Sensor and Simulation Notes

Note 310

28 March 1988

Dual Sources on Boundaries

Carl E. Baum

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Abstract

Duality is an important concept in electromagnetics involving a symmetry in the equations between electric and magnetic parameters. Using the combined field, one has a compact form for stating various electromagnetic theorems. In this paper this is first extended to a dual equivalence principle in which electric and magnetic surface currents are interchanged on a surface with a corresponding interchange of electric and magnetic fields. Then we consider production of electrostatic and magnetostatic fields using electric potential and electric surface current density respectively to make the two fields have the same spatial distribution in a volume.

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surface currents

I. Introduction

By duality we mean the symmetry in the Maxwell equations between the electric and magnetic fields and between other electric and magnetic parameters. This duality (as in Section 2) can be used to define combined fields, currents, etc. which are particular linear combinations of dual parameters [5]. The combined field can be used to generalize the usual reciprocity and Poynting vector theorems [4]. It furthermore simplifies the consideration of the Babinet principle and the definition of complementary (dual) planar structures [3].

This paper extends the concept of dual electromagnetic structures to include devices intended to produce certain electromagnetic field distributions. In particular we consider sources on a boundary surface S which produce dual electromagnetic fields in the enclosed volume V. First, we consider the general dual equivalence principle involving both electric and magnetic surface currents on S. Then restricting our attention to only electric boundary conditions (potentials or electric surface currents) on S we find conditions on these for dual electric and magnetic fields in V. Explicit formulas are exhibited for special cases involving circular cylinders and spheres.

II. Dual Equivalence Principle

Write the Maxwell equations in the combined-field form [1]

$$\left[\nabla \times - \frac{qj}{c} \frac{\partial}{\partial t}\right] \vec{E}_q = qj \; Z_o \; \vec{J}_q$$

$$\vec{E}_q \equiv \vec{E} + qj \; Z_o \; \vec{H} \quad \text{(combined field)}$$

$$\vec{J}_q \equiv \vec{J} + \frac{qj}{Z_o} \vec{J}_h \quad \text{(combined current density)}$$

$$q = \pm 1 \quad \text{(separation index)}$$

$$j = + \sqrt{-1} \quad \text{(unit imaginary)} \tag{2.1}$$

$$Z_o \equiv \sqrt{\frac{\mu_o}{\varepsilon_o}} \quad \text{(wave impedance of free space)}$$

$$c = \frac{1}{\sqrt{\mu_o \; \varepsilon_o}} \quad \text{(speed of light in free space)}$$

$$\rho_q = \rho + \frac{qj}{Z_o} \rho_h \quad \text{(combined charge density)}$$

$$\nabla \cdot \vec{J}_q = - \frac{\partial}{\partial t} \rho_q \quad \text{(combined continuity equation)}$$

$$\nabla \cdot \vec{E}_q = \frac{1}{\varepsilon_o} \rho_q$$

As in Figure 2.1 let there be some closed surface S with unit normal vector (outward) $\vec{1}_S$ with position $\vec{r}$ taken as $\vec{r}_s$ on S. The boundary condition on S is taken as

$$\vec{1}_S\left(\vec{r}_s\right) \times \left[\vec{E}_q^{(ex)}\left(\vec{r}_{s_+}\right) - \vec{E}_q^{(in)}\left(\vec{r}_{s_-}\right)\right] = qj\ Z_o\ \vec{J}_{s_q}\left(\vec{r}_s\right)$$

$$\vec{J}_{s_q} = \vec{J}_s + \frac{qj}{Z_o}\vec{J}_{h_s} \quad \text{(combined surface current density)} \tag{2.2}$$

with "ex" meaning exterior and "in" meaning interior. The + and - subscripts on $\vec{r}_s$ indicate limits being taken from the exterior and interior respectively. The interior volume is designated V.

The field equivalence principle constructs a solution of the Maxwell equations of the form

$$\vec{E}_q(\vec{r},t) = \begin{cases} \vec{E}_q^{(inc)}(\vec{r},t) & \text{for } \vec{r} \in V \\ \vec{0} & \text{for } \vec{r} \notin V \cup S \end{cases}$$

$$\vec{J}_{s_q}^{(inc)}\left(\vec{r}_s,t\right) = \frac{qj}{Z_o}\vec{1}_S\left(\vec{r}_s\right) \times \vec{E}_q^{(inc)}\left(\vec{r}_s,t\right) \tag{2.3}$$

where $\vec{E}_q^{(inc)}$ is any solution of the free-space Maxwell equations with no currents (electric or magnetic) in the V S. This is a compact statement of the Love field equivalence principle for fields interior to a volume [6]. Writing out the above boundary surface currents we have

$$\vec{J}_s^{(inc)}\left(\vec{r}_s,t\right) = -\vec{1}_S\left(\vec{r}_s\right) \times \vec{H}^{(inc)}\left(\vec{r}_s,t\right)$$

$$\vec{J}_{h_s}^{(inc)}\left(\vec{r}_s,t\right) = \vec{1}_S\left(\vec{r}_s\right) \times \vec{E}^{(inc)}\left(\vec{r}_s,t\right) \tag{2.4}$$

