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Canonical Examples for High-Frequency Propagation on Unit Cell of Wave-Launcher Array

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Abstract

This paper extends the available analytic solutions for a periodic array of wave launchers. These solutions are based on the high-frequency approximation of the multiconductor transmission-line equations. The examples here are for some profiles of the characteristic-impedance matrix (2 X 2) for two conductors (plus reference). Comparing the solutions for different profiles one can begin to optimize the profile.

transmission lines, waveguides

Contents

I. Introduction

A recent paper [1] considers a special case of a unit cell of a periodic array of wave launchers. This is based on a two-wire (plus reference) transmission-line model. Using the general results of [4] the high-frequency or early-time solution is obtained via a matrix first-order differential equation. The matrix solution (known as the matrizant) can be written in closed form (common mathematical functions) for the particular case of the characteristic-impedance-matrix variation along the transmission line chosen there [4]. Note that given a specified 2 x 2 matrix (within certain realizability restrictions) one can determine the dimensions of the wave-launcher unit cell including the position and size of the launcher plates within the cell [2].

The present paper extends the solutions available for various profiles (dependence on the longitudinal or z coordinate) of this characteristic-impedance matrix. Treating the square root of a matrix (in the appendices) explicit expressions for the symmetric 2 x 2 case are found. These are used to express the general results for the matrizant via a quadrature (section 5). Here the general constraint that the two conductors join at the end of the wave launcher (aperture plane) is applied. The initial condition of a step voltage on conductor 1 and no voltage on conductor 2 is also applied.

Sections 6 through 8 consider specific profiles. First the previous case in [1] is readily solved using the present general expressions. This involves a rather simple normalized characteristic-impedance matrix of the form

$$\left(F_{n,m}(\zeta)\right) = \begin{pmatrix} 1 & \zeta \\ \zeta & 1 \end{pmatrix}$$
$$0 \le \zeta \le 1 \tag{1.1}$$
$$\zeta \equiv \text{normalized spatial coordinate}$$

Second we make the upper left element (self impedance of conductor 1) a function of ζ with a quadratic variation as

$$\left(F_{n,m}(\zeta)\right) = \begin{pmatrix} \alpha + (\alpha - 1)\zeta^2 & \zeta \\ \zeta & 1 \end{pmatrix} \tag{1.2}$$
$$\alpha > 0$$

and find the general solution for the early-time voltage at the aperture plane. Some improvement is found. Third we investigate a hybrid wave launcher consisting of a first section which is a decoupled transmission-line transformer with

$$\left(F_{n,m}(\xi)\right)=\begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix}$$
$$\alpha<\xi<1 \tag{1.3}$$

Where ν is some function of ξ varying from α to 1. This is followed by a wave-launcher section of the form in (1.1) where $\xi=1$ corresponds to $\zeta=0$. It is found that the ratio of the early-time voltage at the aperture plane to the voltage at the aperture plane to the source voltage can be made 1 provided

$$\alpha=\frac{1}{2} \tag{1.4}$$

II. General Properties of the High-Frequency Solution

The high-frequency behavior of lossless multiconductor transmission lines consisting of (perfect) conductors in uniform media takes the form in retarded time of [4]

$$\left(V_n(z)\right) = \left(\phi_{n,m}(z)\right) \cdot \left(V_n(0)\right) \tag{2.1}$$

where the matrizant is the solution of the first order matrix differential equation

$$\begin{aligned}
&\frac{d}{dz}\left(\phi_{n,m}(z)\right) = \left(A_{n,m}(z)\right) \cdot \left(\phi_{n,m}(z)\right) \\
&\left(\phi_{n,m}(0)\right) = \left(1_{n,m}\right) \\
&\left(A_{n,m}(z)\right) = \frac{1}{2}\left[\frac{d}{dz}\left(Z_{c_{n,m}}(z)\right)\right] \cdot \left(Y_{c_{n,m}}(z)\right) \\
&\qquad = -\frac{1}{2}\left(Z_{c_{n,m}}(z)\right) \cdot \left[\frac{d}{dz}\left(Y_{c_{n,m}}(z)\right)\right]
\end{aligned} \tag{2.2}$$

Here the characteristic impedance matrix is frequency independent, and hence so is the matrizant $\left(\phi_{n,m}(z)\right)$ which is a real function of the real coordinate z. The voltage vector $\left(V_n(z)\right)$ can also be a function of frequency through $\left(V_n(0)\right)$, but we ignore this trivial aspect here. For later purposes $\left(V_n(0)\right)$ is taken as a vector step function and $\left(V_n(z)\right)$ gives the initial amplitude of the propagating step.

As discussed in [4], the matrizant has certain properties:

$$\begin{aligned}
&\det\left(\phi_{n,m}(z)\right) = e^{\int_0^z tr\left(\left(A_{n,m}(z')\right)\right)dz'} \\
&tr\left(\left(A_{n,m}(z)\right)\right) \equiv \text{trace of } \left(A_{n,m}(z)\right) \\
&\qquad = \sum_{n=1}^{N} A_{n,n}(z) \\
&\qquad = \sum_{n=1}^{N} \chi_\beta\left(\left(A_{n,n}(z)\right)\right) \\
&\chi_\beta\left(\left(A_{n,m}(z)\right)\right) \equiv \text{eigenvalue of } \left(A_{n,m}(z)\right)
\end{aligned} \tag{2.3}$$

Also with the sufficient condition that

$$\left(A_{n,m}(z)\right) = \sum_{p=1}^{P} d_p(z)\left(d_{n,m}\right)_p \tag{2.4}$$

$d_p(z) \equiv$ scalar functions of z

$\left(d_{n,m}\right)_p \equiv$ constant NxN matrices, every pair of which commutes

then the matrizant can be written as

$$\left(\phi_{n,m}(z)\right) = e^{\sum_{p=1}^{P} \int_o^z d_p(z')dz'\left(d_{n,m}\right)_p}$$

$$= \prod_{p=1}^{P} e^{\int_o^z d_p(z')dz'\left(d_{n,m}\right)_p}$$

$$= e^{\int_o^z \left(A_{n,m}(z')\right)dz'} \tag{2.5}$$

An example for $N = 2$ is given in [1] for which the above decomposition applies and the integrals can be evaluated in closed form.

III. Normalized High-Frequency Solution

As in [4] the foregoing is extended by normalizing the voltage vector (or combined voltage vector) via

$$
\begin{aligned}
\left(z_{c_{n,m}}(z)\right) &= \left(Z_{c_{n,m}}(z)\right)^{\frac{1}{2}} \\
\left(y_{c_{n,m}}(z)\right) &= \left(Y_{c_{n,m}}(z)\right)^{\frac{1}{2}}
\end{aligned}
\tag{3.1}
$$

where the square root is taken in the positive sense as in appendix A. Note that assuming reciprocal media all the above matrices are symmetric. Define a normalized voltage vector as

$$
\left(w_n(z)\right) \equiv \left(y_{c_{n,m}}(z)\right)\cdot\left(V_n(z)\right) \tag{3.2}
$$

Then (2.1) is transformed to

$$
\left(w_n(z)\right) \equiv \left(\Phi_{n,m}(z)\right)\cdot\left(w_n(0)\right) \tag{3.3}
$$

where

$$
\begin{aligned}
\left(\Phi_{n,m}(z)\right) &= \left(y_{c_{n,m}}(z)\right)\cdot\left(\phi_{n,m}(z)\right)\cdot\left(z_{c_{n,m}}(0)\right) \\
\left(\Phi_{n,m}(0)\right) &= \left(1_{n,m}\right)
\end{aligned}
\tag{3.4}
$$

The corresponding matrizant differential equation is

$$
\begin{aligned}
\frac{d}{dz}\left(\Phi_{n,m}(z)\right) &= \left(C_{n,m}(z)\right)\cdot\left(\Phi_{n,m}(z)\right) \\
\left(C_{n,m}(z)\right) &= -\frac{1}{2}\left[\frac{d}{dz}\left(y_{c_{n,m}}(z)\right)\right]\cdot\left(z_{c_{n,m}}(z)\right)+\frac{1}{2}\left(z_{c_{n,m}}(z)\right)\cdot\left[\frac{d}{dz}\left(y_{c_{n,m}}(z)\right)\right] \\
&= \frac{1}{2}\left(y_{c_{n,m}}(z)\right)\cdot\left[\frac{d}{dz}\left(z_{c_{n,m}}(z)\right)\right]-\frac{1}{2}\left[\frac{d}{dz}\left(z_{c_{n,m}}(z)\right)\right]\cdot\left(y_{c_{n,m}}(z)\right)
\end{aligned}
\tag{3.5}
$$

Symmetric matrices in (3.1) imply

$$
\left(C_{n,m}(z)\right) = -\left(C_{n,m}(z)\right)^T \tag{3.6}
$$

i.e., this matrix is skew symmetric as well as real. Since the diagonal elements are zero we have

$$tr\left(\left(C_{n,m}(z)\right)\right) = \sum_{n=1}^{N} C_{n,m}(z) = 0$$
$$= \sum_{n=1}^{N} \mathrm{X}_{\beta}\left(\left(C_{n,m}(z)\right)\right)$$
$$\mathrm{X}_{\beta}\left(\left(C_{n,m}(z)\right)\right) \equiv \text{ eigenvalue of } \left(C_{n,m}(z)\right) \tag{3.7}$$
$$\det\left(\left(\Phi_{n,m}(z)\right)\right) = 1$$

This normalized matrizant has special properties as seen from

$$\begin{aligned}\frac{d}{dz}\left[\left(\Phi_{n,m}(z)\right)^T \cdot \left(\Phi_{n,m}(z)\right)\right] &= \left(\Phi_{n,m}(z)\right)^T \cdot \left[\frac{d}{dz}\left(\Phi_{n,m}(z)\right)\right] + \left[\frac{d}{dz}\left(\Phi_{n,m}(z)\right)^T\right] \cdot \left(\Phi_{n,m}(z)\right) \\ &= \left(\Phi_{n,m}(z)\right)^T \cdot \left(C_{n,m}(z)\right) \cdot \left(\Phi_{n,m}(z)\right) + \left(\Phi_{n,m}(z)\right)^T \cdot \left(C_{n,m}(z)\right)^T \cdot \left(\Phi_{n,m}(z)\right) \\ &= \left(\Phi_{n,m}(z)\right)^T \cdot \left(C_{n,m}(z)\right) \cdot \left(\Phi_{n,m}(z)\right) - \left(\Phi_{n,m}(z)\right)^T \cdot \left(C_{n,m}(z)\right) \cdot \left(\Phi_{n,m}(z)\right) \\ &= \left(0_{n,m}\right)\end{aligned} \tag{3.8}$$

where we have used (3.6) and the transpose of the matrizant equation (3.5). Integrating and imposing the value at $z = 0$ from (3.4) gives

$$\begin{aligned}\left(\Phi_{n,m}(z)\right)^T \cdot \left(\Phi_{n,m}(z)\right) &= \left(1_{n,m}\right) \\ \left(\Phi_{n,m}(z)\right)^{-1} &= \left(\Phi_{n,m}(z)\right)^T\end{aligned} \tag{3.9}$$

Noting that this matrizant is real valued we now have that it is unitary [5, 8].

