_s)\,dS'\right]\frac{d\,f(t)}{cdt} \tag{4.11}$$

which is a vector wave of constant polarization. If the temporal part of (4.10) varied differently over the aperture, then one would have a smaller peak of $\vec{U}$ as

$$\overleftrightarrow{1}_z \cdot \vec{E}(\vec{r}'_s,t) = \vec{E}_t(\vec{r}'_s)\; f\left(t - \frac{\vec{1}_r \cdot \vec{r}'_s}{c} + \tau(\vec{r}'_s)\right)$$
$$\vec{U}(\vec{r}_s,t) = \frac{1}{2\pi}\int_P \vec{E}_t(\vec{r}'_s)\frac{\partial}{c\partial t}f(t-\tau(\vec{r}'_s))\,dS' \tag{4.12}$$

Maximizing this over all time, one can consider the x component for which

$$\sup_t\left|U_x(\vec{1}_r,t)\right| \le \frac{1}{2\pi}\left|\int_P E_{tx}(\vec{r}'_s)\,dS'\right|\left[\sup_t\left|\frac{\partial}{c\partial t}\;f(t+\tau(\vec{r}'_s))\right|\right]$$
$$= \frac{1}{2\pi}\left|\int_P E_{tx}(\vec{r}'_s)\,dS'\right|\left[\sup_t\left|\frac{\partial f(t)}{\partial\partial t}\right|\right] \tag{4.13}$$

provided E_{tx} has the same sign over all A, with equality occurring if τ is independent of $\vec{r}'_s$. The same applies to the x component and thereby to the maximum of $|\vec{U}(\vec{1}_r,t)|$. Applying various orders of temporal differentiation or integration to $\vec{U}$ (or convolving with more general temporal functions), and then maximizing over t the same general result occurs with τ independent of $\vec{r}'_s$. With this constraint the general result in (4.11) describes the far field as a function of time. Note the consistency of the maximization constraint in time domain (4.10) and frequency domain (phase as in (4.8) and (4.9)). This is an example of how norm concepts as in Appendix A can be applied in frequency domain for temporal quantities, and conversely.

5. Comparing Waveforms

Suppose that it is desired to have an antenna driven by a temporal source give some far field $\vec{V}_1^{(f)}(\vec{1}_r,t)$. This might even be some kind of specification. It might apply for some particular observer direction (e.g., $\vec{1}_r = \vec{1}_z$) or over some range (solid angle) of $\vec{1}_r$. Suppose also that one has another far field $\vec{V}_2^{(f)}(\vec{1}_r,t)$ which characterizes some antenna/source combination (whether from calculation or measurement). How should we compare these two? Should we ask if $\vec{V}_2^{(f)}$ is larger than $\vec{V}_1^{(f)}$ and, if so, in what sense? There are issues of time, frequency, and polarization to consider.

Another approach is to determine if $\vec{V}_2^{(f)} - \vec{V}_1^{(f)}$ is small in some sense. In terms of norms this would often be done as a relative error of the form

$$\upsilon_2 \equiv \frac{\left\| \vec{V}_2^{(f)}(\vec{1}_r,t) - \vec{V}_1^{(f)}(\vec{1}_r,t) \right\|}{\left\| \vec{V}_1^{(f)}(\vec{1}_r,t) \right\|} \tag{5.1}$$

which is ideally small compared to unity. Note that the norms in the numerator and denominator should be in the same sense, making υ_2 dimensionless.

Suppose, however, that $\vec{V}_2^{(f)}$ is exactly like $\vec{V}_1^{(f)}$, but twice as large (for all times) with the same polarization. Should such a result be deemed unacceptable, or should we multiply $\vec{V}_2^{(f)}$ by some real scalar χ which gives a best fit to $\vec{V}_1^{(f)}$ in the sense of smallest υ_2 in (5.1)? One should also shift time by some real τ in $\vec{V}_2^{(f)}$ since the definition of $t = 0$ is arbitrary anyway. What about polarization? For a given $\vec{1}_r$ one could rotate the antenna around this axis and thereby rotate $\vec{V}_2^{(f)}$ at the observer for a better fit to $\vec{V}_1^{(f)}$. If, however, one is interested in the performance over some range of $\vec{1}_r$ there are limitations on the rotation of the antenna to rotations about at most two axes (O_3^+ group) which can be represented by an orthogonal dyadic $\overleftrightarrow{\Omega}^{+}$ (proper rotations, i.e., no reflections). This is in effect a rotation of the coordinates as

$$\begin{aligned} &\vec{r}_2 = \overleftrightarrow{\Omega}^{+} \cdot \vec{r}_1 \\ &|\vec{r}_2| = |\vec{r}_1| \text{ with both coordinate sets right handed} \\ &\det(\overleftrightarrow{\Omega}^{+}) = 1 \end{aligned} \tag{5.2}$$