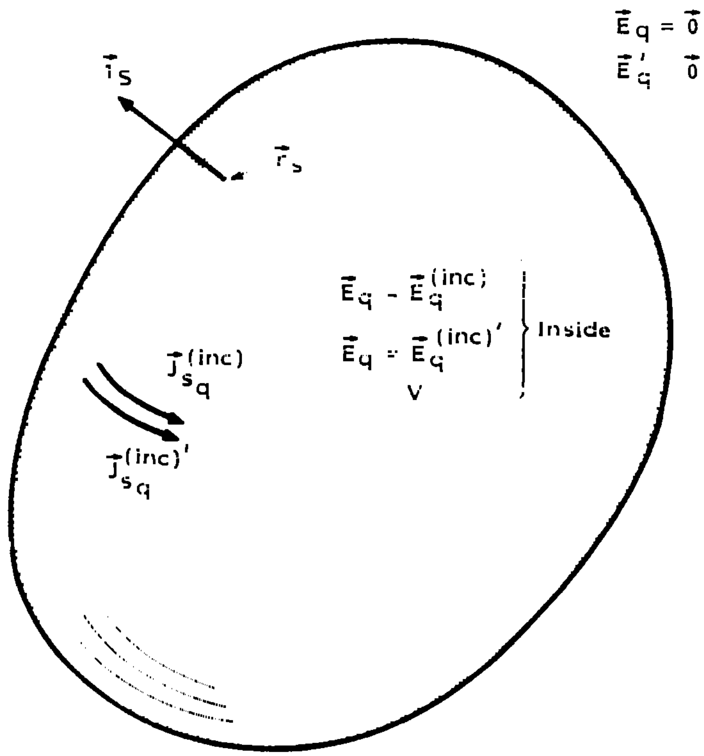

Figure 2.1. Dual Equivalence Principle

In our discussion of the generalized Babinet principle, the concept of a complement or dual (designated by a prime) can be related to the original quantity as [3]

$$\Gamma'_q = -qj\Gamma$$
$$\Gamma_q = qj\Gamma' \qquad (2.5)$$

This shows the interchange of electric and magnetic parameters (duality) to be simply thought of as multiplication by the unit imaginary. Except for constants this shows that

(electric parameter)' ⟷ + (magnetic parameter)

(magnetic parameter)' ⟷ - (electric parameter) (2.6)

Now take the dual of the field equivalence principle so that

$$\vec{E}'_q(\vec{r},t) = \begin{cases} \vec{E}_q^{(inc)'}(\vec{r},t) & \text{for } \vec{r} \in V \\ \vec{0} & \text{for } \vec{r} \notin V \end{cases}$$

$$\vec{J}_{s_q}^{(inc)'}\left(\vec{r}_s,t\right) = \frac{qj}{Z_o} \vec{1}_S\left(\vec{r}_s\right) \times \vec{E}_q^{(inc)'}\left(\vec{r}_s,t\right)$$

$$\vec{E}'_q(\vec{r},t) = -qj\,\vec{E}_q(\vec{r},t) \tag{2.7}$$

$$\vec{J}_{s_q}^{(inc)'}\left(\vec{r}_s,t\right) = -qj\,\vec{J}_{s_q}^{(inc)}\left(\vec{r}_s,t\right)$$

This says that if one interchanges the sources on S as

$$\vec{J}_s^{(inc)'}\left(\vec{r}_s,t\right) = \frac{1}{Z_o} \vec{J}_{h_s}^{(inc)}\left(\vec{r}_s,t\right)$$
$$\vec{J}_{h_s}^{(inc)'}\left(\vec{r}_s,t\right) = -Z_o\,\vec{J}_s^{(inc)}\left(\vec{r}_s,t\right) \tag{2.8}$$

The resultant fields are similarly interchanged as

$$\vec{E}'(\vec{r},t) = Z_o\,\vec{H}(\vec{r},t)$$
$$\vec{H}'(\vec{r},t) = -\frac{1}{Z_o}\,\vec{E}(\vec{r},t) \tag{2.9}$$

The fields are still zero outside S, but are the dual (or complement) of the original fields inside S.

III. Electrostatic and Magnetostatic Boundary Value Problems

Instead of having electric and magnetic surface currents on the boundary we are normally dealing with electric currents for magnetostatic problems and electric charges for electrostatic problems. (Note that by use of electric dipoles and magnetic dipoles (loops) one can have both electric and equivalent magnetic currents to approximate the required distributions for the field equivalence principle as in the PARTES concept [1].) In the more traditional form of static devices, say for producing uniform electric or magnetic fields, fields are produced both inside and outside of S and only electric current or charge is used.