From (3.3) we now have

$$\left(w_n(z)\right) = \left(\Phi_{n,m}(z)\right) \cdot \left(w_n(0)\right) = \left(w_n(0)\right) \cdot \left(\Phi_{n,m}(z)\right)^T$$

$$\begin{aligned}\left(w_n(z)\right) \cdot \left(w_n(z)\right) &= \left(w_n(0)\right) \cdot \left(\Phi_{n,m}(z)\right)^T \cdot \left(\Phi_{n,m}(z)\right) \cdot \left(w_n(0)\right) \\ &= \left(w_n(0)\right) \cdot \left(w_n(0)\right) \\ \left|\left(w_n(z)\right)\right| &= \left|\left(w_n(0)\right)\right|\end{aligned} \tag{3.10}$$

so that the magnitude of $\left(w_n(z)\right)$ is conserved along the transmission line. This can be interpreted as conservation of power in the wavefront since we have

$$
\begin{aligned}
(w_n(0))\cdot(w_n(0)) &= (V_n(0))\cdot\left(y_{c_{n,m}}(0)\right)\cdot\left(y_{c_{n,m}}(0)\right)\cdot(V_n(0)) \\
&= (V_n(0))\cdot\left(Y_{c_{n,m}}(0)\right)\cdot(V_n(0)) = (V_n(0))\cdot(I_n(0)) \\
&= (w_n(z))\cdot(w_n(z)) = (V_n(z))\cdot\left(y_{c_{n,m}}(z)\right)\cdot\left(y_{c_{n,m}}(z)\right)\cdot(V_n(z)) \\
&= (V_n(z))\cdot\left(Y_{c_{n,m}}(z)\right)\cdot(V_n(z)) = (V_n(z))\cdot(I_n(z))
\end{aligned}
\tag{3.11}
$$

The fact that $\left(\Phi_{n,m}(z)\right)$ is unitary (and real) implies various things [5, 8]. As indicated in (3.10) it is length preserving when dot multiplying vectors. The columns are a set of N mutually orthogonal unit vectors, i.e., form an orthonormal set. Likewise the rows also form an orthonormal set. The eigenvalues as in (3.7) all have magnitude 1. Note that in general this matrizant is not symmetric so that there are different left and right eigenvectors (which may be complex) forming a biorthonormal set as in appendix A.

Defining

$$
\begin{aligned}
&\Lambda_\beta(z) = \text{ eigenvalues of } \left(\Phi_{n,m}(z)\right) \\
&\quad \beta = 1,2,\ldots,N
\end{aligned}
\tag{3.12}
$$

we have

$$
\begin{aligned}
&\left|\Lambda_\beta(z)\right| = 1 \quad \text{for all } \beta \\
&\det\left(\left(\Phi_{n,m}(z)\right)\right) = \prod_{\beta=1}^{N}\Lambda_\beta(z) = 1 \\
&tr\left(\left(\Phi_{n,m}(z)\right)\right) = \sum_{n=1}^{N}\Phi_{n,n}(z) \quad \text{(real)} \\
&\qquad = \sum_{\beta=1}^{N}\Lambda_\beta(z) \\
&\left|tr\left(\left(\Phi_{n,m}(z)\right)\right)\right| \le \sum_{n=1}^{N}\left|\Lambda_\beta(z)\right| = N \\
&\qquad -N \le tr\left(\left(\Phi_{n,m}(z)\right)\right) \le N
\end{aligned}
\tag{3.13}
$$

Extending to eigenvectors we have

$$
\begin{aligned}
&\left(\Phi_{n,m}(z)\right)\cdot\left(R_n(z)\right)_\beta = \Lambda_\beta(z)\left(R_n(z)\right)_\beta \\
&\left(L_n(z)\right)_\beta\cdot\left(\Phi_{n,m}(z)\right) = \Lambda_\beta(z)\left(L_n(z)\right)_\beta
\end{aligned}
\tag{3.14}
$$

Using the unitary property

$$(R_n(z))_\beta = \Lambda_\beta(z)\left(\Phi_{n,m}(z)\right)^T \cdot (R_n(z))_\beta = \Lambda_\beta(z)(R_n(z))_\beta \cdot \left(\Phi_{n,m}(z)\right)$$

$$(L_n(z))_\beta = \Lambda_\beta(z)(L_n(z))_\beta \cdot \left(\Phi_{n,m}(z)\right)^T = \Lambda_\beta(z)\left(\Phi_{n,m}(z)\right) \cdot (L_n(z))_\beta \tag{3.15}$$

This shows that for each β_1 there is a β_2 (which may or may not equal β_1) with

$$\begin{aligned}(R_n(z))_{\beta_1} &= (L_n(z))_{\beta_2} \\ \Lambda_{\beta_1}(z) &= \Lambda^{-1}_{\beta_2}(z) = \Lambda^*_{\beta_2}(z)\end{aligned} \tag{3.16}$$

with the last relation (conjugate) implied by the unit magnitude of the eigenvalues. So the eigenvalues come in conjugate pairs.

Considering the eigenvector equation for conjugate eigenvalues as in (3.16) we have

$$\begin{aligned}\left(\Phi_{n,m}(z)\right) \cdot (R_n(z))_{\beta_1} &= \Lambda_{\beta_1}(z)(R_n(z))_{\beta_1} \\ \left(\Phi_{n,m}(z)\right) \cdot (R_n(z))^*_{\beta_1} &= \Lambda^*_{\beta_1}(z)\,(R_n(z))^*_{\beta_1} \\ &= \Lambda_{\beta_2}(z)\,(R_n(z))^*_{\beta_1}\end{aligned} \tag{3.17}$$

This shows that we can set

$$\begin{aligned}(R_n(z))_{\beta_1} &= (R_n(z))^*_{\beta_2} = (L_n(z))_{\beta_2} \\ (L_n(z))_{\beta_1} &= (L_n(z))^*_{\beta_2} = (R_n(z))_{\beta_1}\end{aligned} \tag{3.18}$$

with the second of these coming from doing as in (3.17) with left eigenvectors. So the eigenvectors corresponding to conjugate eigenvalues (paired if not purely real) are themselves mutually conjugate. The roles of left and right eigenvectors are interchanged on interchanging conjugate eigenvalues. Also left and right eigenvectors are mutually conjugate for each β.

A special case is that of real eigenvalues for which we can take $\beta_1 = \beta_2$ in the foregoing and find

$$\begin{aligned}\Lambda_\beta(z) &= \pm 1 \\ (R_n(z))_\beta &= (L_n(z))_\beta = \text{a real vector} \\ (R_n(z))_\beta \cdot (R_n(z))_\beta &= 1 = (L_n(z))_\beta \cdot (L_n(z))_\beta\end{aligned} \tag{3.19}$$

IV. Case of $N = 2$

Summarizing from [4] we have

$$\left(C_{n,m}(z)\right) = h(z)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$g(z) \equiv \int_0^z h(z')dz' \tag{4.1}$$

$$\left(\Phi_{n,m}(z)\right) \equiv e^{\int_0^z \left(C_{n,m}(z')\right)dz'} = e^{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g(z)}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(g(z)) + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sin(g(z))$$

$$= \begin{pmatrix} \cos(g(z)) & \sin(g(z)) \\ -\sin(g(z)) & \cos(g(z)) \end{pmatrix}$$

$$h(z) = \frac{1}{2}\frac{\left[z_{c_{2,2}} - z_{c_{1,1}}\right]\frac{d}{dz}z_{c_{1,2}}(z) - \left[\frac{d}{dz}z_{c_{2,2}}(z) - \frac{d}{dz}z_{c_{1,1}}(z)\right]z_{c_{1,2}}(z)}{z_{c_{1,1}}(z)z_{c_{2,2}}(z) - z^2_{c_{1,2}}(z)}$$

Knowing that $\left(\Phi_{n,m}(z)\right)$ is a real unitary matrix now we have for the eigenvalues

$$\Lambda_\beta(z) \equiv \text{ eigenvalues of } \left(\Phi_{n,m}(z)\right)$$

$$\det\left(\left(\Phi_{n,m}(z)\right)\right) = 1 = \Lambda_1(z)\Lambda_2(z)$$

$$tr\left(\left(\Phi_{n,m}(z)\right)\right) = 2\cos(g(z)) = \Lambda_1(z) + \Lambda_2(z) \tag{4.2}$$

$$\left|\Lambda_1(z)\right| = \left|\Lambda_2(z)\right| = 1$$

$$\Lambda_2(z) = \Lambda_1^*(z) = \Lambda_1^{-1}(z)$$

$$\Lambda_{\substack{1\\2}}(z) = \cos(g(z)) \pm j\sin(g(z)) = e^{\pm jg(z)}$$

Going on to eigenvectors we have the right eigenvector equation

$$\left(\Phi_{n,m}(z)\right)\cdot\left(R_n\right)_\beta = \Lambda_\beta(z)\left(R_n\right)_\beta \tag{4.3}$$

which in component form is

$$\left[\cos(g(z)) - \Lambda_\beta(z)\right]R_{1;\beta} + \sin(g(z))R_{2;\beta} = 0$$
$$-\sin(g(z))R_{1;\beta} + \left[\cos(g(z)) - \Lambda_\beta(z)\right]R_{2;\beta} = 0 \tag{4.4}$$

Substituting for the eigenvalues gives

$$\mp j\sin(g(z))R_{1;\beta} + \sin(g(z))R_{2;\beta} = 0$$
$$R_{2;\beta} = \pm jR_{1;\beta}$$
$$(R_n)_\pm = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm j \end{pmatrix} \tag{4.5}$$

Similarly for the left eigenvectors one can repeat the above and obtain

$$(L_n)_\pm = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \mp j \end{pmatrix} \tag{4.6}$$

As can be verified these right and left eigenvectors are a biorthonormal set. Note that the eigenvectors are independent of z.

The eigendyads are

$$(R_n)_\pm(L_n)_\pm = \frac{1}{2}\begin{pmatrix} 1 & \mp j \\ \pm j & 1 \end{pmatrix} \quad . \tag{4.7}$$

The diagonal form of $(\Phi_{n,m}(z))$ is now

$$(\Phi_{n,m}(z)) = \sum_{\beta=1}^{2}\Lambda_\beta(z)(R_n)_\pm(L_n)_\pm$$
$$= \frac{e^{jg(z)}}{2}\begin{pmatrix} 1 & -j \\ j & 1 \end{pmatrix} + \frac{e^{-jg(z)}}{2}\begin{pmatrix} 1 & j \\ -j & 1 \end{pmatrix} \tag{4.8}$$

V. General Result for Case of Two Conductors Representing Unit Cell of Wave Launcher

A special case of N = 2 is considered in [1]. Here we have a two-conductor (plus reference) transmission-line model of a unit cell of a periodic array of wave launchers. The symmetry of the problem has planar electric and magnetic boundaries, simplifying the analysis somewhat. The discussion and illustrations of section II of that paper are directly applicable here.

The characteristic-impedance matrix for the unit cell, or for one quarter of the cell (considering the electric and magnetic boundaries which symmetrically divide the cell), has the form

$$\begin{aligned}\left(Z_{c_{n,m}}\right) &= Z_o\left(f_{g_{n,m}}\right)\\ Z_o &= \sqrt{\frac{\mu_o}{\epsilon_o}} = \text{ characteristic impedance of free space}\end{aligned} \tag{5.1}$$

The geometric-impedance-factor matrix $\left(f_{g_{n,m}}\right)$ is symmetric (reciprocity) and has non-negative eigenvalues. It can also be used for the per-unit-length inductance and capacitance matrices since we assume the medium surrounding the perfect conductors is uniform and isotropic.

