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Design of Two-Dimensional EM Lenses Via Differential Geometric Scaling

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Abstract

In transient line synthesis, there are a variety of ways of specifying inhomogeneous media such that simple electromagnetic waves, such as TEM waves, can propagate in the medium. In the simplified situation considered here, we have a two-dimensional problem which $u_3 = z$, and either the formal electric field or formal magnetic field has only a u_3 component while the remaining field has only a u_2 component with both field components a function of u_1 only. The uniform TEM wave then propagates in the u_1 direction. Two-dimensional lenses can then be specified and these lenses are suitable for launching TEM waves on two parallel perfectly conducting plates.

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Abstract

In transient line synthesis, there are a variety of ways of specifying inhomogeneous media such that simple electromagnetic waves, such as TEM waves, can propagate in the medium. In the simplified situation considered here, we have a two-dimensional problem in which $u_3 = z$, and either the formal electric field or formal magnetic field has only a u_3 component while the remaining field has only a u_2 component with both field components a function of u_1 only. The uniform TEM wave then propagates in the u_1 direction. Two-dimensional lenses can then be specified and these lenses are suitable for launching TEM waves on two parallel perfectly conducting plates.

TEM waves, inhomogeneous media, electric fields, magnetic fields

1 Introduction

The study of inhomogeneous TEM plane waves which propagate on ideal cylindrical transmission lines with two or more independent perfectly conducting boundaries leads to the study of lens transition regions. These types of inhomogeneous media can be used to define lenses for transitioning TEM waves, without reflections or distortions, between conical and cylindrical transmission lines. While there are practical limitations (e.g., the properties of materials used to obtain the desired permittivity and permeability of the medium) perfect characteristics are not really necessary. The differential geometric approach to the design of lens transitions was initiated by C. E. Baum (see [1,2,3,4]), and in essence it is a scaling method. This method creates an equivalence between two classes of electromagnetic problems. The first EM problem, called the formal problem, has a simple geometry and medium and simple wave. The second EM problem, which is the real world or lens problem, consists of a more complicated geometry and medium and known wave. Thus the differential geometric scaling method transforms an EM problem by a coordinate change, and is a method that is well known in mechanics and fluid dynamics.

In Section 2, formal operators and fields are introduced along with Maxwell's equations. In Section 3, the case of two-dimensional TEM waves is discussed. In practice, there will be many possible ways to choose coordinates to form an orthogonal coordinate system from which one can extract the scale factors and specify the properties of a lens medium. In Section 4, the differential geometric concepts that are needed to specify a lens are introduced and the key results are discussed. In Section 5 several examples of two-dimensional lenses are given. These lenses might be appropriate for launching TEM waves between wide perfectly conducting parallel sheets. Regions corresponding to the condition $h > 1$ should be excluded. Finally, in Section 6 our results are summarized.

2 Formal Operators and Fields and Maxwell's Equations

As in [1] we consider an orthogonal curvilinear coordinate system (u_1, u_2, u_3) with unit vectors $\vec{1}_1, \vec{1}_2, \vec{1}_3$, with line element

$$(d\ell)^2 = h_1^2(du_1)^2 + h_2^2(du_2)^2 + h_3^2(du_3)^2 . \tag{2.1}$$

The scale factors are given by

$$h_i^2 = \left(\frac{\partial x}{\partial u_i}\right)^2 + \left(\frac{\partial y}{\partial u_i}\right)^2 + \left(\frac{\partial z}{\partial u_i}\right)^2 , \; i = 1, 2, 3 \tag{2.2}$$

where (x, y, z) are rectangular Cartesian coordinates, and the h_i are taken as positive. We define, as in [1], the following:

$$\begin{aligned}
(\alpha_{i,j}) &= \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix} \qquad (\beta_{i,j}) = \begin{pmatrix} h_2h_3 & 0 & 0 \\ 0 & h_1h_3 & 0 \\ 0 & 0 & h_1h_2 \end{pmatrix} \\
(\gamma_{i,j}) &= \begin{pmatrix} \frac{h_2h_3}{h_1} & 0 & 0 \\ 0 & \frac{h_1h_3}{h_2} & 0 \\ 0 & 0 & \frac{h_1h_2}{h_3} \end{pmatrix} \\
&= (\alpha_{i,j})^{-1} \cdot (\beta_{i,j}) = (\beta_{i,j}) \cdot (\alpha_{i,j})^{-1} .
\end{aligned} \tag{2.3}$$

With respect to the u_i coordinates, gradient, curl, and divergence are

$$\begin{aligned}
\nabla f &= \sum_{i=1}^{3} \frac{1}{h_i} \frac{\partial f}{\partial u_i} \vec{1}_i \qquad \nabla \times \vec{X} = \begin{vmatrix} \frac{1}{h_2h_3}\vec{1}_1 & \frac{1}{h_1h_3}\vec{1}_2 & \frac{1}{h_1h_2}\vec{1}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1X_1 & h_2X_2 & h_3X_3 \end{vmatrix} \\
\nabla \cdot \vec{Y} &= \frac{1}{h_1h_2h_3} \left\{ \frac{\partial}{\partial u_1}(h_2h_3Y_1) + \frac{\partial}{\partial u_2}(h_1h_3Y_2) + \frac{\partial}{\partial u_3}(h_1h_2Y_3) \right\} .
\end{aligned} \tag{2.4}$$

The X_i are called physical components of $\vec{X}$ which has the representation

$$\vec{X} = \sum_{i=1}^{3} X_i \vec{1}_i . \tag{2.5}$$

Formal vectors and operators may be defined as follows. These objects are denoted by attaching a prime to the usual symbols. Thus, for vectors ($\vec{E}$ and $\vec{H}$) which are subject to curl we define

$$\begin{aligned}
\vec{X}' &\equiv \sum_{i=1}^{3} X_i' \vec{1}_i = \sum_{i=1}^{3} h_i X_i \vec{1}_i \\
X_i' &= h_i X_i \,.
\end{aligned} \tag{2.6}$$

