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Self-Complementary Array Antennas

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Abstract

The concept of a self-complementary antenna is important in antenna theory in that it has an input impedance which is half the free-space wave impedance, independent of frequency. This paper generalizes such antennas to include antenna arrays with the same property. By imposition of additional symmetries associated with the two-dimensional space groups (discrete translation, rotation, and axial reflection) one can make the array radiate (at least normal to the array plane) with a frequency-independent polarization. Symmetry also allows there to be two independent orthogonal linear polarizations on boresight.

I. Introduction

Antenna arrays have some useful properties, particularly when it comes to steering the beam (in transmission and/or reception) without physically moving the antenna elements. This is a commonly used technique in designing narrow-beam antennas. It can also be used for radiating and/or receiving fast electromagnetic transients (timed array) [3]. Here our concern is with planar arrays suitable for illuminating a planar aperture just in front of the array with an approximate plane wave so as to focus the aperture at infinity in some desired direction. As discussed in [3] it is very important that the elements in the array be connected together, so that for low frequencies (such that the wavelength is large compared to the element spacing) the current can continuously flow from one element to the next. The elements can be thought of as unit cells in a periodic array, and there are various possible shapes the cells and the materials and sources therein can take [3 (and references therein)].

Symmetry plays a key role in the present discussion. For present purposes the array will be analyzed as though it were infinite in two dimensions. It will be taken as periodic in these two dimensions, which together with the point symmetries (rotation and reflection) give space groups [11]. In addition, the array will be planar, so that the symmetry of self complementarity can be applied, giving some new simple results for antenna arrays.

II. Planar Complementary Structures

Consider some ensemble of perfectly conducting sheets, or more general sheet impedances, together with sources (voltage or current) on a plane S which we take as the $z = 0$ plane with coordinates x and y from the usual Cartesian (x, y, z) system. in its original form [7] (also known as the Babinet principle) one replaces perfectly conducting sheets by free space (or sheets of zero sheet admittance) and conversely. For example, a disk becomes an aperture and conversely. Associated with the interchange of electric and magnetic fields between the two complementary structures (duality), there is an interchange of voltage and current sources, leading to the well-known relation that the product of the input impedances (for a single source in each case) for the two structures is just $Z_0^2 / 4$ where

$$\begin{aligned} Z_0 &= \sqrt{\frac{\mu_0}{\varepsilon_0}} \equiv \text{free space wave impedance} \\ &\simeq 3.76.73\,\Omega \end{aligned} \tag{2.1}$$

Of course this relation can apply to antennas in uniform isotropic media other than free space by a substitution of the wave impedance for such media.

As discussed in [5, 6, 11] the idea of a complementary structure can be generalized to include sheet impedances (both scalar and dyadic). Using a superscript c to indicate the complementary problem we have

$$\begin{aligned} &\tilde{\overleftrightarrow{Y}}_s(\vec{r}_s, s) = \frac{2}{Z_0}\tilde{\overleftrightarrow{y}}_s(\vec{r}_s, s) \equiv \text{sheet admittance } (2\times 2 \text{ dyadic}) \\ &\tilde{\overleftrightarrow{Y}}_s^{(c)}(\vec{r}_s, s) = \frac{2}{Z_0}\tilde{\overleftrightarrow{y}}_s^{(c)}(\vec{r}_s, s) \equiv \text{complementary sheet admittance } (2\times 2 \text{ dyadic}) \\ &\tilde{\overleftrightarrow{Y}}_s^{(c)}(\vec{r}_s, s) = \tilde{\tau}_d \cdot \tilde{\overleftrightarrow{y}}_s^{-1}(\vec{r}_s, s) \cdot \tilde{\tau}_d^T \\ &\qquad \tilde{\tau}_d = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \pi / 2 \text{ rotation} \\ &\vec{r}_s = (x, y, 0) \equiv \text{ coordinates on } z = 0 \text{ plane } (S) \\ &\sim \equiv 2 - \text{sided Laplace transform (over time)} \\ &s \equiv \Omega + j\omega \equiv \text{ Laplace - transform variable or complex frequency} \end{aligned} \tag{2.2}$$

Note that the inverse of the 2×2 dyadics is used in a two-dimensional sense since there are no z components defined, e.g.,

$$\tilde{\tilde{Y}}_s(\vec{r}_s,s)\cdot\tilde{\tilde{Y}}_s^{-1}(\vec{r}_s,s)=\tilde{\tilde{Y}}_s^{-1}(\vec{r}_s,s)\cdot\tilde{\tilde{Y}}_s(\vec{r}_s,s)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\equiv\overleftrightarrow{1}_z \tag{2.3}$$

Here the transverse-to-z identity is written in 2×2 form, but it can also be written in 3×3 form by insertion of zeros for third row and third column.

The above relations are derived from the transformation of the fields to the dual fields in going from the original problem to the complementary problem via

$$\begin{aligned}
&\vec{E}_q(\vec{r},t)=\vec{E}(\vec{r},t)+qjZ_0\vec{H}(\vec{r},t)\equiv \text{ combined field}\\
&\vec{E}_q^{(d)}(\vec{r},t)=-qj\vec{E}(\vec{r},t)\equiv \text{ dual (or complementary) combined field}\\
&\vec{E}^{(d)}(\vec{r},t)=Z_0\vec{H}(\vec{r},t)\,,\;\vec{H}^{(d)}(\vec{r},t)=-\frac{1}{Z_0}\vec{E}(\vec{r},t)\\
&\qquad q=\pm1 \text{ (separation index)}
\end{aligned} \tag{2.4}$$

Combine this with

$$\tilde{\vec{J}}_s(\vec{r}_s,s)=\tilde{\tilde{Y}}_s(\vec{r}_s,s)\cdot\tilde{\vec{E}}_s(\vec{r}_s,s)\equiv \text{ surface current density on } z=0 \text{ plane} \tag{2.5}$$

where a subscript s denotes the tangential components of the field at S, in the case of the magnetic field (discontinuous through S) this being taken at $z=0_+$ (the $+z$ side of S). Noting that since the fields are symmetric [4, 11] with respect to S the surface current density also satisfies the boundary condition on S as

$$\vec{J}_s(\vec{r}_s,t)=2\overleftrightarrow{\tau}_d\cdot\vec{H}_s(\vec{r},t) \tag{2.6}$$

Applying (2.4) through (2.6) for both original and complementary quantities gives the complementary sheet admittance in (2.2).