Assuming that the wave-launcher plates are flat and that in a cross section of the unit cell they are parallel to the electric boundaries we have for an arbitrary cross section

$$\begin{aligned}\left(f_{g_{n,m}}\right) &= \begin{pmatrix} f_{g_{1,1}} & \frac{b'}{a} \\ \frac{b'}{a} & \frac{b}{a} \end{pmatrix}\\ 2a &\equiv \text{ width of unit cell}\\ 2b &\equiv \text{ height of unit cell}\\ 2a' &\equiv \text{width of plate}\\ 2b' &\equiv \text{ spacing of wave-launcher plates}\\ 0 &\le a' \le a\\ 0 &\le b' \le b\end{aligned} \tag{5.2}$$

Define a normalized form as

$$\left(F_{n,m}\right) \equiv \frac{a}{b}\left(f_{g_{n,m}}\right) = \begin{pmatrix} F_{1,1} & \frac{b'}{b} \\ \frac{b'}{b} & 1 \end{pmatrix} \tag{5.3}$$

Define a normalized coordinate ζ such that our region of concern is

$$0 \le \zeta \le 1 \tag{5.4}$$

In the previous paper this was taken as

$$\zeta \equiv \frac{z}{\ell} + 1$$
$$\ell \equiv \text{ length of wavelaunchers} \tag{5.5}$$

While the development of the solution of the differential equation in sections II and III of the present paper as well as in [4] is in terms of the real spatial coordinate z (meters) note that in (2.2) this variable can be changed to some other, say ζ, merely by replacing d/dz by $d/d\zeta$ wherever it appears in (2.2), in effect renormalizing $\left(A_{n,m}\right)$. In the exponential form of the solution as in (2.5) the dz' becomes $d\zeta'$, and the integral is over a range of ζ' which we take as $0 \le \zeta' \le \zeta$. Similar comments apply to the normalized form as in (3.5). Furthermore $\left(F_{n,m}\right)$ can be substituted for $\left(Z_{c_{n,m}}\right)$ and $\left(F_{n,m}\right)^{-1}$ for $\left(Y_{c_{n,m}}\right)$ wherever they appear in these solutions, i.e. in (2.2), (2.5), and (3.5).

Now in (5.2) let us have

$$\left.\frac{b'}{b}\right|_{\zeta=0} = 0$$
$$\left.\frac{b'}{b}\right|_{\zeta=1} = 1 \tag{5.6}$$

so that as in [1] the wave-launcher starts from some apex (relatively) small spacing and expands to fill the unit cell at the aperture plane where the wave launcher makes contact with adjacent wave launchers. Now we could choose b'/b as some monotonic function of ζ subject to (5.5), but by a change of variable this can become ζ, i.e. let

$$\frac{b'}{b} \equiv \zeta \tag{5.7}$$

This means that ζ need not be a simple linear function of z as in (5.5).

Let us now write (5.3) as

$$\left(F_{n,m}(\zeta)\right) = \begin{pmatrix} \nu & \zeta \\ \zeta & 1 \end{pmatrix} \tag{5.8}$$

where ν is some yet-to-be-specified function of ζ. Noting that

$$\det\left(\left(F_{n,m}(\zeta)\right)\right) = \nu - \zeta^2 \geq 0 \tag{5.9}$$

we must constrain

$$\nu \geq \zeta^2 \ \text{ for } 0 \leq \zeta \leq 1 \tag{5.10}$$

At the aperture plane as discussed in [1] let us have the wave launcher width $2a'$ also fill the unit cell giving

$$\nu\big|_{\zeta=1} = 1 \tag{5.11}$$

At the apex choose some value, say

$$\nu\big|_{\zeta=0} = \alpha > 0 \tag{5.12}$$

Then let ν generally take the form of some monotonic function of ζ with

$$\nu > \text{ greater of } \left[\alpha, \zeta^2\right] \text{ for } \ 0 < \zeta < 1 \tag{5.13}$$

subject to (5.11) and (5.12). If $0 < \alpha < 1$ (the case of interest) then ν is a monotonically increasing function of ζ. As discussed in [2] for given values of ν (or $F_{1,1}$), b'/b (or ζ), and b/a one can find the required a'/a, thereby generating the contour of the edge of the launcher plate.

Let us define

$$\left(H_{n,m}(\zeta)\right) = \begin{pmatrix} H_{1,1}(\zeta) & H_{1,2}(\zeta) \\ H_{2,1}(\zeta) & H_{2,2}(\zeta) \end{pmatrix} \equiv \left(F_{n,m}(\zeta)\right)^{\frac{1}{2}} = \begin{pmatrix} \nu & \zeta \\ \zeta & 1 \end{pmatrix}^{\frac{1}{2}}$$
$$H_{2,1}(\zeta) = H_{1,2}(\zeta) \tag{5.14}$$

where the square root of this matrix is discussed in appendix B. Replacing the square root of the characteristic impedance matrix $\left(\text{i.e., } \left(z_{c_{n,m}}\right)\right)$ by $\left(H_{n,m}\right)$ and similarly for the inverse and changing the coordinate to ζ allows us to apply section 4 for the normalized high-frequency solution for N = 2 as

$$\left(C_{n,m}(\zeta)\right) = h(\zeta)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$g(\zeta) \equiv \int_o^{\zeta} h(\zeta) d\zeta$$

$$\left(\Phi_{n,m}(\zeta)\right) = \begin{pmatrix} \cos(g(\zeta)) & \sin(g(\zeta)) \\ -\sin(g(\zeta)) & \cos(g(\zeta)) \end{pmatrix} \tag{5.15}$$

$$h(\zeta) = \frac{1}{2}\frac{\left[H_{2,2}(\zeta) - H_{1,1}(\zeta)\right]\frac{d}{d\zeta}H_{1,2}(\zeta) - \left[\frac{d}{d\zeta}H_{2,2}(\zeta) - \frac{d}{d\zeta}H_{1,1}(\zeta)\right]H_{1,2}(\zeta)}{H_{1,1}(\zeta)H_{2,2}(\zeta) - H_{1,2}^2(\zeta)}$$

Now one can try to integrate $h(\zeta)$, but first let us consider the simple solution. From (3.10) and (3.11) we have

$$\left|w_n(\zeta)\right| = \left|w_n(0)\right| \tag{5.16}$$

At $\zeta = 0$ we have as in [1]

$$\left(V_n(0)\right) = V_0\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{5.17}$$

i.e., a voltage is applied to the launcher plates, but none to the outer electric boundaries of the unit cell. Noting

$$\left(Y_{c_{n,m}}(\zeta)\right) = Z_0^{-1}\frac{a}{b}\left(F_{n,m}(\zeta)\right)^{-1} = Z_0^{-1}\frac{a}{b}\left[v - \zeta^2\right]^{-1}\begin{pmatrix} 1 & -\zeta \\ -\zeta & v \end{pmatrix} \tag{5.18}$$

we then have

$$\begin{aligned} \left(Y_{c_{n,m}}(0)\right) &= Z_0^{-1}\frac{a}{b}\begin{pmatrix} \alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix} \\ \left(w_n(0)\right)\cdot\left(w_n(0)\right) &= \left(V_n(0)\right)\cdot\left(Y_{c_{n,m}}(0)\right)\cdot\left(V_n(0)\right) \\ &= \frac{a}{b}\frac{V_0^2}{Z_0}\alpha^{-1} \end{aligned} \tag{5.19}$$

Furthermore we have

$$
\begin{aligned}
(w_n(0)) &= \left(Y_{c_{n,m}}(0)\right)^{\frac{1}{2}} \cdot (V_n(0)) \\
&= V_0\left[\frac{1}{Z_0}\frac{a}{b}\right]^{\frac{1}{2}}\begin{pmatrix}\alpha^{-\frac{1}{2}} & 0 \\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix} \\
&= V_0\left[\frac{1}{\alpha Z_0}\frac{a}{b}\right]^{\frac{1}{2}}\begin{pmatrix}1\\0\end{pmatrix}
\end{aligned}
\tag{5.20}
$$

so that only $w_1(0)$ is non zero.

Considering the boundary at $\zeta = 1$ note that

$$
\begin{aligned}
(V_n(\zeta)) &= \left(z_{c_{n,m}}(\zeta)\right)\cdot(w_n(\zeta)) \\
&= \left[Z_0\frac{b}{a}\right]^{\frac{1}{2}}\left(F_{n,m}(\zeta)\right)^{\frac{1}{2}}\cdot(w_n(\zeta)) \\
&= \left[Z_0\frac{b}{a}\right]^{\frac{1}{2}}\left(H_{n,m}(\zeta)\right)\cdot(w_n(\zeta))
\end{aligned}
\tag{5.21}
$$

From (B.20) and B.21) with $\zeta = 1$ implying $v = 1$ we have

$$
\left(H_{n,m}(1)\right) = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}
\tag{5.22}
$$

which gives

$$
V_1(1) = V_2(1) = \left[\frac{Z_0}{2}\frac{b}{a}\right]^{\frac{1}{2}}\left[w_1(1) + w_2(1)\right]
\tag{5.23}
$$

The equality of the two voltages at $\zeta = 1$ is consistent with [1] and the fact that the two conductors meet there. As discussed in [1] at $\zeta = 1$ the primary wave fills the unit cell and matches without reflection through the aperture plane with amplitude $V_1(1)$. There is a secondary wave between the launcher plates and outer electric boundaries which totally reflects from the short-circuit junction of the launcher plates with the outer electric boundaries at $\zeta = 1$.

From (3.3), (3.1), (5.15), and (5.20) our solution is now

$$
\begin{aligned}
(w_n(1)) &= \begin{pmatrix} \cos(g(1)) & \sin(g(1)) \\ -\sin(g(1)) & \cos(g(1)) \end{pmatrix} \cdot (w_n(0)) \\
&= V_0\left[\frac{1}{\alpha Z_0}\frac{a}{b}\right]^{\frac{1}{2}} \begin{pmatrix} \cos(g(1)) \\ -\sin(g(1)) \end{pmatrix} \\
V_1(1) = V_2(1) &= V_0[2\alpha]^{-\frac{1}{2}}\left[\cos(g(1)) - \sin(g(1))\right] \\
&= V_0\alpha^{-\frac{1}{2}}\cos\left(g(1)+\frac{\pi}{4}\right)
\end{aligned}
\tag{5.24}
$$

This leaves the problem to one of the evaluation of g(1).