Thus one might define

$$\vec{V}_3^{(f)}(\vec{1}_r,t) \equiv \chi \overleftrightarrow{\Omega}^{+T} \cdot \vec{V}_2^{(f)}\left(\overleftrightarrow{\Omega}^{+T} \cdot \vec{1}_r, t-\tau\right)$$

$$v_3 \equiv \frac{\left\| \vec{V}_3^{(f)}(\vec{1}_r,t) - \vec{V}_1^{(f)}(\vec{1}_r,t) \right\|}{\left\| \vec{V}_1^{(f)}(\vec{1}_r,t) \right\|} \tag{5.3}$$

where τ, χ, and $\overleftrightarrow{\Omega}^{+}$ are adjusted to minimize v_3. One might also require that χ be greater than some positive number such as .9 or 1.

One might use various norms in (5.3). As discussed in Appendix A, these could be norms over time, or over frequency using the two-sided-Laplace/Fourier transforms of the vector temporal waveforms. One can minimize v_3 for some particular observer direction $\vec{1}_r$ or for some range of $\vec{1}_r$. In general one then also defines the norm to appropriately range over $\vec{1}_r$ on the unit sphere (some range of (θ, ϕ)). Various weights can also be included in the norm over space (the unit sphere).

As discussed in Appendix B, one can have weighted frequency-domain norms of temporal waveforms. Suppose that one has some important range of radian frequencies $0 \le \omega_\ell \le \omega \le \omega_h$. Then one might define some filter function $\tilde{g}(s)$, e.g.,

$$\tilde{g}(s) = g_0 \left[1 + \frac{s}{\omega_h}\right]^{-1} \frac{s}{\omega_\ell} \left[1 + \frac{s}{\omega_\ell}\right]^{-1} \tag{5.4}$$

which attenuates the spectrum outside the desired range of frequencies. This approach could be appropriate if spectral content and smoothness through the frequency band were important, as in the case of waveforms for target identification [14, 22].

6.0 Concluding Remarks

In order to compare radiated vector temporal waveforms it is necessary to reduce the infinite dimensional problem (the points on the time axis) to a manageable (preferably small) number of parameters. Norms are quite suitable for the purpose, giving a single non-negative real number. However, there are various possible norms, such as for peak (∞-norm) and for square root of energy (2-norm). These need to be chosen for their appropriateness to the particular antenna and pulser design problem under consideration. For cases in which the spectral properties of the radiated pulse are important, one can use weighted norms of the two-sided-Laplace/Fourier transform of the pulse, this also being an acceptable temporal norm. This leads to ways of specifying antenna/pulser performance so that one can quantitatively compare various designs to each other and to some desired ideal performance.

Appendix A. Frequency-Domain Norms of Temporal Waveforms

Consider some suitably well behaved temporal vector waveform $\vec{F}(t)$. It has various kinds of norms which are all defined to have the properties [12, 17]

$$\|\vec{F}(t)\| \begin{cases} = 0 \text{ iff } \vec{F}(t) \equiv \vec{0} \text{ or has zero "measure" per the particular norm} \\ > 0 \text{ otherwise} \end{cases}$$

$$\|\alpha\, \vec{F}(t)\| = |\alpha|\, \|\vec{F}(t)\| \;,\; \alpha \equiv \text{a complex scalar} \tag{A.1}$$

$$\|\vec{F}_1(t) + \vec{F}_2(t)\| \leq \|\vec{F}_1(t)\| + \|\vec{F}_2(t)\|$$

Note that scalar waveforms are just vectors with one component. While here we indicate the vectors as 3-vectors in the usual 3-dimensional space, the concepts apply to more general N-component vectors.