The X_i' are the covariant components of $\vec{X}$. The contravariant components of a vector $\vec{Y}$ (vectors such as $\vec{D}, \vec{B}, \vec{J}$ subject to diverence) are given by

$$\vec{Y} = \sum Y_i \vec{1}_i = h_2 h_3 Y_1 \vec{1}_1 + h_1 h_3 Y_2 \vec{1}_2 + h_1 h_2 Y_3 \vec{1}_3 \,. \tag{2.7}$$

The formal operators are then defined by,

$$\begin{aligned}
\nabla' f' &= \sum_{i=1}^{3} \frac{\partial f'}{\partial u_i} \vec{1}_i \\
\nabla' \times \vec{X}' &= \begin{vmatrix} \vec{1}_1 & \vec{1}_2 & \vec{1}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ X_1' & X_2' & X_3' \end{vmatrix} \\
\nabla' \cdot \vec{Y}' &= \frac{\partial Y_1'}{\partial u_1} + \frac{\partial Y_2'}{\partial u_2} + \frac{\partial Y_3'}{\partial u_3} \,.
\end{aligned} \tag{2.8}$$

Maxwell's equations are given by

$$\begin{aligned}
\nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\
\nabla \times \vec{H} &= \vec{J} + \frac{\partial \vec{D}}{\partial t} \\
\nabla \cdot \vec{D} &= \rho \\
\nabla \cdot \vec{B} &= 0
\end{aligned} \tag{2.9}$$

together with the constitutive relations

$$\begin{aligned}
\vec{D} &= (\varepsilon_{i,j}) \cdot \vec{E} \\
\vec{B} &= (\mu_{i,j}) \cdot \vec{H}
\end{aligned}$$

and continuity equation

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} .$$

The matrices $(\varepsilon_{i,j})$ and $(\mu_{i,j})$, which describe permittivity and permeability, are assumed frequency independent and thereby real valued and may be dependent on position. The equations above can be expressed in terms of the u_i coordinates. Formal electromagnetic quantities are defined by

$$\begin{aligned} \vec{E}' &= (\alpha_{i,j}) \cdot \vec{E} \\ \vec{H}' &= (\alpha_{i,j}) \cdot \vec{H} \\ E_i' &= h_i E_i \ , \ H_i' = h_i H_i \ , \ i = 1, 2, 3 \ . \end{aligned} \tag{2.10}$$

Since $\vec{B}, \vec{D}$, and $\vec{J}$ arise in divergence equations, we define

$$\begin{aligned} \vec{B}' &= (\beta_{i,j}) \cdot \vec{B} \\ \vec{D}' &= (\beta_{i,j}) \cdot \vec{D} \\ \vec{J}' &= (\beta_{i,j}) \cdot \vec{J} \\ B_i' &= \frac{h_1 h_2 h_3}{h_i} B_i \\ D_i' &= \frac{h_1 h_2 h_3}{h_i} D_i \\ J_i' &= \frac{h_1 h_2 h_3}{h_i} J_i \ . \end{aligned} \tag{2.11}$$

If we require

$$\begin{aligned} \vec{D}' &= (\varepsilon_{i,j}') \cdot \vec{E}' \\ \vec{B}' &= (\mu_{i,j}') \cdot \vec{H}' \end{aligned} \tag{2.12}$$

then Maxwell's equations and the above equations lead to definitions of the formal permittivity and permeability. These are

$$\begin{aligned} (\varepsilon_{i,j}') &= (\beta_{i,j}) \cdot (\varepsilon_{i,j}) \cdot (\alpha_{i,j})^{-1} \\ (\mu_{i,j}') &= (\beta_{i,j}) \cdot (\mu_{i,j}) \cdot (\alpha_{i,j})^{-1} \end{aligned} \tag{2.13}$$

and hence if $(\varepsilon_{i,j})$, $(\mu_{i,j})$ are diagonal,

$$\begin{aligned} (\varepsilon'_{i,j}) &= (\gamma_{i,j}) \cdot (\varepsilon_{i,j}) \\ (\mu'_{i,j}) &= (\gamma_{i,j}) \cdot (\mu_{i,j}) \ . \end{aligned} \tag{2.14}$$

Maxwell's equations can now be expressed in terms of formal fields and operators as:

$$\begin{aligned} \nabla' \times \vec{E}' &= -\frac{\partial \vec{B}'}{\partial t} \\ \nabla' \times \vec{H}' &= \vec{J}' + \frac{\partial \vec{D}'}{\partial t} \\ \nabla' \cdot \vec{D}' &= \rho' \\ \nabla' \cdot \vec{B}' &= 0 \\ \vec{D}' &= (\varepsilon'_{i,j}) \cdot \vec{E}' \\ \vec{B}' &= (\mu'_{i,j}) \cdot \vec{H}' \\ \nabla' \cdot \vec{J}' &= -\frac{\partial \rho'}{\partial t} \ . \end{aligned} \tag{2.15}$$

3 Two-Dimensional TEM Waves

If, in our (u_1, u_2, u_3) coordinate system we let $u_3 = z$ (and hence $h_3 = 1$) and suppose that either the formal electric field or formal magnetic field has only a u_3 component while the remaining field has only a u_2 component with both components a function of u_1 only, then we have a uniform TEM wave which propagates in the u_1 direction. Thus, for example, if the electric field is parallel to the z axis, we assume

$$\begin{aligned} \vec{E}' &= E'_{3_0} f(t - u_1/c')\vec{1}_3 \\ \vec{H}' &= H'_{2_0} f(t - u_1/c')\vec{1}_2 \end{aligned} \tag{3.1}$$

where

$$\vec{E}'_{3_0} = -\sqrt{\frac{\mu'_2}{\varepsilon'_3}} H'_{2_0} \tag{3.2}$$

$$c' = 1\Big/\sqrt{\mu_2'\varepsilon_3'}$$

and $E'_{3_0}, H'_{2_0}, \mu_2'$, and ε_3' are all independent of the coordinates u_i.