Collecting from (B.20), B.22), (B.23), and (5.15) we have

$$
h(\zeta) = \frac{1}{2}\frac{\left[H_{2,2}(\zeta) - H_{1,1}(\zeta)\right]\frac{d}{d\zeta}H_{1,2}(\zeta) - H_{1,2}(\zeta)\frac{d}{d\zeta}\left[H_{2,2}(\zeta) - H_{1,1}(\zeta)\right]}{H_{1,1}(\zeta)H_{2,2}(\zeta) - H_{1,2}^2(\zeta)}
$$

$$
\begin{aligned}
H_{1,1}(\zeta)H_{2,2}(\zeta) - H_{1,2}^2(\zeta) &= \det\left(\left(H_{n,m}(\zeta)\right)\right) = \left[\det\left(\left(F_{n,m}(\zeta)\right)\right)\right]^{\frac{1}{2}} \\
&= \left[\nu - \zeta^2\right]^{\frac{1}{2}} \\
H_{2,2}(\zeta) - H_{1,1}(\zeta) &= \frac{1-\nu}{\zeta}H_{1,2}(\zeta) \\
H_{1,2}^2(\zeta) &= \frac{\zeta^2}{\psi}\left\{1+\nu-2\left[\nu-\zeta^2\right]^{\frac{1}{2}}\right\} \\
\psi &= (1-\nu)^2 + 4\zeta^2
\end{aligned}
\tag{5.25}
$$

Substituting we have

$$h(\zeta)=\frac{1}{2}\left[\nu-\zeta^2\right]^{-\frac{1}{2}}\left\{\frac{1-\nu}{\zeta}H_{1,2}(\zeta)\frac{d}{d\zeta}H_{1,2}(\zeta)-H_{1,2}(\zeta)\frac{d}{d\zeta}\left[\frac{1-\nu}{\zeta}H_{1,2}(\zeta)\right]\right\}$$

$$=\frac{1}{2}\left[\nu-\zeta^2\right]^{-\frac{1}{2}}\left\{-H_{1,2}^2(\zeta)\frac{d}{d\zeta}\left[\frac{1-\nu}{\zeta}\right]\right\}$$

$$=\frac{1}{2}\left[\nu-\zeta^2\right]^{-\frac{1}{2}}\left\{\frac{1-\nu}{\zeta^2}+\frac{1}{\zeta}\frac{d\nu}{d\zeta}\right\}H_{1,2}^2(\zeta)$$

$$=\frac{1}{2}\psi^{-1}\left[\nu-\zeta^2\right]^{-\frac{1}{2}}\left\{1-\nu+\zeta\frac{d\nu}{d\zeta}\right\}\left\{1+\nu-2\left[\nu-\zeta^2\right]^{\frac{1}{2}}\right\}$$

$$=\frac{1}{2}\left[\left(1-\nu^2\right)+4\zeta^2\right]^{-1}\left[\nu-\zeta^2\right]^{-\frac{1}{2}}\left\{1-\nu+\zeta\frac{d\nu}{d\zeta}\right\}\left\{1+\nu-2\left[\nu-\zeta^2\right]^{\frac{1}{2}}\right\} \quad (5.26)$$

VI. Case of $\nu \equiv 1$

Consider the simple case of

$$\alpha = 1,\ \nu = 1 \tag{6.1}$$

From (5.26) we have,

$$h(\zeta) = 0,\ g(1) = 0 \tag{6.2}$$

Then from (5.24) we have

$$V_1(1) = \frac{V_0}{\sqrt{2}} \tag{6.3}$$

This corresponds to the case in appendix C. It also corresponds to the result in [1] which was solved in a different way.

VII. Case of $v \equiv \alpha + (1-\alpha)\zeta^2$

Now consistent with (5.10) through (5.13) let us choose a quadratic form for v as

$$v \equiv \alpha + (1-\alpha)\zeta^2 \tag{7.1}$$

Then in (5.26) we have the terms

$$\begin{aligned} \psi &= (1-v)^2 + 4\zeta^2 = (1-\alpha)^2\left[1-\zeta^2\right]^2 + 4\zeta^2 \\ v - \zeta^2 &= \alpha\left[1-\zeta^2\right] \\ 1 - v + \zeta\frac{dv}{d\zeta} &= (1-\alpha)\left[1+\zeta^2\right] \end{aligned} \tag{7.2}$$

Combining we have

$$\begin{aligned} h(\zeta) &= \frac{1-\alpha}{2\sqrt{\alpha}}\,\psi^{-1}\left[1-\zeta^2\right]^{-\frac{1}{2}}\left[1+\zeta^2\right]\left\{\left[1+\zeta^2\right]+\alpha\left[1-\zeta^2\right]-2\sqrt{\alpha}\left[1-\zeta^2\right]^{\frac{1}{2}}\right\} \\ &= \frac{1-\alpha}{2\sqrt{\alpha}}h_1(\zeta) + \frac{(1-\alpha)\sqrt{\alpha}}{2}h_2(\zeta) - (1-\alpha)h_3(\zeta) \\ h_1(\zeta) &\equiv \psi^{-1}\left[1-\zeta^2\right]^{-\frac{1}{2}}\left[1+\zeta^2\right]^2 \\ h_2(\zeta) &\equiv \psi^{-1}\left[1-\zeta^2\right]^{\frac{1}{2}}\left[1+\zeta^2\right] \\ h_3(\zeta) &\equiv \psi^{-1}\left[1+\zeta^2\right] \end{aligned} \tag{7.3}$$

What we need is

$$\begin{aligned} g(1) &= \int_0^1 h(\zeta)d\zeta = \frac{1-\alpha}{2\sqrt{\alpha}}g_1 + \frac{(1-\alpha)\sqrt{\alpha}}{2}g_2 - (1-\alpha)g_3 \\ g_1 &\equiv \int_0^1 h_1(\zeta)d\zeta \\ g_2 &\equiv \int_0^1 h_2(\zeta)d\zeta \\ g_3 &\equiv \int_0^1 h_3(\zeta)d\zeta \end{aligned} \tag{7.4}$$

These are defined in such a way that the integrands h_n are positive over the range of integration, and hence all the g_n are positive.

Taking the first of these integrals

$$g_1 \equiv \int_0^1 \psi^{-1}\left[1-\zeta^2\right]^{-\frac{1}{2}}\left[1+\zeta^2\right]^2 d\zeta \tag{7.5}$$

make a change of variables

$$\begin{aligned} &u=\zeta\left[1-\zeta^2\right]^{-\frac{1}{2}} \ , \ \ \zeta=u\left[1+u^2\right]^{-\frac{1}{2}} \\ &\left[1-\zeta^2\right]^{-\frac{1}{2}}\mathrm{d}\zeta \ =\left[1+u^2\right]^{-1}du \end{aligned} \tag{7.6}$$

This gives

$$\begin{aligned} &1-\zeta^2=\left[1+u^2\right]^{-1} \\ &1+\zeta^2=\left[1+u^2\right]^{-1}\left[1+2u^2\right] \\ &\quad \psi=\left[1+u^2\right]^{-2}\left[(1-\alpha)^2+4u^2+4u^4\right] \end{aligned} \tag{7.7}$$

and the integral becomes

$$\begin{aligned} g_1&=\int_0^\infty\left[(1-\alpha)^2+4u^2+4u^4\right]^{-1}\left[1+u^2\right]^{-1}\left[1+2u^2\right]^2du \\ &=\frac{1}{2}\int_{-\infty}^{\infty}\left[(1-\alpha)^2+4u^2+4u^4\right]^{-1}\left[1+u^2\right]^{-1}\left[1+2u^2\right]^2du \end{aligned} \tag{7.8}$$

This is now in a form to be evaluated by the residue theorem. Consider a contour along the real axis of the u plane returning in positive (counter clockwise) sense in the upper half of the u plane. Since the integrand on this contour is $0\left(u^{-2}\right)$ as $|u|\to\infty$ there is no contribution and

$$g_1=2\pi j\sum \ \text{(residues in upper half } u \text{ plane)} \tag{7.9}$$

The integrand has three conjugate pole pairs at $\pm u_n$ with

$$\begin{aligned} &u_1=j\left|\frac{1+\Delta}{2}\right|^{\frac{1}{2}} \\ &u_2=j\left|\frac{1-\Delta}{2}\right|^{\frac{1}{2}} \\ &u_3=j \end{aligned} \tag{7.10}$$

where (positive square root)

$$\Delta = \left[1-(1-\alpha)^2\right]^{\frac{1}{2}} \tag{7.11}$$

with the restriction

$$0 \le \alpha \le 2 \tag{7.12}$$

This can be extended for α outside this range by allowing Δ to be imaginary.

Labelling the residues as r_n we have

$$\begin{aligned} r_1 &= \frac{1}{2}\left\{\frac{d}{du}\left[(1-\alpha)^2+4u^2+4u^4\right]\right\}^{-1}\left[1+u^2\right]^{-1}\left[1+2u^2\right]^2\Bigg|_{u=u_1} \\ &= \frac{1}{16u}\left[1+u2\right]^{-1}\left[1+2u^2\right]\Bigg|_{u=u_1} \\ &= j\frac{\sqrt{2}}{8}\frac{\Delta}{1-\Delta}\left[1+\Delta\right]^{-\frac{1}{2}} \end{aligned} \tag{7.13}$$

For r_2 we merely replace Δ by $-\Delta$ giving

$$r_2 = -j\frac{\sqrt{2}}{8}\frac{\Delta}{1+\Delta}\left[1-\Delta\right]^{-\frac{1}{2}} \tag{7.14}$$

For r_3 we have

$$\begin{aligned} r_3 &= \frac{1}{2}\left\{\frac{d}{du}\left[1+u^2\right]\right\}^{-1}\left[(1-\alpha)^2+4u^2+4u^4\right]^{-1}\left[1+2u^2\right]^2\Bigg|_{u=u_3} \\ &= \frac{1}{4u}\left[(1-\alpha)^2+4u^2+4u^4\right]^{-1}\left[1+2u^2\right]^2\Bigg|_{u=u_3} \\ &= -j\frac{1}{4(1-\alpha)^2} \end{aligned} \tag{7.15}$$

Combining these residues gives

$$
\begin{aligned}
g_1 &= 2\pi j \sum_{n=1}^{3} r_n \\
&= -\pi \frac{\sqrt{2}}{4} \frac{\Delta}{1-\Delta}[1+\Delta]^{-\frac{1}{2}} + \pi \frac{\sqrt{2}}{4} \frac{\Delta}{1+\Delta}[1-\Delta]^{-\frac{1}{2}} + \frac{\pi}{2(1-\alpha)^2} \\
&= \pi \frac{\sqrt{2}}{4} \Delta \left\{ -[1-\Delta]^{-1}[1+\Delta]^{-\frac{1}{2}} + [1+\Delta]^{-1}[1-\Delta]^{-\frac{1}{2}} \right\} + \frac{\pi}{2(1-\alpha)^2} \\
&= \pi \frac{\sqrt{2}}{4} \frac{\Delta}{1-\Delta^2} \left\{ -[1+\Delta]^{\frac{1}{2}} + [1-\Delta]^{\frac{1}{2}} \right\} + \frac{\pi}{2(1-\alpha)^2} \\
&= \pi \frac{\sqrt{2}}{4} \frac{\Delta}{(1-\alpha)^2} \left\{ -[1+\Delta]^{\frac{1}{2}} + [1-\Delta]^{\frac{1}{2}} \right\} + \frac{\pi}{2(1-\alpha)^2} \\
&= \frac{\pi}{2(1-\alpha)^2} \left\{ \frac{\sqrt{2}}{2} \Delta \left\{ -[1+\Delta]^{\frac{1}{2}} + [1-\Delta]^{\frac{1}{2}} \right\} + 1 \right\}
\end{aligned} \tag{7.16}
$$

The second integral is

$$
\begin{aligned}
g_2 &= \int_0^1 \psi^{-1} \left[1-\zeta^2\right]^{\frac{1}{2}} \left[1+\zeta^2\right] d\zeta \\
&= \int_0^\infty \left[(1-\alpha)^2 + 4u^2 + 4u^4\right]^{-1} \left[1+u^2\right]^{-1} \left[1+2u^2\right] du \\
&= \frac{1}{2} \int_{-\infty}^{\infty} \left[(1-\alpha)^2 + 4u^2 + 4u^4\right]^{-1} \left[1+u^2\right]^{-1} \left[1+2u^2\right] du
\end{aligned} \tag{7.17}
$$

with the same substitutions as for g_1. The integrand has the same conjugate pole pairs as in (7.10). Let us evaluate g_2 in the same manner as g_1 by closing the contour in the upper half plane. Note for g_2 that the integrand is $0\left(u^{-4}\right)$ as $|u| \to \infty$.