For the electrostatic problem consider a potential function $\Phi_e^{(0)}(\vec{r})$. On the boundary surface we have

$$\Phi_e^{(0)}\left(\vec{r}_s\right) \equiv \Phi_e\left(\vec{r}_s\right) \tag{3.1}$$

This naturally divides into two problems

$$\begin{aligned} &\Phi_e^{(in)}(\vec{r}) \quad \text{for} \quad \vec{r} \in V \cup S \\ &\Phi_e^{(ex)}(\vec{r}) \quad \text{for} \quad \vec{r} \notin V \\ &\Phi_e^{(in)}\left(\vec{r}_s\right) = \Phi_e^{(ex)}\left(\vec{r}_s\right) = \Phi_e^{(0)}\left(\vec{r}_s\right) \end{aligned} \tag{3.2}$$

The electric field is

$$\vec{E}(\vec{r}) = -\nabla \Phi_e(\vec{r}) \tag{3.3}$$

On S one can also specify the tangential electric field as the boundary condition as

$$\begin{aligned} &\vec{E}_s\left(\vec{r}_s\right) = \overleftrightarrow{1}_t\left(\vec{r}_s\right) \cdot \vec{E}\left(\vec{r}_s\right) = -\nabla_s \Phi_e^{(0)}\left(\vec{r}_s\right) \\ &\overleftrightarrow{1}_t\left(\vec{r}_s\right) = \overleftrightarrow{1} - \vec{1}_S\left(\vec{r}_s\right) \vec{1}_S\left(\vec{r}_s\right) \equiv \text{transverse dyad} \end{aligned} \tag{3.4}$$

where ∇_s is the surface gradient [10]. Note that it is important that $\vec{E}_s$ take this form so that it is associated with a conservative potential function and has no associated magnetic field. This is related to the fact that

$$\nabla \times \vec{E}(\vec{r}) = -\nabla \times \left[\nabla \Phi_e(\vec{r})\right] \equiv \vec{0} \quad \text{for} \quad \vec{r} \notin S$$

$$\vec{1}_S(\vec{r}_s) \cdot \left[\nabla_s \times \vec{E}_s(\vec{r}_s)\right] = \vec{1}_S(\vec{r}_s) \cdot \left[\nabla_s \times \left[\nabla \Phi_e^{(0)}(\vec{r}_s)\right]\right] = 0 \tag{3.5}$$

For the magnetostatic problem consider two potential functions

$$\Phi_h^{(in)}(\vec{r}) \quad \text{for} \quad \vec{r} \in V$$

$$\Phi_h^{(ex)}(\vec{r}) \quad \text{for} \quad \vec{r} \notin V \cup S \tag{3.6}$$

These are not in general equal on S. The magnetic field is

$$\vec{H}(\vec{r}) = \nabla \Phi_h(\vec{r}) \tag{3.7}$$

with the boundary condition that

$$\vec{1}_S(\vec{r}_s) \cdot \vec{H}\left(\vec{r}_{s_-}\right) = \vec{1}_S(\vec{r}_s) \cdot \vec{H}\left(\vec{r}_{s_+}\right) \tag{3.8}$$

i.e. that the normal component of the magnetic field is continuous through S, and that

$$\overleftrightarrow{1}_t(\vec{r}_s) \cdot \left[\vec{H}^{(in)}\left(\vec{r}_{s_-}\right) - \vec{H}^{(ex)}\left(\vec{r}_{s_+}\right)\right] = \vec{1}_S(\vec{r}_s) \times \vec{J}_s(\vec{r}_s) \tag{3.9}$$

i.e. that the tangential component of the magnetic field is discontinuous through S by an amount related to the surface current density.

As in Figure 3.1 consider some contour C_h, the boundary of S_h, and note

$$\oint_{C_h} \vec{H}(\vec{r}) \cdot d\vec{\ell}_n = \int_{S_h} \vec{J}(\vec{r}) \cdot \vec{1}_{S_h} dS_h$$

$$= -\int_{C_i} \left[\vec{1}_S\left(\vec{r}_s\right) \times \vec{J}_s\left(\vec{r}_s\right)\right] \cdot d\vec{\ell}_i \tag{3.10}$$

Now the gradient of a potential function has zero curl, implying zero current. This requires the two potentials as in (3.6) for the two regions with no currents. In crossing S there is a discontinuity between the two potential functions

$$\Phi_h^{(0)}\left(\vec{r}_s\right) \equiv \Phi_h^{(in)}\left(\vec{r}_{s_-}\right) - \Phi_h^{(ex)}\left(\vec{r}_{s_+}\right) \tag{3.11}$$

In traversing the contour C_h it is the discontinuity in the potential function which gives the enclosed current (since the contour integral of a gradient of a potential function is the difference of the potential between the end points). Hence we have

$$\oint_{C_h} \vec{H}(\vec{r}) \cdot d\vec{\ell}_n = \Phi_h^{(ex)}\left(\vec{r}_{s_{2_+}}\right) - \Phi_h^{(ex)}\left(\vec{r}_{s_{1_+}}\right) + \Phi_h^{(in)}\left(\vec{r}_{s_{1_-}}\right) - \Phi_h^{(in)}\left(\vec{r}_{s_{2_-}}\right)$$

$$= \Phi_h^{(0)}\left(\vec{r}_{s_1}\right) - \Phi_h^{(0)}\left(\vec{r}_{s_2}\right)$$

$$= -\int_{C_i} \left[\vec{1}_S\left(\vec{r}_s\right) \times \vec{J}_s\left(\vec{r}_s\right) \cdot d\vec{\ell}_i\right] \tag{3.12}$$