Similarly if the magnetic field is parallel to the z-axis the wave is assumed to be of the form

$$\begin{aligned} \vec{E}' &= E'_{2_0} g(t - u_1/c')\vec{1}_2 \\ \vec{H}' &= H'_{3_0} g(t - u_1/c')\vec{1}_3 \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} E'_{2_0} &= \sqrt{\frac{\mu_3'}{\varepsilon_2'}} H'_{3_0} \\ c' &= 1\Big/\sqrt{\mu_3'\varepsilon_2'} \end{aligned} \tag{3.4}$$

and E'_{2_0}, H'_{3_0}, u_3', and ε_2' are all independent of the coordinates.

Note that when the electric field is parallel to the z-axis perfectly conducting planar sheets can be placed perpendicular to this axis and used as boundaries for the TEM wave. Likewise when the magnetic field is in the direction of the z-axis, the electric field is perpendicular to surfaces of constant u_2 and hence perfectly conducting sheets can be placed along the surfaces of constant u_2 and used as boundaries for the TEM wave.

Let us also note that for the constitutive parameters that

$$\begin{aligned} (\varepsilon_{ij}) &= \varepsilon(\delta_{ij}) \\ (\mu_{ij}) &= \mu(\delta_{ij}) \end{aligned} \tag{3.5}$$

with zero conductivity. Thus we have an isotropic, but generally inhomogeneous, medium. The formal constitutive parameters are given by matrices

$$(\varepsilon'_{ij}) = \begin{pmatrix} \varepsilon_1' & 0 & 0 \\ 0 & \varepsilon_2' & 0 \\ 0 & 0 & \varepsilon_3' \end{pmatrix} \tag{3.6}$$

$$(\mu'_{ij}) = \begin{pmatrix} \mu'_1 & 0 & 0 \\ 0 & \mu'_2 & 0 \\ 0 & 0 & \mu'_3 \end{pmatrix}$$

and since the formal fields have only one component, only one entry in each of the matrices (3.6) will be significant. Since the ε'_i and μ'_i are assumed to be independent of position the medium is homogeneous. Moreover we also have

$$\begin{aligned} (\varepsilon'_{ij}) &= \varepsilon(\gamma_{ij}) \\ (\mu'_{ij}) &= \mu(\gamma_{ij}) \end{aligned} \tag{3.7}$$

where

$$(\gamma_{ij}) = \begin{pmatrix} h_2/h_1 & 0 & 0 \\ 0 & h_1/h_2 & 0 \\ 0 & 0 & h_1 h_2 \end{pmatrix}$$

since $h_3 = 1$.

Thus from the equations (3.5) through (3.7) we obtain

$$\begin{aligned} \varepsilon &= \frac{\varepsilon'_3}{h_1 h_2} \\ \mu &= \frac{\mu'_2 h_2}{h_1} \end{aligned} \tag{3.8}$$

when the electric field is parallel to the z-axis. On the other hand, when the magnetic field is in the z-direction we have

$$\begin{aligned} \varepsilon &= \frac{\varepsilon'_2 h_2}{h_1} \\ \mu &= \frac{\mu'_3}{h_1 h_2}\,. \end{aligned} \tag{3.9}$$

We next note that once an orthogonal curvilinear coordinate system (u_1, u_2) has been chosen the scale factors h_1 and h_2 are determined. Thus if the formal constitutive parameters are assigned their free space values the permittivity ε and permeability μ are known. Since we must require $\varepsilon \geq \varepsilon_0$ and $\mu \geq \mu_0$ we may have to exclude certain spatial regions from consideration. These details are discussed in the next section. Specific examples of two-dimensional lens appropriate for launching TEM waves between wide perfectly conducting parallel sheets are given in the following section.

Finally, we note that the two-dimensional cases studied in this paper are part of the more general problem of using a differential geometric approach to transient lens synthesis. In [4] several other cases are investigated. In particular, we may restrict to inhomogeneous isotropic media with field components in all three coordinate directions. The case of inhomogeneous TEM waves with field components in only two directions is also studied in [4].

4 Differential Geometry

Recall that a regular surface has associated with it a metric form (called the first fundamental form)

$$(d\ell)^2 = \sum_{i,j=1}^{2} g_{ij}(du_i du_j) \tag{4.1}$$

which in the case of orthogonal curvilinear coordinates is just

$$(d\ell)^2 = g_{11}(du_1)^2 + g_{22}(du_2)^2 \,. \tag{4.2}$$

In this event we let $g_{11} = h_1^2$ and $g_{22} = h_2^2$ where the h_1 and h_2 are the scale factors referred to in the previous section. Some special cases of interest include semi-geodesic coordinates ($h_1 = 1$) and isothermic coordinates ($h_1 = h_2$). We now discuss these special cases.

The most obvious examples of semi-geodesic coordinates for which the associated metric form is

$$(d\ell)^2 = (du_1)^2 + h^2(du_2)^2 \tag{4.3}$$

include orthogonal Cartesian coordinates in the plane and polar coordinates in the plane. The spherical coordinates on the unit sphere are also semi-geodesic coordinates, since

$$(d\ell)^2 = (d\theta)^2 + \sin^2(\theta)(d\phi)^2 \tag{4.4}$$

when $x = \sin(\theta)\cos(\phi)$, $y = \sin(\theta)\sin(\phi)$, $z = \cos(\theta)$. The Gaussian curvature associated with (4.3) is given by

$$K = -\frac{1}{h}\frac{\partial^2 h}{\partial u_1^2} \tag{4.5}$$

and hence for the unit sphere $K = 1$. Finally, if (u, v) are semi-geodesic coordinates then $h_1 = 1$ and $h_2 = h_2(u, v)$ and so our constitutive parameters have the form

$$\begin{aligned} \varepsilon &= \frac{\varepsilon_3'}{h_2} \\ \mu &= \mu_2' h_2 \end{aligned} \tag{4.6}$$

when the electric field is parallel to the z-axis, and when the magnetic field is parallel to the z-axis,

$$\begin{aligned} \varepsilon &= \varepsilon_2' h_2 \\ \mu &= \frac{\mu_3'}{h_2}. \end{aligned} \tag{4.7}$$