III. Self Complementarity

An important application of the idea of complementarity occurs if the complementary antenna is in some appropriate sense the same as the original antenna. such an antenna is referred to as self complementary, and in the simple form of a single source has an input impedance of $Z_0 / 2$ (frequency independent). The sense in which the complementary antenna is the same as the original one is basically that one can be transformed into the other by a geometrical operation which preserves distances between points, including combinations of rotation, reflection, and translation, i.e., the space groups [11].

A simple example is indicated in fig. 3.1. Here the operation is rotation by $\pi/2$ to rotate the original antenna into its (self) complement. This is referred to as C_{2c} symmetry, a self-complementary rotation group [11]. Rotation by $\pi/2$ gives back the original antenna. Note that the antenna conductors extend in principle to infinity (but are truncated at some finite radius in practice). There are also indicated four patches with sheet admittances of $2 / Z_0$ (or $y_s = 1$) which rotate into each other on taking the complement, this value of admittance being itself its own complement. As discussed in [5, 6, 11] there are also special cases of dyadic sheet admittance which are self complementary. Such patches need not be self complementary but can alternate with their complement on $\pi/2$ rotation.

Note in fig. 3.1 the contours C_e and C_h for integrating the electric and magnetic fields across the (ideally small) source region to give voltage and current respectively. With $\pi/2$ rotation the role of these two contours is interchanged (including an appropriate sign reversal) for voltage and current. It is possible to have self-complementary structures where the rotation angle ϕ_c is given by

$$\phi_c = \frac{\pi}{N} , N = 2,3,4,... \tag{3.1}$$

giving C_{Nc} symmetry. The resulting antennas also have the symmetry of the subgroup C_N (N-fold rotation axis). However, as discussed in [6] these are not in general self complementary as far as sources are concerned. There are N terminals or N-1 ports (terminal pairs) for defining voltages and currents. As discussed in [8] the resulting impedance matrix is still calculable (and frequency independent) based on symmetry considerations. In the present paper our concern is centered on single terminal pairs which, with two-dimensional rotation as the geometrical part of the symmetry operation, gives C_{2c} as the symmetry of concern.

Another symmetry operation in the plane S to consider is reflection R. Defining the usual cylindrical coordinates (Ψ, ϕ, z) with

