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Sensor and Simulation Notes

Note 352

December 1992

Uniform Isotropic Dielectric Equal-Time Lenses for Matching Combinations of Plane and Spherical Waves

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Abstract

This paper considers the general design of dielectric lenses for transitioning between various spherical waves, including plane waves as special cases, such as one may wish to have in antenna and transmission systems for fast-rising electromagnetic pulses. For transitioning spherical and plane waves through a single boundary surface, this surface is a prolate spheroid or one sheet of a hyperboloid of two sheets. For the case of two spherical waves this surface is a fourth order polynomial equation. For multiple media with two (or more) boundary surfaces these solutions can be applied to each boundary to generate various lens designs of interest.

1 Introduction

In transitioning between various spherical waves, a general technique involves a lens in which the propagation speeds (in general different) in two or more media are used to control the arrival times on the wavefronts of interest. In uniform dielectric media spherical wavefronts, including plane wavefronts as limiting cases, are of interest. Such lenses have application in TEM transmission systems such as coaxial waveguides (cables) [2], and antennas such as IRAs (impulse radiating antennas) [1]. In these cases one requires that pulses with very fast risetimes (compared to cross-section transit times) be treated as waves (in general three dimensional). The problem is then one of matching these waves from one region to another with a minimum of distortion and reflection.

In this paper we treat the lens from an equal-time point of view, matching from spherical wavefronts in two uniform dielectric media along their common boundary. This introduces some reflection in general at the boundary, and so such lenses are not "perfect" transitioning devices for TEM waves in the sense of [5]. However, in some cases the reflections can be small and the wave passing into the second medium can usefully approximate the desired dispersionless TEM wave. In a more detailed consideration, such as in [2] one needs to consider the paths of the conductors (for the desired TEM wave) through the lens medium and the resulting transmission-line impedances in the various media to optimize the matching of the waves.

2 Diverging Spherical Wave to Plane Wave

In [2] the general problem of launching a plane wave (in a medium of low permittivity) from a diverging spherical wave (in a medium of greater permittivity) has been considered. There the specific problem involved a coaxial guiding system. However, the result for the lens shape is more general, as it relies only on the equal-time requirement, and not the impedance considereations for the transmission structure. Here, the result is extended.

As in Fig. 2.1 we have a spherical wave propagating outward from

$$\vec{r}_1 = z_{1_0}\vec{1}_z, \ \ (r_1, \theta_1) = (0, \theta_1), \ \ (\Psi, z) = (0, z_{1_0}) \tag{2.1}$$

On the lens boundary we have

$$z_b - z_{1_0} = r_b \cos(\theta_b), \ \ \Psi_b = r_b \sin(\theta_b) \tag{2.2}$$

The equal-time requirement to some aperture plane ($z = z_a$) is

$$\begin{aligned} \sqrt{\epsilon_1} r_b + \sqrt{\epsilon_2}[z_a - z_b] &= \text{constant} \\ &= \sqrt{\epsilon_1}[z_{b_0} - z_{1_0}] + \sqrt{\epsilon_2}[z_a - z_{b_0}] \end{aligned} \tag{2.3}$$

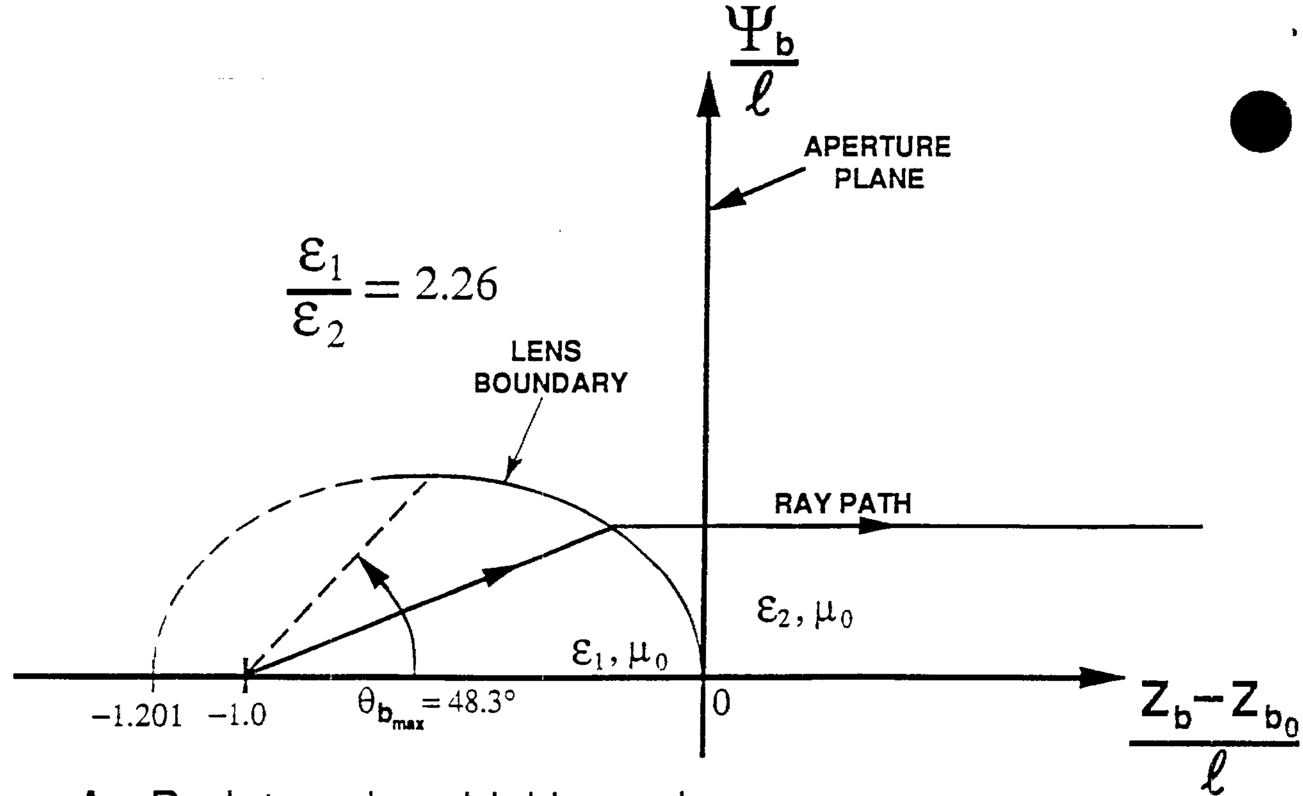
where the constant has been evaluated by considering the ray along the z axis, z_{b_0} being the z coordinate of the intersection of the lens boundary with the z axis. Rearranging

$$\begin{aligned} \sqrt{\epsilon_1} r_b - \sqrt{\epsilon_2}[z_b - z_{b_0}] &= \sqrt{\epsilon_1}\ell \\ \ell &\equiv z_{b_0} - z_{1_0} \end{aligned} \tag{2.4}$$

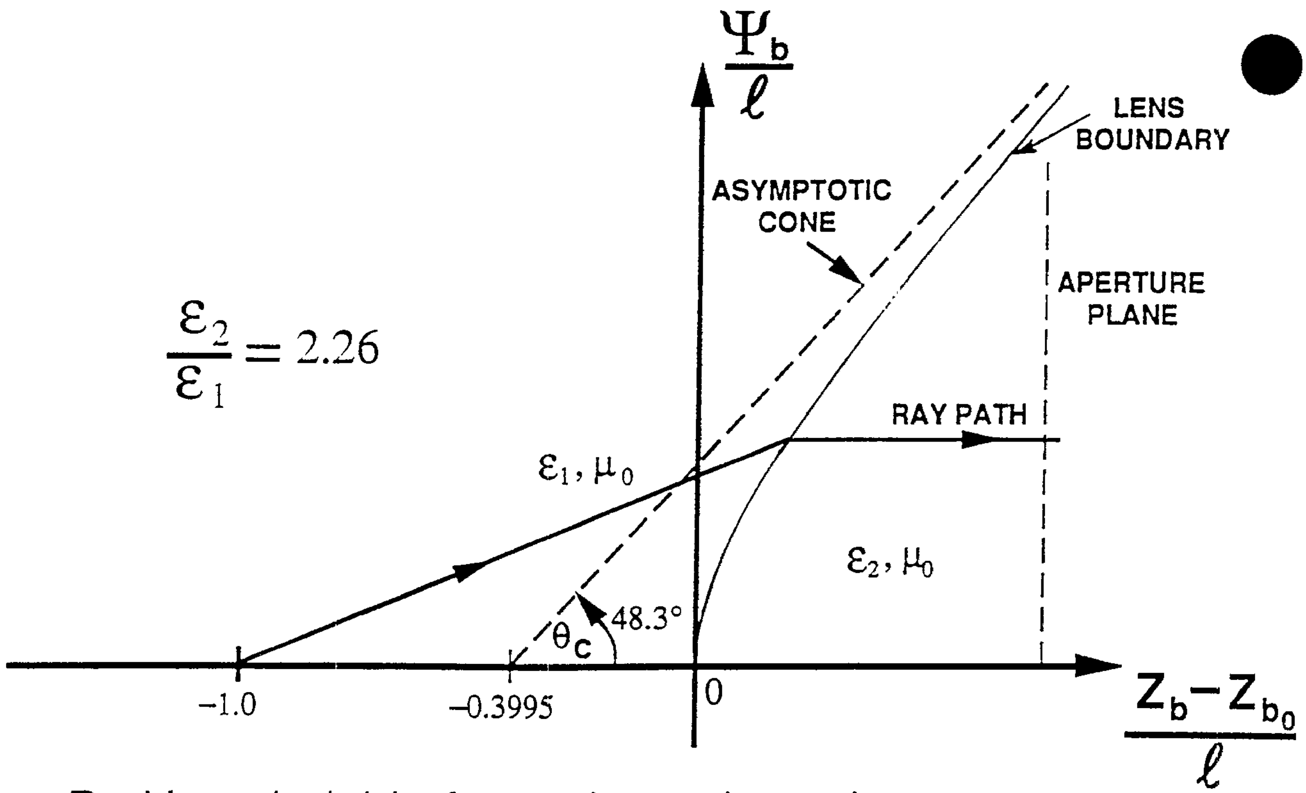
where ℓ is the convenient scaling length as in [2].

Combining (2.2) and (2.4) we have

$$\begin{aligned} \frac{r_b}{\ell} &= \sqrt{\frac{\epsilon_2}{\epsilon_1}} \frac{z_b - z_{b_0}}{\ell} + 1 = \frac{\sqrt{\frac{\epsilon_2}{\epsilon_1}} - 1}{\sqrt{\frac{\epsilon_2}{\epsilon_1}} - \cos(\theta_b)} \\ &= \frac{\Psi_b}{\ell} \sin^{-1}(\theta_b) = [\frac{z_b - z_{b_0}}{\ell} + 1] \cos^{-1}(\theta_b) \end{aligned} \tag{2.5}$$



A. Prolate spheroidal boundary



B. Hyperboloid-of-two-sheets boundary

Fig 2.1: Lens Boundary for Converting Expanding Spherical Wave into Plane Wave

Converting to cylindrical coordinates (2.2) gives

$$\left(\frac{r_b}{\ell}\right)^2 = \left(\frac{\Psi_b}{\ell}\right)^2 + \left[\frac{z_b - z_{b_0}}{\ell} + 1\right]^2 \tag{2.6}$$

which with (2.5) gives

$$\left(\frac{\Psi_b}{\ell}\right)^2 + \left[\frac{z_b - z_{b_0}}{\ell} + 1\right]^2 = \left[\sqrt{\frac{\epsilon_2}{\epsilon_1}}\frac{z_b - z_{b_0}}{\ell} + 1\right]^2 \tag{2.7}$$

as the equation for the lens boundary.

