Sensor and Simulation Notes

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Relation Between the Differential Geometry Method and Transit-Time and Differential-Impedance Matching in Synthesizing Lenses for Inhomogeneous TEM Plane Waves

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Abstract

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An alternative approach to transient lens design is one which might be termed a dif- ntial impedance-matching and differential transit-time conservation approach. Firstly, erential impedances must be matched at all lens-waveguide boundaries so that a TEM e may be transmitted from one region to another without reflections. Secondly, in er that a wave be transmitted undistorted, a plane wave front in one region should into a plane wave front in another region and consequently the travel time for waves owing different paths must be conserved. As a result a system of ordinary nonlinear ferential equations will usually arise, and solutions to this system will specify the lens ometry (shape) and physics (material).

In this paper the relation between these two approaches is studied. In the case of e differential geometric scaling method one finds that for a TEM wave propagating one of the coordinate directions that both transit time conservation and differential pedance-matching is obtained at the boundaries of the lens. On the other hand, if one

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Abstract

This paper concerns the design of lenses for TEM plane waves, such as might exist on certain types of transmission lines. The medium (or transition region) can be used to specify lenses for transitioning TEM waves, without reflection or distortion, between these transmission lines. The desired transmission is to be frequency independent and the lens design is based on frequency independent solutions of Maxwell's equations. As such, these lens regions are suitable for transitioning broad-band transient waves.

The first approach to the design of transition regions is a differential geometric one. This method is a scaling method which creates an equivalence between two classes of electromagnetic problems. The first EM problem has a simple geometry and medium and simple wave. It is called the formal problem. The second EM problem, which is the real world or lens problem, consists of a more complicated geometry and medium and known wave. Thus the differential geometric scaling method transforms an EM problem by a coordinate change, and it is a method that is well known in mechanics and fluid dynamics.

An alternative approach to transient lens design is one which might be termed a differential impedance-matching and differential transit-time conservation approach. Firstly, differential impedances must be matched at all lens-waveguide boundaries so that a TEM wave may be transmitted from one region to another without reflections. Secondly, in order that a wave be transmitted undistorted, a plane wave front in one region should go into a plane wave front in another region and consequently the travel time for waves following different paths must be conserved. As a result a system of ordinary nonlinear differential equations will usually arise, and solutions to this system will specify the lens geometry (shape) and physics (material).

In this paper the relation between these two approaches is studied. In the case of the differential geometric scaling method one finds that for a TEM wave propagating in one of the coordinate directions that both transit time conservation and differential impedance-matching is obtained at the boundaries of the lens. On the other hand, if one

starts with the impedance-matching and transit-time requirements, then arriving at the differential geometric conditions requires considerable study. This involves first considering the general case of transport of waves through a set of ducts connecting surfaces which form the boundaries for a lens.

One then examines use of these ducts to reorder positions on a wavefront and then considers two possible ways of arriving at an EM lens design. In the first approach, all magnetic walls of ducts are removed to form an electric hyperduct (E 2-lens) which is called a jacket. The ensemble of jackets then leads to a hyperjacket (E 3-lens) and removal of all intermediate electric walls then leads to an EM lens. The second approach is an exact dual of the first in that one starts with the removal of all intermediate electric walls to finally arrive at an EM lens.

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Foreword

To

The Inhabitants of SPACE IN GENERAL

And H. C. IN PARTICULAR

This Work is Dedicated

By a Humble Native of Flatland

In the Hope that

Even as he was Initiated into the Mysteries

Of THREE Dimensions

Having been previously conversant

With ONLY TWO

So the Citizens of that Celestial Region

May aspire yet higher and higher

To the Secrets of FOUR FIVE OR EVEN SIX Dimensions

Thereby contributing

To the Enlargement of THE IMAGINATION

And the possible Development

Of that most rare and excellent Gift of MODESTY

Among the Superior Races

Of SOLID HUMANITY

from Flatland (circa 1880)

by Edwin A. Abott

1 Introduction

One approach to EM lens design, developed by Baum in [1], for transitioning TEM waves between certain kinds of transmission lines involves a differential geometric scaling technique. The basic idea in this approach is the creation of a class of electromagnetic problems, each having a complicated geometry and medium, which are equivalent under the scaling to an electromagnetic problem having a simple geometry and medium. The latter problem we might term the formal problem while the former problem is our lens or real-world problem. Solutions to Maxwell's equations can then be used in specifying various types of EM lenses for transitioning TEM waves, without distortion or reflection, between certain types of transmission lines. For example, in [1] Baum has given examples of inhomogeneous lenses based on bispherical and toroidal coordinate systems. These lenses, which can be thought of as converging and diverging lenses, may be used to transition TEM waves between conical and/or cylindrical transmission lines. Further work by Stone (see [2]) on this problem resulted in a general procedure for these types of lenses.

An alternative approach to transient lens design is one which might be termed a differential-impedance-matching and transit-time-conservation approach. Firstly, differential impedances must be matched at all waveguide and lens boundaries so that a TEM wave may be transmitted from one region to another without reflections. Secondly, in order that a wave be transmitted undistorted, a wave in one region should go into a wave in another region and consequently the travel times for waves following different paths must be conserved. As a result a system of ordinary nonlinear differential equations will usually arise. Solutions to this system will then specify the lens geometry (shape) and physics (material). This approach has been described by Baum et al. (see [3]) in a problem in which a lens is inserted between two cylindrical coaxial waveguides of different size. Another example occurs in a paper by Baum and Stone ([4]) where the design of a certain anisotropic lens suitable for launching TEM waves on a conducting circular conical system is specified. Still another example, of interest because no system of differential equations arises, appears in another paper by Baum and Stone ([5]). In that paper, another anisotropic lens for transitioning plane waves between media of different permittivities is specified.

Thus in considering these two approaches to EM lens design it is entirely natural to ask such questions as (a) will similar results be obtained by these two approaches in a particular design problem, (b) are the two methods equivalent in any sense, and (c) is it possible to develop a set of axioms so that the two approaches are unified. In this paper we address the equivalence problem. In Section 2 we show how the scaling technique leads to the differential-impedance-matching and differential transit-time conservation approach. The more difficult problem of arriving at the differential geometric scaling method from the impedance matching and transit time approach is discussed in Sections 3 through 6. In these sections we begin by considering the general case of transport of waves through a set of ducts, and the use of these ducts to reorder positions on a wavefront. Restrictions to the cases of no magnetic currents on magnetic boundaries and no additional electric boundaries are then considered and we are then led to the equations describing the differential geometric scaling method. The paper concludes with a summary in Section 7.