In the case of isothermic coordinates, $h_1 = h_2$ and examples of surfaces for which the metric form is

$$(d\ell)^2 = h^2[(du)^2 + (dv)^2] \tag{4.8}$$

include such surfaces as helicoids and catenoids, and also a sphere with parametrization given by

$$\begin{aligned} x &= \frac{4a^2 u}{4a^2 + u^2 + v^2} \\ y &= \frac{4a^2 v}{4a^2 + u^2 + v}, \\ z &= \left(\frac{4a^2 - u^2 - v^2}{4a^2 + u^2 + v^2}\right) a \end{aligned} \tag{4.9}$$

for which

$$(d\ell)^2 = \frac{16a^4}{(4a^2 + u^2 + v^2)}((du)^2 + (dv)^2) \ . \tag{4.10}$$

The Gaussian curvature for a surface with metric form (4.8) is

$$K = -\frac{1}{h^2}\nabla^2(\ell n(h)) \ . \tag{4.11}$$

We note that the metric form (4.8) is obtainable through conformal mappings of the plane. That is, if $p = x + iy$ is a complex variable and $q = F(p)$ is a conformal mapping, then it is easily checked that

$$(d\ell)^2 = \left|\frac{dp}{dq}\right|^2 ((du)^2 + (dv)^2) \tag{4.12}$$

where $q = u + iv$, and hence $h_1 = h_2 = \left|\frac{dp}{dq}\right| = h$. In the case that $h_1 = h_2$, the equations for the constitutive parameters have the form

$$\begin{aligned} \varepsilon &= \frac{\varepsilon_3'}{h^2} \\ \mu &= \mu_2' \end{aligned} \tag{4.13}$$

for $\vec{E}$ in the z-direction, and

$$\begin{aligned} \varepsilon &= \varepsilon_2' \\ \mu &= \frac{\mu_3'}{h^2} \end{aligned} \tag{4.14}$$

for $\vec{H}$ in the z-direction.

In either of these cases with $h_1 = h_2 = h$ the choice of free space values for the formal constitutive parameters leads to the restriction that $h \leq 1$ since $\varepsilon \geq \varepsilon_0$ and $\mu \geq \mu_0$ must hold. Thus regions corresponding to $h > 1$ would have to be excluded from considerations. In the next section, we consider some specific examples of two-dimensional lenses. These examples arise from conformal mappings of the complex plane. We now make some further observations on this situation.

If $q = F(p) = u + iv$ is a conformal mapping of $S \subset C$, the complex plane, to $S' \subset C$, we recall the following facts. First, if we let

$$p = x + iy \tag{4.15}$$

then

$$\begin{aligned} x &= f(u,v) \\ y &= g(u,v) \end{aligned}$$

and

$$p = x + iy = f(u,v) + ig(u,v) \equiv G(q)$$

where G is the inverse of F. Then

$$\frac{dp}{dq} = \frac{\partial f}{\partial u} + i\frac{\partial g}{\partial u} = \frac{\partial g}{\partial v} - i\frac{\partial f}{\partial v} \tag{4.16}$$

since G is differentiable. Moreover

$$h^2 = \left|\frac{dp}{dq}\right|^2 = \left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial g}{\partial u}\right)^2 = \left(\frac{\partial f}{\partial v}\right)^2 + \left(\frac{\partial g}{\partial v}\right)^2 \tag{4.17}$$

and $h^2 = h_1^2 = h_2^2$ with

$$(d\ell)^2 = h^2[(du)^2 + (dv)^2]\ . \tag{4.18}$$

Moreover, by a direct calculation or from the fact that the Gaussian curvature $\mathcal{K}$ vanishes identically, we have

$$\frac{\partial^2 \ell n(h)}{\partial u^2} + \frac{\partial^2 \ell n(h)}{\partial v^2} = 0\ , \tag{4.19}$$

(i.e., $\ell n(h)$ is a harmonic function of (u,v)). Thus conformal maps force the condition that $h_1 = h_2$. Conversely, if we choose a metric form as in (3.18), for some orthogonal transformation of the form

$$\begin{aligned} x &= f(u,v) \\ y &= g(u,v) \end{aligned} \tag{4.20}$$

then we must have

$$\frac{\partial f}{\partial u}\frac{\partial f}{\partial v}+\frac{\partial g}{\partial u}\frac{\partial g}{\partial v}=0 \tag{4.21}$$

because of the orthogonality, and we must also have equations (4.17) valid. Thus if we set

$$\alpha=\frac{\frac{\partial f}{\partial u}}{\frac{\partial g}{\partial u}}=-\frac{\frac{\partial g}{\partial v}}{\frac{\partial f}{\partial v}} \tag{4.22}$$

(which is another way of expressing the orthogonality) then

$$(\alpha^2+1)\left(\frac{\partial g}{\partial u}\right)^2=(\alpha^2+1)\left(\frac{\partial f}{\partial v}\right)^2 . \tag{4.23}$$

But $\alpha^2+1\neq 0$, since $\alpha=\pm i$ leads to a linear relation between f and g. Thus we must have

$$\left(\frac{\partial g}{\partial u}\right)^2=\left(\frac{\partial f}{\partial v}\right)^2 \tag{4.24}$$

and as a consequence either

$$\frac{\partial f}{\partial u}=\frac{\partial g}{\partial v} \quad\text{and}\quad \frac{\partial f}{\partial v}=-\frac{\partial g}{\partial u} \tag{4.25}$$

or

$$\frac{\partial f}{\partial u}=-\frac{\partial g}{\partial v} \quad\text{and}\quad \frac{\partial f}{\partial v}=\frac{\partial g}{\partial u} . \tag{4.26}$$