A. Case of $\epsilon_1 > \epsilon_2$

This is the case considered in [2]. It corresponds to Fig. 2.1A. Rearranging (2.7) as

$$\left(\frac{\Psi_b}{\ell}\right)^2 + 2\left[1 - \sqrt{\frac{\epsilon_2}{\epsilon_1}}\right]\frac{z_b - z_{b_0}}{\ell} + \left[1 - \frac{\epsilon_2}{\epsilon_1}\right]\left[\frac{z_b - z_{b_0}}{\ell}\right]^2 = 0 \tag{2.8}$$

This is the equation of a prolate spheroid [4]. It interects the z axis at

$$\frac{z - z_{b_0}}{\ell} = 0, \ -2\left[1 + \sqrt{\frac{\epsilon_2}{\epsilon_1}}\right]^{-1} \tag{2.9}$$

It reaches maximum cylindrical radius $\Psi_{b_{max}}$ at

$$\begin{aligned} &\left.\frac{z_b - z_{b_0}}{\ell}\right|_{\Psi_b = \Psi_{b_{max}}} = -\left[1 + \sqrt{\frac{\epsilon_2}{\epsilon_1}}\right]^{-1} = -\frac{a}{\ell} \\ &a \equiv \text{ major radius} \end{aligned} \tag{2.10}$$

$$\begin{aligned} &\frac{\Psi_{b_{max}}}{\ell} = \frac{\left[1 - \frac{\epsilon_2}{\epsilon_1}\right]^{\frac{1}{2}}}{1 + \sqrt{\frac{\epsilon_2}{\epsilon_1}}} = \frac{1 - \sqrt{\frac{\epsilon_2}{\epsilon_1}}}{\left[1 - \frac{\epsilon_2}{\epsilon_1}\right]^{\frac{1}{2}}} = \frac{b}{\ell} \\ &b \equiv \text{ minor radius} \end{aligned}$$

Another common parameter of a prolate spheroid is the eccentricity

$$\begin{aligned} &\alpha = \left\{1 - \left(\frac{b}{a}\right)^2\right\}^{\frac{1}{2}} = \sqrt{\frac{\epsilon_2}{\epsilon_1}} \\ &\frac{b}{a} = \left[1 - \frac{\epsilon_2}{\epsilon_1}\right]^{\frac{1}{2}} \end{aligned} \tag{2.11}$$

It has foci on the z axis at

$$z - z_{b_0} = \begin{cases} -a[1+\alpha] & = -\ell \\ -a[1-\alpha] & = -\dfrac{1-\sqrt{\dfrac{\epsilon_2}{\epsilon_1}}}{1+\sqrt{\dfrac{\epsilon_2}{\epsilon_1}}} \end{cases}, \tag{2.12}$$

the first of these being the launch point for the spherical wave, and the second being a focus for the reflection (ideally small) of the spherical wave from the lens boundary.

As indicated in Fig. 2.1A one need use only a certain portion of the prolate spheroid for the lens. In particular the lens boundary for $z - z_{b_0} < -a$ where the lens radius is maximum is not useful for focusing in the forward ($+z$) direction. In an angular sense θ_1 is maximum on the boundary at

$$\theta_{b_{max}} = \arctan\left(\left[\frac{\epsilon_1}{\epsilon_2} - 1\right]^{\frac{1}{2}}\right) \tag{2.13}$$

There are some limiting cases. For ϵ_1/ϵ_2 near 1 we have

$$\frac{a}{\ell} \to \frac{1}{2},\ \frac{b}{\ell} \to 0,\ \theta_{b_{max}} \to 0 \text{ as } \frac{\epsilon_1}{\epsilon_2} \to 1_+ \tag{2.14}$$

For large ϵ_1/ϵ_2 we have

$$\frac{a}{\ell} \to 1,\ \frac{b}{\ell} \to 1,\ \theta_{b_{max}} \to \frac{\pi}{2} \text{ as } \frac{\epsilon_1}{\epsilon_2} \to \infty \tag{2.15}$$

which is a sphere.

B. Case of $\epsilon_2 > \epsilon_1$

Now consider the converse case in which one is launching the spherical wave from a low permittivity medium into a plane wave in a high permittivity medium. This corresponds to Fig. 2.1B. Rearrange (2.7) as

$$\begin{aligned} \left(\frac{\Psi_b}{\ell}\right)^2 &= \left[\frac{\epsilon_2}{\epsilon_1} - 1\right]\left[\frac{z_b - z_{b_0}}{\ell}\right]^2 + 2\left[\sqrt{\frac{\epsilon_2}{\epsilon_1}} - 1\right]\frac{z_b - z_{b_0}}{\ell} \\ &= \left[\frac{\epsilon_2}{\epsilon_1} - 1\right]\left\{\left[\frac{z_b - z_{b_0}}{\ell} + \left[\sqrt{\frac{\epsilon_2}{\epsilon_1}} + 1\right]^{-1}\right]^2 - \left[\sqrt{\frac{\epsilon_2}{\epsilon_1}} + 1\right]^{-2}\right\} \end{aligned} \tag{2.16}$$

This is the equation of a hyperboloid of two sheets [4]. Here we consider only the sheet in the positive z direction.

This body of revolution about the z axis intersects the z axis at z_{b_0} (the other sheet intersection being as in (2.9)). For large positive z this surface is asymptotic to a circular cone given by

$$\frac{\Psi_b}{\ell} = \left[\frac{\epsilon_2}{\epsilon_1} - 1\right]^{\frac{1}{2}} \left\{\frac{z_b - z_{b_0}}{\ell} + \left[\sqrt{\frac{\epsilon_2}{\epsilon_1}} + 1\right]^{-1} + O\left(\left(\frac{z_b - z_{b_0}}{\ell}\right)^{-1}\right)\right\}$$
$$\text{as } z_b \to +\infty \tag{2.17}$$

This cone has apex on the z axis at z_c given by

$$\frac{z_c - z_{b_0}}{\ell} = -\left[\sqrt{\frac{\epsilon_2}{\epsilon_1}} + 1\right]^{-1} \tag{2.18}$$

(comparable to (2.10)) with a half-cone angle

$$\theta_c = \arctan\left(\left[\frac{\epsilon_2}{\epsilon_1} - 1\right]^{\frac{1}{2}}\right) \tag{2.19}$$

as indicated in Fig. 2.1B.

There are some limiting cases. For ϵ_2/ϵ_1 near 1 we have

$$\frac{z_c - z_{b_0}}{\ell} \to -\frac{1}{2},\ \theta_c \to 0 \text{ as } \frac{\epsilon_2}{\epsilon_1} \to 1_+ \tag{2.20}$$

For large ϵ_2/ϵ_1 we have

$$\frac{z_c - z_{b_0}}{\ell} \to 0,\ \ \theta_c \to \frac{\pi}{2} \text{ as } \frac{\epsilon_2}{\epsilon_1} \to \infty \tag{2.21}$$

which is a plane of constant $z(= z_{b_0})$.

C. Some Comments

As a practical matter one is often concerned with launching a plane wave into a medium of permittivity ϵ_0 (free space). For a transient dispersionless wave in medium 1 we need a constant (frequency-independent and lossless) ϵ_1 which is physically then required to be larger than ϵ_0 so that the frequency-independent propagation speed is less than c, the speed of light. Such a situation implies Case A, the prolate spheroidal lens. For special cases where one wishes to launch a plane wave into a medium of permittivity $\epsilon_2 > \epsilon_0$ (such as earth, water, etc.) then the hyperboloidal lens may be of interest.

The reader can note that the wave-propagation can be reversed. Then in Fig. 2.1 a plane wave is propagating in medium 2 in the $-z$ direction. On passing through the lens boundary it is converted into a spherical wave converging (focusing) to $z = z_{1_0}$ on the z axis in medium 1.

3 Plane Wave to Diverging Spherical Wave

Now consider the complementary problem (to that of Section 2), the launching of a diverging spherical wave from an incoming plane wave as illustrated in Fig. 3.1. In this case the spherical wave in medium 2 is propagating outward from

$$\vec{r}_2 = z_{2_0}\vec{1}_z, \ \ (r_2, \theta_2) = (0, \theta_2), \ \ (\Psi, z) = (0, z_{2_0}) \tag{3.1}$$

which is actually located in medium 1. Matching the plane and spherical waves on the lens boundary we have

$$z_b - z_{b_0} = r_b \cos(\theta_b), \ \ \Psi_b = r_b \sin(\theta_b) \tag{3.2}$$

The equal-time requirement is now from a source plane ($z = z_s$) to an aperture sphere ($r_2 = r_a$) which gives

$$\begin{aligned} \sqrt{\epsilon_1}[z_b - z_s] + \sqrt{\epsilon_2}[r_a - r_b] &= \text{ constant} \\ &= \sqrt{\epsilon_1}[z_{b_0} - z_s] + \sqrt{\epsilon_2}[r_a - (z_{b_0} - z_{2_0})] \end{aligned} \tag{3.3}$$

where the constant is evaluated by considering the ray along the z axis, z_{b_0} again being the z coordinate of the intersection of the lens boundary with the z axis. Rearranging

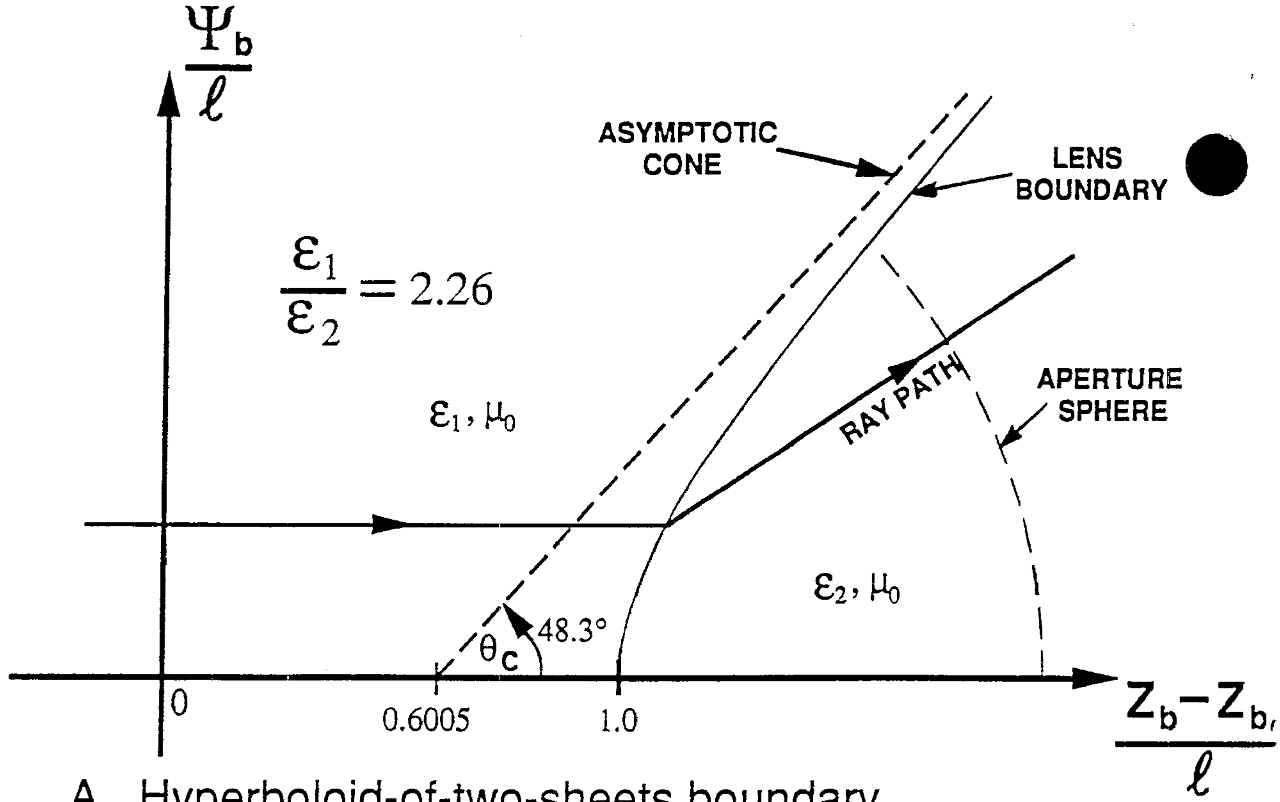
$$\begin{aligned} \sqrt{\epsilon_2} r_b - \sqrt{\epsilon_1}[z_b - z_{b_0}] &= \sqrt{\epsilon_2}\ell \\ \ell &\equiv z_{b_0} - z_{2_0} \end{aligned} \tag{3.4}$$

where ℓ is now redefined (compared to Section 2) as the distance (along the z axis) between the lens boundary and the virtual focal point of the spherical wave.