The most commonly used norm is the p-norm defined by

$$\|\vec{F}(t)\|_{pt} \equiv \|\vec{F}_n(t)\|_{pt} \equiv \left\{ \int_{-\infty}^{\infty} \left|\vec{F}(t)\right|^p dt \right\}^{\frac{1}{p}} \tag{A.2}$$

$$= \| \, \|\vec{F}(t)\|_{pv} \, \|_{pt} = \| \, \|\vec{F}(t)\|_{pt} \, \|_{pv}$$

where the subscript "t" (for temporal) refers to the temporal integration, and the subscript "v" (for vector) refers to the summation over vector components. Commonly used are the 2-norm

$$\|\vec{F}(t)\|_{2t} = \left\{ \int_{-\infty}^{\infty} \vec{F}(t) \cdot \vec{F}^{\,*}(t)\, dt \right\}^{\frac{1}{2}} \tag{A.3}$$

and ∞-norm (or peak)

$$\|\vec{F}(t)\|_{\infty t} = \sup_{t,n} |F_n(t)| \tag{A.4}$$

In [13] there is introduced the m-norm (with "m" symbolizing maximum magnitude) as

$$\|\vec{F}(t)\|_{mt} \equiv \| \, \|\vec{F}(t)\|_{2v} \, \|_{\infty t} = \sup_t \left|\vec{F}(t)\right|$$

$$= \left\{ \| \vec{F}(t) \cdot \vec{F}^{\,*}(t) \|_{\infty t} \right\}^{\frac{1}{2}} \tag{A.5}$$

For vector functions this gives the peak of the vector magnitude over all time. Note the use of conjugate (*); for cases of real-valued vectors this is not required. Furthermore t (time) is taken as real valued.

With the two-sided Laplace (or Fourier) transform

$$
\begin{aligned}
\tilde{\vec{F}}(s) &\equiv \int_{-\infty}^{\infty} \vec{F}(t)\, e^{-st} dt \qquad \text{(transform)} \\
\vec{F}(t) &\equiv \frac{1}{2\pi j} \int_{\mathrm{Br}} \tilde{\vec{F}}(t)\, e^{st} ds \quad \text{(inverse transform)} \\
s &\equiv \Omega + j\omega \equiv \text{Laplace-transform variable or complex frequency} \\
\mathrm{Br} &\equiv \text{Bromwich contour in strip of convergence}
\end{aligned}
\tag{A.6}
$$

we can describe $\vec{F}(t)$ by $\tilde{\vec{F}}(s)$ and consider norms of $\tilde{\vec{F}}(s)$. Of course, let us then restrict $\vec{F}(t)$ such that its transform exists and has a strip of convergence for defining the inverse transform. This is part of what is meant by "suitably behaved". Note also that isolated points of $\vec{F}(t)$ which are discontinuous on both sides are assumed not to exist since their temporal integrals are zero and thereby contribute zero to the transform.

Consider now the norm of $\tilde{\vec{F}}(s)$ and its use as a norm of $\vec{F}(t)$. For a norm of $\tilde{\vec{F}}(s)$ the properties in (A.1) must apply with t replaced by s. A subscript "s" or "ω" as appropriate will be used to distinguish such norms. Suppose there is such a norm with

$$
\begin{aligned}
&\| \tilde{\vec{F}}(s) \|_s \begin{cases} = 0 \textit{ iff } \tilde{\vec{F}}(s) \equiv \vec{0} \\ > 0 \text{ otherwise} \end{cases} \\
&\| \alpha\, \tilde{\vec{F}}(s) \|_s = |\alpha|\, \| \tilde{\vec{F}}(s) \|_s \,, \ \alpha \equiv \text{a complex scalar} \\
&\| \tilde{\vec{F}}_1(s) + \tilde{\vec{F}}_2(s) \|_s \le \| \tilde{\vec{F}}_1(s) \|_s + \| \tilde{\vec{F}}_2(s) \|_s
\end{aligned}
\tag{A.7}
$$

Within our "suitably behaved" requirement $\tilde{\vec{F}}(s) = \vec{0}$ is equivalent to $\vec{F}(t) = \vec{0}$, so the first property in (A.7) is equivalent to the first property in (A.1). The second property is also equivalent in frequency and time due to the linearity of the Laplace/Fourier transform. Consider the third property of (A.7) and note that if the frequency-domain norm satisfies this, and the frequency-domain norm is used to *define* the temporal norm, then the third property in (A.1) is automatically satisfied (by definition). Therefore, norms of the transforms are norms of the temporal functions (and conversely).