Labelling the residues as r_n' we have

$$
\begin{aligned}
r_1' &= \frac{1}{2} \left\{ \frac{d}{du} \left[(1-\alpha)^2 + 4u^2 + 4u^4\right] \right\}^{-1} \left[1+u^2\right]^{-1} \left[1+2u^2\right] \Bigg|_{u=u_1} \\
&= \frac{1}{16u} \left[1+u^2\right]^{-1} \Bigg|_{u=u_1} \\
&= -j \frac{\sqrt{2}}{8} [1-\Delta]^{-1} [1+\Delta]^{-\frac{1}{2}}
\end{aligned} \tag{7.18}
$$

For r_2' replacing Δ by $-\Delta$ we have

$$r_2' = -j\frac{\sqrt{2}}{8}[1+\Delta]^{-1}[1-\Delta]^{-\frac{1}{2}} \tag{7.19}$$

For r_3' we have

$$\begin{aligned}
r_3' &= \frac{1}{2}\left\{\frac{d}{du}\left[1+u^2\right]^{-1}\right\}^{-1}\left[(1-\alpha)^2+4u^2+4u^4\right]^{-1}\left[1+2u^2\right]\Bigg|_{u=u_3} \\
&= \frac{1}{4u}\left[(1-\alpha)^2+4u^2+4u^4\right]^{-1}\left[1+2u^2\right]\Bigg|_{u=u_3} \\
&= j\frac{1}{4(1-\alpha)^2}
\end{aligned} \tag{7.20}$$

Combining these residues gives

$$\begin{aligned}
g_2 &= 2\pi j\sum_{n=1}^{3} r_n' \\
&= \pi\frac{\sqrt{2}}{4}[1-\Delta]^{-1}[1+\Delta]^{-\frac{1}{2}} + \frac{\pi\sqrt{2}}{4}[1+\Delta]^{-1} + [1-\Delta]^{-\frac{1}{2}} - \frac{\pi}{2(1-\alpha)^2} \\
&= \pi\frac{\sqrt{2}}{4}\left[1-\Delta^2\right]^{-1}\left\{[1+\Delta]^{\frac{1}{2}} + [1-\Delta]^{\frac{1}{2}}\right\} - \frac{\pi}{2(1-\alpha)^2} \\
&= \pi\frac{\sqrt{2}}{4}(1-\alpha)^2\left\{[1+\Delta]^{\frac{1}{2}} + [1-\Delta]^{\frac{1}{2}}\right\} - \frac{\pi}{2(1-\alpha)^2} \\
&= \frac{\pi}{2(1-\alpha)^2}\left\{\frac{\sqrt{2}}{2}\left\{[1+\Delta]^{\frac{1}{2}} + [1-\Delta]^{\frac{1}{2}}\right\} - 1\right\}
\end{aligned} \tag{7.21}$$

Combining the first two terms in (7.4), for convenience,

$$\frac{1-\alpha}{2\sqrt{\alpha}}g_1+\frac{(1-\alpha)\sqrt{\alpha}}{2}g_2=\frac{1-\alpha}{2\sqrt{\alpha}}\{g_1+\alpha g_2\}$$

$$=\frac{1-\alpha}{2\sqrt{\alpha}}\frac{\pi}{2(1-\alpha)^2}\left\{\frac{\sqrt{2}}{2}\left\{[-\Delta+\alpha][1+\Delta]^{\frac{1}{2}}+[\Delta+\alpha][1-\Delta]^{\frac{1}{2}}\right\}+(1-\alpha)\right\}$$

$$=\frac{\pi}{4(1-\alpha)\sqrt{2}}\left\{\frac{\sqrt{2}}{2}\left\{[1-\Delta][1+\Delta]^{\frac{1}{2}}-[1-\alpha][1+\Delta]^{\frac{1}{2}}+[1+\Delta][1-\Delta]^{\frac{1}{2}}-[1-\alpha][1-\Delta]^{\frac{1}{2}}\right\}+(1-\alpha)\right\}$$

$$=\frac{\pi}{4(1-\alpha)\sqrt{2}}\left\{\frac{\sqrt{2}}{2}[1-\Delta]^{\frac{1}{2}}\left\{[1+\Delta]^{\frac{1}{2}}+[1-\Delta]^{\frac{1}{2}}\right\}-\frac{\sqrt{2}}{2}[1-\alpha]\left\{[1+\Delta]^{\frac{1}{2}}+[1-\Delta]^{\frac{1}{2}}\right\}+(1-\alpha)\right\}$$

$$=\frac{\pi}{4\sqrt{2}}\left\{\frac{\sqrt{2}}{2}[\mathrm{sign}(1-\alpha)-1]\left\{[1+\Delta]^{\frac{1}{2}}+[1-\Delta]^{\frac{1}{2}}\right\}+1\right\} \tag{7.22}$$

The third integral is

$$g_3=\int_0^1 \psi^{-1}\left[1+\zeta^2\right]d\zeta$$

$$=\int_0^1\left[(1+\alpha)^2\left[1-\zeta^2\right]^2+4\zeta^2\right]^{-1}\left[1+\zeta^2\right]d\zeta \tag{7.23}$$

The integrand has two conjugate pole pairs at $\pm\zeta_n$ with

$$\zeta_1=j\frac{1+\Delta}{|1-\alpha|}$$

$$\zeta_2=j\frac{1-\Delta}{|1-\alpha|}=-\zeta_1^{-1} \tag{7.24}$$

subject to (7.11) and (7.12).

Labelling the residues as R_n we have

$$R_1 = \left\{ \frac{d}{d\zeta}\left[(1-\alpha)^2\left[1-\zeta^2\right]^2 + 4\zeta^2 \right] \right\}^{-1} \left[1+\zeta^2\right] \Bigg|_{\zeta=\zeta_1}$$

$$= \frac{1}{4\zeta}\left[-(1-\alpha)^2\left[1-\zeta^2\right] + 2 \right]^{-1} \left[1+\zeta^2\right] \Bigg|_{\zeta=\zeta_1}$$

$$= j\frac{|1-\alpha|}{8\Delta(1+\Delta)}\left\{ 1 - \left[\frac{1+\Delta}{1-\alpha}\right]^2 \right\}$$

$$= j\frac{|1-\alpha|}{4\Delta(1+\Delta)}\left\{ 1 - \frac{1+\Delta}{(1-\alpha)^2} \right\} \tag{7.25}$$

Similarly for R_2 we need but change Δ to $-\Delta$ as in (7.24) in going from ζ_1 to ζ_2 giving

$$R_2 = -j\frac{|1-\alpha|}{8\Delta(1-\Delta)}\left\{ 1 - \left[\frac{1-\Delta}{1-\alpha}\right]^2 \right\}$$

$$= -j\frac{|1-\alpha|}{4\Delta(1-\Delta)}\left\{ 1 - \frac{1-\Delta}{(1-\alpha)^2} \right\} \tag{7.26}$$

The integrand can now be written as

$$h_3(\zeta) = \frac{R_1}{\zeta-\zeta_1} + \frac{R_1^*}{\zeta-\zeta_1^*} + \frac{R_2}{\zeta-\zeta_2} + \frac{R_2^*}{\zeta-\zeta_2^*}$$

$$= \frac{2R_1\zeta_1}{\zeta^2-\zeta_1^2} + \frac{2R_2\zeta_2}{\zeta^2-\zeta_2^2}$$

$$= \frac{2R_1\zeta_1}{\zeta^2+|\zeta_1|^2} + \frac{2R_2\zeta_2}{\zeta^2+|\zeta_2|^2} \tag{7.27}$$

where we have used the fact that the ζ_n and R_n are pure imaginary. Note also that $h_3(\zeta)$ is $0\left(\zeta^{-2}\right)$ as $\zeta \to \infty$ so there is no additional polynomial term in (7.27). Integrating we have [6, 7]

$$
\begin{aligned}
g_3 &= \int_0^1 h_3(\zeta)d\zeta \\
&= \frac{2R_1\zeta_1}{|\zeta_1|}\arctan\left(\frac{1}{|\zeta_1|}\right)+\frac{2R_2\zeta_2}{|\zeta_2|}\arctan\left(\frac{1}{|\zeta_2|}\right) \\
&= -\frac{|1-\alpha|}{2\Delta(1+\Delta)}\left\{1-\frac{1+\Delta}{(1-\alpha)^2}\right\}\arctan\left(\frac{1}{|\zeta_1|}\right)+\frac{|1-\alpha|}{2\Delta(1-\Delta)}\left\{1-\frac{1-\Delta}{(1-\alpha)^2}\right\}\arctan\left(\frac{1}{|\zeta_2|}\right) \\
&= \left[2\Delta|1-\alpha|\right]^{-1}\left\{-\frac{(1-\alpha)^2-(1+\Delta)}{1+\Delta}\arctan\left(\frac{1}{|\zeta_1|}\right)+\frac{(1-\alpha)^2-(1-\Delta)}{1-\Delta}\arctan\left(\frac{1}{|\zeta_1|}\right)\right\} \\
&= \left[2\Delta|1-\alpha|\right]^{-1}\left\{-\frac{(1-\alpha)^2(1-\Delta)-\left(1-\Delta^2\right)}{1-\Delta^2}\arctan\left(\frac{1}{|\zeta_1|}\right)+\frac{(1-\alpha)^2(1+\Delta)-(1-\Delta)^2}{1-\Delta^2}\arctan\left(\frac{1}{|\zeta_2|}\right)\right\} \\
&= \left[2|1-\alpha|\right]^{-1}\left\{\arctan\left(\frac{1}{|\zeta_1|}\right)+\arctan\left(\frac{1}{|\zeta_2|}\right)\right\} \\
&= \frac{\pi}{4|1-\alpha|}
\end{aligned} \tag{7.28}
$$

with the last result following from the reciprocal relation of $|\zeta_1|$ and $|\zeta_2|$.