2 Differential Geometry Approach Leading to Differential-Transit-Time and Differential-Impedance Matching

2.1 Formal Operators and Fields

Summarizing the results in [1] we consider an orthogonal curvilinear coordinate system (u_1, u_2, u_3) with unit vectors $\vec{1}_1, \vec{1}_2, \vec{1}_3$, with line element

$$(d\ell)^2 = h_1^2(du_1)^2 + h_2^2(du_2)^2 + h_3^2(du_3)^2 \,. \tag{2.1}$$

The scale factors are given by

$$h_i^2 = \left(\frac{\partial x}{\partial u_i}\right)^2 + \left(\frac{\partial y}{\partial u_i}\right)^2 + \left(\frac{\partial z}{\partial u_i}\right)^2 \,,\; i = 1,2,3 \tag{2.2}$$

where (x, y, z) are rectangular Cartesian coordinates, and the h_i are taken as positive. We define, as in [1], the following:

$$(\alpha_{i,j}) = \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix} \qquad (\beta_{i,j}) = \begin{pmatrix} h_2h_3 & 0 & 0 \\ 0 & h_1h_3 & 0 \\ 0 & 0 & h_1h_2 \end{pmatrix} \tag{2.3}$$

$$\begin{aligned} (\gamma_{i,j}) &= \begin{pmatrix} \frac{h_2h_3}{h_1} & 0 & 0 \\ 0 & \frac{h_1h_3}{h_2} & 0 \\ 0 & 0 & \frac{h_1h_2}{h_3} \end{pmatrix} \\ &= (\alpha_{i,j})^{-1} \cdot (\beta_{i,j}) = (\beta_{i,j}) \cdot (\alpha_{i,j})^{-1} \,. \end{aligned}$$

With respect to the u_i coordinates, gradient, curl, and divergence are

$$\nabla f = \sum_{i=1}^{3} \frac{1}{h_i} \frac{\partial f}{\partial u_i} \vec{1}_i \qquad \nabla \times \vec{X} = \begin{vmatrix} \frac{1}{h_2h_3}\vec{1}_1 & \frac{1}{h_1h_3}\vec{1}_2 & \frac{1}{h_1h_2}\vec{1}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1X_1 & h_2X_2 & h_3X_3 \end{vmatrix} \tag{2.4}$$

$$\nabla \cdot \vec{Y} = \frac{1}{h_1h_2h_3} \left\{ \frac{\partial}{\partial u_1}(h_2h_3Y_1) + \frac{\partial}{\partial u_2}(h_1h_3Y_2) + \frac{\partial}{\partial u_3}(h_1h_2Y_3) \right\} \,.$$

The X_i are called physical components of $\vec{X}$ which has the representation

$$\vec{X} = \sum_{i=1}^{3} X_i \vec{1}_i \,. \tag{2.5}$$

Formal vectors and operators may be defined as follows. These objects are denoted by attaching a prime to the usual symbols. Thus, for vectors ($\vec{E}$ and $\vec{H}$) which are subject to curl we define

$$\begin{aligned} \vec{X}' &\equiv \sum_{i=1}^{3} X_i' \vec{1}_i = \sum_{i=1}^{3} h_i X_i \vec{1}_i \\ X_i' &= h_i X_i \,. \end{aligned} \tag{2.6}$$

The X_i' are the covariant components of $\vec{X}$. The contravariant components of a vector $\vec{Y}$ (vectors such as $\vec{D}, \vec{B}, \vec{J}$ subject to diverence) are given by

$$\vec{Y} = \sum Y_i \vec{1}_i = h_2 h_3 Y_1 \vec{1}_1 + h_1 h_3 Y_2 \vec{1}_2 + h_1 h_2 Y_3 \vec{1}_3 \,. \tag{2.7}$$

The formal operators are then defined by

$$\begin{aligned} \nabla' f' &= \sum_{i=1}^{3} \frac{\partial f'}{\partial u_i} \vec{1}_i \\ \nabla' \times \vec{X}' &= \begin{vmatrix} \vec{1}_1 & \vec{1}_2 & \vec{1}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ X_1' & X_2' & X_3' \end{vmatrix} \\ \nabla' \cdot \vec{Y}' &= \frac{\partial Y_1'}{\partial u_1} + \frac{\partial Y_2'}{\partial u_2} + \frac{\partial Y_3'}{\partial u_3} \,. \end{aligned} \tag{2.8}$$

2.2 Maxwell's Equations

Maxwell's equations are given by

$$\begin{aligned} \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \nabla \times \vec{H} &= \vec{J} + \frac{\partial \vec{D}}{\partial t} \\ \nabla \cdot \vec{D} &= \rho \\ \nabla \cdot \vec{B} &= 0 \end{aligned} \tag{2.9}$$

$$\nabla \cdot \vec{D} = \rho$$
$$\nabla \cdot \vec{B} = 0$$

together with the constitutive relations

$$\vec{D} = (\varepsilon_{i,j}) \cdot \vec{E}$$
$$\vec{B} = (\mu_{i,j}) \cdot \vec{H}$$

and continuity equation

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} .$$

The matrices $(\varepsilon_{i,j})$ and $(\mu_{i,j})$, which describe permittivity and permeability, are assumed frequency independent and thereby real valued and may be dependent on position. The equations above can be expressed in terms of the u_i coordinates. Formal electromagnetic quantities are defined by

$$\begin{aligned} \vec{E}' &= (\alpha_{i,j}) \cdot \vec{E} \\ \vec{H}' &= (\alpha_{i,j}) \cdot \vec{H} \\ E_i' &= h_i E_i \ , \ H_i' = h_i H_i \ , \ i = 1,2,3 \ . \end{aligned} \tag{2.10}$$

Since $\vec{B}, \vec{D}$, and $\vec{J}$ arise in divergence equations, we define

$$\begin{aligned} \vec{B}' &= (\beta_{i,j}) \cdot \vec{B} \\ \vec{D}' &= (\beta_{i,j}) \cdot \vec{D} \\ \vec{J}' &= (\beta_{i,j}) \cdot \vec{J} \\ B_i' &= \frac{h_1 h_2 h_3}{h_i} B_i \\ D_i' &= \frac{h_1 h_2 h_3}{h_i} D_i \\ J_i' &= \frac{h_1 h_2 h_3}{h_i} J_i \ . \end{aligned} \tag{2.11}$$