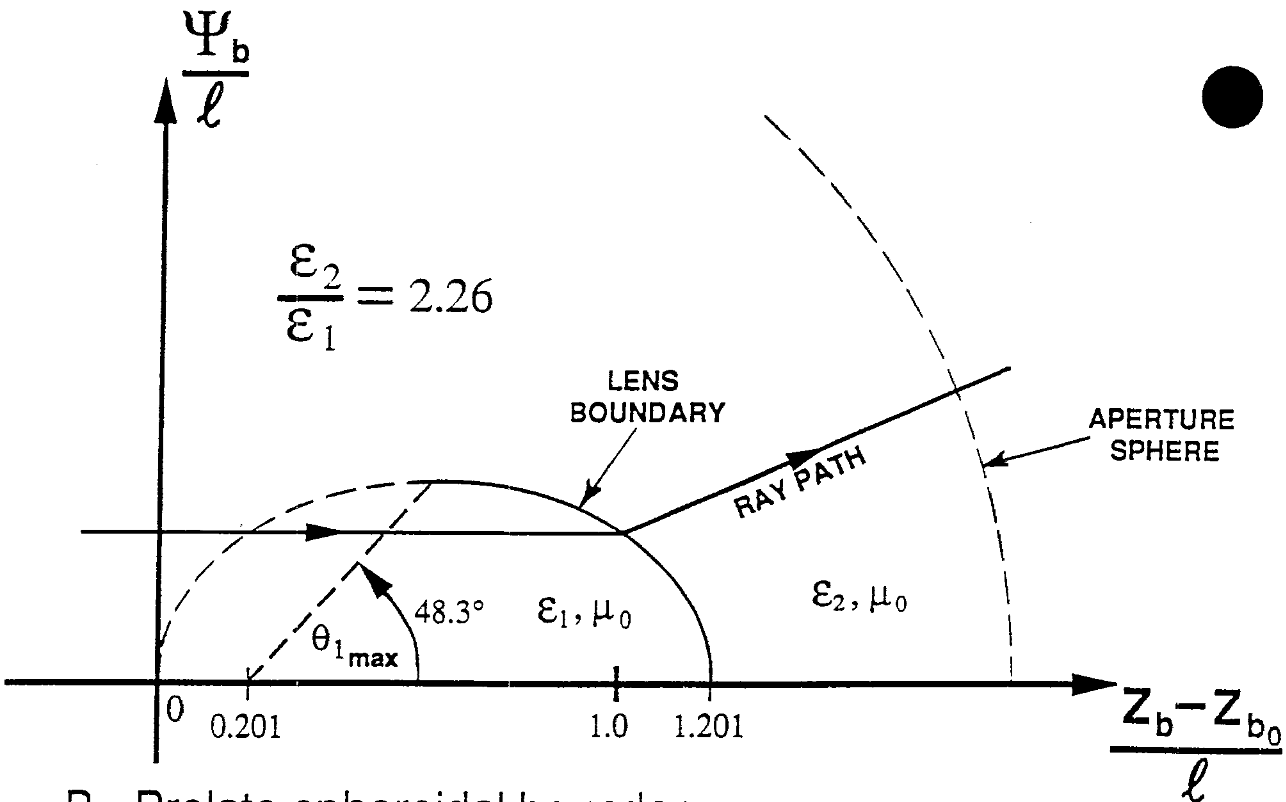
Comparing (3.4) to (2.4), note that they are the same except for the interchange of subscripts 1 and 2. With this relationship then (2.7) is rewritten for the present case as

$$\left(\frac{\Psi_b}{\ell}\right)^2 + \left[\frac{z_b - z_{b_0}}{\ell} + 1\right]^2 = \left[\sqrt{\frac{\epsilon_1}{\epsilon_2}}\frac{z_b - z_{b_0}}{\ell} + 1\right]^2 \tag{3.5}$$

where the only difference is the inversion of the ratio of the permittivities.



A. Hyperboloid-of-two-sheets boundary



B. Prolate spheroidal boundary

Fig 3.1: Lens Boundary for Converting Plane Wave into Expanding Spherical Wave

A. Case of $\epsilon_1 > \epsilon_2$

This case is illustrated in Fig. 3.1A. Note that, due to the interchange of the permittivities in (3.5) as compared to Section 2, the corresponding boundary is a hyperboloid of two sheets (as in Fig. 2.1B). The corresponding equation is

$$\left(\frac{\Psi_b}{\ell}\right)^2 = \left[\frac{\epsilon_1}{\epsilon_2} - 1\right]\left\{\left[\frac{z_b - z_{b_0}}{\ell} + \left[\sqrt{\frac{\epsilon_1}{\epsilon_2}} + 1\right]^{-1}\right]^2 - \left[\sqrt{\frac{\epsilon_1}{\epsilon_2}} + 1\right]^{-2}\right\} \tag{3.6}$$

where here we consider only the sheet in the positive z direction. Again this sheet is asymptotic to a circular cone with apex at z_c given by

$$\frac{z_c - z_{b_0}}{\ell} = -\left[\sqrt{\frac{\epsilon_1}{\epsilon_2}} + 1\right]^{-1} \tag{3.7}$$

with a half cone angle

$$\theta_c = \arctan\left(\left[\frac{\epsilon_1}{\epsilon_2} - 1\right]^{\frac{1}{2}}\right) \tag{3.8}$$

In a similar manner all the formulae of Section 2B can be simply converted to the present case.

B. Case of $\epsilon_2 > \epsilon_1$

This case is illustrated in Fig. 3.2B. Again due to the interchange (from Section 2) in the permittivities in the lens-boundary equation this is a prolate spheroid given by

$$\left(\frac{\Psi_b}{\ell}\right)^2 + 2\left[1 - \sqrt{\frac{\epsilon_1}{\epsilon_2}}\right]\frac{z_b - z_{b_0}}{\ell} + \left[1 - \frac{\epsilon_1}{\epsilon_2}\right]\left[\frac{z_b - z_{b_0}}{\ell}\right]^2 = 0 \tag{3.9}$$

It intersects the z axis at

$$\frac{z - z_{b_0}}{\ell} = 0, \quad -2\left[1 + \sqrt{\frac{\epsilon_1}{\epsilon_2}}\right]^{-1} \tag{3.10}$$

The maximum value of θ_2 on the boundary is

$$\theta_{b_{max}} = \arctan\left(\left[\frac{\epsilon_2}{\epsilon_1} - 1\right]^{\frac{1}{2}}\right) \tag{3.11}$$

In a similar manner all the formulae of Section 2A can be simply converted to the present case.

C. Some Comments

Note that in both cases illustrated in Fig. 3.2 the diverging spherical wave has a restricted range of angles θ_2 centered on the z axis. In particular for finite permittivities $\theta_2 < \pi/2$, i.e. the diverging rays do not fill a half space (with boundary as a plane of constant z).

Again the wave-propagation direction can be reversed. A spherical wave in medium 2 propagating toward z_{2_0} on the z axis is converted into a plane wave in medium 1 propagating in the $-z$ direction.

4 Diverging Spherical Wave to Diverging Spherical Wave

A. General Considerations

The more general case, as indicated in Fig. 4.1, has a spherical wave in medium 1 propagating outward from

$$\vec{r}_1 = z_{1_0}\vec{1}_2, \ \ (r_1, \theta_1) = (0, \theta_1), \ \ (\Psi, z) = (0, z_{1_0})$$
$$\ell_1 \equiv z_{b_0} - z_{1_0} \tag{4.1}$$

There is also a spherical wave propagating in medium 2 outward from a position in medium 1 as

$$\vec{r}_2 = z_{2_0}\vec{1}_2, \ \ (r_2, \theta_2) = (0, \theta_2), \ \ (\Psi, z) = (0, z_{2_0})$$
$$\ell_2 \equiv z_{b_0} - z_{2_0} \tag{4.2}$$

Here, as before, z_{b_0} is the intersection of the lens boundary with the z axis (symmetry axis), a subscript b on coordinates applying to the lens boundary. Introduce a convenient scaling length as

$$\ell_0 \equiv \left[\ell_1^{-1} + \ell_2^{-2}\right]^{-1} \tag{4.3}$$

which follows the usual focal-length formula.

Relating the coordinates on the boundary we have

$$\begin{aligned} z_b - z_{1_0} &= r_{1_b}\cos(\theta_{1_b}) = z_b - z_{b_0} + \ell_1 \\ z_b - z_{2_0} &= r_{2_b}\cos(\theta_{2_b}) = z_b - z_{b_0} + \ell_2 \\ \Psi_b &= r_{1_b}\sin(\theta_{1_b}) = r_{2_b}\sin(\theta_{2_b}) \end{aligned} \tag{4.4}$$

These are manipulated into the form

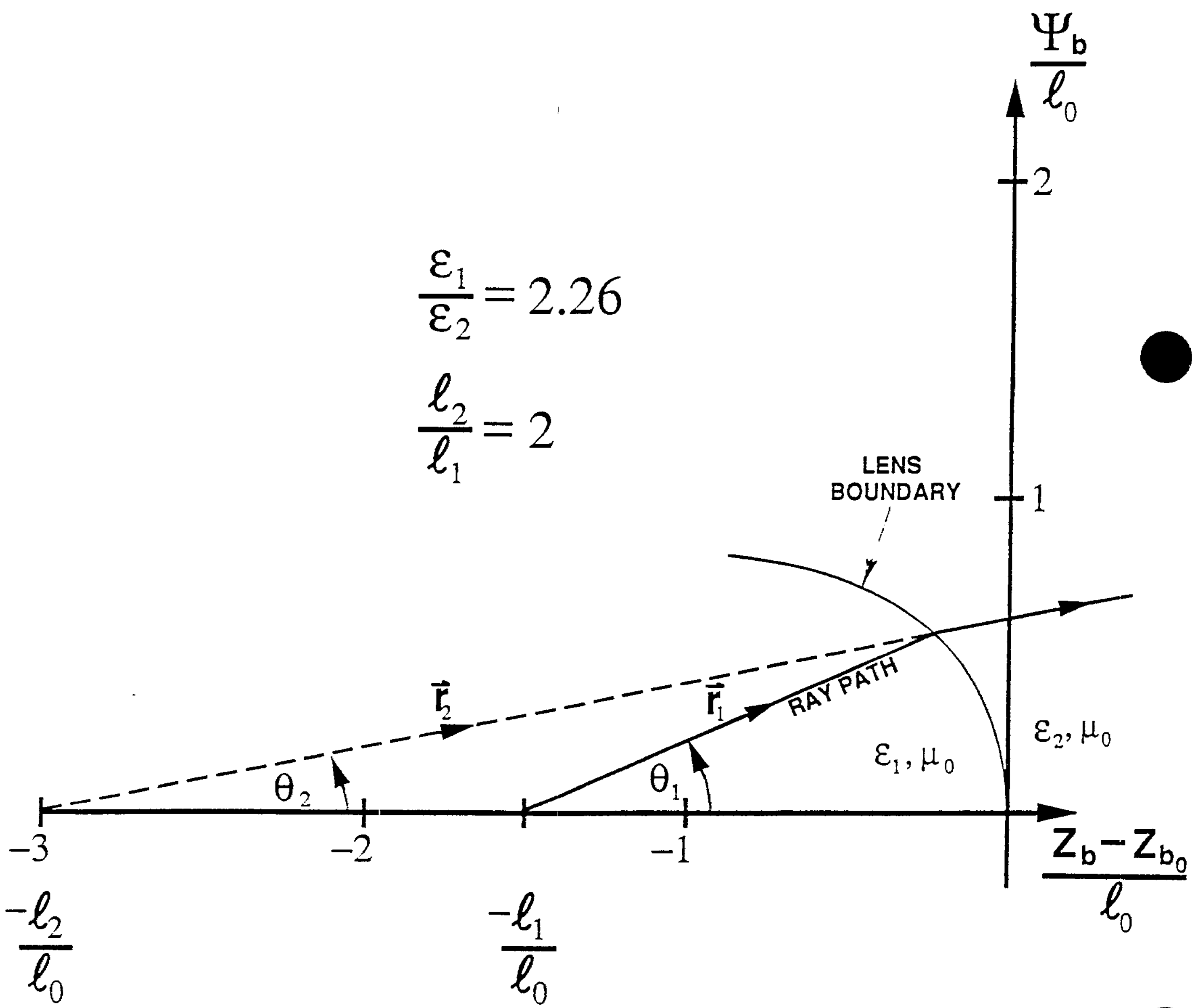


Fig 4.1: Two Diverging Spherical Waves

$$
\begin{aligned}
z_b - z_{b_0} &= r_{1_b}\cos(\theta_{1_b}) - \ell_1 = r_{2_b}\cos(\theta_{2_b}) \\
&= \Psi_b\cot(\theta_{1_b}) - \ell_1 = \Psi_b\cot(\theta_{2_b}) - \ell_2
\end{aligned} \tag{4.5}
$$

For the equal-time condition we have the time from $z_{1_0}\vec{1}_z$ to the lens boundary as

$$
t_1 = \sqrt{\mu_0\epsilon_1}\,r_{1_b} \tag{4.6}
$$

and from there to the spherical wavefront of radius r_2 (centered on $z_{2_0}\vec{1}_z$) as

$$
t_2 = \sqrt{\mu_0\epsilon_2}[r_{2_b} - r_2] \tag{4.7}
$$

Then we set

$$
\begin{aligned}
t_0 \quad &= t_1 + t_2 = \text{ constant (for fixed } r_2) \\
&= \sqrt{\mu_0\epsilon_1}\,r_{1_b} + \sqrt{\mu_0\epsilon_2}[r_2 - r_{r_b}] \\
&= \sqrt{\mu_0\epsilon_1}\,\ell_1 + \sqrt{\mu_0\epsilon_2}[r_2 - \ell_2]
\end{aligned} \tag{4.8}
$$

where the last form is found by evaluating the time along the z axis as the ray path. So we have

$$
\begin{aligned}
&\sqrt{\epsilon_1}[r_{1_b} - \ell_1] = \sqrt{\epsilon_2}[r_{2_b} - \ell_2] \\
&\sqrt{\epsilon_1}\left\{\left[[z_b - z_{b_0} + \ell_1]^2 + \Psi_b^2\right]^{\frac{1}{2}} - \ell_1\right\} = \sqrt{\epsilon_2}\left\{\left[[z_b - z_{b_0} + \ell_2]^2 + \Psi_b^2\right]^{\frac{1}{2}} - \ell_2\right\}
\end{aligned} \tag{4.9}
$$

as the equal-time condition. Note the symmetry in this equation with the interchange of the subscripts 1 and 2. By clearing the square roots one finds that this is in general a quartic equation (including fourth powers of coordinates).