Now in (7.4) we have

$$
-(1-\alpha)g_3 = -\frac{\pi}{4}\text{sign}(1-\alpha) \tag{7.29}
$$

Combining with (7.22) gives the result for (7.4) as

$$
\begin{aligned}
g(1) &= \frac{1-\alpha}{2\sqrt{\alpha}}g_1+\frac{(1-\alpha)\sqrt{\alpha}}{2}g_2-(1-\alpha)g_3 \\
&= \frac{\pi}{4}\left\{\frac{\sqrt{2}}{2\sqrt{\alpha}}\left[\text{sign}(1-\alpha)-1\right]\left\{[1+\Delta]^{\frac{1}{2}}+[1-\Delta]^{\frac{1}{2}}\right\}+\alpha^{-\frac{1}{2}}-\text{sign}(1-\alpha)\right\}
\end{aligned} \tag{7.30}
$$

Note for $0<\alpha<1$ we have a simpler result

$$
g(1) = \frac{\pi}{4}\left\{\alpha^{-\frac{1}{2}}-1\right\} \tag{7.31}
$$

Note that

$$
\lim_{\alpha\to 1} g(1) = 0 \tag{7.32}
$$

and that

$$g(1) > 0 \quad \text{for } 0 < \alpha < 1 \tag{7.33}$$

From (5.24) we have

$$\begin{aligned}\frac{V_1(1)}{V_0} &= [2\alpha]^{-\frac{1}{2}}\left[\cos(g(1)) - \sin(g(1))\right] \\ &= \alpha^{-\frac{1}{2}} \cos\left(g(1) + \frac{\pi}{4}\right)\end{aligned} \tag{7.34}$$

For the case of $0 < \alpha < 1$ we have

$$\begin{aligned}\frac{V_1(1)}{V_0} &= \alpha^{-\frac{1}{2}} \cos\left(\frac{\pi}{4\sqrt{\alpha}}\right) \\ &= [2\alpha]^{-\frac{1}{2}}\left[\cos\left(\frac{\pi}{4}\left(\alpha^{-\frac{1}{2}} - 1\right)\right) - \sin\left(\frac{\pi}{4}\left(\alpha^{-\frac{1}{2}} - 1\right)\right)\right]\end{aligned} \tag{7.35}$$

For α near (but below) 1 we have

$$\begin{aligned}\frac{V_1(1)}{V_0} &= [2\alpha]^{-\frac{1}{2}}\left[1 - \frac{\pi}{4}\left(\alpha^{-\frac{1}{2}} - 1\right) + O\left(\left(\alpha^{-\frac{1}{2}} - 1\right)^2\right)\right] \\ &= \frac{1}{\sqrt{2}}\left[1 + \left[1 - \frac{\pi}{4}\right]\left(\alpha^{-\frac{1}{2}} - 1\right) + O\left(\left(\alpha^{-\frac{1}{2}} - 1\right)^2\right)\right] \text{ as } \alpha \to 1\end{aligned} \tag{7.36}$$

Note that decreasing α from 1 initially increases $V_1(1)$, which may be desirable. The solution goes to a maximum at $\alpha \simeq .83$ where $V_1/V_0 \simeq .714$. This is an increase from $V_1/V_0 \simeq .707$ at $\alpha = 1$. The increase is not very much.

VIII. Case of Hybrid Wave Launcher

Now consider a special case consisting of two sections. In the first section let us transform V_1 from V_0 to $V_0\sqrt{2}$ without coupling to the second conductor. Here the first conductor is confined to a small height (ideally zero) so that there is no coupling to the second conductor. The section is the wave launcher proper in which $V_0\sqrt{2}$ is reduced to V_0 in the aperture plane.

A. Decoupled Transmission-Line-Transformer Section

For this section let the normalized coordinate be ξ with

$$0 \le \xi \le 1 \tag{8.1}$$

Choose the normalized impedance matrix as

$$\begin{aligned} \left(F_{n,m}(\xi)\right) &\equiv \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} \\ \left(H_{n,m}(\xi)\right) &\equiv \left(F_{n,m}(\xi)\right)^{\frac{1}{2}} = \begin{pmatrix} v^{\frac{1}{2}} & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{8.2}$$

with positive square root. Let

$$\begin{aligned} v\big|_{\xi=0} &= \alpha > 0 \\ v\big|_{\xi=1} &= 1 \end{aligned} \tag{8.3}$$

Note the absence of the off-diagonal terms. This means no coupling between V_1 and V_2 on the two-conductor transmission line. The variation of v over the range in (8.1) is not critical for the present analysis although as a practical matter one may wish that this variation be monotonic.

Now in $\left(H_{n,m}(\xi)\right)$ we have

$$H_{1,2}(\xi) = 0 \tag{8.4}$$

implying in (5.15)

$$\begin{aligned} & h(\xi) = 0 \ , \ g(1) = 0 \\ & \left(\Phi_{n,m}(1)\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{8.5}$$

Then we have from (3.2) and (3.3)

$$\left(V_n(\xi)\right)\Big|_{\xi=1} = \left(H_{n,m}(\xi)\right)\Big|_{\xi=1} \cdot \left(\Phi_{n,m}(\xi)\right)\Big|_{\xi=1} \cdot \left(H_{n,m}(\xi)\right)^{-1}\Big|_{\xi=0} \cdot \left(V_n(\xi)\right)\Big|_{\xi=0}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \alpha^{-\frac{1}{2}} & 0 \\ 0 & 1 \end{pmatrix} \cdot \left(V_n(\xi)\right)\Big|_{\xi=0}$$
$$= \begin{pmatrix} \alpha^{-\frac{1}{2}} & 0 \\ 0 & 1 \end{pmatrix} \cdot \left(V_n(\xi)\right)\Big|_{\xi=0} \tag{8.6}$$

With our initial condition as in (5.17) we have

$$\left(V_n(\xi)\right)\Big|_{\xi=1} = \begin{pmatrix} \alpha^{-\frac{1}{2}} & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} V_0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{V_0}{\sqrt{\alpha}} \\ 0 \end{pmatrix} \tag{8.7}$$

Our transformer section has then increased the voltage on conductor 1 by a factor $\alpha^{-\frac{1}{2}}$ and has left zero potential on conductor 2.

B. Wave-Launcher Section

At $\xi = 1$ we begin a new section of the two-conductor system. Relabel the coordinate here as ζ with

$$0 \le \zeta \le 1 \tag{8.8}$$

where $\zeta = 0$ corresponds to $\xi = 1$. Choose the normalized impedance matrix as

$$\left(F_{n,m}(\zeta)\right) = \begin{pmatrix} 1 & \zeta \\ \zeta & 1 \end{pmatrix} = \left(H_{n,m}(\zeta)\right)^2$$
$$\left(H_{n,m}(\zeta)\right) = \begin{pmatrix} H_{1,1}(\zeta) & H_{1,2}(\zeta) \\ H_{2,1}(\zeta) & H_{2,2}(\zeta) \end{pmatrix} \tag{8.9}$$

with the elements of the square root as discussed in appendix C. In this case we have

$$H_{2,2}(\zeta) - H_{1,1}(\zeta) = 0$$
$$h(\zeta) = 0, \quad g(1) = 0 \tag{8.10}$$
$$\left(\Phi_{n,m}(1)\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This is also the case considered in [1].

The solution has

$$\left(V_n(\zeta)\right)\Big|_{\zeta=1} = \left(H_{n,m}(\zeta)\right)\Big|_{\zeta=1} \cdot \left(\Phi_{n,m}(\zeta)\right)\Big|_{\zeta=1} \cdot \left(H_{n,m}(\zeta)\right)^{-1}\Big|_{\zeta=0} \cdot \left(V_n(\zeta)\right)\Big|_{\zeta=0}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \left(V_n(\zeta)\right)\Big|_{\zeta=0}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \left(V_n(\zeta)\right)\Big|_{\zeta=0} \tag{8.11}$$

Noting that at $\xi = 1$ which is the same as $\zeta = 0$ we have

$$\left(V_n(\xi)\right)\Big|_{\xi=1} = \left(V_n(\zeta)\right)\Big|_{\zeta=0} = \begin{pmatrix} \frac{V_0}{\sqrt{\alpha}} \\ 0 \end{pmatrix} \tag{8.12}$$

This result relies on the fact that there is no (high-frequency) reflection at the interface since

$$\left(F_{n,m}(\xi)\right)\Big|_{\xi=1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \left(F_{n,m}(\zeta)\right)\Big|_{\zeta=0} \tag{8.13}$$

Then we have at $\zeta = 1$

$$\left(V_n(\zeta)\right)\Big|_{\zeta=1} = \frac{V_0}{\sqrt{2\alpha}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{8.14}$$

so that

$$\frac{V_1(\zeta)}{V_0}\Bigg|_{\zeta=1} = \frac{1}{\sqrt{2\alpha}} \tag{8.15}$$

This can be set to be 1 provided

$$\alpha = \frac{1}{2} \tag{8.16}$$

This gives a unity high-frequency transfer function through the hybrid wave launcher.

C. Some Comments

Remember that the present analysis is based on a high-frequency approximation. As discussed in section 5 there is a certain arbitrariness in the definition of the coordinate along the transmission line (z or ζ or ξ). By a transformation of ζ to some $f(\zeta)$ varying between 0 and 1 one gets the same answer for the high-frequency transfer function for source voltage to wave launched from the aperture plane.

Here two sections of the wave launcher are considered. How long should each be? By a scaling argument one can normalize frequency or time by a characteristic length, say the length of one of these sections. Perhaps one can better use the total length ($\xi = 0$ to $\zeta = 1$) as a scaling length. Even so this leaves the relative length of the two sections as an open question for optimization. The high-frequency approximation is not adequate for this purpose.

IX. Concluding Remarks

Now we have a few canonical forms of wave launcher with analytic high-frequency solutions. By comparing the results for the different cases one has some appreciation of how various aspects of the launcher profiles influence the high-frequency behavior. Perhaps other solutions can also be obtained.

Having the high-frequency solution does not tell us everything. There are intermediate frequencies/times of concern. Also note that a high-frequency transmission-line model breaks down when the wavelength becomes less than the cross section of the unit cell.

Appendix A: Functions of Diagonalizable Matrices

Consider an N x N square matrix $\left(a_{n,m}\right)$ with eigenvalues λ_β, right eigenvectors $\left(r_n\right)_\beta$, and left eigenvectors $\left(\ell_n\right)_\beta$ defined by

$$\begin{aligned}&\left(a_{n,m}\right)\cdot\left(r_n\right)_\beta = \lambda_\beta\left(r_n\right)_\beta\\&\left(\ell_n\right)_\beta\cdot\left(a_{n,m}\right) = \lambda_\beta\left(\ell_n\right)_\beta\end{aligned} \tag{A.1}$$

with biorthogonalization condition

$$\begin{aligned}&\left(\ell_n\right)_{\beta_1}\cdot\left(r_n\right)_{\beta_2} = 1_{\beta_1,\beta_2} \equiv \begin{cases}1 \text{ } for \text{ } \beta_1 = \beta_2\\0 \text{ } for \text{ } \beta_1 \neq \beta_2\end{cases}\\&\beta_1, \beta_2 = 1, 2, \ldots, N\end{aligned} \tag{A.2}$$

A sufficient condition for (A.2) is that the N eigenvalues λ_β be all distinct. As discussed in [3,5] all we really need is N linearly independent eigenvectors (say the $\left(r_n\right)_\beta$). This is also called a matrix of simple structure. However, this is often not a necessary condition. We have the characteristic polynomial

$$\begin{aligned}\Delta(\lambda) &\equiv \det\left(\left(a_{n,m}\right) - \lambda\left(1_{n,m}\right)\right)\\&= \sum_{p=0}^{N} a_p\lambda^p\\&= a_N\prod_{\beta=1}^{N}\left(\lambda - \lambda_\beta\right)\\&= \prod_{\beta=1}^{N}\left(\lambda_\beta - \lambda\right)\end{aligned} \tag{A.3}$$

with some of the coefficients

$$a_N = (-1)^N$$

$$a_{N-1} = (-1)^{N-1} tr\left(\left(a_{n,m}\right)\right)$$

$$= (-1)^{N-1} \sum_{p=1}^{N} a_{p,p}$$

$$= (-1)^{N-1} \sum_{\beta=1}^{N} \lambda_\beta$$

$$a_o = \det\left(\left(a_{n,m}\right)\right)$$

$$= \prod_{\beta=1}^{N} \lambda_\beta \tag{A.4}$$

and the general a_p given in [3,8]. Setting the characteristic polynomial to zero gives the characteristic equation for determination of the eigenvalues

$$\Delta\left(\lambda_\beta\right) = 0 \tag{A.5}$$

The matrix now has the dyadic representation

$$\left(a_{n,m}\right) = \sum_{\beta=1}^{N} \lambda_\beta (r_n)_\beta (\ell_n)_\beta \tag{A.6}$$

There are matrices which are not so diagonalizable and can be handled by the Jordan form [5]. For present purposes we only consider matrices diagonalizable as above.