If we require

$$\begin{aligned} \vec{D}' &= (\varepsilon'_{i,j}) \cdot \vec{E}' \\ \vec{B}' &= (\mu'_{i,j}) \cdot \vec{H}' \end{aligned} \tag{2.12}$$

then Maxwell's equations and the above equations lead to definitions of the formal permittivity and permeability. These are

$$\begin{aligned} (\varepsilon'_{i,j}) &= (\beta_{i,j}) \cdot (\varepsilon_{i,j}) \cdot (\alpha_{i,j})^{-1} \\ (\mu'_{i,j}) &= (\beta_{i,j}) \cdot (\mu_{i,j}) \cdot (\alpha_{i,j})^{-1} \end{aligned} \tag{2.13}$$

and hence if $(\varepsilon_{i,j})$, $(\mu_{i,j})$ are diagonal,

$$\begin{aligned} (\varepsilon'_{i,j}) &= (\gamma_{i,j}) \cdot (\varepsilon_{i,j}) \\ (\mu'_{i,j}) &= (\gamma_{i,j}) \cdot (\mu_{i,j}) . \end{aligned} \tag{2.14}$$

Maxwell's equations can now be expressed in terms of formal fields and operators as:

$$\begin{aligned} \nabla' \times \vec{E}' &= -\frac{\partial \vec{B}'}{\partial t} \\ \nabla' \times \vec{H}' &= \vec{J}' + \frac{\partial \vec{D}'}{\partial t} \\ \nabla' \cdot \vec{D}' &= \rho' \\ \nabla' \cdot \vec{B}' &= 0 \\ \vec{D}' &= (\varepsilon'_{i,j}) \cdot \vec{E}' \\ \vec{B}' &= (\mu'_{i,j}) \cdot \vec{H}' \\ \nabla' \cdot \vec{J}' &= -\frac{\partial \rho'}{\partial t} . \end{aligned} \tag{2.15}$$

2.3 Restriction to Inhomogeneous Isotropic Media with Field Components in all Three Coordinate Directions

If we restrict our consideration to inhomogeneous isotropic media (in the real coordinates), then the constitutive matrices assume the form

$$\begin{aligned} (\varepsilon_{i,j}) &= \varepsilon(1_{i,j}) \\ (\mu_{i,j}) &= \mu(1_{i,j}) \end{aligned} \tag{2.16}$$

where ε and μ are positive, real-valued scalar functions of position. Hence

$$\begin{aligned} (\varepsilon'_{i,j}) &= \varepsilon(\gamma_{i,j}) \\ (\mu'_{i,j}) &= \mu(\gamma_{i,j}) \ . \end{aligned} \tag{2.17}$$

We may now consider some possible forms for the diagonal matrices $(\varepsilon'_{i,j})$ and $(\mu'_{i,j})$. These in general should be fairly simple so that the formal electromagnetic fields have desired forms.

If both $\vec{E}'$ and $\vec{H}'$ have all three formal components and the constitutive parameters have the form

$$\begin{aligned} (\varepsilon'_{i,j}) &= \varepsilon'(1_{i,j}) \\ (\mu'_{i,j}) &= \mu'(1_{i,j}) \end{aligned} \tag{2.18}$$

where ε', μ' are constants, then we have a formal homogeneous medium. Hence (2.18), (2.17) and (2.3) yield

$$\frac{h_2 h_3}{h_1} = \frac{h_1 h_3}{h_2} = \frac{h_1 h_2}{h_3} \tag{2.19}$$

and hence

$$h_1 = h_2 = h_3 \equiv h \tag{2.20}$$

and also

$$\begin{aligned} \varepsilon h &= \varepsilon' \\ \mu h &= \mu' \ . \end{aligned} \tag{2.21}$$

Therefore εh and μh are constant. However, we do not have the freedom to pick any function of the u_i for our scale factor h since the h_i must satisfy the Lamé equations [1]. The general result is that there are only two possible forms for h. In the first case h must be a constant, which implies that the u_i form a Cartesian system of coordinates and also that the ε and μ are constants. Hence in the first case we have a homogeneous medium. In the second case an inhomogeneous medium is obtained with

$$\frac{\varepsilon}{\varepsilon'} = \frac{\mu}{\mu'} = \frac{1}{h} = \frac{a^2}{x^2 + y^2 + z^2} \tag{2.22}$$

where $a \neq 0$ is a real constant. This type of h corresponds to 6-sphere coordinates (inversion of Cartesian coordinates). The class of inhomogeneous media obtained is restricted to spherically stratified media.

2.4 Waves With Field Components Only in the $\vec{1}_1$ and $\vec{1}_2$ Directions (Inhomogeneous TEM Waves)

If we now restrict our attention to inhomogeneous TEM waves which propagate in the $+u_3$ direction and which have no field components in this direction and if the formal constitutive parameters have the form

$$(\varepsilon_{i,j})' = \begin{pmatrix} \varepsilon' & 0 & 0 \\ 0 & \varepsilon' & 0 \\ 0 & 0 & \varepsilon_3' \end{pmatrix}, \; (\mu_{i,j}) = \begin{pmatrix} \mu' & 0 & 0 \\ 0 & \mu' & 0 \\ 0 & 0 & \mu_3' \end{pmatrix} \tag{2.23}$$

then the dependence of ε_3' and μ_3' on the coordinates is irrelevant since $E_3' = 0$ and $H_3' = 0$. If we choose ε' and μ' as constants, the medium is formally isotropic and homogeneous. The formal fields have the form

$$\begin{aligned} E_1' &= E_{1_0}'(y_1, u_2) f(t - u_3/c') = h_1 E_1 \\ E_2' &= E_{2_0}'(u_1, u_2) f(t - u_3/c') = h_2 E_2 \end{aligned}$$

$$
\begin{aligned}
E'_3 &= 0 \\
H'_1 &= H'_{1_0}(u_1,u_2)f(t-u_3/c') = h_1 H_1 \\
H'_2 &= H'_{2_0}(u_1,u_2)f(t-u_3/c') = h_2 H_2 \\
H'_3 &= 0 \\
\mu' &= \text{constant}\ ,\ \varepsilon' = \text{constant} \\
Z'_0 &= \sqrt{\frac{\mu'}{\varepsilon'}} = \text{constant} \\
c' &= \frac{1}{\sqrt{\mu'\varepsilon'}} = \text{constant}
\end{aligned} \tag{2.24}
$$

and the choice of $f(t-u_3/c')$ specifies the waveform. Since $\vec{E}' \cdot \vec{H}' = 0$, we find

$$
\frac{E'_1}{H'_2} = -\frac{E'_2}{H'_1} = Z'_0 \tag{2.25}
$$

where

$$
Z'_0 = \sqrt{\frac{\mu'}{\varepsilon'}} \tag{2.26}
$$

is the formal wave impedance.