In a symbolic way this result of frequency-domain norms applying to temporal waveforms can be stated as

$$\| F(t) \|_t \equiv \| \int_{-\infty}^{\infty} F(t)\, e^{-st}\, dt \|_s \tag{A.8}$$

with this defining the temporal norm by a frequency-domain norm. One can also define a frequency-domain norm by

$$\| \tilde{F}(s) \|_s \equiv \| \frac{1}{2\pi j} \int_{\mathrm{Br}} \tilde{F}(s)\, e^{st}\, ds \|_t \tag{A.9}$$

as a dual procedure.

This relation between norms in time and frequency domains is illustrated by the 2-norm. Considering real vector-valued functions we have [12, 17]

$$\begin{aligned} \| \vec{F}(t) \|_{2t} &= \left\{ \int_{-\infty}^{\infty} \vec{F}(t) \cdot \vec{F}(t)\; dt \right\}^{\frac{1}{2}} = \frac{1}{\sqrt{2\pi}} \left\{ \int_{Br} \vec{\tilde{F}}(s) \cdot \vec{\tilde{F}}(-s)\; d\omega \right\}^{\frac{1}{2}} \\ &= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{\infty} \vec{\tilde{F}}(j\omega) \cdot \vec{\tilde{F}}^{*}(j\omega)\; d\omega \right\}^{\frac{1}{2}} = \frac{1}{\sqrt{2\pi}} \| \vec{\tilde{F}}(j\omega) \|_{2\omega} \end{aligned} \tag{A.10}$$

showing that the 2-norm is the same in time and frequency domains except for a multiplicative constant. Note the requirement that $\vec{\tilde{F}}(s)$ and $\vec{\tilde{F}}(-s)$ have a common strip of convergence which includes the $j\omega$ axis.

The above can be generalized to complex temporal vector functions, such as the combined field in Section 2, from [11] as

$$\vec{E}_q(\vec{r},t) = \vec{E}(\vec{r},t) + qj\; Z_0\, \vec{H}(\vec{r},t)$$

$$\| \vec{E}_q(\vec{r},t) \|_{2t} = \left\{ \int_{-\infty}^{\infty} \vec{E}_q(\vec{r},t) \cdot \vec{E}_q^{*}(\vec{r},t)\, dt \right\}^{\frac{1}{2}} = \left\{ \int_{-\infty}^{\infty} \vec{E}_q(\vec{r},t) \cdot \vec{E}_{-q}(\vec{r},t)\, dt \right\}^{\frac{1}{2}}$$

$$= \left\{ \int_{-\infty}^{\infty} \left[\vec{E}(\vec{r},t) \cdot \vec{E}(\vec{r},t) + Z_0^2\, \vec{H}(\vec{r},t) \cdot \vec{H}(\vec{r},t) \right] dt \right\}^{\frac{1}{2}}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{\infty} \vec{\tilde{E}}_q(\vec{r},t) \cdot \vec{\tilde{E}}_{-q}(\vec{r},-s)\, ds \right\}^{\frac{1}{2}} = \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{\infty} \vec{\tilde{E}}_q(\vec{r},j\omega) \cdot \vec{\tilde{E}}^{*}_q(\vec{r},j\omega)\, d\omega \right\}^{\frac{1}{2}} \tag{A.11}$$

$$= \frac{1}{\sqrt{2\pi}} \, \| \vec{\tilde{E}}_q(\vec{r},j\omega) \|_{2\omega}$$

This is a generalization of the well-known Parseval theorem to complex vector functions.