Both sets of equations (4.25) and (4.26) are the Cauchy-Riemann equations, and hence if (4.25) holds, then $f+ig$ is an analytic function of $u+iv$. Similarly, if (4.26) holds, then $f-ig$ is an analytic function of $u+iv$. Hence

$$f(u,v)\pm ig(u,v)=G(u+iv) \tag{4.27}$$

and we can then state the following result. If $u=$ constant and $v=$ constant form a system of isometric coordinates, then all other isometric systems are given by

$$\begin{aligned} Re[G(u+iv)] &= f(u,v) \\ Im[G(u+iv)] &= g(u,v) \end{aligned} \tag{4.28}$$

where $Re[G], Im[G]$ are the real and imaginary parts of an arbitrary analytic function of $u + iv$. Hence taking $h_1 = h_2 = h$ for any orthogonal coordinates u, v enforces the analyticity of the mapping $q = F(p)$, when h satisfies (4.17).

As a concluding remark, if we are given a metric for the plane of the form

$$(d\ell)^2 = h^2[(du)^2 + (dv)^2] \tag{4.29}$$

for some arbitrary choice of $h = h(u, v)$ we can ask if there is a conformal map $q = F(p) = u + iv$ for which $h = \left|\frac{dp}{dq}\right|$. Since the existence of such a map implies $\ell n(h)$ is harmonic in u and v, the condition that

$$\frac{\partial^2}{\partial u^2}\ell n(h) + \frac{\partial^2}{\partial v^2}\ell n(h) \neq 0 \tag{4.30}$$

would imply that such a map does not exist. In general, one can assert that the necessary and sufficient conditions that an arbitrary set of $h_i, i = 1, 2, 3$ will correspond to a triply orthogonal set of surfaces in Euclidean 3-space are given by a set of six partial differential equations, which are usually referred to as the Lamé equations ([5]). Since we are considering metrics in a plane, these equations reduce to a single equation, namely

$$\frac{\partial^2}{\partial u^2}\ell n(h) + \frac{\partial^2}{\partial v^2}\ell n(h) = 0\ . \tag{4.31}$$

Since the Gaussian curvature associated with a surface whose metric is

$$(d\ell)^2 = h_1^2(du)^2 + h_2^2(dv)^2 \tag{4.32}$$

is given by

$$K = -\frac{1}{h_1 h_2}\left[\frac{\partial}{\partial u}\left(\frac{1}{h_1}\frac{\partial h_2}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1}{h_2}\frac{\partial h_1}{\partial v}\right)\right] \tag{4.33}$$

the result (4.31) is not surprising. Moreover the Lamé equations themselves are obtained by using the vanishing of the Riemann curvature tensor, R_{ijkl}, as integrability conditions. Let us consider some further examples. Let

$$(d\ell)^2 = e^{2u}[(du)^2 + (dv)^2] \tag{4.34}$$

and we ask if there is a conformal map $p = G(q)$ leading to this metric form. Since

$$\nabla^2 \ell n(h) = 0 \tag{4.35}$$

the answer is yes, and so $\left|\frac{dp}{dq}\right| = e^u$ and we have the problem of finding $G(q)$. In this case, the answer is obvious since we can write

$$\frac{dp}{dq} = e^u e^{iv} = e^{u+iv} = e^q \tag{4.36}$$

and so

$$p = G(q) = e^q \ . \tag{4.37}$$

On the other hand, if

$$(d\ell)^2 = (u^2 + v^2)[(du)^2 + (dv)^2] \tag{4.38}$$

is a given form, then clearly

$$\nabla^2 \ell n(h) = \frac{1}{2}\nabla^2 \ell n(u^2 + v^2) = 0 \tag{4.39}$$

and so we have the existence of a conformal map $p = G(q)$ with

$$\left|\frac{dp}{dq}\right| = \sqrt{u^2 + v^2} \tag{4.40}$$

and one can then obtain

$$p = \frac{1}{2}q^2 \ . \tag{4.41}$$

The point is, however, that if we know a conformal map $p = G(q)$ exists for a metric form

$$(d\ell)^2 = h^2[(du)^2 + (dv)^2] \tag{4.42}$$

then

$$\begin{aligned} \left|\frac{dp}{dq}\right| &= h(u, v) \\ &= \sqrt{\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial g}{\partial u}\right)^2} \\ &= \sqrt{\left(\frac{\partial f}{\partial v}\right)^2 + \left(\frac{\partial g}{\partial v}\right)^2} \end{aligned} \tag{4.43}$$

and actually obtaining the functions f and g for which

$$p = G(q) = f(u,v) + ig(u,v) \tag{4.44}$$

may present some difficulties.

As a final example, if we have a form

$$(d\ell)^2 = u^4[(du)^2 + (dv)^2] \tag{4.45}$$

then $\nabla^2 \ell n(u^2) = -\frac{4}{u^2} \neq 0$, and so there is no conformal map leading to the form (4.45).