These results require that the conductors forming the transmission line intersect surfaces of constant u_3 in such a fashion that the surfaces are represented in terms of only their u_1 and u_2 coordinates [1].

If we assume that $(\varepsilon_{i,j})$ and $(\mu_{i,j})$ correspond to isotropic but inhomogeneous media, i.e.,

$$
\begin{aligned}
(\varepsilon_{i,j}) &= \varepsilon(1_{i,j}) \\
(\mu_{i,j}) &= \mu(1_{i,j})
\end{aligned} \tag{2.27}
$$

where ε and μ may be positive valued functions of the coordinates, then (2.17) and (2.23) yield

$$
\frac{h_2 h_3}{h_1} = \frac{h_3 h_1}{h_2} = \frac{\varepsilon'}{\varepsilon} = \frac{\mu'}{\mu} \tag{2.28}
$$

and

$$
\frac{h_1 h_2}{h_3} = \frac{\varepsilon'_3}{\varepsilon} = \frac{\mu'_3}{\mu} \ .
$$

Hence

$$h_1 = h_2 = h \tag{2.29}$$

and

$$\begin{aligned} \varepsilon h_3 &= \varepsilon' \\ \mu h_3 &= \mu' \end{aligned} \tag{2.30}$$

Therefore εh_3 and μh_3 are constant and the formal wave impedance is the same as the physical wave impedance since (2.30) implies

$$\sqrt{\frac{\mu'}{\varepsilon'}} = \sqrt{\frac{\mu}{\varepsilon}} = Z_0' = Z_0 \; . \tag{2.31}$$

Note that since ε' and μ' are arbitrary constants any orthogonal curvilinear coordinate system for which $h_1 = h_2$ apparently determines our ε and μ since

$$\begin{aligned} \varepsilon &= \frac{\varepsilon'}{h_3} \\ \mu &= \frac{\mu'}{h_3} \end{aligned} \tag{2.32}$$

However, a differential geometric fact [6] leads to a further restriction on our coordinate system, namely, that if $h_1 = h_2$, then surfaces of constant u_3 can only be spheres or planes.

Moreover, for rotational orthogonal curvilinear coordinate systems (v_1, ϕ, v_3) which have been used to construct coordinates (u_1, u_2, u_3) in our examples we must also have surfaces of constant v_3 as spheres or planes which requires h_ϕ / h_{v_1} to be independent of v_3 [1,6]. In our examples one method of construction of the u_i relies on the equations

$$\begin{aligned} u_1 &= \lambda(v_1)\cos(\phi) \\ u_2 &= \lambda(v_1)\sin(\phi) \\ u_3 &= \xi(v_3) \end{aligned} \tag{2.33}$$

for which h_ϕ / h_{v_1} is a function of v_1 alone and where there are choices which must be made for $\lambda(v_1)$ and $\xi(v_3)$.

We can now verify, in this particular case of no field components in the u_3 direction, that our scaling apprach leads to differential-transit-time conservation and differential-impedance matching throughout the lens (as well as at lens-transmission-line boundaries). For the (u_1, u_2, u_3) coordinates in the lens transition region, all waveforms contain factors, as in (2.24), of the form $f(t - u_3/c')$, and hence we must have transit-time conservation. Our equations (2.30) have also shown that

$$h_3\sqrt{\mu\varepsilon} = \sqrt{\mu'\varepsilon'} = \text{ constant} \tag{2.34}$$

which confirms the above observation concerning transit-time conservation.

A special case of the above has

$$u_3 = r \ , \ h_3 = 1 \tag{2.35}$$

in the usual spherical (r, θ, ϕ) coordinate system. In this case we can have

$$\begin{aligned} \mu &= \mu' = \text{ constant} \\ \varepsilon &= \varepsilon' = \text{ constant} \ . \end{aligned} \tag{2.36}$$

This leads to the case of a conical transmission line as in [7] which has a uniform propagation medium (free space if one prefers).

If we next consider an impedance expression of the form

$$\begin{aligned} Z_d = \frac{\Delta V_1}{\Delta I_2} &= \frac{-\int_{u_1}^{u_1+\Delta u_1} E_1 d\ell_1}{-\int_{u_2}^{u_2+\Delta u_2} H_2 d\ell_2} \\ d\ell_1 &= h_1 du_1 \\ d\ell_2 &= h_2 du_2 \end{aligned} \tag{2.37}$$

on surfaces of constant u_3, where E_1 is evaluated along curves of constant u_1 and H_2 is evaluated along curves of constant u_2, then (2.37) is

$$Z_d = \frac{E_1 h_1 \Delta u_1}{H_2 h_2 \Delta u_2} = Z_0' \frac{\Delta u_1}{\Delta u_2} \ . \tag{2.38}$$

Since Z_0' is not a function of u_3 then the differential impedance is matched all along u_3. In the analysis leading up to (2.31) we have actually found that $\sqrt{\mu/\varepsilon}$ is a constant. Thus impedances can be matched at lens boundaries by choosing $\sqrt{\mu/\varepsilon}$ the same as that of the assumed uniform medium on the other side of a lens boundary of the form u_3 = constant.

Previous results have established examples of lenses with this type of inhomogeneous TEM wave [1]. There is a converging lens using bispherical coordinates, a diverging lens using toroidal lens, and a bending lens using cylindrical coordinates. In each case the transmission-line conductors pass right through the lens on appropriate coordinate surfaces. At the two lens boundaries (surfaces of constant u_3 in these examples) both $\vec{E}$ and $\vec{H}$ match through the boundaries, in these cases the fields being tangential to the constant u_3 boundaries.