$$x = \Psi \cos(\phi) , y = \Psi \sin(\phi) \tag{3.2}$$

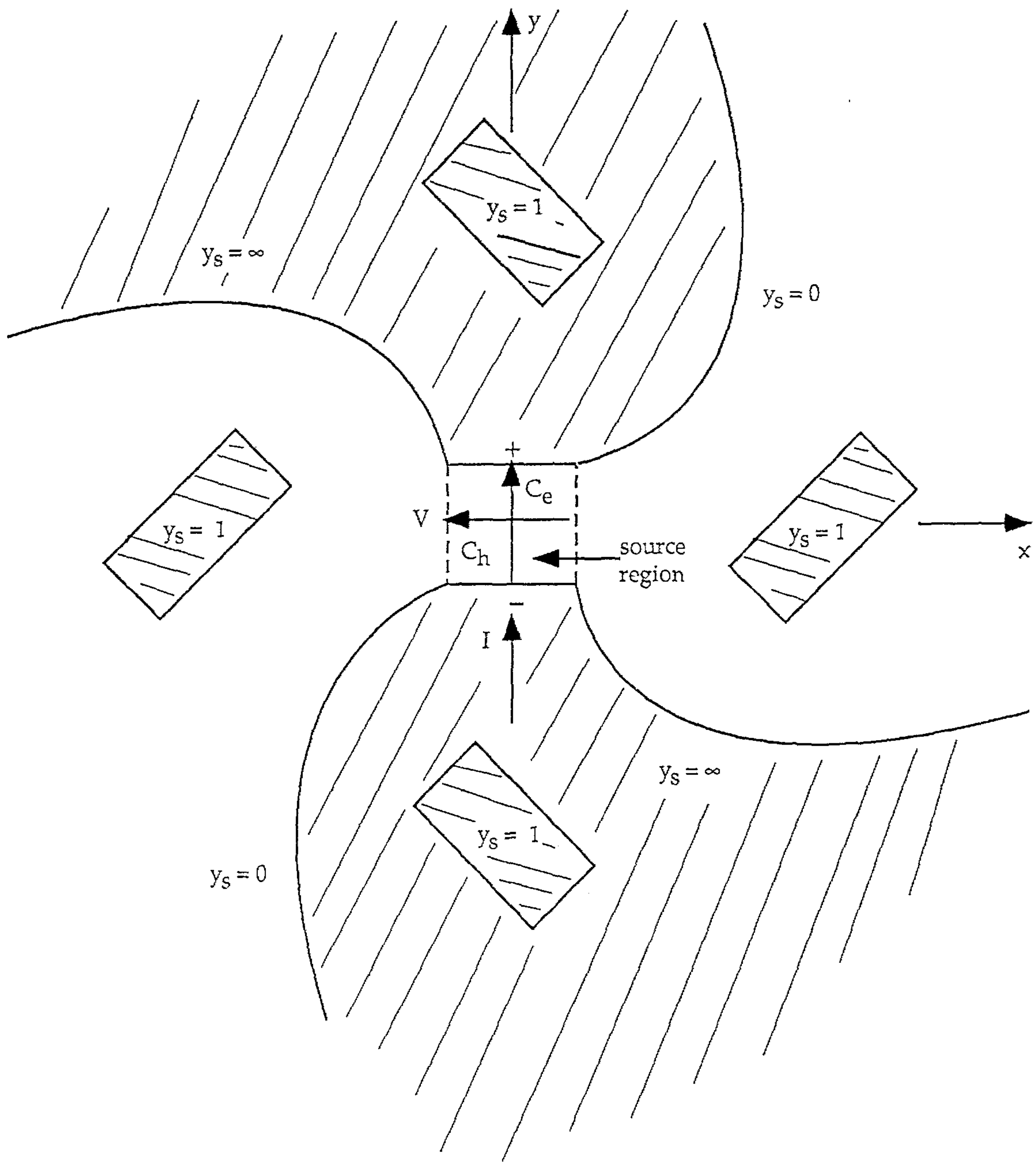


Fig. 3.1. C_{2c} Self-Complementary Antenna

then consider lines in the (x, y) plane defined by constant ϕ; these can also be considered as planes perpendicular to S by extension in the $\pm z$ direction. Consider P_1 at $\phi = \pi/4, 5\pi/4$ as indicated in fig. 3.2. Reflection R_1 through this plane gives the complementary antenna provided each perfectly conducting position reflects to a free space position and conversely provided P_1 is also a boundary between two such regions. (Note that perfect conductors can alternate on both sides of P_1, a more general situation than in fig. 3.2.) In addition another plane P_2 at $\phi = 3\pi/4, 7\pi/4$ becomes such a reflection plane (reflection R_2). Again note that in these reflections the contours C_e and C_h are interchanged. With these constraints note that what has been produced is a special case of the C_{2c} symmetry discussed previously. Now the planes P_1 and P_2 defining reflection into the complement produce reflection symmetry planes P_x ($x = 0$ plane) and P_y ($y = 0$ plane) in the original antenna. Besides rotation symmetry C_2 the two axial symmetry planes P_x and P_y give C_{2a} symmetry (four group elements) and the self-complementary property on $\pi/2$ rotation gives C_{2ac} symmetry (eight group elements). Note that the self-complementary scalar sheet admittance $2/Z_0$ (or $y_s = 1$) replicates itself on $\pi/4$ rotation and has reflection symmetry with respect to P_x and P_y as well as P_1 and P_2. Furthermore, such patches need not be self complementary, but can alternate with their complements provided the reflection symmetry is maintained.

This reflection-self-complementary antenna has a useful property in that the electric field away from S is polarized in the y direction on both P_x and P_y. More generally the fields are symmetric with respect to P_x, and antisymmetric with respect to P_y [4, 11]. If one restricts the observer to locations on the z axis only one symmetry plane (P_x or P_y) is required to assure a y polarization. However, the additional constraint of self complementarity makes both P_x and P_y symmetry planes, and gives the nice property of $Z_0/2$ input impedance. One can go a step further and use only perfect conductors and free space to give a flat-plate cone (of infinite size) which has the additional nice property that the radiated time-domain waveform is the same as the voltage waveform attached to the terminal pairs, the fields being a spherical TEM wave [10].

There are other special cases of reflection self complementarity. As discussed in [8] one can have a two-port structure with the two ports separated on the same reflection plane. (This can be extended to an arbitrary number of ports (collinear ports) on this reflection line or plane.) In this case there are some special results for the 2×2 impedance matrix. In a more general sense one can construct various self-complementary antenna structures with multiple ports (a finite number) involving rotation and/or reflection. Depending on the case specifics there may be certain requirements to match certain of the sources at particular ports.

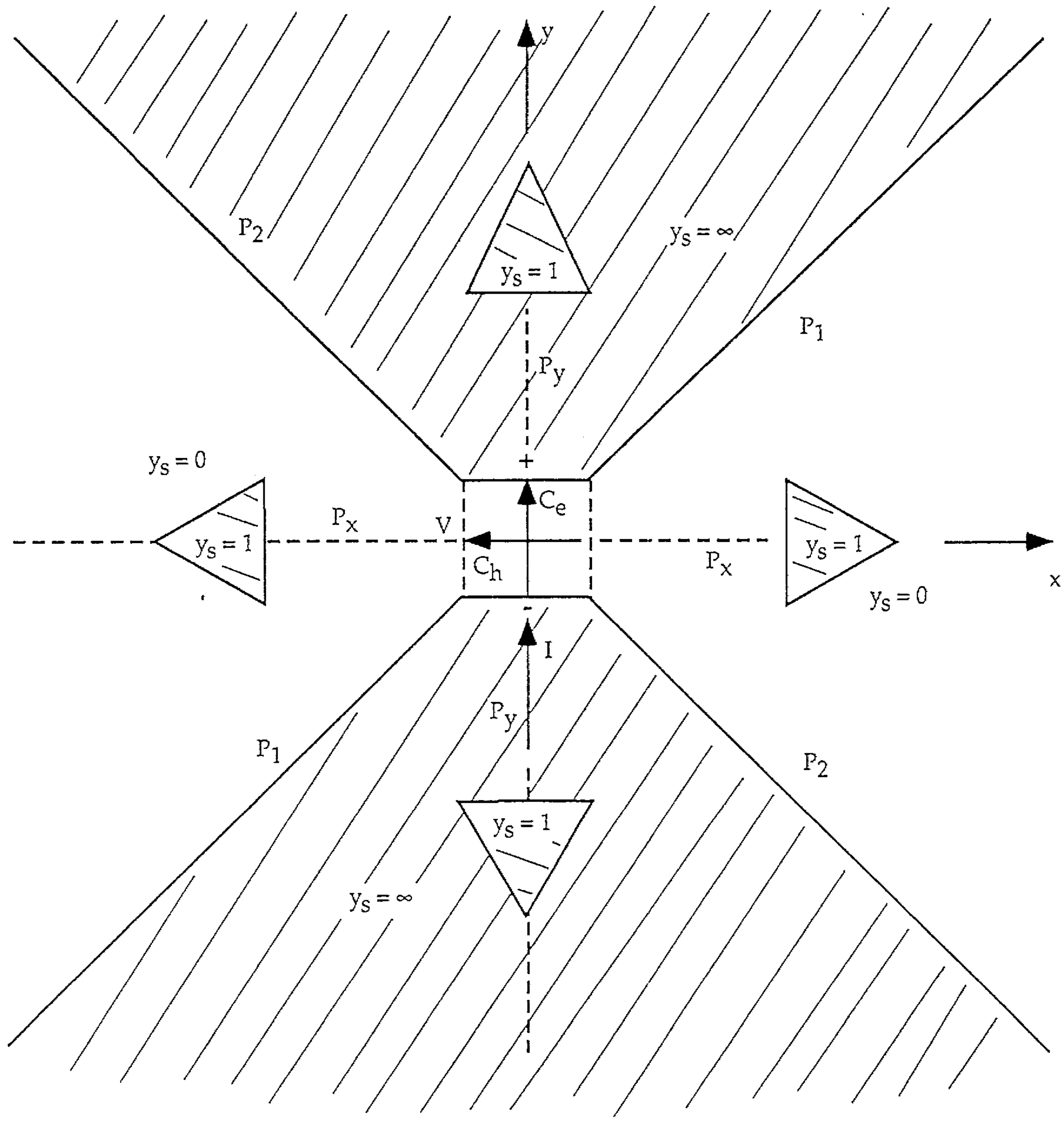


Fig. 3.2. C_{2ac} Self-Complementary Antenna

IV. Two-Dimensional Periodic Array Antennas

Now consider array antennas such as in [3] modeled as two-dimensionally infinite periodic structures. For this section the array need not be planar, but can have depth in the z direction. There is no assumption that S (or any plane parallel to it) is a symmetry plane. In [9] such a structure is referred to as a *network pattern*. In the next section, when self-complementary planar structures are considered where S is a symmetry plane, such a structure is referred to as a *layer* in [9].