Consider some function f of a complex variable λ defined by a power series

$$f(\lambda) = \sum_{p=0}^{\infty} f_p \lambda^p \ for\, |\lambda| < \lambda_0$$

$$\lambda_0 \equiv \text{radius of convergence (series not converging for } |\lambda| > \lambda_0\text{)} \tag{A.7}$$

Then we have a way to define a function of a square matrix by [5]

$$f\left(\left(a_{n,m}\right)\right) \equiv \sum_{p=0}^{\infty} f_p \left(a_{n,m}\right)^p \quad \text{for all } \left|\lambda_\beta\right| < \lambda_0 \tag{A.8}$$

since integer powers of a matrix are well defined. Using the dyadic form in (A.6) we have

$$(a_{n,m})^p = \sum_{\beta=1}^{N} \lambda_\beta^p \, (r_n)_\beta \, (\ell_n)_\beta \tag{A.9}$$

from which we obtain

$$f\big((a_{n,m})\big) = \sum_{\beta=1}^{N} f(\lambda_\beta)(r_n)_\beta(\ell_n)_\beta \tag{A.10}$$

Here we have used (A.7) to identify the infinite series for each of the $f(\lambda_\beta)$. While this is limited to $|\lambda| < \lambda_0$, we can use (A.10) for our (extended) definition provided $f(\lambda)$ is defined some other way for $|\lambda| \geq \lambda_0$. In this form we can consider fractional powers, logarithms, etc. of matrices with N linearly independent eigenvectors provided we take care in defining the intended branch of the multiple valued function with which we are dealing.

Some special cases of interest are

$$\begin{aligned}
(a_{n,m})^{-1} &= \sum_{\beta=1}^{N} \lambda_\beta^{-1} \, (r_n)_\beta \, (\ell_n)_\beta \quad \text{(inverse)} \\
(1_{n,m}) &= \sum_{\beta=1}^{N} (r_n)_\beta(\ell_n)_\beta \\
&= \sum_{\beta=1}^{N} (\ell_n)_\beta(r_n)_\beta \quad \text{(identity)} \\
(a_{n,m})^T &= \sum_{\beta=1}^{N} \lambda_\beta \, (\ell_n)_\beta \, (r_n)_\beta \quad \text{(transpose)}
\end{aligned} \tag{A.11}$$

At this point it can be noted that the Cayley-Hamilton theorem states [5]

$$\Delta\big((a_{n,m})\big) = \sum_{p=0}^{N} a_p \, (a_{n,m})^p = (0_{n,m}) \tag{A.12}$$

For this result the matrix need not be diagonalizable. This is readily related to matrices as in (A.6) by

$$\Delta\big((a_{n,m})\big) = \sum_{p=0}^{N} \Delta(\lambda_\beta) \, (r_n)_\beta \, (\ell_n)_\beta = (0_{n,m}) \tag{A.13}$$

using the result (A.5).

If the matrix is symmetric we have

$$\begin{gathered}
\left(a_{n,m}\right)^T \equiv \left(a_{n,m}\right) \\
a_{n,m} \equiv a_{m,n} \\
\left(a_{n,m}\right)\cdot\left(x_n\right)_\beta = \left(x_n\right)_\beta \cdot \left(a_{n,m}\right) = \lambda_\beta \left(x_n\right)_\beta \\
\left(r_n\right)_\beta = \left(\ell_n\right)_\beta \equiv \left(x_n\right)_\beta \\
\left(x_n\right)_{\beta_1}\cdot\left(x_n\right)_{\beta_2} = 1_{\beta_1,\beta_2} \\
\left(a_{n,m}\right) = \sum_{\beta=1}^{N} \lambda_\beta \left(x_n\right)_\beta \left(x_n\right)_\beta \\
f\left(\left(a_{n,m}\right)\right) = \sum_{\beta=1}^{N} f\left(\lambda_\beta\right) \left(x_n\right)_\beta \left(x_n\right)_\beta
\end{gathered} \tag{A.14}$$

so that only one set (orthonormal) of eigenvectors is needed. Again a sufficient condition is that the $\left(x_n\right)_\beta$ are a set of N linearly independent eigenvectors. Note that a function of a symmetric matrix is itself symmetric.

If the matrix is Hermetian we have

$$\begin{gathered}
\left(a_{n,m}\right)^\dagger \equiv \left(a_{n,m}\right) \\
\dagger \equiv T^* \\
a_{n,m} \equiv a^*_{n,m} \\
\left(a_{n,m}\right)\cdot\left(x_n\right)_\beta = \lambda_\beta \left(x_n\right)_\beta \\
\left(x_n\right)^*_\beta \cdot \left(a_{n,m}\right) = \lambda_\beta \left(x_n\right)^*_\beta \\
\lambda_\beta \text{ real for } \beta = 1, 2, \ldots, N \\
\left(x_n\right)^*_{\beta_1}\cdot\left(x_n\right)_{\beta_2} = 1_{\beta_1,\beta_2} \\
\left(a_{n,m}\right) = \sum_{\beta=1}^{N} \lambda_\beta \left(x_n\right)_\beta \left(x_n\right)^*_\beta \\
f\left(\left(a_{n,m}\right)\right) = \sum_{\beta=1}^{N} f\left(\lambda_\beta\right) \left(x_n\right)_\beta \left(x_n\right)^*_\beta
\end{gathered} \tag{A.15}$$

Assuming that f(λ) is real for real λ then such a function of a Hermetian matrix is itself a Hermetian matrix. Again only one set of eigenvectors is needed. Note that Hermetian matrices are always diagonalizable [8].

For convenient reference we have some special cases of matrix functions as

$$f\left(\lambda\left(1_{n,m}\right)\right)=f(\lambda)\left(1_{n,m}\right) \qquad \text{(identity)}$$

$$f\left(\begin{pmatrix}\lambda_{1_{\lambda_1}} & & 0\\ & & \\ 0 & & \lambda_N\end{pmatrix}\right)=\begin{pmatrix}f(\lambda_1)_{f(\lambda_2)} & & 0\\ & & \\ 0 & & f(\lambda_N)\end{pmatrix} \qquad \text{(diagonal matrix)} \tag{A.16}$$

For 2 x 2 matrices we have the special cases (from [1, 4]

$$\begin{aligned}
e^{\lambda\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}} &= \begin{pmatrix}\cos(\lambda) & \sin(\lambda)\\ -\sin(\lambda) & \cos(\lambda)\end{pmatrix}\\
&= \cos(\lambda)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\sin(\lambda)\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\\
e^{\lambda\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}} &= \begin{pmatrix}\cosh(\lambda) & \sinh(\lambda)\\ -\sinh(\lambda) & \cosh(\lambda)\end{pmatrix}\\
&= \cosh(\lambda)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\sinh(\lambda)\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}
\end{aligned} \tag{A.17}$$

Note that these last cases both have determinant equal to 1. Using these as canonical forms trigonometric, hyperbolic, and logarithmic functions (and inverse functions) can be found for these two matrices.

Appendix B: Square Root of a Symmetric 2 x 2 Matrix

Let us now work out in detail the square root of a 2 x 2 matrix under certain restrictions. The general form is given in (A.10) where f is interpreted as square root. This requires one to define the proper branch of this function. For real and positive λ_β this can be taken as the positive square root by convention. This can be extended to complex λ_β by mapping phase (or arg) from $-\pi$ to π into $-\pi/2$ to $\pi/2$. Note that, if desired, various conventions ($\pm$) can be followed for the various λ_β, since when the resulting matrix is squared the original matrix is reproduced. So for diagonalizable matrices there are 2^N different values of the square root (assuming all are non zero). For 2 x 2 matrices there are four square roots, but here we consider only one, the positive or p.r. (positive real) square root since we are normally dealing with impedance or admittance matrices.

Beginning with a general 2 x 2 matrix we have

$$\left(a_{n,m}\right)^{\frac{1}{2}} = a_{2,2}^{\frac{1}{2}} \begin{pmatrix} \dfrac{a_{1,1}}{a_{2,2}} & \dfrac{a_{1,2}}{a_{2,2}} \\ \dfrac{a_{2,1}}{a_{2,2}} & 1 \end{pmatrix}^{\frac{1}{2}} \tag{B.1}$$

so that if $a_{2,2}$ is non zero and its square root is defined only three general matrix elements need be considered. If $\left(a_{n,m}\right)$ is also symmetric then we only need to consider matrices of the form

$$\left(F_{n,m}\right) = \begin{pmatrix} \nu & \zeta \\ \zeta & 1 \end{pmatrix} \tag{B.2}$$

Here our concern is for real elements, and constraining the determinant real and non-negative gives

$$\begin{aligned} \det\left(\left(F_{n,m}\right)\right) &= \nu - \zeta^2 \geq 0 \\ \nu &\geq \zeta^2 \end{aligned} \tag{B.3}$$

Now define

$$\left(H_{n,m}\right) \equiv \left(F_{n,m}\right)^{\frac{1}{2}} \tag{B.4}$$

with the square root positive as discussed above. Note that

$$\det\left(\left(H_{n,m}\right)\right) = H_{1,1}\,H_{2,2} - H_{1,2}^2$$
$$= \left[\det\left(\left(F_{n,m}\right)\right)\right]^{\frac{1}{2}} \qquad \text{(positive square root)}$$
$$= \left[\nu - \zeta^2\right]^{\frac{1}{2}}$$
$$\left(H_{n,m}\right) = \left(H_{n,m}\right)^T$$
$$H_{2,1} = H_{1,2} \tag{B.5}$$

where we have used the fact that a function (square root) of a symmetric diagonalizable matrix is itself symmetric and that $\det\left(\left(F_{n,m}\right)\right)$ is non negative.