2.5 Waves With Electric Field Only in $\vec{1}_1$ Direction and Magnetic Field Only in $\vec{1}_2$ Direction (Homogeneous TEM Waves)

If we further restrict our TEM waves to the form

$$\begin{aligned}
E_1' &= h_1 E_1 = E_0' f(t - u_3/c') \\
E_2' &= E_3' = 0 \\
H_2' &= h_2 H_2 = \frac{E_0'}{Z_0'} f(t - u_3/c') \\
H_1' &= H_3' = 0 \\
\mu' &= \text{constant}\ ,\ \varepsilon' = \text{constant} \\
Z_0' &= \sqrt{\frac{\mu'}{\varepsilon'}} = \text{constant} \\
c' &= \frac{1}{\sqrt{\mu'\varepsilon'}} = \text{constant}
\end{aligned} \tag{2.39}$$

the formal permittivity and permeability must satisfy equations of the form

$$
\begin{aligned}
(\varepsilon'_{i,j}) &= (\beta_{i,j}) \cdot (\varepsilon_{i,j}) \cdot (\alpha_{i,j})^{-1} \\
(\mu'_{i,j}) &= (\beta_{i,j}) \cdot (\mu_{i,j}) \cdot (\alpha_{i,j})^{-1} \\
(\varepsilon_{i,j}) &= \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix} \\
(\mu_{i,j}) &= \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix} .
\end{aligned}
\tag{2.40}
$$

With the only field components as E_1 and H_2 only ε_1 and μ_2 in the above are significant, the other permittivity and permeability components being unspecified. Thus we obtain

$$
\begin{aligned}
\varepsilon'_1 &= \frac{h_2 h_3}{h_1} \varepsilon_1 \\
\mu'_2 &= \frac{h_1 h_3}{h_2} \mu_2
\end{aligned}
\tag{2.41}
$$

Since ε'_1 and μ'_2 are constants we then obtain

$$
\begin{aligned}
Z'_0 &= \sqrt{\frac{\mu'_2}{\varepsilon'_1}} = \frac{h_1}{h_2}\sqrt{\frac{\mu_2}{\varepsilon_1}} = \text{ constant} \\
\frac{1}{c'} &= \sqrt{\mu'_2 \varepsilon'_1} = h_3 \sqrt{\mu_2 \varepsilon_1} = \text{ constant}
\end{aligned}
\tag{2.42}
$$

as our basic constraints.

An interesting special case of this is if one coordinate is "straight" and "parallel", i.e. corresponds to a Cartesian coordinate. If this coordinate is taken as u_1 (in the $\vec{E}$ direction) then we have

$$
u_1 = y \ , \ h_1 = 1 \ . \tag{2.43}
$$

If in addition we require a constant permeability (such as μ_0) then

$$
\mu_2 = \text{constant}
$$

$$h_2\sqrt{\varepsilon_1} = \text{constant} \tag{2.44}$$
$$h_3\sqrt{\varepsilon_1} = \text{constant}$$

implying

$$\frac{h_2}{h_3} = \text{constant} . \tag{2.45}$$

If this last constant is taken as unity then the u_2, u_3 coordinates are expressible as a conformal transformation (in the x, z plane) with

$$\begin{aligned} h_2 &= h_3 \equiv h \\ \varepsilon_1 &= \frac{\text{constant}}{h^2} . \end{aligned} \tag{2.46}$$

An example of such a lens is given in [1 (Section IX)]. Assumptions other than constant permeability are also possible but do not give so simple results as above.

Alternatively one might choose u_2 (in the $\vec{H}$ direction) as a Cartesian coordinate giving

$$u_2 = x \ , \ h_2 = 1 . \tag{2.47}$$

If in addition we require a constant permittivity (such as ε_0) then

$$\begin{aligned} \varepsilon_1 &= \text{constant} \\ h_1\sqrt{\mu_2} &= \text{constant} \\ h_3\sqrt{\mu_3} &= \text{constant} \end{aligned} \tag{2.48}$$

implying

$$\frac{h_1}{h_3} = \text{constant} . \tag{2.49}$$

Again, taking this last constant as unity leads to the u_1, u_3 coordinates being expressible as a conformal transformation (in the y, z plane) with

$$\begin{aligned} h_1 &= h_3 \equiv h \\ \mu_2 &= \frac{\text{constant}}{h^2} . \end{aligned} \tag{2.50}$$

An example of such a lens is also given in [1 (Section IX)]. Again, other than constant permittivities are also possible in the above.

2.6 Summary

In summary, then, if we start with generalized orthogonal curvilinear coordinates (u_1, u_2, u_3) with scale factors h_1, h_2, h_3 given by (2.2), and a TEM waveform propagating in one of the coordinate directions (for example, of the form (2.24)), then clearly a transit time conservation condition is satisfied both globally and locally as waves follow different paths through a lens meeting each u_3 surface at the same time during the transit. Moreover, in the cases we have considered we have noted that $\sqrt{\mu/\varepsilon}(h_1/h_2)$ is a constant and hence not a function of u_3. Thus impedances are matched differentially through the lens. That is, "differential geometry implies differential-impedance-matching and differential-transit-time conservation". The progression of the differential geometric cases here is summarized in Table 2.1.

Case	Results
Inhomogeneous Isotropic Media: General Case With All Three Field Components	$h_1 = h_2 = h_3 = h$ only Cartesian $(h = 1)$ and 6-sphere $\left(h = \frac{x^2 + y^2 + z^2}{a^2}\right)$ coordinates $\varepsilon h = \varepsilon'$, $\mu h = \mu'$, $\sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu'}{\varepsilon'}} =$ constant
Inhomogeneous TEM Waves $\left(\text{Propagation in } \vec{1}_3 \text{ Direction}\right.$ With $\left.\frac{E'}{H'} = \sqrt{\frac{\mu'}{\varepsilon'}}\right)$	$h_1 = h_2 = h$ u_3 surfaces spheres or planes $\varepsilon h_3 = \varepsilon'$, $\mu h_3 = \mu'$ $\sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu'}{\varepsilon'}} =$ constant
Homogeneous TEM Waves $\left(\text{Propagation in } \vec{1}_3 \text{ Direction}\right.$ With $\frac{E'}{H'} = \sqrt{\frac{\mu'}{\varepsilon'}}$ $\vec{E}$ in $\vec{1}_1$ direction $\left.\vec{H} \text{ in } \vec{1}_2 \text{ direction}\right)$	$\frac{h_1}{h_2}\sqrt{\frac{\mu_2}{\varepsilon_1}} =$ constant $h_3\sqrt{\mu_2\varepsilon_1} =$ constant
A. $u_1 = y$, $h_1 = 1$ $\mu_2 =$ constant (not necessary)	$h_2 = h_3 = h$ $\varepsilon_1 = \frac{\text{constant}}{h^2}$
B. $u_2 = x$, $h_2 = 1$ $\varepsilon_1 =$ constant (not necessary)	$h_1 = h_3 = h$ $\mu_2 = \frac{\text{constant}}{h^2}$