5 Examples of Two-Dimensional Lenses

For our first example of a two-dimensional lens, we consider, as in [1], the conformal transformation

$$\begin{aligned} q &= \frac{a}{\pi}\ell n[e^{\pi p/a} - 1] \\ p &= \frac{a}{\pi}\ell n[e^{\pi q/a} + 1]\,. \end{aligned} \tag{5.1}$$

Then

$$\begin{aligned} u &= \frac{a}{2\pi}\ell n\left[e^{2\pi x/a} - 2e^{\pi x/a}\cos\left(\frac{\pi y}{a}\right) + 1\right] \\ v &= \frac{a}{\pi}\arctan\left[\frac{e^{\pi x/a}\sin(\pi y/a)}{e^{\pi x/a}\cos(\pi y/a) - 1}\right] + ak\ ,\ k = 0, \pm 1 \end{aligned} \tag{5.2}$$

while

$$\begin{aligned} x &= \frac{a}{2\pi}\ell n\left[e^{2\pi u/a} + 2e^{\pi u/a}\cos\left(\frac{\pi}{a}v\right) + 1\right] \\ y &= \frac{a}{\pi}\arctan\left[\frac{e^{\pi u/a}\sin\left(\frac{\pi v}{a}\right)}{e^{\pi u/a}\cos\left(\frac{\pi v}{a}\right) + 1}\right] + ak\,. \end{aligned} \tag{5.3}$$

So, for example, $v/a = 1/2$ maps into $e^{\pi x/a} = \sec(\pi y/a)$. For a detailed plot, see Figure 5.1. The transformation describes the potential distribution around a uniformly

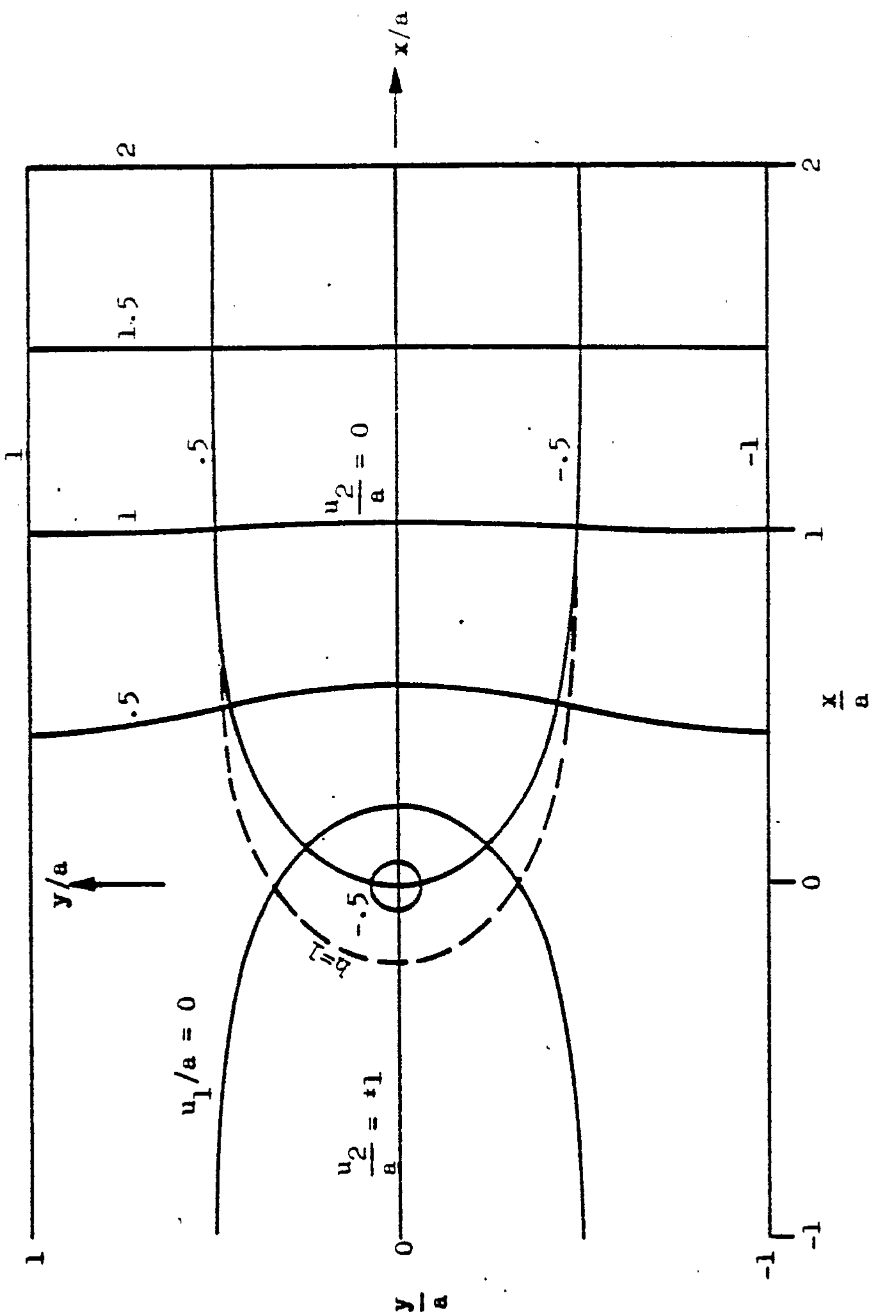


Figure 5.1. Coordinates for First Example

charged wire grid (in a homogeneous medium) terminating a uniform electric field for $x \gg 0$ (see [1]). As usual, a is a parameter which can be used to scale dimensions.

The scale factor h given by $\left|\frac{dp}{dq}\right|$ is

$$h^2 = 1 - 2e^{-\pi x/a} \cos\left(\frac{\pi y}{a}\right) + e^{-2\pi x/a} . \tag{5.4}$$

Since those regions corresponding to $h \leq 1$ are the only ones of interest, we find the contour for $h = 1$ from (5.4) by taking

$$2\cos\left(\frac{\pi y}{a}\right) = e^{-\pi x/a} . \tag{5.5}$$

In terms of the coordinates (x, y) we find

$$h^2 = \left[1 + 2e^{-\pi u/a} \cos\left(\frac{\pi v}{a}\right) + e^{-2\pi u/a}\right]^{-1} . \tag{5.6}$$

Thus the condition $|v/a| \leq 1/2$ will guarantee that $h \leq 1$ will hold. Hence perfectly conducting boundaries could be placed on surfaces of constant v. Note finally that as $x \to \infty$, $h \to 1$, and so one of the constitutive parameters of the lens is the same as free space, while the other parameter tends to the free space value as $x \to \infty$. Hence for x large in value the lens material can be shortened without significantly distorting the TEM wave.