For convenience (4.9) can be put in parametric form by normalizing distances to ℓ_0 and permittivities to $\sqrt{\epsilon_1\epsilon_2}$ as

$$
\begin{gathered}
\sqrt{\epsilon_n}\left\{\left[[z_b - z_{b_0} + \ell_n]^2 + \Psi_b^2\right]^{\frac{1}{2}} - \ell_n\right\} = \chi\ell_0[\epsilon_1\epsilon_2]^{\frac{1}{4}} \\
n = 1, 2, \;\; \chi \equiv \text{ parameter}
\end{gathered} \tag{4.10}
$$

Varying χ generates the lens surface, noting that for each χ there are two equations with coordinate unknowns $(z_b - z_{b_0})/\ell_0$ and Ψ_b/ℓ_0. One still needs to specify ϵ_1/ϵ_2 and ℓ_1/ℓ_2 (or ℓ_1/ℓ_0) to give a particular lens surface. Note the particular point common to all lens surfaces

$$z_b = z_{b_0}, \quad \Psi_b = 0, \quad \chi = 0 \tag{4.11}$$

To simplify the computation of the surface shape rewrite (4.10) as

$$\begin{aligned} \xi_n &= \frac{[\epsilon_1\epsilon_2]^{\frac{1}{4}}}{\epsilon_n^{\frac{1}{2}}}\chi + \frac{\ell_n}{\ell_0} \\ &= \frac{1}{\ell_0}\left\{[z_b - z_{b_0} + \ell_n]^2 + \Psi_b^2\right\}^{\frac{1}{2}} \end{aligned} \tag{4.12}$$

Then eliminate Ψ_b as

$$\begin{aligned} \xi_2^2 - \xi_1^2 &= \frac{1}{\ell_0^2}\left\{[z_b - z_{b_0} + \ell_2]^2 - [z_b - z_{b_0} + \ell_1]^2\right\} \\ &= 2\frac{\ell_2 - \ell_1}{\ell_0}\frac{z_b - z_0}{\ell_0} + \frac{\ell_2^2 - \ell_1^2}{\ell_0^2} \end{aligned} \tag{4.13}$$

where we also have

$$\begin{aligned} \xi_2^2 - \xi_1^2 = &\left[\left[\frac{\epsilon_1}{\epsilon_2}\right]^{\frac{1}{2}} - \left[\frac{\epsilon_2}{\epsilon_1}\right]^{\frac{1}{2}}\right]\chi^2 + 2\left[\frac{\ell_2}{\ell_0}\left[\frac{\epsilon_1}{\epsilon_2}\right]^{\frac{1}{4}} - \frac{\ell_1}{\ell_0}\left[\frac{\epsilon_2}{\epsilon_1}\right]^{\frac{1}{4}}\right]\chi \\ &+ \frac{\ell_2^2 - \ell_1^2}{\ell_0^2} \end{aligned} \tag{4.14}$$

These combine to give

$$\frac{z_b - z_{b_0}}{\ell_0} = \frac{1}{2}\frac{\ell_0}{\ell_2 - \ell_1}\left\{\left[\left[\frac{\epsilon_1}{\epsilon_2}\right]^{\frac{1}{2}} - \left[\frac{\epsilon_2}{\epsilon_1}\right]^{\frac{1}{2}}\right]\chi^2 + 2\left[\frac{\ell_2}{\ell_0}\left[\frac{\epsilon_1}{\epsilon_2}\right]^{\frac{1}{4}} - \frac{\ell_1}{\ell_0}\left[\frac{\epsilon_2}{\epsilon_1}\right]^{\frac{1}{4}}\right]\chi\right\} \tag{4.15}$$

Note the special cases

$$\begin{aligned} &z_b = z_{b_0} \\ &\chi = \begin{cases} \chi_1 = 0 \\ \chi_2 = 2\left[\left[\frac{\epsilon_2}{\epsilon_1}\right]^{\frac{1}{2}} - \left[\frac{\epsilon_1}{\epsilon_2}\right]^{\frac{1}{2}}\right]^{-1}\left[\frac{\ell_2}{\ell_0}\left[\frac{\epsilon_1}{\epsilon_2}\right]^{\frac{1}{4}} - \frac{\ell_1}{\ell_0}\left[\frac{\epsilon_2}{\epsilon_1}\right]^{\frac{1}{4}}\right] \end{cases} \end{aligned} \tag{4.16}$$

Then from (4.12) we have

$$\begin{aligned}\left[\frac{\Psi_b}{\ell_0}\right]^2 &= \xi_n^2 - \frac{1}{\ell_0^2}[z_b - z_{b_0} + \ell_n]^2 \\ &= \frac{[\epsilon_1\epsilon_2]^{\frac{1}{2}}}{\epsilon_n}\chi^2 + 2\frac{[\epsilon_1\epsilon_2]^{\frac{1}{4}}}{\epsilon_n^{\frac{1}{2}}}\frac{\ell_n}{\ell_0}\chi \\ &- \left[\frac{z_b - z_{b_0}}{\ell_0}\right]^2 - 2\frac{\ell_n}{\ell_0}\frac{z_b - z_{b_0}}{\ell_0}\end{aligned} \tag{4.17}$$

Since this is the same for both values of n we can average them (for greater symmetry in the expression) giving

$$\left[\frac{\Psi_b}{\ell_0}\right]^2 = \frac{1}{2}\left[\left[\frac{\epsilon_2}{\epsilon_1}\right]^{\frac{1}{2}} + \left[\frac{\epsilon_1}{\epsilon_2}\right]^{\frac{1}{2}}\right]\chi^2 + \left[\frac{\ell_2}{\ell_0}\left[\frac{\epsilon_1}{\epsilon_2}\right]^{\frac{1}{4}} + \frac{\ell_1}{\ell_0}\left[\frac{\epsilon_2}{\epsilon_1}\right]^{\frac{1}{4}}\right]\chi$$
$$- \left[\frac{z_b - z_{b_0}}{\ell_0}\right]^2 - \frac{\ell_2 + \ell_1}{\ell_0}\frac{z_b - z_{b_0}}{\ell_0} \tag{4.18}$$

At this point with (4.15) we have explicit representations for Ψ_b and $z_b - z_{b_0}$ in terms of χ. Now we can see that in general Ψ_b has terms ranging from χ through χ^4, while $z_b - z_{b_0}$ has χ and χ^2 terms. Carrying the substitution a little further gives

$$\left[\frac{\Psi_b}{\ell_0}\right]^2 = \frac{\ell_0}{\ell_2 - \ell_1}\left[\frac{\ell_2}{\ell_0}\left[\frac{\epsilon_2}{\epsilon_1}\right]^{\frac{1}{2}} - \frac{\ell_1}{\ell_0}\left[\frac{\epsilon_1}{\epsilon_2}\right]^{\frac{1}{2}}\right]\chi^2 + \frac{2\ell_1\ell_2}{\ell_0(\ell_2 - \ell_1)}\left[\left[\frac{\epsilon_2}{\epsilon_1}\right]^{\frac{1}{4}} - \left[\frac{\epsilon_1}{\epsilon_2}\right]^{\frac{1}{4}}\right]\chi$$
$$- \left[\frac{z_b - z_{b_0}}{\ell_0}\right]^2 \tag{4.19}$$

with the χ term explicit. Since Ψ_b^2 must be positive then sign of χ depends on the coefficient of χ in the above equation. This information can be combined with (4.16) to see if the χ_2 root exists (in which case the lens surface crosses the $z - z_{b_0}$ plane). With (4.15) and (4.19) one can carry the analysis further to eliminate χ and obtain a quartic equation relating Ψ_b and z_b. This is found in Appendix A.

Assuming that the coefficients of χ in (4.15) and (4.19) are non zero, then we have

$$\begin{aligned}\frac{\ell_0}{z_b - z_{b_0}}\left[\frac{\Psi_b}{\ell_0}\right]^2 &= \left[\frac{\ell_2}{\ell_0}\left[\frac{\epsilon_1}{\epsilon_2}\right]^{\frac{1}{4}} - \frac{\ell_2}{\ell_0}\left[\frac{\epsilon_2}{\epsilon_1}\right]^{\frac{1}{4}}\right]^{-1}\frac{2\ell_1\ell_2}{\ell_0^2}\left[\left[\frac{\epsilon_2}{\epsilon_1}\right]^{\frac{1}{4}} - \left[\frac{\epsilon_1}{\epsilon_2}\right]^{\frac{1}{4}}\right] \\ &+ O(\chi)\chi \text{ as } \chi \to 0 \\ &= 2\left[\frac{\ell_0}{\ell_1}\left[\frac{\epsilon_1}{\epsilon_2}\right]^{\frac{1}{4}} - \frac{\ell_0}{\ell_2}\left[\frac{\epsilon_2}{\epsilon_1}\right]^{\frac{1}{4}}\right]^{-1}\left[\left[\frac{\epsilon_2}{\epsilon_1}\right]^{\frac{1}{4}} - \left[\frac{\epsilon_1}{\epsilon_2}\right]^{\frac{1}{4}}\right] \\ &+ O(\chi) \text{ as } \chi \to 0\end{aligned} \tag{4.20}$$

If this expression is positive the lens surface is concave to the right and if negative concave to the left (near the z axis). Fitting this to a sphere, the center is at $z_c \vec{1}_z$ with

$$\frac{z_c - z_{b_0}}{\ell_0} = \left[\frac{\ell_0}{\ell_1}\left[\frac{\epsilon_1}{\epsilon_2}\right]^{\frac{1}{4}} - \frac{\ell_0}{\ell_2}\left[\frac{\epsilon_2}{\epsilon_1}\right]^{\frac{1}{4}}\right]^{-1}\left[\left[\frac{\epsilon_2}{\epsilon_1}\right]^{\frac{1}{4}} - \left[\frac{\epsilon_1}{\epsilon_2}\right]^{\frac{1}{4}}\right] \tag{4.21}$$

To illustrate this solution consider a special permittivity ratio $\epsilon_2/\epsilon_1 = 2.26$ and let ℓ_2/ℓ_1 be a parameter, giving the family of lens surfaces in Fig. 4.2.

B. Special Spherical Lens Surface

As a special case choose

$$\sqrt{\epsilon_1}\ell_1 = \sqrt{\epsilon_2}\ell_2 \tag{4.22}$$

which can be described as equal electrical distances to the two wave centers in terms of the respective medium permittivities. Then (4.9) gives

$$\left[\frac{z_b - z_{b_0}}{\ell_1} + 1\right]^2 + \left[\frac{\Psi_b}{\ell_1}\right]^2 = \left[\frac{z_b - z_{b_0}}{\ell_2} + 1\right]^2 + \left[\frac{\Psi_b}{\ell_2}\right]^2 \tag{4.23}$$

This is conveniently a quadratic equation which can be rearranged as

$$[[z_b - z_{b_0}] + \ell_0]^2 + \Psi_b^2 = \ell_0^2 \tag{4.24}$$

which is a sphere of radius ℓ_0 centered on $-\ell_0\vec{1}_z$ as in Fig. 4.3.