Begin with the two-dimensional translation group T_2 given by [11]

$$T_2 = \left\{ T_2(\vec{\xi}\,) \middle| \vec{r} \rightarrow \vec{r} + \vec{\xi} \right\}$$
$$\vec{\xi} = \sum_{\ell=1}^{2} p_\ell \vec{\xi}_\ell \quad , \quad p_\ell \equiv \text{ integers} \tag{4.1}$$

The two vectors $\vec{\xi}_1$ and $\vec{\xi}_2$ (real and linearly independent) can be thought of as basis vectors for a two-dimensional space (plane) which we can take without loss of generality as any plane of constant z (such as S). Then we have

$$\vec{\xi}_\ell \cdot \vec{1}_z = 0 \, , \, \ell = 1,2 \tag{4.2}$$

By T_2 symmetry we then mean that on each plane of constant z, whatever is at some point $\vec{r}$ is the same at $\vec{r} + \vec{\xi}$ for all integers p_1 and p_2.

Now without loss of generality choose

$$\vec{\xi}_1 = a \vec{1}_x \tag{4.3}$$

where a is some scaling distance. The choice of $\vec{\xi}_2$ (magnitude and angle with respect to $\vec{\xi}_1$) gives the different kinds of symmetry (five kinds of parallelogram systems in [9]) that characterize the unit cells in our antenna array. The first symmetry system is based on a square with

$$\vec{\xi}_2 = a \vec{1}_y = \vec{1}_z \times \vec{\xi}_1 \tag{4.4}$$

This is the system of interest here since it is readily applicable to dual polarization (say in x and y directions independently) and in the next section is readily adaptable to self complementarity. Another important system is based on unit cells of equilateral triangles, including the case of regular hexagons. In this case $\left|\vec{\xi}_2\right| = a$ but $\vec{\xi}_2$ has an angle of $\pi/6$ with respect to $\vec{\xi}_1$. Examples of all the above as transient antenna arrays are given in [1]. Note that equilateral triangles, squares, and regular hexagons are the only regular polygons which can uniformly divide up the plane, completely filling it. The remaining systems are based on the rhombus ($\left|\vec{\xi}_2\right| = a$ but $\vec{\xi}_2$ at an arbitrary angle with respect to $\vec{\xi}_1$), rectangle, and oblique parallelogram ($\vec{\xi}_2$ arbitrary but not parallel or perpendicular to $\vec{\xi}_1$).

The space groups (E_2 in two dimensions) are formed by adjoining rotations and reflections to the translation groups [11]. Of course, only rotations and reflections compatible with the particular type of two-dimensional translation (selected from the five kinds of parallelogram systems) are allowed. Including only reflections in planes perpendicular to S, then [9] enumerates seven symmetry classes. One way to look at these different kinds of symmetries is to visualize the various decorative tile patterns that have been developed over the centuries. Patterning antenna arrays after these could lead to many strange-looking examples. Out of all these possibilities one needs to look for those with desirable electromagnetics properties.

Limiting ourselves for the present to square unit cells, one can envision various compatible rotations and reflections. The square itself has C_4 symmetry and can admit four axial symmetry planes giving C_{4a} symmetry. Of course, any subgroup of C_{4a} is then permissible. One can also have rotation, reflection, and translation of the geometry ongoing from one unit cell to another in a systematic way. However, this can also be considered from the point of view of a larger unit cell which can be shifted periodically as in (4.1) to form the entire array. For the present let each unit cell of size $a \times a$ (square) be identical, at least in geometry. As is usual in phased arrays (frequency domain), timed arrays can have sources turned on at different times in the form of a line sweeping across the array (at a speed faster than the speed of light c) to produce an approximate plane wave leaving the array with some arbitrary direction of propagation. As a special case all the sources on a plane of constant z can be turned on at the same time to make $\vec{1}_z$ the direction of propagation. Note that this direction of propagation is the principal one (main beam) since there can also be sidelobes (grating lobes) depending on a and each frequency in the pulse.

The discussion in Section III regarding polarization is applicable here as well. The unit cell can be designed with a symmetry plane so as to make the electric field linearly polarized. Centering one unit cell on $(x,y) = (0,0)$, then P_x or P_y as symmetry planes gives this property. For the example in fig. 3.2 the polarization is in the y direction in both time- and frequency-independent senses. Of course, the unit cell with x and y both ranging between $-a/2$ and $a/2$ needs to be superimposed on this example with planes P_1 andP_2 intersecting the four corners of the square. With the structure terminated at the unit cell boundaries, the resulting unit cell geometry is repeated by translation (per (4.1)) in both x and y directions. Then the symmetry plane, say P_x, is also repeated at $x = 0, \pm a, \pm 2a, \ldots$ giving an infinite number of such "vertical" symmetry planes. (In addition $x = \pm a/2, \pm 3a/2, \pm 5a/2, \ldots$ also become symmetry planes.) If one wishes an electric field with only a y component on P_x (i.e., the $x = 0$ plane) it is sufficient to maintain a symmetric field with respect to P_x. The geometry having reflection symmetry with respect to P_x, it is also important that the sources (amplitude, waveform, turn-on time) also are symmetric with respect to P_x. As a special case all the sources can be identical, but at the present stage of the argument they need only be symmetric with respect to P_x.

Similarly, if one wishes an electric field with only a y component on P_y (i.e., $y = 0$ plane) it is sufficient to maintain an antisymmetric field with respect to P_y. this requires that the geometry have reflection symmetry with respect to P_y (and with identical unit cells, reflection symmetry with respect to symmetry planes at $y = 0, \pm a, \pm 2a, \ldots$ as well as $y = \pm a/2, \pm 3a/2, \pm 5a/2, \ldots$), and antisymmetric sources with respect to P_y.