The normal $\vec{H}$ continuity in (3.8) becomes

$$\vec{1}_S\left(\vec{r}_s\right) \cdot \nabla\Phi_h^{(in)}\left(\vec{r}_{s_-}\right) = \vec{1}_S\left(\vec{r}_s\right) \cdot \nabla\Phi^{(ex)}\left(\vec{r}_{s_+}\right) \tag{3.13}$$

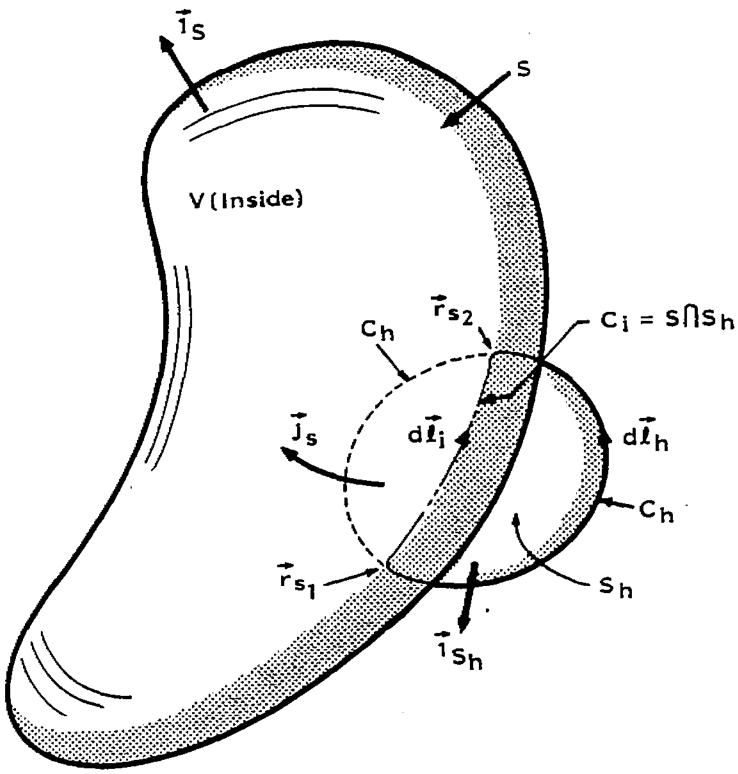

Figure 3.1. Magnetostatic Boundary Value Problem

while the tangential $\vec{H}$ discontinuity in (3.9) becomes

$$\begin{aligned}\nabla_s \Phi_h^{(0)}\left(\vec{r}_s\right) &= \overleftrightarrow{1}_t\left(\vec{r}_s\right) \cdot \left[\nabla\Phi_h^{(in)}\left(\vec{r}_{s_-}\right) - \nabla\Phi_h^{(ex)}\left(\vec{r}_{s_+}\right)\right] \\ &= \vec{1}_S\left(\vec{r}_s\right) \times \vec{J}_s\left(\vec{r}_s\right)\end{aligned} \tag{3.14}$$

So this says consistent with (3.12) that the surface current density can be derived from the surface gradient of a surface potential function. Note that [10]

$$\begin{aligned}\nabla_s \cdot \vec{J}_s\left(\vec{r}_s\right) &= \nabla_s \cdot \left[\vec{1}_S\left(\vec{r}_s\right) \times \nabla_s \Phi_h^{(0)}\left(\vec{r}_s\right)\right] \\ &= 0\end{aligned} \tag{3.15}$$

For a magnetostatic problem this zero-divergence condition is required to give zero electric field.

So we see now that both electrostatic and magnetostatic boundary value problems are describable in terms of a potential function on S or its surface gradient. While the boundary condition has the same general form there is an important difference. The electrostatic potential is continuous through S assuming the specified potential on S. The magnetostatic potential is discontinuous through S by the specified potential on S.

IV. Comparison for Circular Cylinder

In cylindrical (Ψ,ϕ,z) coordinates we have

$$x = \Psi \cos(\phi)$$
$$y = \Psi \sin(\phi) \tag{4.1}$$

The general form of a potential (solution of the Laplace equation) is [9]

$$\Phi = \sum_{m,{e \atop o}} \int_\zeta \left[a_{m,{e \atop o}}(\zeta)\ I_m(\zeta\Psi) + a'_{m,{e \atop o}}(\zeta)\ \kappa_m(\zeta\Psi) \right] \left\{ {\cos(m\phi) \atop \sin(m\phi)} \right\} e^{\zeta z}\, d\zeta$$
$$+ \left[a_o\ \ell n(\Psi) + a'_o \right] \left[z + a''_o \right] \tag{4.2}$$

where continuity with respect to ϕ for $0 \leq \phi \leq 2\pi$ has been assumed. This general form introduces the complication of the cylindrical Bessel functions (modified), especially if ζ is allowed to vary.