Noting that

$$\frac{\epsilon_2}{\epsilon_1} = \left[\frac{\ell_1}{\ell_2}\right]^2 \tag{4.25}$$

then only one parameter ratio needs to be specified to constrain the relative permeabilities and relative wave centers (foci). Note that

$$\begin{array}{l} 1 \leq \dfrac{\ell_2}{\ell_1} \leq \infty \\ \text{(converging lens)} \end{array} \quad \Rightarrow \left\{ \begin{array}{l} \ell_0 \leq \ell_1 \leq 2\ell_0, \;\; \infty \geq \ell_2 \geq 2\ell_0 \\ 0 \leq \dfrac{\epsilon_2}{\epsilon_1} \leq 1 \end{array} \right. \tag{4.26}$$

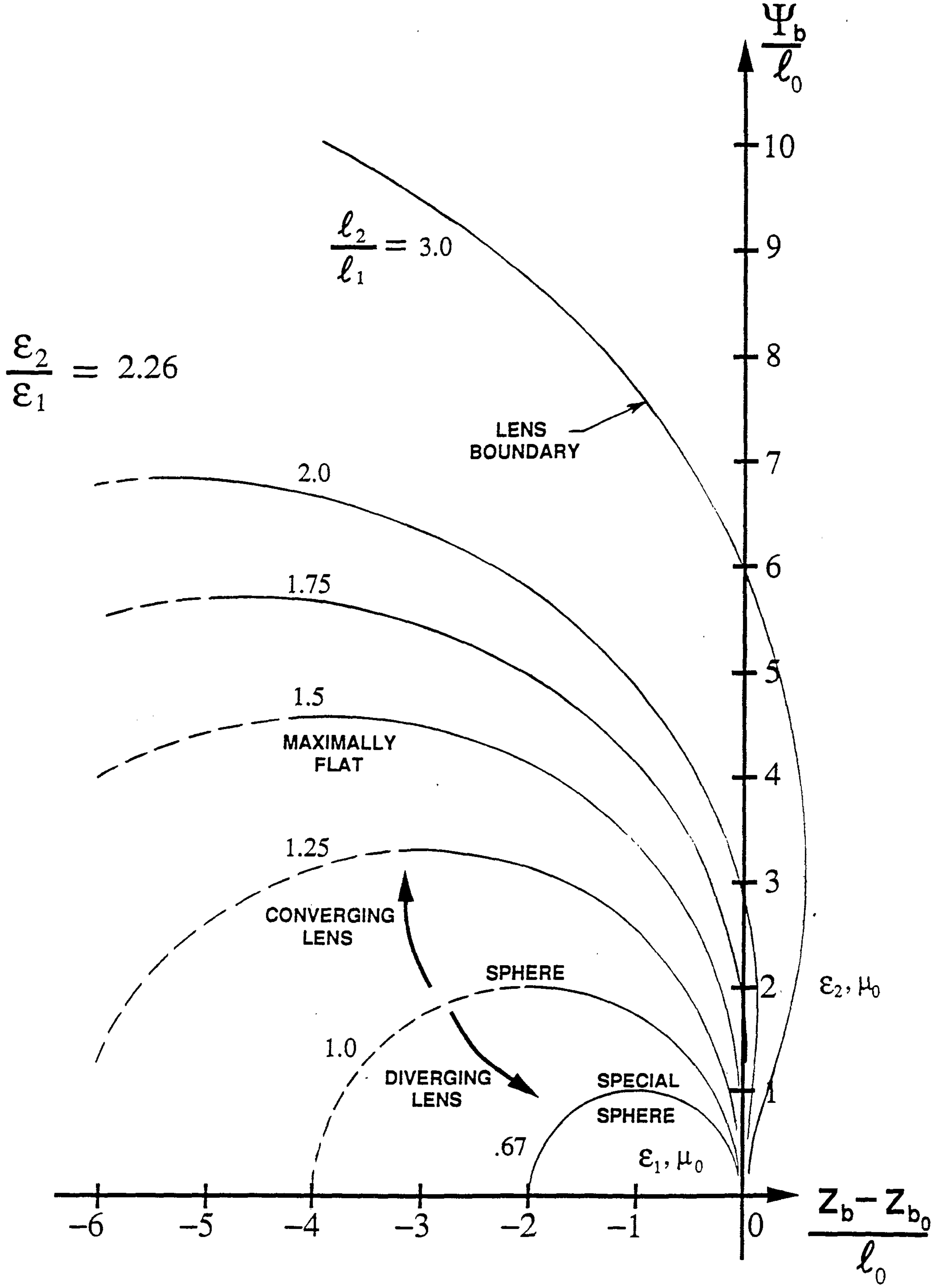


Fig 4.2: Lens Surface for Fixed Permittivity Ratio

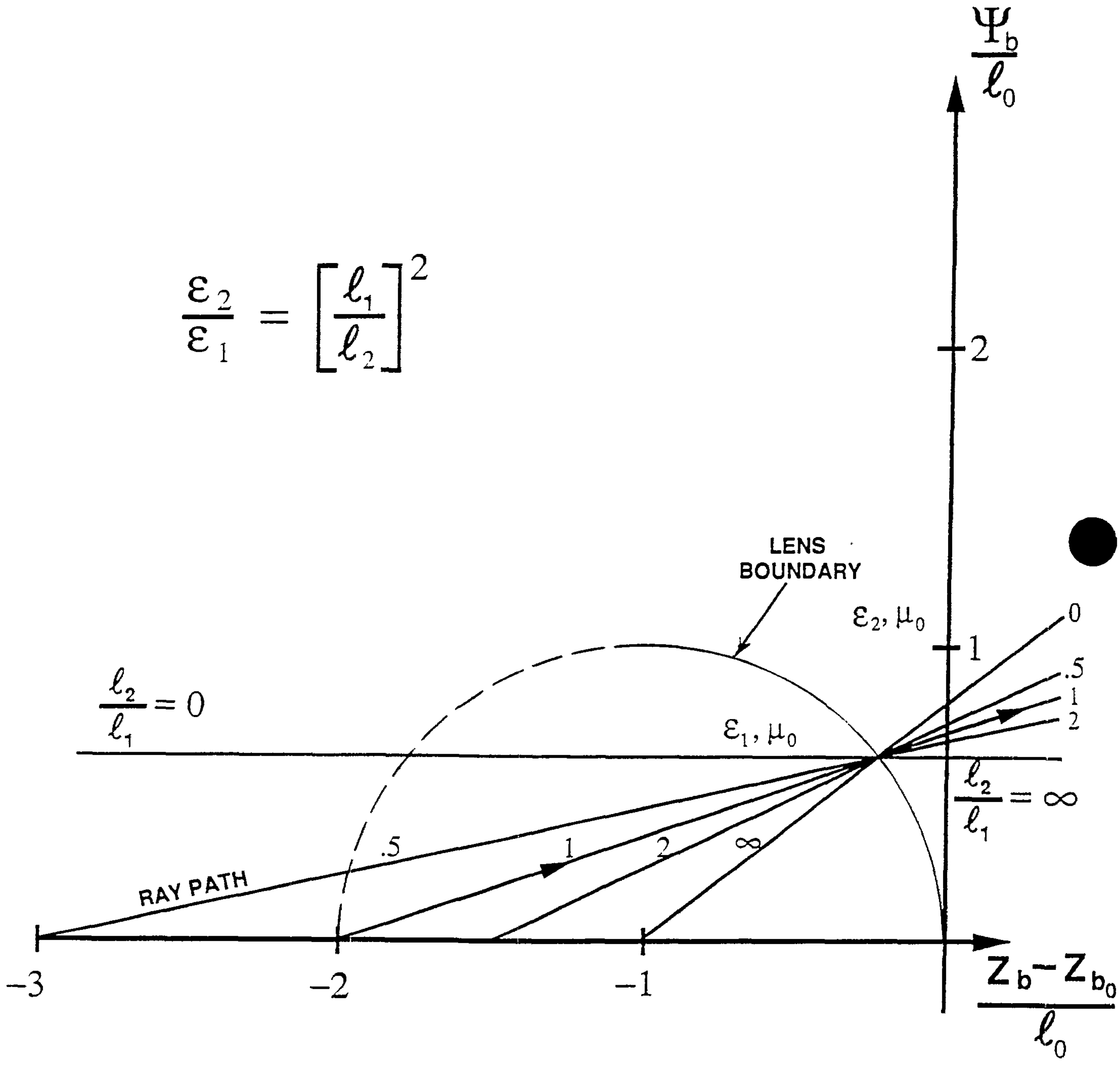


Fig 4.3: Special Spherical Lens Surface

and

$$0 \leq \frac{\ell_2}{\ell_1} \leq 1 \quad \text{(diverging lens)} \quad \Rightarrow \begin{cases} 2\ell_0 \leq \ell_1 \leq \infty \;, 2\ell_0 \geq \ell_2 \geq \ell_0 \\ 1 \leq \dfrac{\epsilon_2}{\epsilon_1} \leq \infty \end{cases} \tag{4.27}$$

C. Maximally Flat Lens Surface

Another special case is found by setting the coefficient of χ in (4.15) to zero in the representation of $z_b - z_{b_0}$ as

$$\begin{aligned} &\frac{\ell_2}{\ell_0}\left[\frac{\epsilon_1}{\epsilon_2}\right]^{\frac{1}{4}} - \frac{\ell_1}{\ell_0}\left[\frac{\epsilon_2}{\epsilon_1}\right]^{\frac{1}{4}} = 0 \\ &\frac{\epsilon_1}{\epsilon_2} = \left[\frac{\ell_1}{\ell_2}\right]^2 \end{aligned} \tag{4.28}$$

This makes $z_b - z_{b_0}$ proportional to χ^2 near the z axis, a case which can be referred to as maximally flat. With this constraint only one parameter ratio needs to be specified to specify the relative permittivities and relative wave centers (foci).

Now the only term in $z_b - z_{b_0}$ is proportional to χ^2 as

$$\begin{aligned} \frac{z_b - z_{b_0}}{\ell_0} &= \frac{1}{2}\frac{\ell_0}{\ell_2 - \ell_1}\left[\left[\frac{\epsilon_1}{\epsilon_2}\right]^{\frac{1}{2}} - \left[\frac{\epsilon_2}{\epsilon_1}\right]^{\frac{1}{2}}\right]\chi^2 \\ &= -\frac{1}{2}\chi^2 \end{aligned} \tag{4.29}$$

which is negative for real χ indicating that the lens surface is concave to the left, never crossing the $z_b = z_{b_0}$ plane. Then (4.19) becomes

$$\left[\frac{\Psi_b}{\ell_0}\right]^2 = \frac{1}{4}\chi^4 + \frac{\ell_1^3 - \ell_2^3}{\ell_2\ell_1(\ell_1 - \ell_2)}\chi^2 + \frac{2[\ell_1\ell_2]^{\frac{1}{2}}}{\ell_0}\chi \tag{4.30}$$

and χ is positive. Near the z axis (small χ) we have

$$\frac{\ell_0}{z_b - z_{b_0}}\left[\frac{\Psi_b}{\ell_0}\right]^4 = -8\frac{\ell_1\ell_2}{\ell_0^2} + O(\chi) \text{ as } \chi \to 0 \tag{4.31}$$

The curvature of the lens surface at the z axis is zero (infinite radius of curvature).

Figure 4.4 shows the lens surface with ℓ_2/ℓ_1 (and hence ϵ_2/ϵ_1) as a parameter. Note from (4.30) the symmetrical roles of ℓ_1 and ℓ_2, so that a particular value of ℓ_2/ℓ_1 corresponds to the same value of ℓ_1/ℓ_2.

D. Nearly Matched Permittivities

Consider now the case that

$$\frac{\epsilon_2}{\epsilon_1} = 1 + \nu \,, \quad |\nu| << 1 \tag{4.32}$$

where ν can be positive or negative depending on which permittivity is larger. Such a case can occur when one medium is free space (air) and the other a low density dielectric (such as foam polyethylene).

For small χ the leading term in (4.15) gives

$$\frac{z_b - z_{b_0}}{\ell_0} = [1 + O(\nu)]\, \chi + O(\chi^2) \text{ as } \nu \to 0 \text{ and } \chi \to 0 \tag{4.33}$$

Note that we are taking $\nu \to 0$ before $\chi \to 0$. Similarly from (4.19) we have

$$\left[\frac{\Psi_b}{\ell_0}\right]^2 = \frac{\ell_1 \ell_2}{\ell_0(\ell_2 - \ell_1)} \left[\nu + O(\nu^2)\right] \chi + O(\chi^2) \text{ as } \nu \to 0 \text{ and } \chi \to 0 \tag{4.34}$$

which means χ takes the sign of $\nu/(\ell_2 - \ell_1)$. Combining these gives

$$\frac{\ell_0}{z_b - z_{b_0}} \left[\frac{\Psi_b}{\ell_0}\right]^2 = \frac{\ell_1 \ell_2}{\ell_0(\ell_2 - \ell_1)} [\nu + O(\nu^2)] + O(\chi) \text{ as } \nu \to 0 \text{ and } \chi \to 0 \tag{4.35}$$

This is the equation of a paraboloid which (for positive ℓ_1 and ℓ_2) is concave to the right for positive $\nu/(\ell_2 - \ell_1)$, and to the left if negative.