In this section the emphasis has been on geometrical symmetry in the context of the space group E_2. The array is based on the translation group T_2 with squares as unit cells which is convenient for dual polarization [1, 3], and which will be used later in the context of self complementarity. For operation in a given linear polarization (y polarized here, but applicable to dual (x and y) polarization as well) there is also the constraint that the unit cell have a symmetry plane (P_x or P_y). In addition the sources should be symmetric or antisymmetric as appropriate with respect to this plane. (For plane-wave illumination of the aperture this constrains the scanning of the beam to be also centered on the same symmetry plane if the symmetry is to be maintained). With both P_x and P_y as symmetry planes then the unit cell has C_{2a} symmetry (2-fold rotation axis with two axial symmetry planes). At the present stage the unit cells can still have depth in the z direction, such as the non-planar conical-TEM launchers in [3]. So these symmetries have application to array antennas whether or not one includes the additional property of self complementarity.

V. Self-Complementary Arrays

In line with the previous discussion consider now a planar array as illustrated in fig. 5.1. This is illustrated for the case that the array produces a y-polarized electric field. Note how the voltage sources and associated currents are labeled with subscripts corresponding to the center $\vec{r}_{n,m}$ of the unit cells, these being the locations of the sources as well, where

$$\vec{r}_{n,m} = na\vec{1}_x + ma\vec{1}_y \ , n,m = 0, \pm 1, \pm 2, \ldots \tag{5.1}$$

Now all conductors, sources, and impedances are on S (the $z = 0$ plane). The sources $V_{n,m}^{(y)}(t)$ and associated currents $I_{n,m}^{(y)}(t)$ can then be considered as infinite-dimensioned matrices if one wishes. Note the superscript y corresponding to the particular polarization with which these sources are associated. Of course one can equally regard the currents as the sources. Note that in taking the complement as in (2.4) the roles of voltages and currents are interchanged.

The unit cells have been given the self-complementary symmetry C_{2ac} discussed in Section III. This is self complementary, not only under $\pi/2$ rotation, but also on reflection through the diagonal planes P_1 and P_2. While one can have $Z_0/2$ sheet impedances (and special dyadic ones as well) with C_{4a} symmetry as in fig. 3.2, let us leave these out at present except for a special case.

At the corners of the unit cells, i.e., at

$$\vec{r} \equiv \vec{r}_{n,m}^{(cor)} = \vec{r}_{n,m} + \frac{a}{2}\vec{1}_x + \frac{a}{2}\vec{1}_y \tag{5.2}$$

there is the question of how the perfectly conducting sheets are interconnected in the ±x directions. In the ±y directions these sheets are continuous across the unit-cell boundaries. In the ±x directions the free-space regions are similarly continuous across the unit-cell boundaries. This leaves the corners for special consideration.

Consider a unit cell as in fig. 5.2A where the perfectly conducting sheets are connected across the corner to the adjacent sheets in the $\pm x$ directions. Modeling these connections as little square regions of perfectly conducting sheet ($y_s = \infty$), the complementary array is illustrated in fig. 5.2B where these little squares are now characterized by free space ($y_s = 0$) which corresponds to lack of connection (i.e., disconnection) of the adjacent perfectly conducting sheets, now is the ±y directions. On rotation about the source point by $\pi/2$ or reflection through P_1 or P_2 (as in fig. 3.2) the complement is clearly not a self complement. So in transforming the sources as

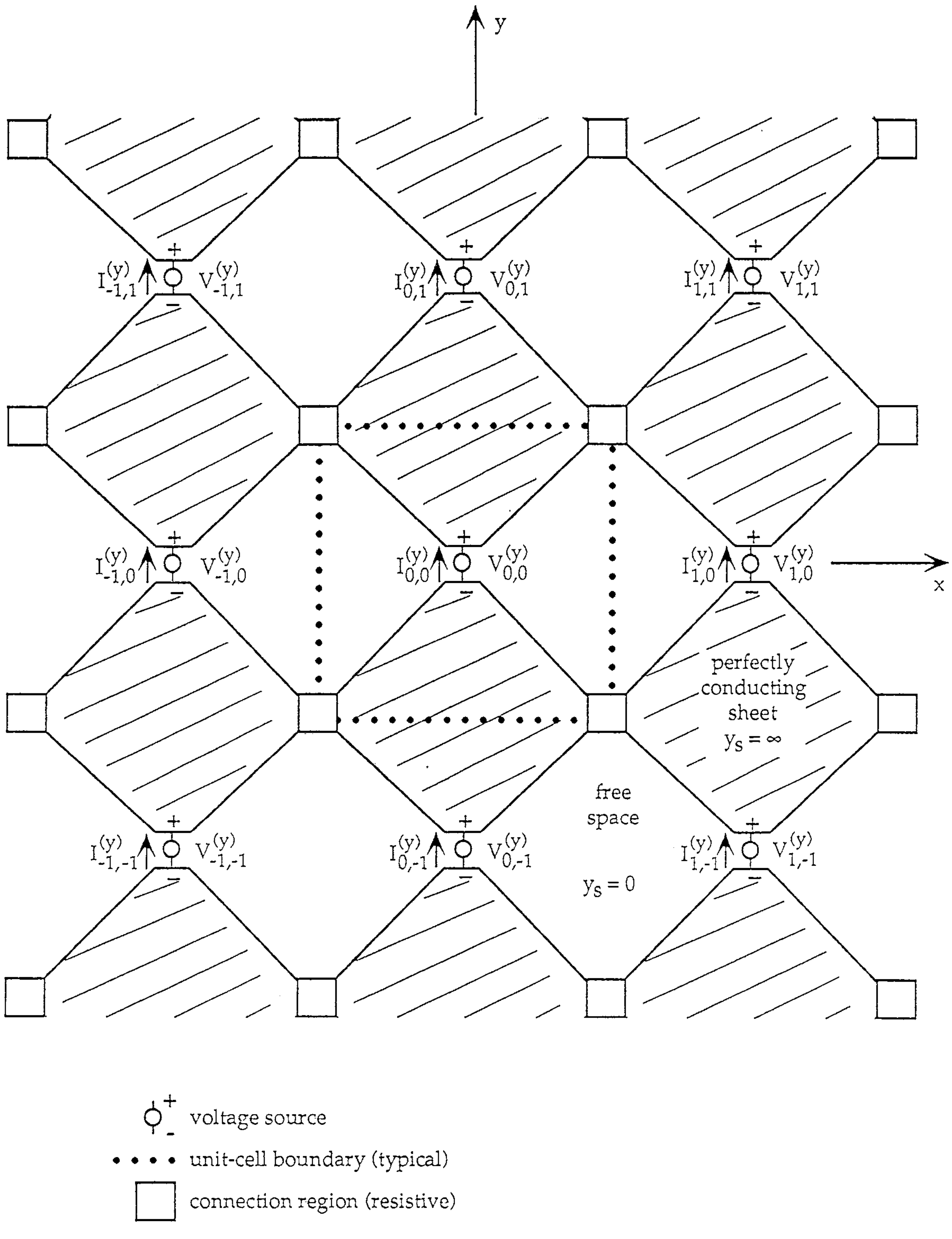
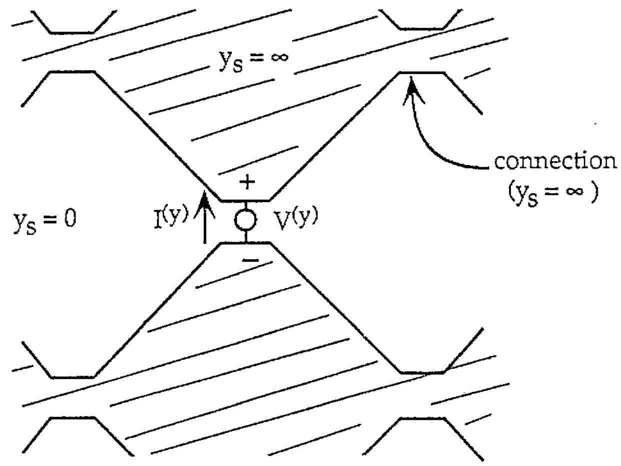
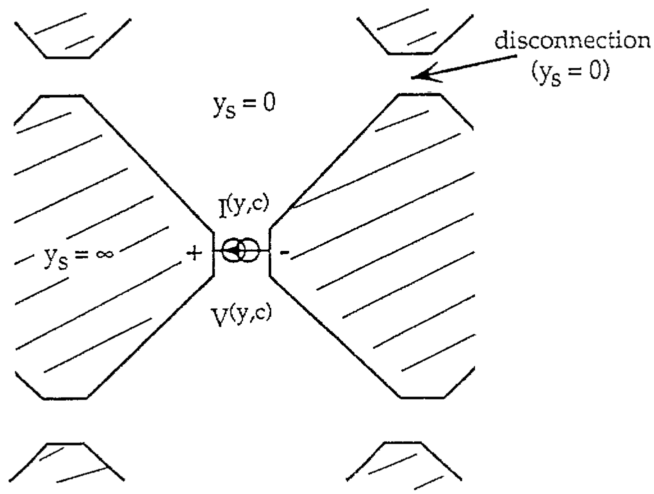