For the special case of Φ independent of z we have

$$\Phi = \sum_{m,{e \atop o}} \left[a_{m,{e \atop o}}\ \Psi^m + a'_{m,{e \atop o}}\ \Psi^{-m} \right] \left\{ {\cos(m\phi) \atop \sin(m\phi)} \right\} \tag{4.3}$$

noting that for m=0 only σ=e is non zero. With respect to ϕ there is the orthogonality condition

$$\int_o^{2\pi} \left\{ {\cos(m\phi) \atop \sin(m\phi)} \right\} \left\{ {\cos(m'\phi) \atop \sin(m'\phi)} \right\} d\phi$$
$$= \pi\left[1 + \left[1_{e,\sigma} - 1_{0,\sigma}\right] 1_{0,m}\right] 1_{m,m'}\, 1_{\sigma,\sigma'}$$
$$= \begin{cases} f_{m,\sigma} & \text{for } m=m',\ \sigma=\sigma' \\ 0 & \text{otherwise} \end{cases} \tag{4.4}$$

The gradient in cylindrical coordinates is

$$\nabla = \vec{1}_\Psi \frac{\partial}{\partial \Psi} + \vec{1}_\phi \frac{1}{\Psi} \frac{\partial}{\partial \phi} + \vec{1}_z \frac{\partial}{\partial z} \tag{4.5}$$

Since Φ is independent of z there is no electric or magnetic field in the z direction in the respective boundary value problems. The z component of the gradient is then not used. Only fields transverse to the z axis are considered.

Let our surface S now be taken as a circular cylinder defined by $\Psi = \Psi_o$. The electrostatic boundary value problem has potentials

$$\Phi_e^{(in)}(\Psi,\phi) = \sum_{m,\sigma} a_{m,\substack{e\\o}} \Psi^m \begin{Bmatrix}\cos(m\phi)\\ \sin(m\phi)\end{Bmatrix} \text{ for } 0 \leq \Psi \leq \Psi_o$$

$$\Phi_e^{(ex)}(\Psi,\phi) = \sum_{m,\sigma} a'_{m,\substack{e\\o}} \Psi^{-m} \begin{Bmatrix}\cos(m\phi)\\ \sin(m\phi)\end{Bmatrix} \text{ for } \Psi_o \leq \Psi \tag{4.6}$$

The boundary condition is

$$\Phi_e^{(0)}(\phi) = \Phi_e^{(in)}\left(\Psi_o,\phi\right) = \Phi_e^{(ex)}\left(\Psi_o,\phi\right)$$

$$\vec{E}_s(\phi) = - \nabla_s \Phi_e^{(0)}(\phi) = - \vec{1}_\phi \frac{1}{\Psi_o} \frac{\partial \Phi_e^{(0)}(\phi)}{\partial \phi} \tag{4.7}$$

Solving for the coefficients we have

$$a_{m,\substack{e\\o}} \Psi_o^{\ m} = a'_{m,\substack{e\\o}} \Psi_o^{\ -m} = \frac{1}{f_{m,\sigma}} \int_o^{2\pi} \Phi_e^{(0)}(\phi) \begin{Bmatrix}\cos(m\phi)\\ \sin(m\phi)\end{Bmatrix} d\phi \tag{4.8}$$

The magnetostatic boundary value problem has potentials (using b for coefficients)

$$\Phi_h^{(in)}(\Psi,\phi) = \sum_{m,\sigma} b_{m,\substack{e\\o}} \; \Psi^m \begin{Bmatrix}\cos(m\phi)\\ \sin(m\phi)\end{Bmatrix} \text{ for } 0 \le \Psi < \Psi_o$$

$$\Phi_h^{(ex)}(\Psi,\phi) = \sum_{m,\sigma} b'_{m,\substack{e\\o}} \; \Psi^{-m} \begin{Bmatrix}\cos(m\phi)\\ \sin(m\phi)\end{Bmatrix} \text{ for } \Psi_o < \Psi \qquad (4.9)$$

The boundary condition is

$$\Phi_h^{(0)}(\phi) = \Phi_h^{(in)}\left(\Psi_o-,\phi\right) - \Phi_h^{(ex)}\left(\Psi_o+,\phi\right)$$

$$\vec{1}_\Psi(\phi) \cdot \nabla\Phi_h^{(in)}\left(\Psi_o-,\phi\right) = \vec{1}_\Psi(\phi) \cdot \nabla\Phi_h^{(ex)}\left(\Psi_o+,\phi\right)$$

$$\nabla_s\Phi_h^{(0)}(\phi) = \overleftrightarrow{1}_t(\phi) \cdot \left[\nabla\Phi_h^{(in)}\left(\Psi_o-,\phi\right) - \nabla\Phi_h^{(ex)}\left(\Psi_o+,\phi\right)\right]$$

$$= \vec{1}_\Psi(\phi) \times \vec{J}_s(\phi) \qquad (4.10)$$

$$\overleftrightarrow{1}_t(\phi) = \vec{1}_\phi(\phi)\,\vec{1}_\phi(\phi) + \vec{1}_z\,\vec{1}_z$$

where the z,z component of the transverse dyad does not enter the computation. Note the use of Ψ_o+ and Ψ_o- for positions just outside and inside S respectively. Matching the normal magnetic field through S gives

$$b_{m,\substack{e\\o}}\,\Psi_o^{m} = -\,b'_{m,\substack{e\\o}}\,\Psi_o^{-m} \qquad (4.11)$$