As a limiting case let $\ell_2 = \infty$ so that the second wave (in medium 2) is a plane wave, giving

$$\begin{gathered} \frac{\ell_2}{\ell_0} = \infty \,, \ \frac{\ell_1}{\ell_0} = 1 \\ \frac{\ell_0}{z_b - z_{b_0}} \left[\frac{\Psi_b}{\ell_0}\right]^2 = \nu + O(\nu^2) + O(\chi) \\ \text{as } \nu \to 0 \text{ and } \chi \to 0 \end{gathered} \tag{4.36}$$

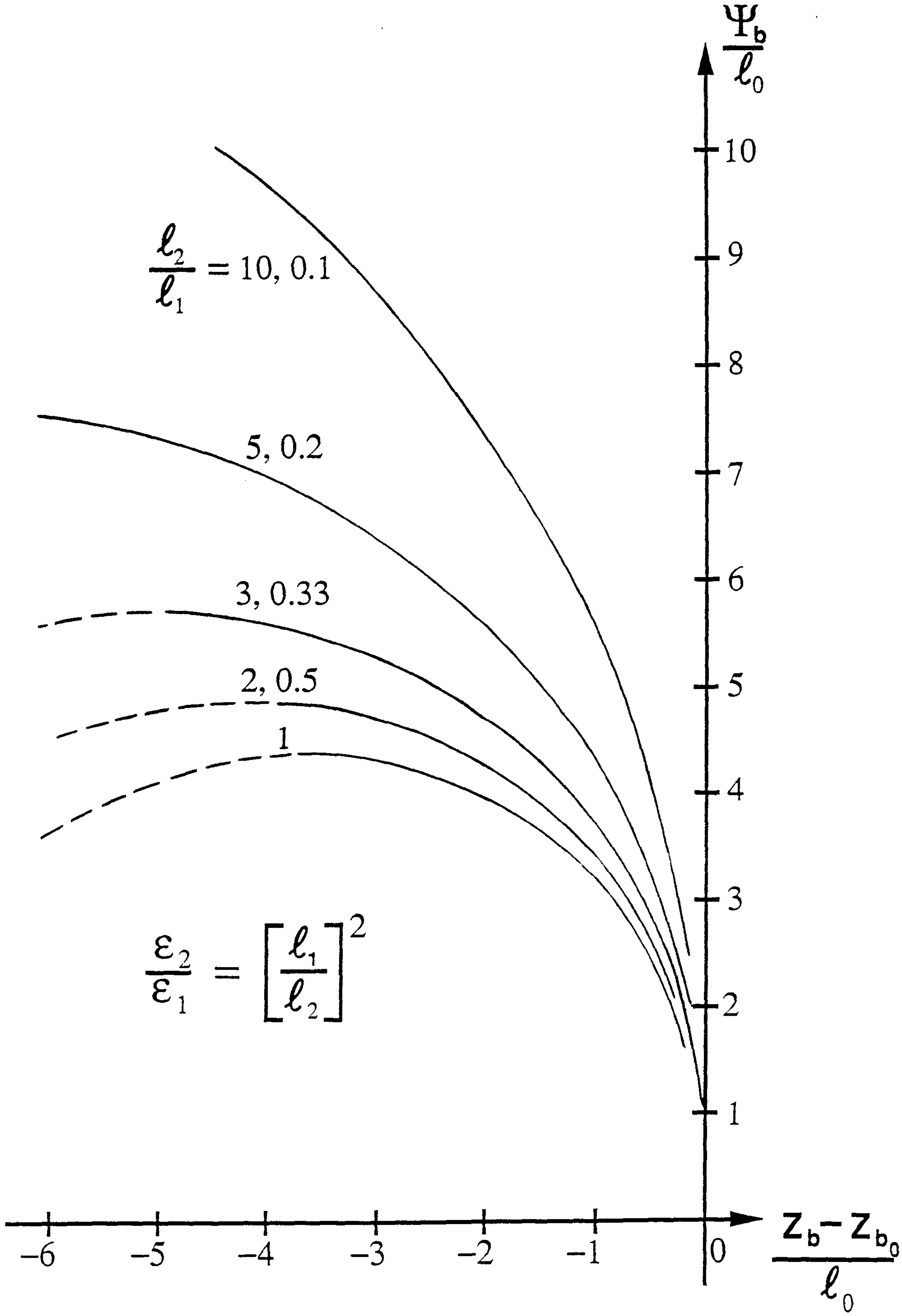


Fig 4.4: Maximally Flat Lens Surfaces

Assuming ν positive the lens surface is concave to the right. For a fixed lens surface

$$\frac{\Psi_b^2}{z_b - z_{b_0}} \simeq \ell_0 \nu \tag{4.37}$$

so that small ν implies large ℓ_0 (focal length). For comparison, if this surface were a reflector [3] we would have

$$\frac{\Psi_b^2}{z_b - z_{b_0}} = 4F \tag{4.38}$$

$F \equiv$ focal length

5 Diverging Spherical Wave to Less Divergent Spherical Wave to Plane Wave

Now let us combine the results of previous sections to have three waves in three media. As illustrated in Figs. 5.1 and 5.2 there are three regions, the second (middle) one being the lens and the first and last (third) having the same electrical properties in the examples. This defines two lens surfaces designated 1 and 2 (superscripts). These may intersect at

$$(\Psi_b^{(1)}, z_b^{(1)}) = (\Psi_i, z_i) = (\Psi_b^{(2)}, z_b^{(2)}) \tag{5.1}$$

depending on the choices of the various parameters. While there are many possible combinations of medium parameters, our primary interest has medium 2 as a lens with $\epsilon_2 > \epsilon_1 = \epsilon_3$.

Tracing the ray back we have a ray specified parallel to the z axis (plane wave) in medium 3. In medium 2 this corresponds to a ray on a radial from $\vec{r}_2 = z_{2_0}\vec{1}_z$. As discussed in Section 2 for $\epsilon_2 > \epsilon_3$ this corresponds to a prolate spheroid for lens surface 2. For $\epsilon_2 < \epsilon_3$ this corresponds to one sheet of a hyperboloid of two sheets. In medium 1 the ray is on a radial from $\vec{r}_1 = z_{1_0}\vec{1}_z$. As discussed in Section 4 lens surface 1 is in general described by a quartic equation. For a specified set of ℓ_1, ℓ_2, the lens thickness $(z_{b_0}^{(2)} - z_{b_0}^{(1)})$ and the permittivities, the two lens surfaces are determined.

The examples in Figs. 5.1 and 5.2 are determined by first specifying ϵ_2/ϵ_1 ($= 2.26$) with $\epsilon_3 = \epsilon_1$. Then choosing ℓ determines z_{2_0} from $z_{b_0}^{(2)}$ as

$$\ell = z_{b_0}^{(2)} - z_{b_0} \tag{5.2}$$

Then choosing $z_b^{(1)}$ and ℓ_2 gives the lens thickness on axis

$$z_b^{(2)} - 2_b^{(1)} = \ell - \ell_2 \tag{5.3}$$

Choosing ℓ_2/ℓ_1 then determines ℓ_1, ℓ_0 and lens surface 1.

$$\frac{\varepsilon_2}{\varepsilon_1} = 2.26 \qquad \frac{\ell_1}{\ell_0} = 1.5$$

$$\frac{\ell}{\ell_2} = 1.5 \qquad \frac{\ell_2}{\ell_0} = 3$$

$$\frac{\ell_2}{\ell_1} = 2$$

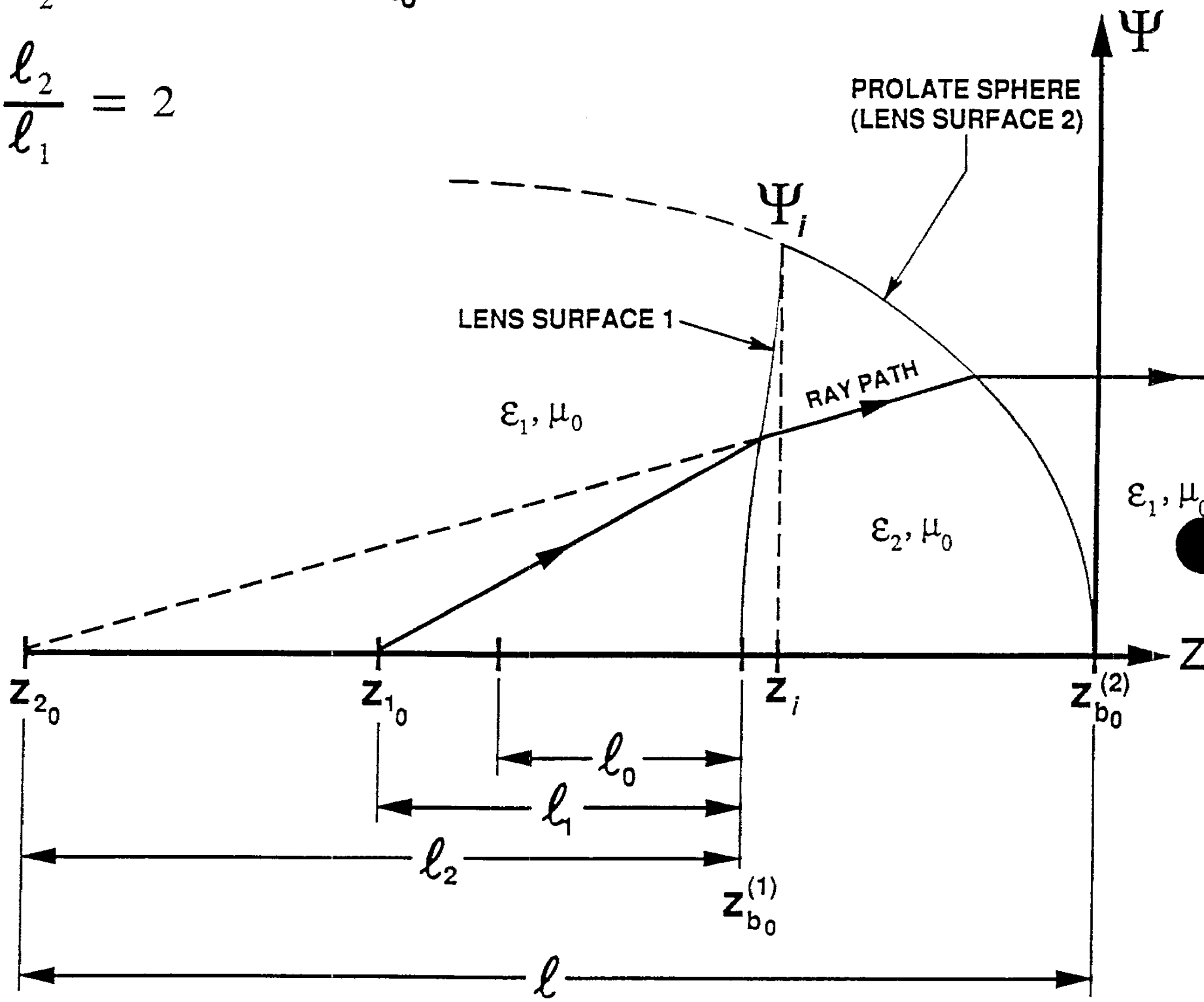


Fig 5.1: Converging Lens Separating Two Identical Media

$$\frac{\varepsilon_2}{\varepsilon_1} = 2.26 \qquad \frac{\ell_1}{\ell_0} = 1.67$$