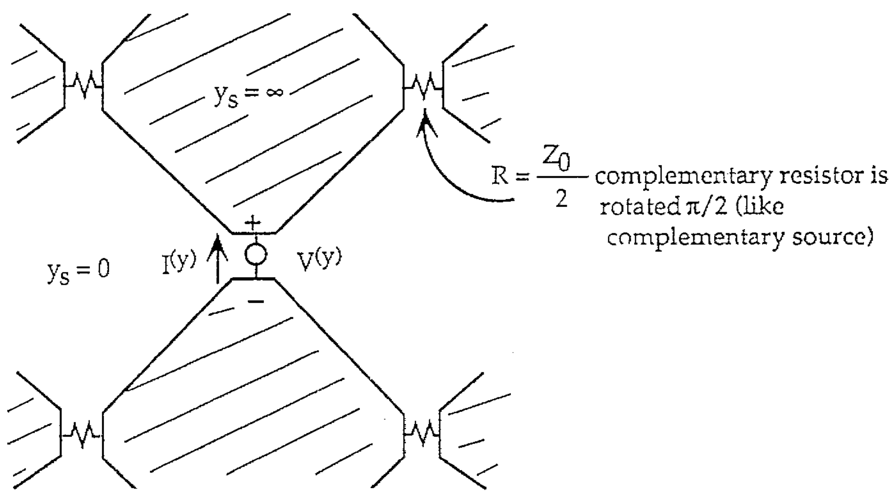
Fig. 5.1. Planar Array with Square Unit Cells with C_{2a} Symmetry



A. Array with connection (zero impedance) at corners



B. Complementary array (disconnection at corners)



C. Self-complementary corner connection (resistive)

Fig. 5.2. Connection at Unit-Cell Boundaries

$$I^{(y,c)}(t) = \frac{2}{Z_0} V^{(y)}(t) \, , \, V^{(y,c)}(t) = \frac{Z_0}{2} I(t) \tag{5.3}$$

the impedances are not the same, but are related by the complementary relationship

$$\begin{gathered} \tilde{Z}(s) \equiv \frac{\tilde{V}^{(y)}(s)}{\tilde{I}} \, , \, \tilde{Z}^{(c)}(s) \equiv \frac{\tilde{V}^{(c)}(s)}{\tilde{I}^{(c)}(s)} \\ \tilde{Z}(s)\tilde{Z}^{(c)}(s) = \frac{Z_0^2}{4} \end{gathered} \tag{5.4}$$

Next, as in fig. 5.2C, replace the above connection (fig. 5.2A) and disconnection (fig. 5.2B) by the self complementary resistive connection of value $Z_0/2$. One can model this by considering again a small square connection region as a sheet resistance of value $Z_0/2$. This is unchanged on taking the complement, but note that the edges of the square which contact regions of $y_s = 0, \infty$ are interchanged (thereby changing the direction of the current) on passing from the original array to its (now self) complement. Thus when symbolized by a resistor $R = Z_0/2$, this resistor is rotated by $\pi/2$ on passing to the complement (just like the sources are rotated).

Note that there are special ideal cases in which such resistors are not required, specifically if no currents pass through the conductive connections (and no voltage appears across the complementary disconnections). Such cases can be achieved by imposition of appropriate translation symmetry among the sources in the array. Specifically one can have the voltages sources $V_{n,m}^{(y)}(t)$ independent of n or m (or both). Independence of n (or identical sources in each row) makes the fields and currents symmetric with respect to symmetry planes at $x = 0, \pm a/2, \pm a, \pm 3a/2, \ldots$. Independence of m (or identical sources in each column) makes the fields and currents antisymmetric with respect to symmetry planes at $y = 0, \pm a/2, \pm a, \pm 3a/2, \ldots$. The first case allows scanning in the $\pm y$ directions (E plane) and the second allows scanning in the $\pm x$ directions (H plane). However, scanning the beam in more general directions does result in currents through connections, for which case the resistive connections may have some advantage.