Our second, and last, example of a coordinate system for a two-dimensional lens is given, as in [1], by the conformal transformation

$$\begin{aligned} q &= \frac{2a}{\pi} \ell n \left[\sinh\left(\frac{\pi p}{2a}\right)\right] \\ p &= \frac{2a}{\pi} \operatorname{arcsinh} \left[e^{\pi q/2a}\right] \end{aligned} \tag{5.7}$$

Thus we have

$$\begin{aligned} u &= \frac{a}{\pi} \ell n \left[\cosh^2\left(\frac{\pi x}{2a}\right) - \cos^2\left(\frac{\pi y}{2a}\right)\right] \\ v &= \frac{2a}{\pi} \arctan\left[\coth\left(\frac{\pi x}{2a}\right) \tan\left(\frac{\pi y}{2a}\right)\right] + 2ak \end{aligned} \tag{5.8}$$

where $k = 0, \pm 1$. The transformation (5.7) describes the potential distribution around a uniformly charged wire grid (in a homogeneous medium) terminating equal but opposite electric fields for $x \gg 0$ and $x \ll 0$. Figure 5.2 illustrates the transformation (5.7).

The scale factor h satisfies

$$h = |1 + e^{-\pi q/a}|^{-1/2} = \left|\tanh\left(\frac{\pi p}{2a}\right)\right| \tag{5.9}$$

and hence

$$h^2 = \frac{1 - 2e^{-\pi x/a}\cos\left(\frac{\pi y}{a}\right) + e^{-2\pi x/a}}{1 + 2e^{-\pi x/a}\cos\left(\frac{\pi y}{a}\right) + e^{-2\pi x/a}} \; . \tag{5.10}$$

Hence if $y = \pm a/2$, the contour for $h = 1$ is obtained. The minimum value of h will occur on $v = 0$ (for which $y = 0$) and at $u = u_0$, where u_0 is the minimum u of interest. Thus

$$h^2\Big|_{\min} = (1 + e^{-\pi u_0/a})^{-1} \tag{5.11}$$

and the maximum ε or μ can be found from (4.6) or (4.7).

Since as $x \to \infty$ we have

$$\begin{aligned} h &\to 1 \\ u &\to x - \frac{2a}{\pi}\ell n(2) \\ v &\to y \end{aligned} \tag{5.12}$$

and so as in the first example the lens material can be terminated at sufficiently large x without significantly distorting the TEM wave.

Figure 5.3 illustrates two-dimensional lens with parallel-plate transmission lines. The cases of either the electric or magnetic field parallel to the z axis are shown. In both cases the conductors and medium are cut off before the singularity on the z axis is reached. Sources to launch a TEM wave might be located at this point. The perfectly conducting sheets and inhomogeneous medium are stopped on surfaces of constant v. While this would result in distortion of the TEM wave, this distortion is minimized if the sheet separation is very small in relation to the sheet width.