$$\frac{\ell}{\ell_2} = 1.5 \qquad \frac{\ell_2}{\ell_0} = 2.5$$

$$\frac{\ell_2}{\ell_1} = 1.5$$

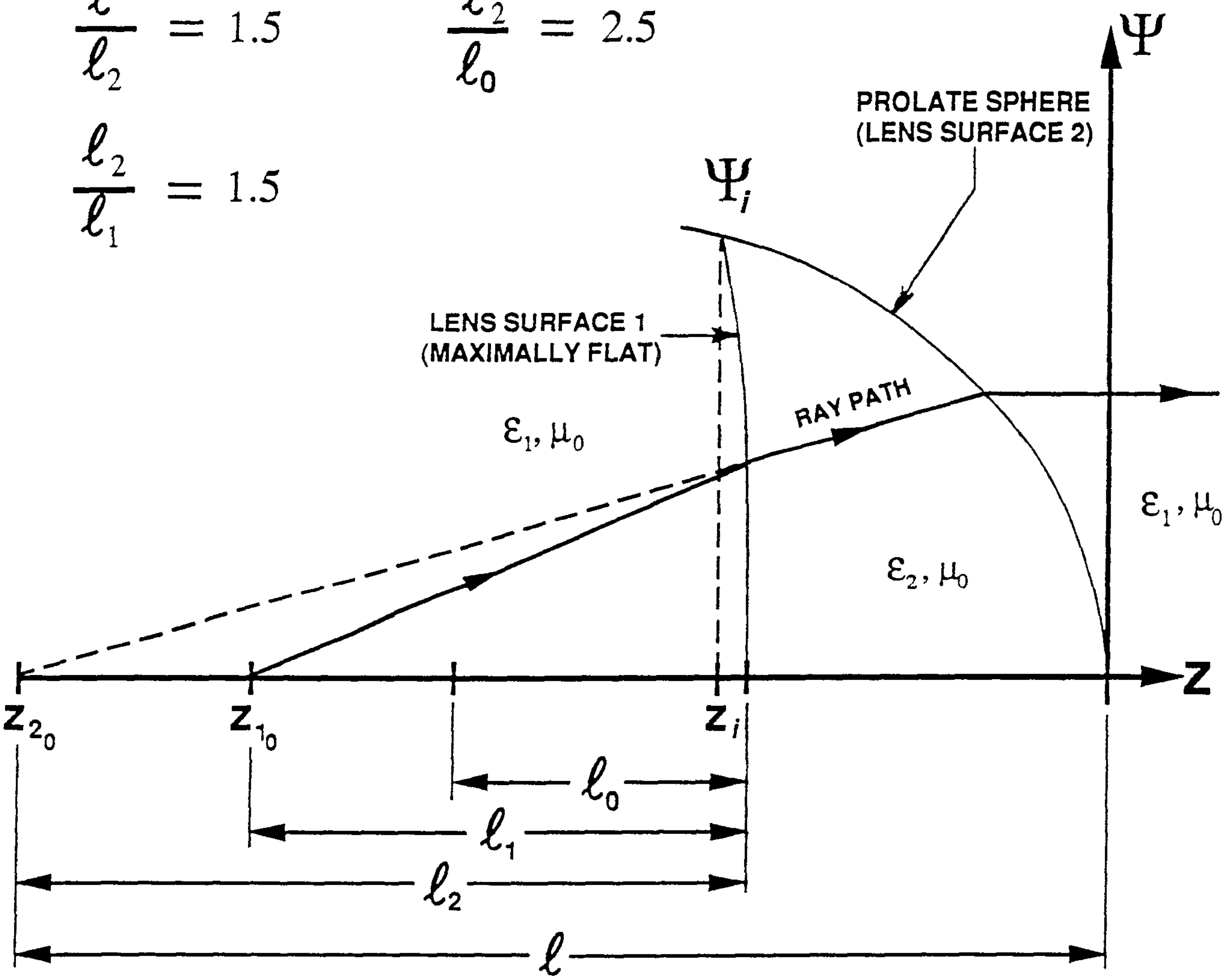


Fig 5.2: Converging Lens with Maximally Flat First Lens Surface

6 Transmission of Waves Through Lens Boundaries

In passing through the lens boundaries, while the equal-time requirement is met, not all of the wave is transmitted. Associated with the discontinuity of the medium parameters there is in general a partial reflection of the wave at each lens boundary. As discussed in the previous section one can have a dielectric lens of permittivity ϵ_2 separating two media of permittivity ϵ_1. With common permeabilities μ_0 the media have wave impedances

$$Z_n = \sqrt{\frac{\mu_0}{\epsilon_n}} \text{ for } n = 1, 2 \tag{6.1}$$

As illustrated in Fig. 6.1, consider the simple problem of two planar lens surfaces with normal wave incidence. Except for the curvature of the lens surfaces in Section 5, this corresponds to the transmission and reflection of the ray along the z axis. At lens surface 1 there is a transmission coefficient (for the electric field) using the usual one-dimensional formula.

$$T_1 = \frac{2Z_2}{Z_1 + Z_2} = 2\epsilon_2^{-\frac{1}{2}} \left[\epsilon_1^{-\frac{1}{2}} + \epsilon_2^{-\frac{1}{2}}\right]^{-1} \tag{6.2}$$

At lens surface 2 we similarly have

$$T_2 = \frac{2Z_1}{Z_1 + Z_2} = 2\epsilon_1^{-\frac{1}{2}} \left[\epsilon_1^{-\frac{1}{2}} + \epsilon_2^{-\frac{1}{2}}\right]^{-1} \tag{6.3}$$

This gives an overall transmission coefficient of

$$T = T_1 T_2 = 4[\epsilon_1\epsilon_2]^{\frac{1}{2}} \left[\epsilon_1^{\frac{1}{2}} + \epsilon_2^{-\frac{1}{2}}\right]^{-2} = 4\left[\left[\frac{\epsilon_2}{\epsilon_1}\right]^{\frac{1}{4}} + \left[\frac{\epsilon_1}{\epsilon_2}\right]^{\frac{1}{4}}\right]^{-2} \tag{6.4}$$

This applies to the first portion of a transient wave reaching medium 3. There are reflections at the lens boundaries which can appear in medium 3 at a later time due to multiple reflections. Since media 1 and 3 have the same constitutive parameters, $T^2 < 1$ represents the relative power transmission (at early times) from medium 1 to medium 3.

Ideally one would like T to be one, such as for the ideal lenses discussed in [5], this being accomplished by controlling the permeability and/or medium inhomogeneity and anisotropy

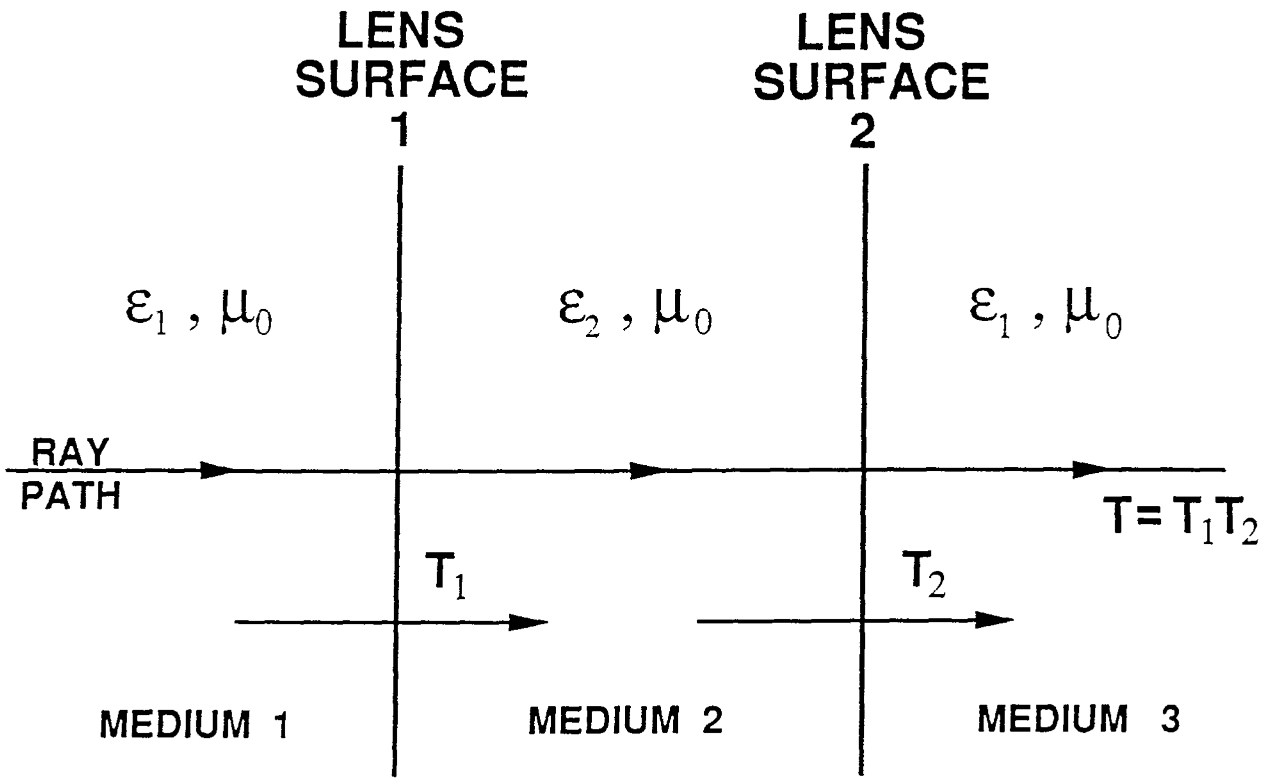


Fig 6.1: Wave Normally Incident on Two Lens Surfaces

as well as special angles (e.g. Brewster angle) at the lens boundary. In the present case one sacrifices ideal performance for simplicity of the lens medium. If the two permittivities are only slightly different so that as in (4.32) we write

$$\frac{\epsilon_2}{\epsilon_1} = 1 + \nu \tag{6.5}$$

For small ν we have

$$\begin{aligned} T &= \cosh^{-2}\left(\ell n\left(\left(\frac{\epsilon_2}{\epsilon_1}\right)^{\frac{1}{4}}\right)\right) = \cosh^{-2}\left(\frac{1}{4}\ell n(1+\nu)\right) \\ &= 1 - \frac{1}{16}\ell n^2(1+\nu) + O(\ell n^4(1+\nu)) = 1 - \frac{1}{16}\nu^2 + O(\nu^3) \text{ as } \nu \to 0 \end{aligned} \tag{6.6}$$

So, for small ν, not much is lost in transmission through the lens. The fractional power removed from the initial wave by the lens-boundary reflections is

$$\begin{aligned} 1 - T^2 &= \frac{1}{8}\ell n^2(1+\nu) + O(\ell n^4(1+\nu)) \\ &= \frac{1}{8}\nu^2 + O(\nu^3) \text{ as } \nu \to 0 \end{aligned} \tag{6.7}$$

7 Plane Wave to Diverging Spherical Wave to More Divergent Spherical Wave

The considerations in Section 5 can readily be extended to lenses for going from a plane wave through a lens in which the wave is spherically expanded to a third medium in which the rate of spherical expansion is even greater as illustrated in Figs. 7.1 and 7.2. Begin with a plane wave in medium 1 where a ray is parallel to the z-axis. In medium 2 the ray is on a radial from $\vec{r}_2 = z_{2_0}\vec{1}_2$. As discussed in Section 3 for $\epsilon_2 > \epsilon_1$ this corresponds to a prolate spheroid for lens surface 1. For $\epsilon_2 < \epsilon_1$ this corresponds to one sheet of a hyperboloid of two sheets. In medium 3 the spherical wave is described by the ray on a radial from $\vec{r}_3 = z_{3_0}\vec{1}_z$ As discussed in Section 4 lens surface 2 is in general described by a quartic equation. For a specified set of ℓ_2, ℓ_3 the lens thickness $(z_{b_0}^{(2)} - z_{b_0}^{(1)})$ and the three permittivities, the two lens surfaces are determined.

Note now that

$$
\begin{aligned}
&\ell_0^{-1} \equiv \ell_2^{-1} + \ell_3^{-1} \\
&\ell_2 \equiv z_{b_0}^{(2)} - z_{2_0}\ , \ \ \ell_3 \equiv z_{b_0}^{(2)} - z_{3_0} \\
&\ell \equiv z_{b_0}^{(1)} - z_{2_0}
\end{aligned}
\tag{7.1}
$$

With our primary interest in $\epsilon_2 > \epsilon_1 = \epsilon_3$, the two lens surfaces have closest approach near the axis where one can choose $z_{b_0}^{(1)} = z_{b_0}^{(2)}$. In some cases the z axis may be in a region where there is no wave (e.g. inside a conductor such as the center conductor of a coaxial structure (cable)). In such cases we can have $z_{b_0}^{(1)} > z_{b_0}^{(2)}$ and the intersections of the two surfaces described as in (5.2).