To diagonalize $\left(F_{n,m}\right)$we have the eigenvalues as

$$\lambda_1\lambda_2 = \det\left(\left(F_{n,m}\right)\right) = \nu - \zeta^2$$
$$\lambda_1\lambda_2 = tr\left(\left(F_{n,m}\right)\right) = 1 + \nu \tag{B.6}$$

This has the solution

$$\lambda_{\frac{1}{2}} = \frac{1 + \nu \pm \psi^{\frac{1}{2}}}{2}$$
$$\psi \equiv (1 - \nu)^2 + 4\zeta^2 \tag{B.7}$$

For the eigenvectors begin with

$$\left(F_{n,m}\right)\cdot\left(x_n\right)_\beta = \lambda_\beta\left(x_n\right)_\beta \tag{B.8}$$

which in component form is

$$\left(\nu - \lambda_\beta\right)x_{1;\beta} + \zeta x_{2;\beta} = 0$$
$$\zeta x_{1;\beta} + \left(1 - \lambda_\beta\right)x_{2;\beta} = 0 \tag{B.9}$$

This gives

$$\frac{x_{1;\beta}}{x_{2;\beta}} = \frac{\zeta}{\lambda_\beta - \nu} = \frac{\lambda_\beta - 1}{\zeta} \tag{B.10}$$

This establishes the relative sizes of the two elements of the eigenvectors. So we take

$$(x_n)_\beta = a_\beta \begin{pmatrix} \zeta \\ \lambda_\beta - \nu \end{pmatrix} = b_\beta \begin{pmatrix} \lambda_\beta - 1 \\ \zeta \end{pmatrix} \tag{B.11}$$

The biorthonormalization condition gives

$$\begin{aligned} (x_n)_\beta \cdot (x_n)_\beta &= 1 = a_\beta^2\left[(\lambda_\beta - \nu)^2 + \zeta^2\right] \\ &= b_\beta^2\left[(\lambda_\beta - 1)^2 + \zeta^2\right] \end{aligned} \tag{B.12}$$

Again with positive square-root convention (and some algebra)

$$\begin{aligned} a_{\frac{1}{2}} &= \left\{\left(\lambda_{\frac{1}{2}} - \nu\right)^2 + \zeta^2\right\}^{-\frac{1}{2}} \\ &= \left\{\frac{1}{2}\left[\psi \pm (1-\nu)\,\psi^{\frac{1}{2}}\right]\right\}^{-\frac{1}{2}} \\ b_{\frac{1}{2}} &= \pm\left\{\left(\lambda_{\frac{1}{2}} - 1\right)^2 + \zeta^2\right\}^{-\frac{1}{2}} \\ &= \pm\left\{\frac{1}{2}\left[\psi \mp (1-\nu)\psi^{\frac{1}{2}}\right]\right\}^{-\frac{1}{2}} \\ b_{\frac{1}{2}} &= \pm\, a_{\frac{2}{1}} \end{aligned} \tag{B.13}$$

The $\pm$ with b_β is chosen for consistency between the a_β and b_β forms in defining the eigenvectors.

Combining the above results gives various forms for the eigenvectors as

$$\begin{aligned} (x_n)_{\frac{1}{2}} &= \left\{\frac{1}{2}\left[\psi \pm (1-\nu)\psi^{\frac{1}{2}}\right]\right\}^{-\frac{1}{2}} \begin{pmatrix} \zeta \\ \dfrac{1-\nu \pm \psi^{\frac{1}{2}}}{2} \end{pmatrix} \quad (a_\beta \text{ form}) \\ &= \left\{\frac{1}{2}\left[\psi \mp (1-\nu)\psi^{\frac{1}{2}}\right]\right\}^{-\frac{1}{2}} \begin{pmatrix} \dfrac{1-\nu \mp \psi^{\frac{1}{2}}}{2} \\ \pm\zeta \end{pmatrix} \quad (b_\beta \text{ form}) \end{aligned} \tag{B.14}$$

In a more symmetrical form we have

$$(x_n)_{\substack{1\\2}} = \sqrt{2}\zeta \begin{pmatrix} \left[\psi \pm (1-\nu)\psi^{\frac{1}{2}}\right]^{-\frac{1}{2}} \\ \pm\left[\psi \mp (1-\nu)\psi^{\frac{1}{2}}\right]^{-\frac{1}{2}} \end{pmatrix} \tag{B.15}$$

With ζ assumed positive we can note the special case for $\nu = 1$ as

$$(x_n)_{\substack{1\\2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} \text{ for } \nu = 1 \tag{B.16}$$

so that the eigenvectors are seen to go to a simple form that is obviously an orthonormal set in this limit. After some algebra one can verify that the two vectors in (B.15) satisfy

$$(x_n)_{\beta_1} \cdot (x_n)_{\beta_2} = 1_{\beta_1,\beta_2} \tag{B.17}$$

i.e., also form an orthonormal set.

Now we can write

$$\begin{aligned} (F_{n,m}) &= \begin{pmatrix} \nu & \zeta \\ \zeta & 1 \end{pmatrix} = \sum_{\beta=1}^{2} \lambda_\beta (x_n)_\beta (x_n)_\beta \\ (H_{n,m}) &= \begin{pmatrix} H_{1,1} & H_{1,2} \\ H_{2,1} & H_{2,2} \end{pmatrix} = \sum_{\beta=1}^{2} \lambda_\beta^{\frac{1}{2}} (x_n)_\beta (x_n)_\beta \end{aligned} \tag{B.18}$$

The dyadic products of eigenvectors are

$$\begin{aligned} (x_n)_{\substack{1\\2}} (x_n)_{\substack{1\\2}} &= 2\zeta^2 \begin{pmatrix} \left[\psi \pm (1-\nu)\psi^{\frac{1}{2}}\right]^{-1} & \pm\frac{1}{2\zeta\psi^{\frac{1}{2}}} \\ \pm\frac{1}{2\zeta\psi^{\frac{1}{2}}} & \left[\psi \mp (1-\nu)\psi^{\frac{1}{2}}\right]^{-1} \end{pmatrix} \\ &= 2\zeta^2 \psi^{-\frac{1}{2}} \begin{pmatrix} \left[\psi^{\frac{1}{2}} \pm (1-\nu)\right]^{-1} & \pm[2\zeta]^{-1} \\ \pm[2\zeta]^{-1} & \left[\psi^{\frac{1}{2}} \mp (1-\nu)\right]^{-1} \end{pmatrix} \end{aligned} \tag{B.19}$$

and the eigenvalues are

$$\lambda_{\substack{1\\2}} = \frac{1+\nu \pm \psi^{\frac{1}{2}}}{2}$$

$$\psi = (1-\nu)^2 + 4\zeta^2 \tag{B.20}$$

One can even write out the components of $\left(H_{n,m}\right)$ as

$$H_{1,1} = 2\zeta^2\psi^{-\frac{1}{2}}\left\{\left[\frac{1+\nu+\psi^{\frac{1}{2}}}{2}\right]^{\frac{1}{2}}\left[\psi^{\frac{1}{2}}+(1-\nu)\right]^{-1}+\left[\frac{1+\nu-\psi^{\frac{1}{2}}}{2}\right]^{\frac{1}{2}}\left[\psi^{\frac{1}{2}}-(1-\nu)\right]^{-1}\right\}$$

$$H_{2,2} = 2\zeta^2\psi^{-\frac{1}{2}}\left\{\left[\frac{1+\nu+\psi^{\frac{1}{2}}}{2}\right]^{\frac{1}{2}}\left[\psi^{\frac{1}{2}}-(1-\nu)\right]^{-1}+\left[\frac{1+\nu-\psi^{\frac{1}{2}}}{2}\right]^{\frac{1}{2}}\left[\psi^{\frac{1}{2}}+(1-\nu)\right]^{-1}\right\} \tag{B.21}$$

$$H_{1,2} = H_{2,1} = \zeta\psi^{-\frac{1}{2}}\left\{\left[\frac{1+\nu+\psi^{\frac{1}{2}}}{2}\right]^{\frac{1}{2}}-\left[\frac{1+\nu-\psi^{\frac{1}{2}}}{2}\right]^{\frac{1}{2}}\right\}$$

This formally completes the square root. There are certain combinations which one can form for convenience

$$\begin{aligned} H_{2,2}+H_{1,1} &= \left[\frac{1+\nu+\psi^{\frac{1}{2}}}{2}\right]^{\frac{1}{2}}+\left[\frac{1+\nu-\psi^{\frac{1}{2}}}{2}\right]^{\frac{1}{2}} \\ &= tr\left(\left(H_{n,m}\right)\right) = \lambda_1^{\frac{1}{2}}+\lambda_2^{\frac{1}{2}} \end{aligned}$$

$$\tag{B.22}$$

$$\begin{aligned} H_{2,2}-H_{1,1} &= (1-\nu)\psi^{-\frac{1}{2}}\left\{\left[\frac{1+\nu+\psi^{\frac{1}{2}}}{2}\right]^{\frac{1}{2}}-\left[\frac{1+\nu-\psi^{\frac{1}{2}}}{2}\right]^{\frac{1}{2}}\right\} \\ &= (1-\nu)\psi^{-\frac{1}{2}}\left\{\lambda_1^{\frac{1}{2}}-\lambda_2^{\frac{1}{2}}\right\} \end{aligned}$$

Further combinations are

$$
\begin{aligned}
H_{2,2}^2 - H_{1,1}^2 &= \left(H_{2,2} + H_{1,1}\right)\left(H_{2,2} - H_{1,1}\right) = 1 - \nu \\
H_{1,2} = H_{2,1} &= \frac{\zeta}{1-\nu}\left(H_{2,2} - H_{1,1}\right) = \zeta\left(H_{2,2} + H_{1,1}\right)^{-1} \\
\det\left(\left(H_{n,m}\right)\right) &= H_{1,1}H_{2,2} - H_{1,2}^2 = \lambda_1^{\frac{1}{2}}\lambda_2^{\frac{1}{2}} = \left[\nu - \zeta^2\right]^{\frac{1}{2}} \\
H_{1,2}^2 &= \frac{\zeta^2}{\psi}\left\{1 + \nu - 2\left[\nu - \zeta^2\right]^{\frac{1}{2}}\right\}
\end{aligned}
\tag{B.23}
$$

Appendix C: Square Root of $\begin{pmatrix} 1 & \zeta \\ \zeta & 1 \end{pmatrix}$

As a special case of interest let us set ν as 1, corresponding to the problem in [1]. Then the results of appendix B simplify considerably. Summarizing for $\left(F_{n,m}\right)$ we have

$$\begin{aligned}
\left(F_{n,m}\right) &= \begin{pmatrix} 1 & \zeta \\ \zeta & 1 \end{pmatrix} \\
\det\left(\left(F_{n,m}\right)\right) &= \lambda_1 \lambda_2 = 1-\zeta^2 \geq 0 \\
tr\left(\left(F_{n,m}\right)\right) &= \lambda_1 + \lambda_2 = 2 \\
\psi &= 4\zeta^2 \\
\lambda_{\frac{1}{2}} &= 1 \pm \zeta \\
\left(x_n\right)_{\frac{1}{2}} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} \\
\left(x_n\right)_{\frac{1}{2}} \left(x_n\right)_{\frac{1}{2}} &= \frac{1}{2} \begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}
\end{aligned} \tag{C.1}$$

The square root (positive) then has

$$\begin{aligned}
\left(H_{n,m}\right) &\equiv \left(F_{n,m}\right)^{\frac{1}{2}} \\
\det\left(\left(H_{n,m}\right)\right) &= \lambda_1^{\frac{1}{2}} \lambda_2^{\frac{1}{2}} = \left[1-\zeta^2\right]^{\frac{1}{2}} \geq 0 \\
tr\left(\left(H_{n,m}\right)\right) &= \lambda_1^{\frac{1}{2}} + \lambda_2^{\frac{1}{2}} = [1+\zeta]^{\frac{1}{2}} + [1-\zeta]^{\frac{1}{2}} \\
\left(H_{n,m}\right) &= \frac{1}{2}[1+\zeta]^{\frac{1}{2}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{2}[1-\zeta]^{\frac{1}{2}} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} [1+\zeta]^{\frac{1}{2}} + [1-\zeta]^{\frac{1}{2}} & [1+\zeta]^{\frac{1}{2}} - [1-\zeta]^{\frac{1}{2}} \\ [1+\zeta]^{\frac{1}{2}} - [1-\zeta]^{\frac{1}{2}} & [1+\zeta]^{\frac{1}{2}} + [1-\zeta]^{\frac{1}{2}} \end{pmatrix} \\
H_{2,2} &= H_{1,1} \\
H_{2,1} &= H_{1,2} = \frac{\zeta}{2} H_{1,1}^{-1}
\end{aligned} \tag{C.2}$$

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