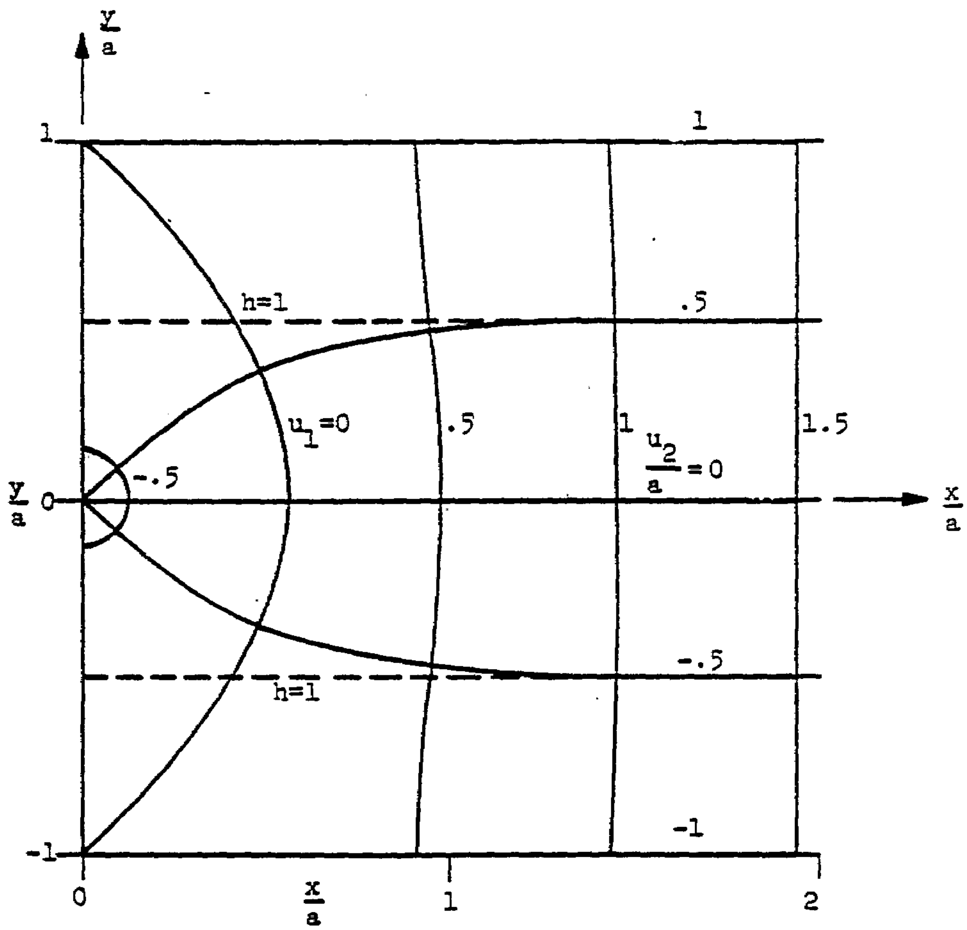
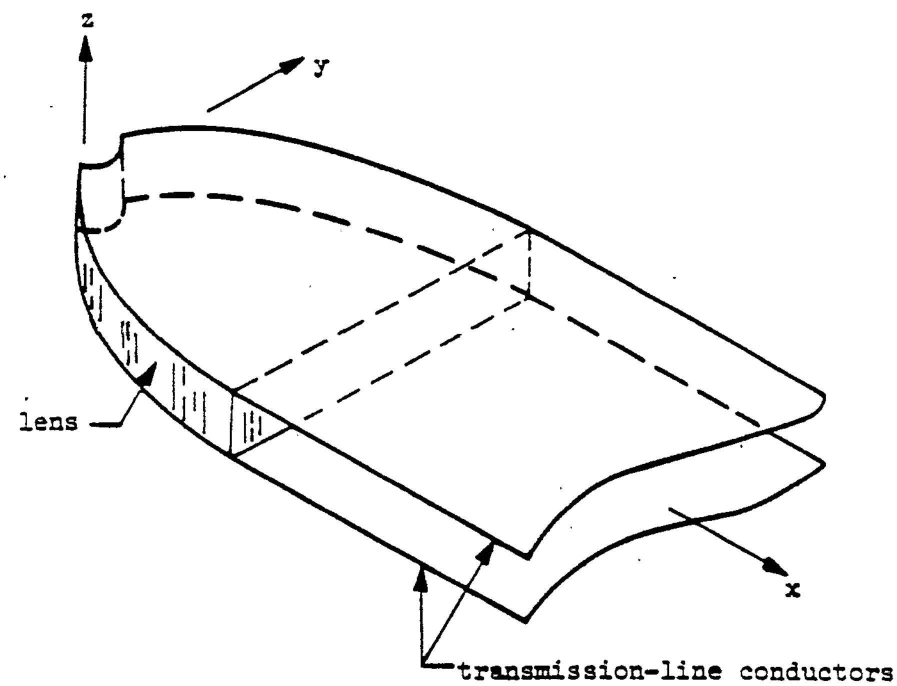
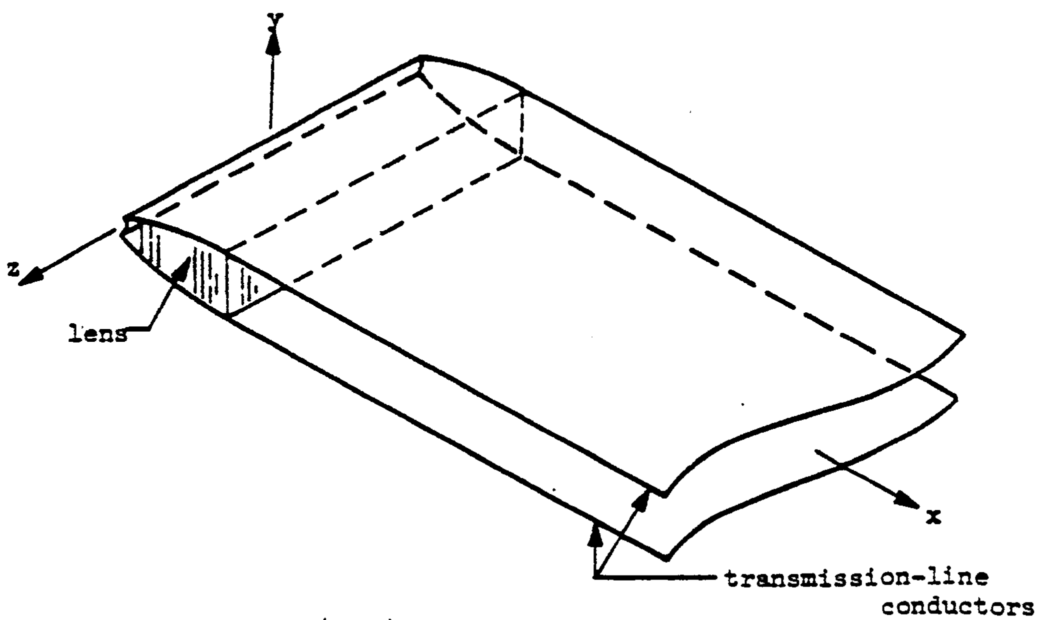


Figure 5.2. Coordinates for Second Example



A. Case 1: $\vec{E}$ parallel to z axis



B. Case 2: $\vec{H}$ parallel to z axis

Figure 5.3. Two-Dimensional Lenses with Transmission Lines

6 Summary

There appear to be many ways of specifying an inhomogeneous medium such that TEM waves can propagate in the medium, which can then be used to define a lens for transmitting a TEM wave without reflection or distortion between certain types of transmission lines. There may, of course, be practical limitations in the realization of such lenses. Ideally, the lens region should be infinite in extent, and so perturbations can be introduced into the desired pure TEM wave by cutting off the lens. Another limitation lies in the characteristics of the materials used to obtain the desired permittivity and permeability of the lens medium. Ideally while we desire frequency independence, perfect characteristics are not really necessary.

In this paper we have given several examples of two-dimensional lenses appropriate for launching TEM waves between wide perfectly conducting parallel sheets.

References

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[2] T. C. Mo, C. H. Papas, and C. E. Baum, "Differential geometry scaling method for electromagnetic field and its applications to coaxial waveguide functions," Sensor and Simulation Note 169, March 1973, Kirtland Air Force Base, Albuquerque, New Mexico 87117. (Also in shorter version as a paper by the same authors under the title, "General scaling method for electromagnetic field with application to a matching problem," J. Math. Physics, vol. 14, pp. 479-483, 1973.)

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[5] L. Eisenhart, *A Treatise on the Differential Geometry of Curves and Surfaces*, Dover Publications, 1960.