For true self complementarity we also need to consider the sources in more detail. As discussed in Section III there are two ways to consider rotation by $\pi/2$ of the original array into its complement. With the coordinate origin chosen at one source $V_{0,0}^{(y)}$ then this source location remains fixed while the others rotate into their complementary sources. In terms of the location of the sources we have in Cartesian form

$$\vec{r}_{n,m} = (na, ma, 0) \equiv \text{ source location}$$

$$\vec{r}^{(c)}_{n,m} = (-ma, na, 0) \equiv \text{ complementary source location} \tag{5.5}$$

$$= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \vec{r}_{n,m}$$

where the 3×3 matrix is the three dimensional form of $\overleftrightarrow{\tau}_d$ as in (2.2).

With the sources now indexed as above the complementary relationship in (5.3) now becomes

$$I^{(y,c)}_{-m,n}(t) = \frac{2}{Z_0} V^{(y,c)}_{n,m}(t) \, , \, V^{(y,c)}_{-m,n}(t) = \frac{Z_0}{2} I_{n,m}(t) \tag{5.6}$$

Rotating another $\pi/2$ gives the original quantities (complement of complement) as

$$\begin{aligned} V^{(y)}_{-n,-m}(t) &= \frac{Z_0}{2} I^{(y,c)}_{-m,n}(t) = V^{(y)}_{n,m}(t) \\ I^{(y)}_{-n,-m}(t) &= \frac{2}{Z_0} V^{(y,c)}_{-m,n}(t) = I^{(y)}_{n,m}(t) \end{aligned} \tag{5.7}$$

So the array sources, when regarded as scalars, have C_2 symmetry which is the same as two-dimensional inversion. If we make the source relationship self complementary then we constrain that the sources be the same on $\pi/2$ rotation as

$$V^{(y,c)}_{-m,n}(t) = V^{(y)}_{n,m}(t) \, , \, I^{(y,c)}_{-m,n}(t) = I^{(y)}_{n,m}(t) \tag{5.8}$$

Continuing this rotation four times to complete the circle gives

$$\begin{aligned} V^{(y)}_{n,m}(t) &= V^{(y)}_{-m,n}(t) = V^{(y)}_{-n,-m}(t) = V^{(y)}_{m,-n}(t) \\ I^{(y)}_{n,m}(t) &= I^{(y)}_{-m,n}(t) = I^{(y)}_{-n,-m}(t) = I^{(y)}_{m,-n}(t) \end{aligned} \tag{5.9}$$

This is a kind of C_4 symmetry if we consider the sources as scalars and ignore their vector orientation $\left(\vec{1}_y\right)$.

Consider the second way, reflection through P_1 and P_2 (fig. 3.2), we have directly the relationship for self complement (say using P_1)

$$V^{(y)}_{n,m}(t) = V^{(y)}_{m,n}(t) \, , \, I^{(y)}_{n,m}(t) = I^{(y)}_{m,n}(t) \tag{5.10}$$

which also gives (5.9) when reflection using P_2 is applied. As discussed previously, the application of both rotation and reflection self complementarity leads to symmetry planes which assure linear polarization from the array of y-oriented sources.

However, things are more complicated than this. In general one can specify either a voltage source or a current source at a given location, but not both. An exception can be found by replacing all sources, except say the one at $\vec{r}_{0,0}$, by $Z_0/2$ resistive loads and noting that with the convention used for power outgoing from sources

$$V_{n,m}^{(y)}(t) = -\frac{Z_0}{2} I_{n,m}^{(y)}(t) \text{ for } (n,m) \neq (0,0) \tag{5.11}$$

That this is self complementary can be readily seen by applying the previous derivation for the $Z_0/2$ resistors at the unit cell corners at $\vec{r}_{n,m}^{(cor)}$ to the $\vec{r}_{n,m}$ locations as well. In this case with only one source (at $\vec{r}_{0,0}$) we have

$$V_{0,0}^{(y)}(t) = \frac{Z_0}{2} I_{0,0}^{(y)}(t) \tag{5.12}$$

giving a completely self-complementary array. This can be generalized by placing a voltage source in series with a $Z_0/2$ resistor at $\vec{r}_{0,0}$ (which or course now drives an impedance of Z_0). Since there is nothing special about source location, and $\vec{r}_{0,0}$ can refer to any source location in the array (translation symmetry), then one can have every source location supplied with such a series combination of $Z_0/2$ and a voltage source, each of which will see an impedance Z_0) when all other voltage sources are turned off. When more than one voltage source is used, however, each source sends, in general, currents through the other sources. Nevertheless this is an interesting kind of self complementarity which may be applicable to a scanning timed array.

A special case of interest has both self-complementary C_{2ac} and two-dimensional translation T_2 symmetries (with the square unit cells) applied to the sources as well as the geometry, i.e.,

$$V_{n,m}^{(y)}(t) = V^{(y)}(t) \, , \, I_{n,m}^{(y)}(t) = I^{(y)}(t) \text{ for all } n,m \tag{5.13}$$

Constraining the voltage sources all to be the same results in equal currents through the sources merely due to the translation symmetry. Then the self-complementary relations in (5.7) reduce to

$$V^{(y)}(t) = \frac{Z_0}{2} I^{(y)}(t) \tag{5.14}$$

for all the sources. This is a very simple and convenient result valid for all times and frequencies (for an infinite array). As discussed in [1, 2] the impedance seen by the sources can be calculated for both high and low frequencies (early and late times) if the sources are excited in a plane-wave sequence such as for launching a wave from the array in a particular direction (i.e., for scanning the beam). Now we see that for the special case of the plane wave propagation perpendicular to the array (the z direction), in which the sources are all in phase (or with identical waveforms, amplitudes, and turn-on times) as in (5.13), then the impedance can be easily calculated for intermediate frequencies (and times) with the wavelength of the order of the unit-cell size as well.

VI. Dual-Polarized Self-Complementary Array

As mentioned previously translation symmetry in the sources, with $V_{n,m}^{(y)}(t)$ independent of n or m (or both), implies that there are no voltages or currents across the connection regions in figs. 5.1 and 5.2. The case of uniform $V^{(y)}$ as in (5.13) is such a case. This implies that we need not use $Z_0/2$ resistors for the connections in the $\pm$x directions at the unit-cell corners. This leaves the corners free for placement of x-polarized sources as indicated in fig. 6.1.