$$\frac{\varepsilon_2}{\varepsilon_1} = 2.26 \qquad \frac{\ell_3}{\ell_0} = 1.5$$

$$\frac{\ell}{\ell_2} = 1 \qquad \frac{\ell_2}{\ell_0} = 3$$

$$\frac{\ell_2}{\ell_3} = 2$$

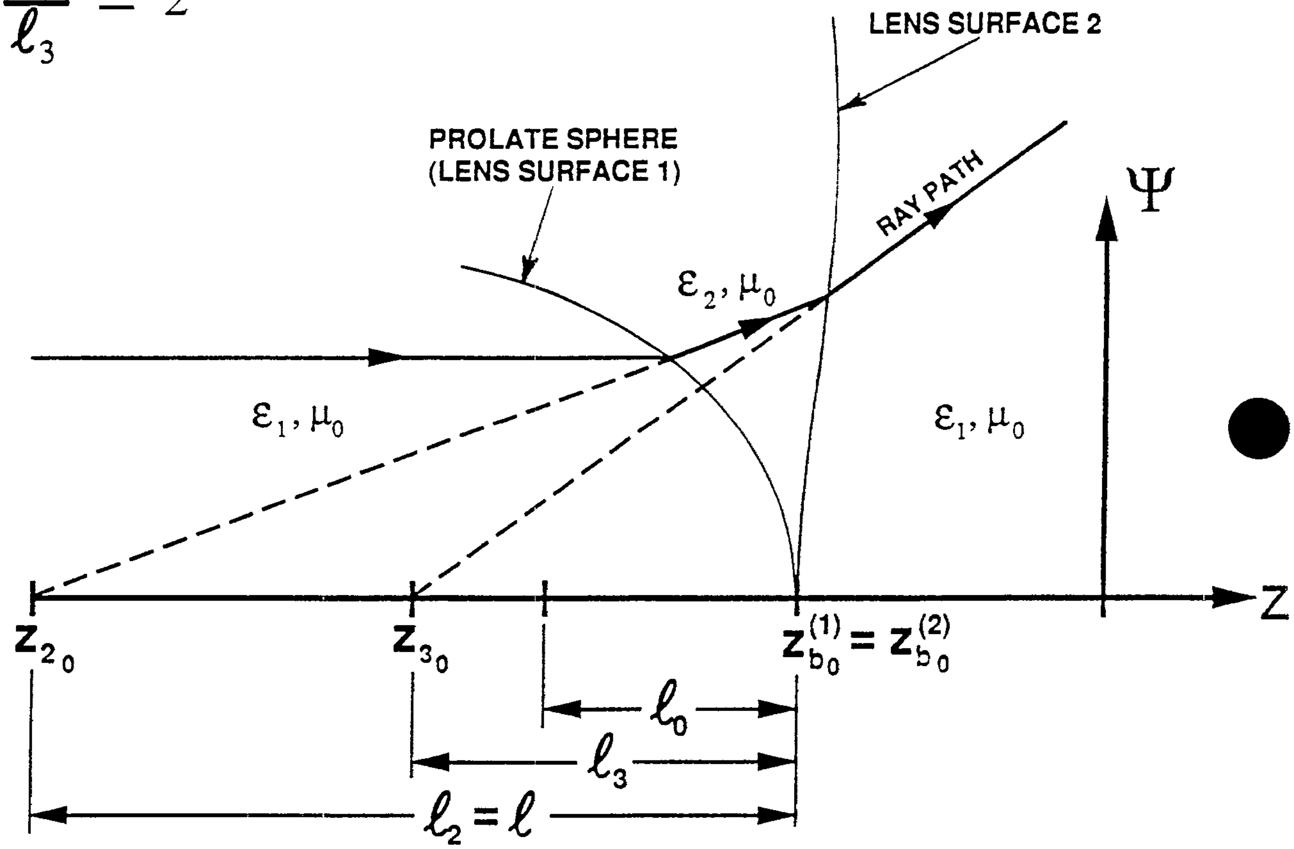


Fig 7.1: Diverging Lens Separating Two Identical Media

$$\frac{\varepsilon_2}{\varepsilon_1} = 2.26 \qquad \frac{\ell_3}{\ell_0} = 1.67$$

$$\frac{\ell}{\ell_2} = 1 \qquad \frac{\ell_2}{\ell_0} = 2.5$$

$$\frac{\ell_2}{\ell_3} = 1.5$$

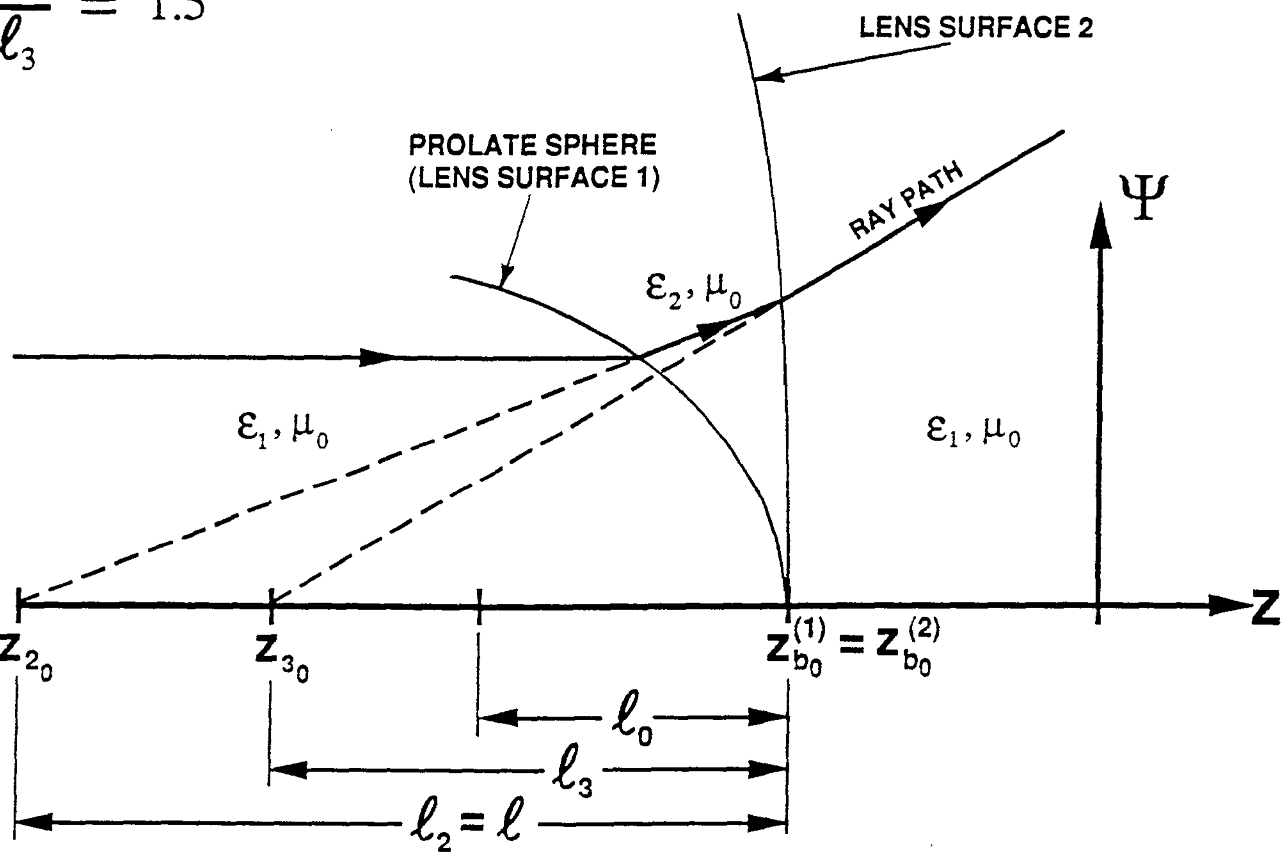


Fig 7.2: Diverging Lens with Maximally Flat Second Lens Surface

8 Concluding Remarks

The case of a single lens surface separating two media of different permittivities gives rather simple results, allowing only for prolate spheres and hyperboloids (one of two sheets), provided one of the two waves is a plane wave. With two surfaces separating three media the situation is in general more complicated. The surface matching two spherical waves is described by a fourth order-polynomial equation. Here we have considered some interesting examples of this, in particular where the first and third media have the same permittivities, and the wave in either medium 1 or 3 (but not both) is a plane wave. However, the results of this paper are more general in that they can be applied to a wider set of situations involving three different permittivities and/or three spherical waves as well. Utilizing the general results in Section 4 and Appendix A one can go from a spherical wave in one medium to another spherical wave in an adjacent medium, and then on to another, etc.

Appendix A: Non-Parametric Form of Lens Surface

Section 4 considers the general case of a lens surface matching two spherical waves. There the general solution is cast in terms of a parameter χ. From (4.15) and (4.19) this is

$$\frac{z_b - z_{b_0}}{\ell_0} = \frac{1}{2}\frac{\ell_0}{\ell_2 - \ell_1}\left\{\left[\left[\frac{\epsilon_1}{\epsilon_2}\right]^{\frac{1}{2}} - \left[\frac{\epsilon_2}{\epsilon_1}\right]^{\frac{1}{2}}\right]\chi^2 + 2\left[\frac{\ell_2}{\ell_0}\left[\frac{\epsilon_1}{\epsilon_2}\right]^{\frac{1}{4}} - \frac{\ell_1}{\ell_0}\left[\frac{\epsilon_2}{\epsilon_1}\right]^{\frac{1}{4}}\right]\chi\right\}$$

$$\left[\frac{\Psi_b}{\ell_0}\right]^2 + \left[\frac{z_b - z_{b_0}}{\ell_0}\right]^2 = \frac{\ell_0}{\ell_2 - \ell_1}\left[\frac{\ell_2}{\ell_0}\left[\frac{\epsilon_2}{\epsilon_1}\right]^{\frac{1}{2}} - \frac{\ell_1}{\ell_0}\left[\frac{\epsilon_1}{\epsilon_2}\right]^{\frac{1}{2}}\right]\chi^2$$

$$+\frac{2\ell_1\ell_2}{\ell_0(\ell_2 - \ell_1)}\left[\left[\frac{\epsilon_2}{\epsilon_1}\right]^{\frac{1}{4}} - \left[\frac{\epsilon_1}{\epsilon_2}\right]^{\frac{1}{4}}\right]\chi \tag{A.1}$$

Rewrite this in the convenient form

$$\begin{aligned}
\frac{z_b - z_{b_0}}{\ell_0} &= C_1\chi^2 + C_2\chi \\
\left[\frac{\Psi_b}{\ell_0}\right]^2 + \left[\frac{z_b - z_{b_0}}{\ell_0}\right]^2 &= C_3\chi^2 + C_4\chi
\end{aligned} \tag{A.2}$$

with the four constants as above.

Solving the first of (A.2) for χ^2 and substituting for χ^2 in the second gives

$$\left[\frac{\Psi_b}{\ell_0}\right]^2 + \left[\frac{z_b - z_{b_0}}{\ell_0}\right]^2 = \frac{C_3}{C_1}\frac{z_b - z_{b_0}}{\ell_0} + \frac{C_1C_4 - C_2C_3}{C_1}\chi \tag{A.3}$$

Solving for χ we have

$$\chi = \frac{C_1}{C_1C_4 - C_2C_3}\left\{\left[\frac{\Psi_b}{\ell_0}\right]^2 + \left[\frac{z_b - z_{b_0}}{\ell_0}\right]^2\right\} - \frac{C_3}{C_1C_4 - C_2C_3}\frac{z_b - z_{b_0}}{\ell_0} \tag{A.4}$$

which can be placed back in the first of (A.2) for both χ^2 and χ to give

$$\begin{aligned}
\frac{z_b - z_{b_0}}{\ell_0} &= \frac{C_1^2}{[C_1C_4 - C_2C_3]^2}\left\{C_1\left\{\left[\frac{\Psi_b}{\ell_0}\right]^2 + \left[\frac{z_b - z_{b_0}}{\ell_0}\right]^2\right\} - C_3\frac{z_b - z_{b_0}}{\ell_0}\right\}^2 \\
&+\frac{C_2}{C_1C_4 - C_2C_3}\left\{C_1\left\{\left[\frac{\Psi_b}{\ell_0}\right]^2 + \left[\frac{z_b - z_{b_0}}{\ell_0}\right]^2\right\} - C_3\frac{z_b - z_{b_0}}{\ell_0}\right\}
\end{aligned} \tag{A.5}$$

With the four constants given, this is a fourth order equation relating Ψ_b/ℓ_0 and $(z_b - z_{b_0})/\ell_0$.

The special combination of constants reduces to

$$
\begin{aligned}
C_1C_4 - C_2C_3 \quad &= \frac{\ell_1\ell_2}{(\ell_2-\ell_1)^2}\left[\left[\frac{\epsilon_1}{\epsilon_2}\right]^{\frac{1}{2}} - \left[\frac{\epsilon_2}{\epsilon_1}\right]^{\frac{1}{2}}\right]\left[\left[\frac{\epsilon_2}{\epsilon_1}\right]^{\frac{1}{4}} - \left[\frac{\epsilon_1}{\epsilon_2}\right]^{\frac{1}{4}}\right] \\
&-\frac{\ell_0^2}{(\ell_2-\ell_1)^2}\left[\frac{\ell_2}{\ell_0}\left[\frac{\epsilon_1}{\epsilon_2}\right]^{\frac{1}{4}} - \frac{\ell_1}{\ell_0}\left[\frac{\epsilon_2}{\epsilon_1}\right]^{\frac{1}{4}}\right]\left[\frac{\ell_2}{\ell_0}\left[\frac{\epsilon_2}{\epsilon_1}\right]^{\frac{1}{2}} - \frac{\ell_1}{\ell_0}\left[\frac{\epsilon_1}{\epsilon_2}\right]^{\frac{1}{2}}\right] \\
&= \frac{\ell_0}{\ell_2-\ell_1}\left\{\frac{\ell_1}{\ell_0}\left[\frac{\epsilon_1}{\epsilon_2}\right]^{\frac{1}{4}} - \frac{\ell_2}{\ell_0}\left[\frac{\epsilon_2}{\epsilon_1}\right]^{\frac{1}{4}}\right\}
\end{aligned}
\tag{A.6}
$$

With C_1, C_2, and C_3 from (A.1) this gives the constants in (A.5).

A special case treated in Section 4 is the special spherical lens given in (4.22). This corresponds to the combination in (A.6) being zero. Another special case is the maximally flat lens surface in (4.28). This corresponds to $C_2 = 0$.

References

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