Table 2.1. Differential-Geometry Progression of Cases

3 General Case of Transport of Waves Through a Set of Ducts

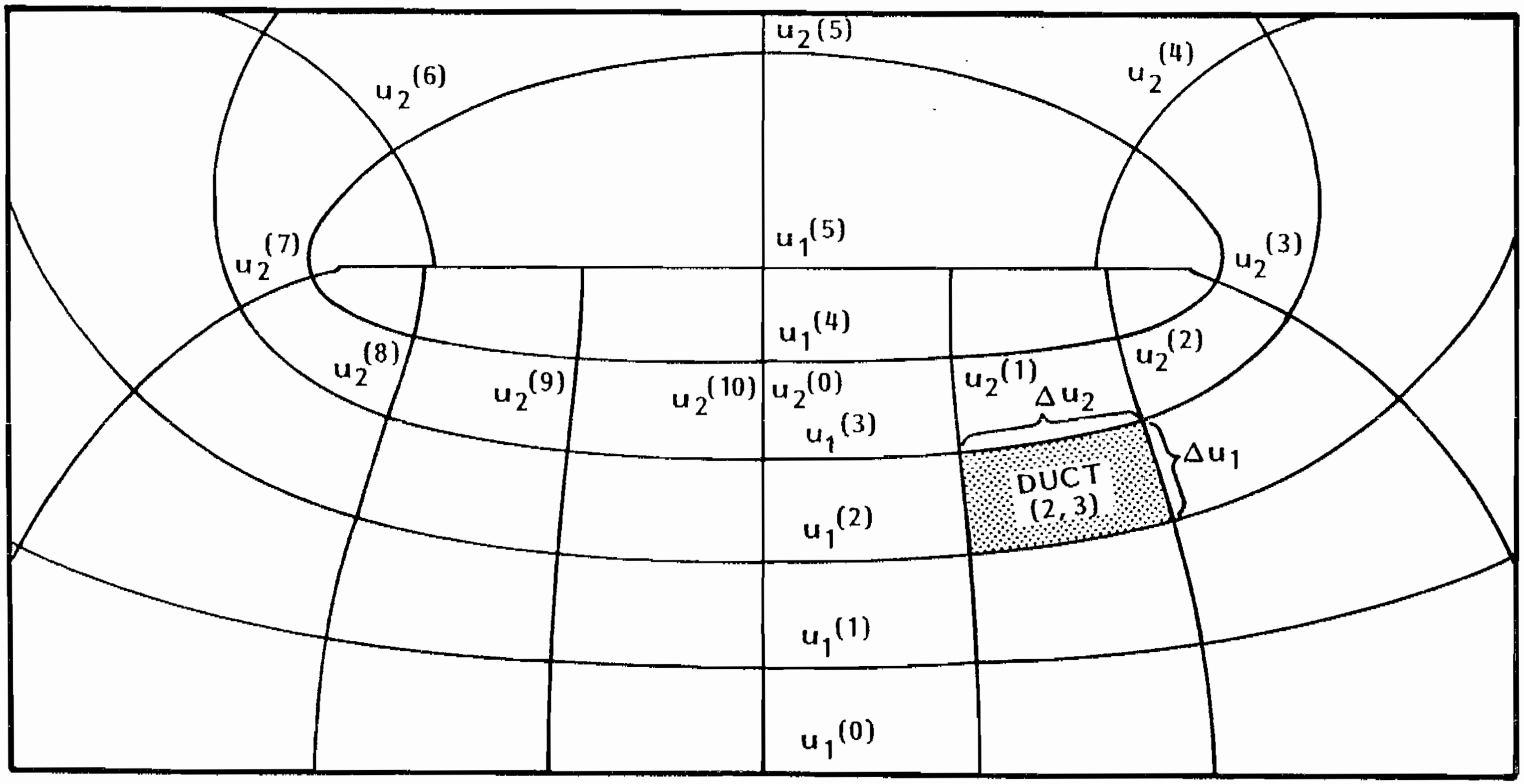
In this section we consider the general case of transport of waves through a set of ducts. Consider some inhomogeneous TEM plane wave such as that existing on a two-conductor transmission line (in a homogeneous isotropic medium). As indicated in Fig. 3.1 this might be a finite-width conducting sheet over a ground plane (ideally infinitely wide) [8,9,10].

As is well known [10] such an inhomogeneous TEM wave can be described by a conformal transformation with a complex potential

$$\begin{aligned} w(\varsigma) &= u_1 + j u_2 \\ \varsigma &= x + jy \\ u_1 &\equiv \text{electric potential} \\ u_2 &\equiv \text{magnetic potential} \\ u_3 &= z \\ \vec{r} &= (x, y, z) \equiv \text{Cartesian coordinates} . \end{aligned} \tag{3.1}$$

Our coordinate system (u_1, u_2, u_3) is thus based on a TEM mode in which u_1 and u_2 correspond to electric and magnetic potentials, and u_3 is a generalized direction of propagation. Thus, for example, if we consider a coaxial cylinder, then u_3 could be chosen as the z-coordinate. We could, of course, have an E-field with u_1, u_2 components, and also an H-field with components in u_1, u_2 directions. However, in this event we would simply redefine our coordinates so as to obtain only an E_1 component in a u_1 direction and an H_2 component in a u_2 direction. In any case we have

$$E_2 = E_3 = 0 \ , \ H_1 = H_3 = 0 \ , \tag{3.2}$$



Example with N = 5, M = 10

Figure 3.1. Potential Distribution for Finite Width Strip Over Infinite Ground Plane

and our wave propagates in the u_3 direction and there is no dependence on u_3 except for a delay.

Now in the conformal transformation as indicated in Fig. 3.1 let us consider a small portion (designated by indices (n, m)) given by

$$
\begin{aligned}
u_1^{(n-1)} &< u_1 < u_1^{(n)} \\
u_1^{(n)} - u_1^{(n-1)} &= \Delta u_1 \quad \text{for} \quad n = 1, 2, \ldots, N \\
u_2^{(m-1)} &< u_2 < u_2^{(m)} \\
u_2^{(m)} - u_2^{(m-1)} &= \Delta u_2 \quad \text{for} \quad m = 1, 2, \ldots, M \, .
\end{aligned}
\tag{3.3}
$$

Associated with this we have incremental voltages and currents as

$$
\begin{aligned}
\Delta V &= -E_1 h_1 \Delta u_1 = -E_1' \Delta u_1 \\
\Delta I &= -H_2 h_2 \Delta u_2 = -H_2' \Delta u_2 \, .
\end{aligned}
\tag{3.4}
$$