As in (5.13) the x-polarized sources are ideally constrained as

$$V_{n,m}^{(x)}(t) = V^{(x)}(t)\,,\; I_{n,m}^{(x)}(t) = I(t) \;\text{ for all } n,m \tag{6.1}$$

where the n,m indices now refer to sources positioned at the corners identified by $\vec{r}_{n,m}^{\,(cor)}$ in (5.2). These are translated by $a/2$ in both coordinates with respect to the y-polarized sources $V^{(y)}$. By rotating and translating the array we can see that the $V^{(y)}$ sources are configured just like the $V^{(x)}$ ones. All the self-complementary properties are the same; the coordinate center for $\pi/2$ rotation can be taken as $\vec{r}_{0,0}^{\,(cor)}$. This gives two ways of defining the unit cells, but with this high order of symmetry they are equivalent. The transfer function from $V^{(x)}$ to $\vec{E}^{(x)} = E^{(x)}\vec{1}_x$ in the far field is the same as for $V^{(y)}$ to $E^{(y)}\vec{1}_y$ by symmetry, being a constant for low frequencies (for an infinite array) and being a time integral at early times. Note that in general we can have

$$V^{(x)}(t) \neq V^{(y)}(t) \tag{6.2}$$

with independent waveforms giving a far-field polarization which can vary with time as one chooses.

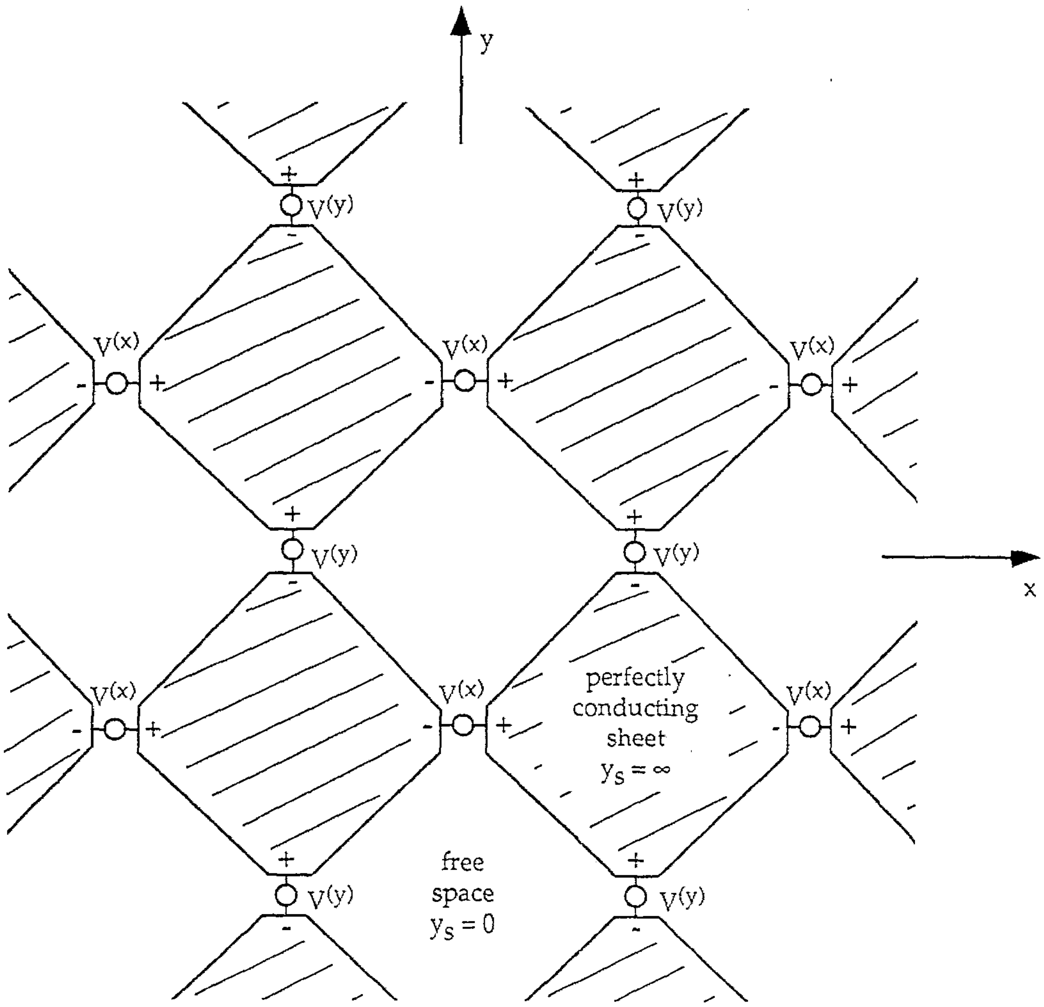


Fig. 6.1. Planar Array with C_{2ac} Symmetry and Dual Polarization

VII. Concluding Remarks

Evidently self complementarity can be applied to the design of array antennas, applicable to both frequency- and time-domain use. This is a special case of the two-dimensional space groups. For various reasons, such as discussed in [3], one may wish to use non-coplanar elements in the unit cells, but one may use the present self-complementary result as an aid in the first-order design of non-coplanar cells. The full self-complementary case with plane-wave illumination (focusing at infinity) is restricted to propagation normal to the array, but for small angular deviation of the beam from the normal to S the impedance results should not change much. There are lots of space groups to explore for designing arrays with various properties. For example, unit cells based on equilateral triangles or regular hexagons can also be configured for multiple polarizations [1]. This whole subject might by analogy be called *antenna crystallography*. Of course, a real antenna array has finite linear dimensions so that infinite periodicity is only an approximation.

In its simplest form as in fig. 6.1 the array antenna has the symmetry of a chessboard with two colors of the squares corresponding to conductors and free space. Carrying the analogy further the white pieces can correspond to the $V^{(x)}$ sources and the black pieces can correspond to the $V^{(y)}$ sources. Of course, the pieces now occupy corners instead of squares (merely a translation). The two players then each have different polarizations.

> "For some minutes Alice stood without speaking, looking out in all directions over the country—and a most curious country it was. There were a number of tiny little brooks running straight across it from side to side, and the ground between was divided up into squares by a number of little green hedges, that reached from brook to brook.
>
> "I declare it's marked out just like a large chess-board!" Alice said at last. "There ought to be some men moving about somewhere—and so there are!" she added in a tone of delight, and her heart began to beat quick with excitement as she went on. "It's a great huge game of chess that's being played—all over the world—if this *is* the world at all, you know. Oh, what fun it is! How I *wish* I was one of them! I wouldn't mind being a Pawn, if only I might join—though of course I should *like* to be a Queen, best."

Lewis Carroll
Through The Looking Glass

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