The potential jump through S is

$$\Phi_h^{(0)}(\phi) = \sum_{m,\sigma} \left[b_{m,\sigma} \Psi_o^{m} - b'_{m,\sigma} \Psi_o^{-m} \right] \begin{Bmatrix} \cos(m\phi) \\ \sin(m\phi) \end{Bmatrix}$$

$$= 2 \sum_{m,\sigma} b_{m,\sigma} \Psi_o^{m} \begin{Bmatrix} \cos(m\phi) \\ \sin(m\phi) \end{Bmatrix} \tag{4.12}$$

Solving for the coefficients we have

$$b_{m,{e \atop o}} \Psi_o^{m} = - b'_{m,{e \atop o}} \Psi_o^{-m} = \frac{1}{2f_{m,\sigma}} \int_o^{2\pi} \Phi_h^{(0)}(\phi) \begin{Bmatrix} \cos(m\phi) \\ \sin(m\phi) \end{Bmatrix} d\phi \tag{4.13}$$

Comparing the electrostatic and magnetostatic problems let us suppose that

$$\Phi_e^{(0)}(\phi) = \Phi_h^{(0)}(\phi) \tag{4.14}$$

Then except for signs arising from definition of the fields from gradients of the respective potentials we have

Electrostatic		Magnetostatic	
$\Phi_e^{(0)}(\phi)$	$\longleftrightarrow$	$\Phi_h^{(0)}(\phi)$	
$\vec{E}_s(\phi)$	$\longleftrightarrow$	$\vec{1}_\Psi(\phi) \times \vec{J}_s(\phi)$	
$\Delta\Phi_e^{(0)}(\phi)$ (across slot in z direction	$\longleftrightarrow$	I (conductor (coil) in z direction)	(4.15)
$\vec{E}(\Psi,\phi)$ (everywhere inside S)	$\longleftrightarrow$	$\vec{H}(\Psi,\phi)$ (everywhere inside S) reduced to 1/2 times electric field	

This is a kind of duality showing that, except for a factor 2, the fields are the same. If we imagine the electrostatic problem consisting of $\Psi=\Psi_0$ as a conducting surface with slots across which voltages are impressed, the dual problem is that of currents on conductors at the slot positions (with no other conductors on S) with the same relative value of currents vs. voltages. Note: the sum of the slot voltages around all closed contours on S must be zero (for a conservative potential), and similarly for the dual currents.

V. Comparison for Sphere

In spherical (r,θ,ϕ) coordinates we have

$$\Psi = r\,\sin(\theta)\ ,\ z = r\,\cos(\theta)$$
$$x = r\,\sin(\theta)\,\cos(\phi)\ ,\ y = r\,\sin(\theta)\,\sin(\phi) \tag{5.1}$$

The general form of a potential (solution of the Laplace equation) is [9]

$$\Phi = \sum_{n,m,\sigma}\left[a_{n,m,{e \atop o}}\, r^{n} + a'_{n,m,{e \atop o}}\, r^{-n-1}\right] Y_{n,m,\sigma}(\theta,\phi)$$
$$Y_{n,m,{e \atop o}}(\theta,\phi) = P_n^{(m)}\left(\cos(\theta)\right)\begin{Bmatrix}\cos(m\phi)\\ \sin(m\phi)\end{Bmatrix} \tag{5.2}$$

noting that for $m=0$ only $\sigma=e$ is non zero. Here continuity with respect to θ,ϕ for all 4π steradians has been assumed. There is the orthogonality condition [2]

$$\int_0^{\pi}\int_0^{2\pi} Y_{n,m,\sigma}(\theta,\phi)\, Y_{n',m',\sigma'}(\theta,\phi)\,\sin(\theta)\,d\phi\,d\theta$$
$$= \frac{2\pi}{2n+1}\,\frac{(n+m)!}{(n-m)!}\left[1 + \left[1_{e,\sigma} - 1_{0,\sigma}\right] 1_{0,m}\right] 1_{n,n'}\, 1_{m,m'}\, 1_{\sigma,\sigma'}$$
$$= \begin{cases} f_{n,m,\sigma} & \text{for } n=n',\ m=m',\ \sigma=\sigma' \\ 0 & \text{otherwise}\end{cases} \tag{5.3}$$

The gradient in spherical coordinates is

$$\nabla = \vec{1}_r\,\frac{\partial}{\partial r} + \vec{1}_\theta\,\frac{1}{r}\,\frac{\partial}{\partial\theta} + \vec{1}_\phi\,\frac{1}{r\,\sin(\theta)}\,\frac{\partial}{\partial\phi} \tag{5.4}$$

Let our surface S now be taken as a sphere defined by $r=r_o$. The electrostatic boundary value problem has potentials.

$$\Phi_e^{(in)}(r,\theta,\phi) = \sum_{n,m,\sigma} a_{n,m,\sigma}\, r^n\, Y_{n,m,\sigma}(\theta,\phi) \text{ for } 0 \le r \le r_o$$

$$\Phi_e^{(ex)}(r,\theta,\phi) = \sum_{n,m,\sigma} a'_{n,m,\sigma}\, r^{-n-1}\, Y_{n,m,\sigma}(\theta,\phi) \text{ for } r_o \le r \tag{5.5}$$