Appendix B. Weighted Frequency-Domain Norms of Temporal Waveforms

Similar to the form in [18], let us define a weighted norm in frequency domain by use of a weighting function $\tilde{g}(s)$, so that (A.7) is applied to $\tilde{g}(s)\,\tilde{\vec{F}}(s)$ and thereby to $g(t) \circ \vec{F}(t)$ where $\circ$ denotes convolution with respect to time. Here $g(t)$ is a scalar, but vector and dyadic forms are also possible. Considering the first property in (A.7), let us restrict $\tilde{g}(s)$ such that $\tilde{g}(s)$ has only isolated zeros and

$$
\begin{aligned}
\tilde{g}(s)\;\tilde{\vec{F}}(s) &\equiv \vec{0} \;\Leftrightarrow\; \tilde{\vec{F}}(s) \equiv \vec{0} \\
\tilde{g}(s) \circ \vec{F}(t) &\equiv \vec{0} \;\Leftrightarrow\; \vec{F}(t) \equiv \vec{0}
\end{aligned}
\tag{B.1}
$$

in the sense previously discussed. Physically, we can think of $\tilde{g}(s)$ as some kind of filter which emphasizes certain frequencies in $\tilde{\vec{F}}(s)$ at our discretion. Thus, one may wish to restrict $\tilde{g}(s)$ such that $g(t)$ is real valued and $g(t)\circ$ is a causal operator, corresponding to a physical filter.

Symbolically we can define

$$
\begin{aligned}
\|\tilde{\vec{F}}(s)\|_{s\tilde{g}} &\equiv \|\tilde{g}(s)\,\tilde{\vec{F}}(s)\|_s \\
\|\vec{F}(t)\|_{tg} &\equiv \|g(t) \circ \vec{F}(t)\|_t
\end{aligned}
\tag{B.2}
$$

as weighted norms in frequency domain, and their equivalents in time domain.

In [7] the D- and I-norms (temporal) are introduced, where by these are meant for our vector waveform

$$
\begin{aligned}
\|\vec{F}(t)\|_{Dt} &\equiv \left\|\frac{d}{dt}\,\vec{F}(t)\right\|_t \\
\|\vec{F}(t)\|_{It} &\equiv \left\|\int_{-\infty}^{t} \vec{F}(t')\,dt'\right\|_t
\end{aligned}
\tag{B.3}
$$

These are norms of $\vec{F}(t)$ provided appropriate restrictions are placed on the waveform. For the D-norm the derivative needs to be suitably bounded. For the I-norm, the integral similarly needs to be bounded and this implies that $\vec{F}(t) \to \vec{0}$ as $t \to -\infty$ in some suitable fashion, such as being identically zero before some finite time.

In frequency domain we have

$$\frac{d}{dt}\vec{F}(t) \rightarrow s\,\tilde{\vec{F}}(s)$$
$$\int_{-\infty}^{t}\vec{F}(t')\,dt' \rightarrow \frac{1}{s}\,\tilde{\vec{F}}(s) \tag{B.4}$$

So in frequency domain we have s and $1/s$ as weighting functions to take the place of $\tilde{g}(s)$ in (B.2). In time domain $g(t)$ takes the forms

$$s \rightarrow \frac{d}{dt}\delta(t) \quad \text{(derivative of a delta function)}$$
$$\frac{1}{s} \rightarrow u(t) \quad \text{(step function)} \tag{B.5}$$
$$\delta(t) = \frac{d}{dt}\,u(t)$$

The delta function and its derivative should be understood in an appropriate limiting sense.

So we define norms of $\vec{F}(t)$ symbolically via

$$\|\vec{F}(t)\|_{Dt} \equiv \|s\,\tilde{\vec{F}}(s)\|_s$$
$$\|\vec{F}(t)\|_{It} \equiv \|\frac{1}{s}\,\tilde{\vec{F}}(s)\|_s \tag{B.6}$$

where the particular form of temporal norm is defined by the form chosen in frequency domain. Note that if we choose a particular form of frequency-domain norm, such as 2-norm, then (A.8) gives

$$\|\vec{F}(t)\|_{D2t} = \frac{1}{\sqrt{2\pi}}\,\|j\omega\,\tilde{\vec{F}}(j\omega)\|_{2\omega} \tag{B.7}$$

which is not the D2-norm (2-norm of the derivative) in time domain, but a constant times it. Similarly we have

$$\|\vec{F}(t)\|_{I2t} = \frac{1}{\sqrt{2\pi}}\,\|\frac{1}{j\omega}\,\tilde{\vec{F}}(j\omega)\|_{2\omega} \tag{B.8}$$

So the norms on the two sides of the equations in (B.6) have, in general, different senses.

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