This gives an associated impedance

$$
Z_d = \frac{\Delta V}{\Delta I} = \frac{h_1 E_1}{h_2 H_2} \frac{\Delta u_1}{\Delta u_2} = \frac{E_1'}{H_2'} \frac{\Delta u_1}{\Delta u_2} \, . \tag{3.5}
$$

The wave impedance is

$$
\begin{aligned}
Z_w &= \frac{h_1 E_1}{h_2 H_2} = \frac{h_1}{h_2} \sqrt{\frac{\mu}{\varepsilon}} \\
\mu &= \text{medium permeability} \\
\varepsilon &= \text{medium permittivity}
\end{aligned}
\tag{3.6}
$$

giving

$$
Z_d = Z_w \frac{\Delta u_1}{\Delta u_2} \, . \tag{3.7}
$$

Note that

$$N\Delta u_1 = V \equiv \text{potential between the two conductors}$$
$$M\Delta u_2 = Z_w I \equiv \text{change in magnetic potential on contour surrounding conducting strip} . \qquad (3.8)$$

The characteristic impedance of the transmission line is

$$Z_c \equiv \frac{V}{I} = Z_w \frac{N\Delta u_1}{M\Delta u_2}$$
$$= Z_w \frac{\text{difference of } u_1 \text{ between conductors}}{\text{change in } u_2 \text{ around a conductor}} . \qquad (3.9)$$

Consider now a small region defined by Δu_1 and Δu_2 as in (3.3). On some surface of constant u_3 designated as S_1 let each such incremental region be the entrance to what we shall call a duct. Such a duct shall be defined so as to have certain properties. We want the wave incident on it at S_1 to pass into each duct without reflection. With electromagnetic waves in each duct considered to be propagating independently from those in other ducts, then the input impedance of each duct must be Z_d, so that the wave incident on S_1 as in Fig. 3.2 is completely transported into the set of $N \times M$ ducts.

This concept of a duct can be realized as illustrated in Fig. 3.3. It is a special kind of transmission line for which we take u_3 (for a given (u_1, u_2) appropriate to the particular duct as defined at its beginning on S_1) as the longitudinal coordinate. Retaining Δu_1 and Δu_2 as the cross section dimensions we have a local impedance

$$Z_d = Z_w \frac{(\Delta u_1) h_1}{(\Delta u_2) h_2} == \sqrt{\frac{\mu}{\varepsilon}} \frac{\Delta u_1}{\Delta u_2} \frac{h_1}{h_2} \qquad (3.10)$$

where $\mu, \varepsilon, \Delta u_1$ and Δu_2 can now be functions of u_3 along the duct. As we desire waves to propagate along a duct without reflection we require

$$Z_d = \sqrt{\frac{\mu}{\varepsilon}} \left(\frac{\Delta u_1}{\Delta u_2}\right) \frac{h_1}{h_2} \neq \text{ function of } u_3 \qquad (3.11)$$