The boundary condition is

$$\Phi_e^{(0)}(\theta,\phi) = \Phi_e^{(in)}\left(r_o,\theta,\phi\right) = \Phi_e^{(ex)}\left(r_o,\theta,\phi\right)$$

$$\vec{E}_s(\theta,\phi) = -\nabla_s\Phi_e^{(0)}(\theta,\phi) = -\vec{1}_\theta\,\frac{1}{r}\,\frac{\partial\Phi_e^{(0)}(\theta,\Phi)}{\partial\theta} - \vec{1}_\theta\,\frac{1}{r\sin(\theta)}\,\frac{\partial\Phi_e^{(0)}(\theta,\phi)}{\partial\phi} \tag{5.6}$$

Solving for the coefficients we have

$$a_{n,m,\sigma}\, r_o^{\,n} = a'_{n,m,o}\, r^{-n-1}$$

$$= \frac{1}{f_{n,m,\sigma}} \int_o^{\pi}\int_o^{2\pi} \Phi_e^{(0)}(\theta,\phi)\, Y_{n,m,\sigma}(\theta,\phi)\, \sin(\theta)\, d\phi\, d\theta \tag{5.7}$$

The magnetostatic boundary value problem has potentials (using b for coefficients)

$$\Phi_h^{(in)}(r,\theta,\phi) = \sum_{n,m,\sigma} b_{n,m,\sigma}\, r^n\, Y_{n,m,\sigma}(\theta,\phi) \text{ for } 0 \le r < r_o$$

$$\Phi_h^{(ex)}(r,\theta,\phi) = \sum_{n,m,\sigma} b'_{n,m,\sigma}\, r^{-n-1}\, Y_{n,m,\sigma}(\theta,\phi) \text{ for } r_o < r \tag{5.8}$$

The boundary condition is

$$\Phi_h^{(0)}(\theta,\phi) = \Phi_h^{(in)}\left(r_o-,\theta,\phi\right) - \Phi_h^{(ex)}\left(r_o+,\theta,\phi\right)$$

$$\vec{1}_r(\theta,\phi) \cdot \nabla\Phi_h^{(in)}\left(r_o-,\theta,\phi\right) = \vec{1}_r(\theta,\phi) \cdot \nabla\Phi_h^{(ex)}\left(r_o+,\theta,\phi\right)$$

$$\nabla_s\Phi_h^{(0)}(\theta,\phi) = \overleftrightarrow{1}_t(\theta,\phi) \cdot \left[\nabla\Phi_h^{(in)}\left(r_o-,\theta,\phi\right) - \nabla\Phi_h^{(ex)}\left(r_o+,\theta,\phi\right)\right]$$

$$= \vec{1}_r(\theta,\phi) \times \vec{J}_s(\theta,\phi) \tag{5.9}$$

$$\overleftrightarrow{1}_t(\theta,\phi) = \vec{1}_\theta(\theta,\phi)\ \vec{1}_\theta(\theta,\phi) + \vec{1}_\phi(\theta,\phi)\ \vec{1}_\phi(\theta,\phi)$$

Note the use of r_o+ and r_o- for positions just outside and inside S respectively. Matching the normal magnetic field through S gives

$$b_{n,m,\sigma}\, n\, r_o^{\,n} = -\, b'_{n,m,\sigma}\,(n+1)\, r_o^{-n-1} \tag{5.10}$$

The potential jump through S is

$$\Phi_h^{(0)}(\theta,\phi) = \sum_{n,m,\sigma} \left[b_{n,m,\sigma}\, r_o^{\,n} - b'_{n,m,\sigma}\, r_o^{-n-1}\right] Y_{n,m,\sigma}(\theta,\phi)$$

$$= \sum_{n,m,\sigma} \frac{2n+1}{n+1}\, b_{n,m,\sigma}\, r_o^{\,n}\, Y_{n,m,\sigma}(\theta,\Phi) \tag{5.11}$$

Solving for the coefficients we have

$$b_{n,m,\sigma}\, r_o^{\,n} = -\left[\frac{n+1}{n}\right] b'_{n,m,\sigma}\, r_o^{\,-n-1} \tag{5.12}$$

$$= \frac{n+1}{2n+1} \frac{1}{f_{n,m,\sigma}} \int_0^{\pi}\int_0^{2\pi} \Phi_h^{(0)}(\theta,\phi)\, Y_{n,m,\sigma}(\theta,\phi) \sin(\theta)\, d\phi\, d\theta$$

Comparing the electrostatic and magnetostatic problems let us suppose that

$$\Phi_e^{(0)}(\theta,\phi) = \Phi_h^{(0)}(\theta,\phi) \tag{5.13}$$

Then except for signs arising from definition of the fields from gradients of the respective potentials we have a difference between the two solutions in that the magnetostatic terms are reduced from the electrostatic terms by a factor $\frac{n+1}{2n+1}$.

Let us now change the assumption and require

$$\Phi_h^{(0)}(\theta,\phi) \equiv \sum_{n,m,\sigma} \frac{2n+1}{n+1}\, a_{n,m,\sigma}\, r_o^{\,n}\, Y_{n,m,\sigma}(\theta,\phi) \tag{5.14}$$

with $a_{n,m,\sigma}$ as in (5.7) to define $\Phi_h^{(0)}(\theta,\phi)$ in terms of $\Phi_e^{(0)}(\theta,\phi)$. Then we have

$$b_{n,m,\sigma} = a_{n,m,\sigma}$$

$$\Phi_h^{(in)}(r,\theta,\phi) = \Phi_e^{(in)}(r,\theta,\phi) \tag{5.15}$$

and we have

Electrostatic		Magnetostatic
$\Phi_e^{(0)}(\theta,\phi)$		$\Phi_h^{(0)}(\theta,\phi)$
$\vec{E}_s(\theta,\phi)$	Coefficients related by (5.7) and (5.13)	$\vec{1}_r(\theta,\phi) \times \vec{J}_s(\theta,\phi)$
$\Delta\Phi_e^{(0)}(\theta,\phi)$ (across slot around S)		I (conductor (coil) around S)
$\vec{E}(r,\theta,\phi)$ (everywhere inside S)	$\longleftrightarrow$	$\vec{H}(r,\theta,\phi)$ (everywhere inside S)

(5.16)

Note that the construction of slots in a spherical bowl or current loops on a spherical surface must be such as to preserve conservative potentials.

It should be noted that these observations can be used to resolve a problem in the relating the designs of spherical devices for producing uniform electrostatic and magnetostatic fields near the coordinate origin $\vec{r}=\vec{0}$. The desired uniform field is that given by

$$n=1, \; m=0, \; \sigma=e \qquad (5.17)$$

for which

$$\Phi_{e_{ideal}}^{(in)}(r,\theta,\phi) = a_{n,0,e}\, r\cos(\theta) = a_{n,0,e}\, z$$

$$\Phi_{h_{ideal}}^{(ex)}(r,\theta,\phi) = b_{n,0,e}\, r\cos(\theta) = b_{n,0,e}\, z \qquad (5.18)$$

$$\vec{E}_{ideal}^{(in)}(r,\theta,\phi) = -\nabla\Phi_{e_{ideal}}^{(in)}(r,\theta,\phi) = - a_{n,0,e}\, \vec{1}_z$$

$$\vec{H}_{ideal}^{(in)}(r,\theta,\phi) = \nabla\Phi_{h_{ideal}}^{(in)}(r,\theta,\phi) = b_{n,0,e}\, \vec{1}_z$$

Now the design problem is to make this the only n=1 term (easily achieved by rotation symmetry with respect to the z axis, the C_∞ group). For higher order n the idea is to make as many as possible for n = 2,3, etc. to be zero with emphasis on the smaller n.

This problem has been solved for the magnetostatic case. There are the well-known Helmholtz (2) and Maxwell (3) coil solutions [8]. This has been generalized to an arbitrary number N of coils [7]. In these cases the coils are at constant values of θ on the spherical surface (rotation symmetry with respect to the z axis) and are symmetrically located (with symmetrical currents) with respect to the x,y plane. For N coils there are N values of θ and N-1 currents to specify (one current being arbitrary and scaling the solution). This allows one to set terms n=2 through n=2N to zero. One can think of this as setting the first 2N-1 derivatives of the magnetic field to zero at the origin.

For the electrostatic problem let there be slots at the same θ values with voltages across the slots corresponding exactly to the currents in the dual coils. Then for n=2 through n=2N the coefficients $a_{n,m,\sigma}$ are zero precisely because the corresponding $b_{n,m,\sigma}$ are zero. The factor of $\frac{n+1}{2n+1}$ does not enter, being multiplied by zero. It does not matter whether we relate $\Phi_h^{(0)}(\theta,\phi)$ to $\Phi_e^{(0)}(\theta,\phi)$ directly as in (5.13) or term by term as in (5.14); the terms for n=2 through 2N are absent. The only difference is the trivial one for the n=1 term for which the $b_{1,0,e}$ is reduced to 2/3 of $a_{1,0,e}$ if (5.13) is used, or no difference if (5.14) is used; this is merely an overall scaling factor. Of course the numerical coefficient of the n=2N+1 term will be in general different if (5.13) is used which results when the slots/voltages are made to correspond to the coils/currents.

A recent paper solved the problem of the 2 slots in a bowl and noted that the optimum solution corresponded exactly to the Helmholtz coil [2]. One can obtain a solution for 3 slots (T.B.A. senior, private communication). Comparing this to that of the Maxwell coil [8] one can note the exact correspondence. Now we have found that the situation is dual (exact correspondence) for all orders N.

VI. Concluding Remarks

Here we have extended somewhat the concept of duality to the requirements for sources on boundaries to give dual fields in some volume V. The use of electric and magnetic surface currents give a direct duality in the form of the dual equivalence principle.

If, however, we restrict ourselves to the situation where the electrostatic problem involves specifying potential (or tangential electric field) on the surface S, and the magnetostatic problem involves specifying surface current density on (or potential jump through) S, then the problem is more complicated. Depending on the shape of S (particularly if S is a surface in a coordinate system for which the Laplace equation solution separates) one can construct dual boundary conditions in the sense of giving the same electric and magnetic field distributions in V. Various coordinate systems (such as prolate spheroidal, oblate spheroidal, ellipsoidal, etc.) might be used for this purpose [9]. For more general shapes it may be possible to construct the dual boundary conditions numerically.

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