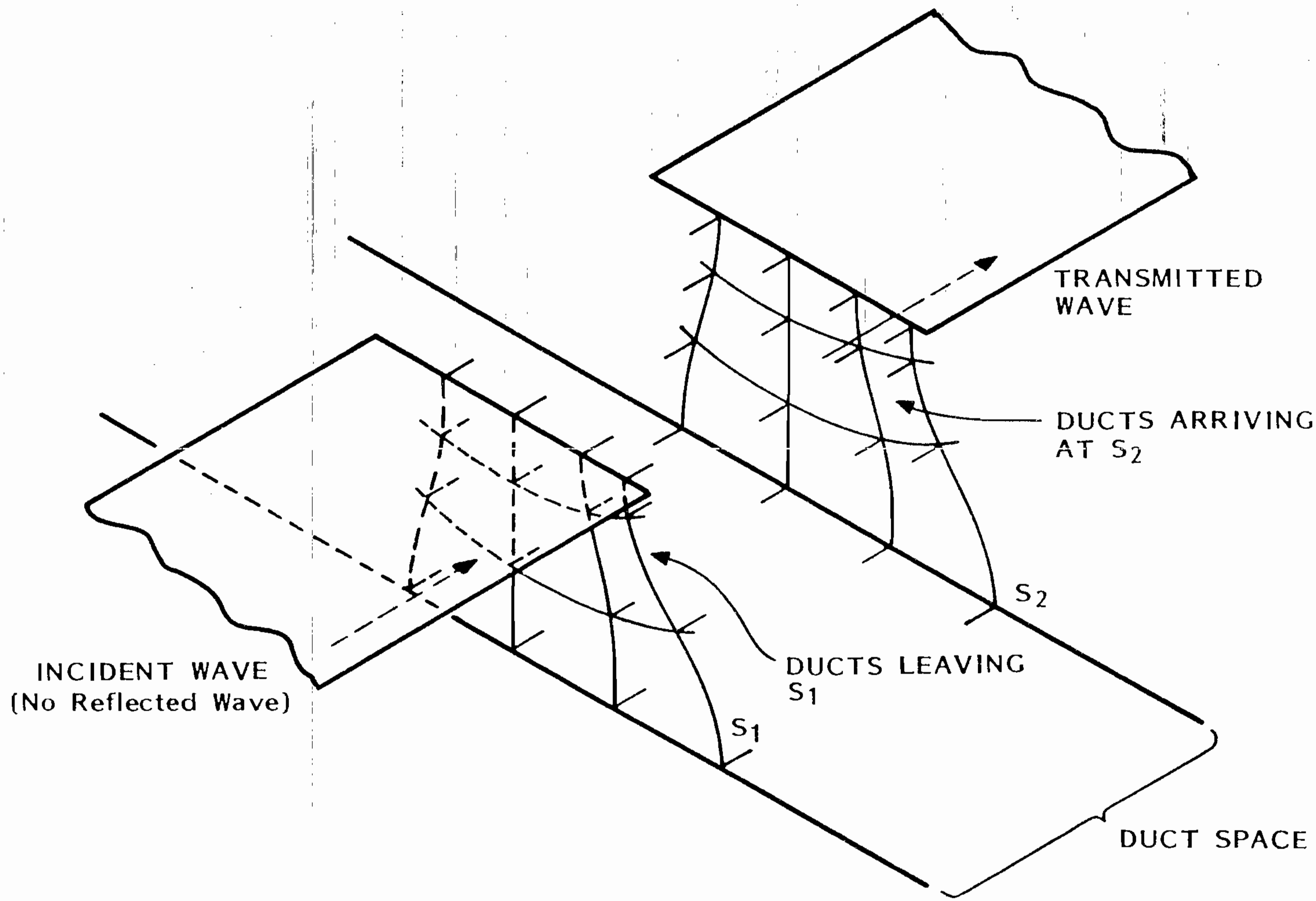


Figure 3.2: Duct Space

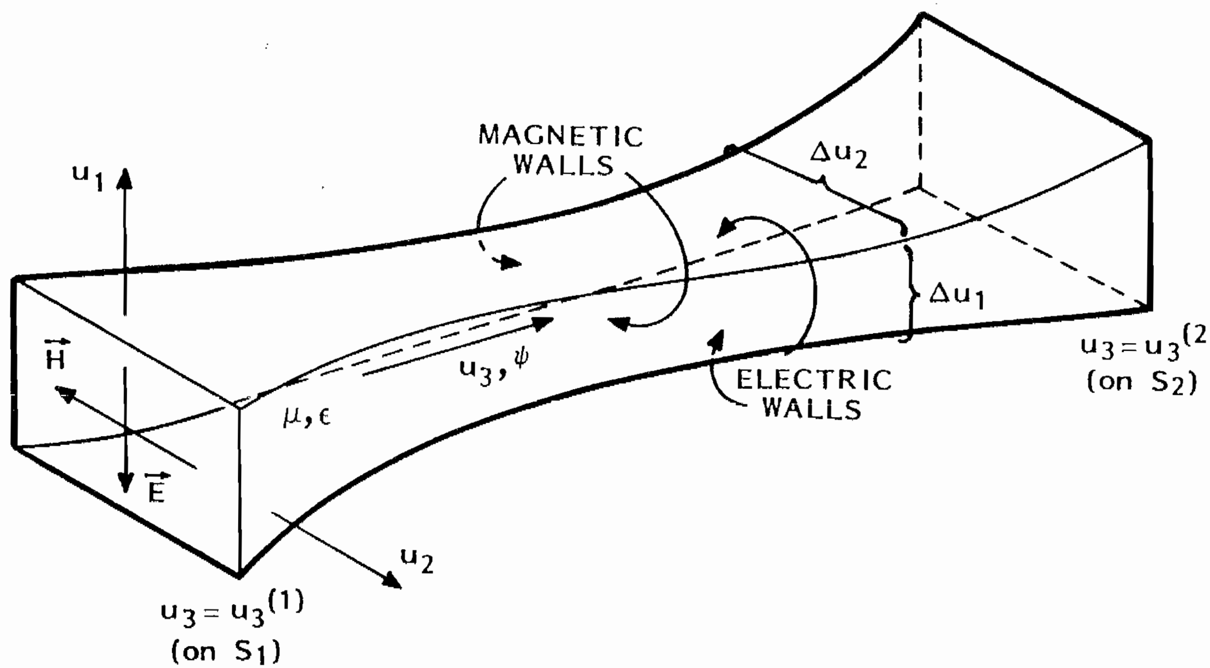


Figure 3.3: Duct (Typical)

constraining the u_3 dependence of these parameters. Thus the constraint given by (3.11) is an impedance matching condition arising from the requirement that the input impedance for each duct be equal to Z_d.

However, suppose these ducts are to be of finite length and to be used to reconstruct a wave on another transmission line at some constant u_3 surface designated S_2 as in Fig. 3.2. Then not only must the duct impedance Z_d match into an appropriate $(\Delta u_1, \Delta u_2)$ on S_2, but also the waves on all the ducts must arrive on S_2 at the same time to reconstruct a TEM mode on the transmission line without generating other modes on the transmission line and sending reflections back into the ducts. If u_3 is arc length in meters, and

$$v(u_3) = \frac{1}{\sqrt{\mu\varepsilon}} \tag{3.12}$$

is the propagation speed in a duct, and

$$\begin{aligned} u_3^{(1)} &\equiv u_3 \text{ on } S_1 (\text{independent of } u_1, u_2) \\ u_3^{(2)} &\equiv u_3 \text{ on } S_2 (\text{independent of } u_1, u_2) \end{aligned} \tag{3.13}$$

then the transit time for the (n,m)th duct is

$$T_{n,m} = \int_{u_3^{(1)}}^{u_3^{(2)}} v^{-1}(u_3') h_3 du_3' \tag{3.14}$$

where now one should note that speed (or velocity) is interpreted with respect to the u_3 coordinate (as is μ and ε) which may or may not be in meters.

In a more general form we can interpret the speed in a duct in terms of an arc-length parameter which we take as ψ with speed taken on a wavefront with ψ in meters as

$$v = \frac{1}{\sqrt{\mu\varepsilon}} \tag{3.15}$$

allowing for some difference between u_3 and ψ. Noting from (2.1) and (2.2) that

$$h_3 = \left| \frac{\partial \psi}{\partial u_3} \right| \Bigg|_{u_1, u_2 \text{ constant}} \tag{3.16}$$

this gives a transit time through a duct

$$
\begin{aligned}
T_{n,m} &= \int_{\psi_1}^{\psi_2} v^{-1}(\psi)d\psi = \int_{\psi_1}^{\psi_2} \sqrt{\mu(\psi)\varepsilon(\psi)}d\psi \\
&= \int_{u_3^{(1)}}^{u_3^{(2)}} \sqrt{\mu\varepsilon}\frac{d\psi}{du_3}du_3 = \int_{u_3^{(1)}}^{u_3^{(2)}} \sqrt{\mu\varepsilon}h_3 du_3 \\
& \text{for } \frac{\partial\psi}{\partial u_3} > 0 \, .
\end{aligned}
\tag{3.17}
$$

Now let

$$
T_{n,m} \neq \text{ function of } (n,m) \text{ or } (u_1, u_2) \, . \tag{3.18}
$$

This gives the constraint that

$$
\int_{u_3^{(1)}}^{u_3^{(2)}} \sqrt{\mu\varepsilon}h_3 du_3 \neq \text{ function of } (n,m) \text{ or } (u_1, u_2) \, . \tag{3.19}
$$

Combined with (3.11) these two relations give the constraints for a duct. The condition (3.19) is a transit-time constraint, while (3.11) is an impedance constraint.

As in Fig. 3.3, one can specify a duct by boundary conditions

$$
\begin{aligned}
\vec{1}_2 \cdot \vec{E} &= E_2 = 0 \text{ on surfaces of constant } u_2 \\
\vec{1}_1 \cdot \vec{H} &= H_1 = 0 \text{ on surfaces of constant } u_1 \, .
\end{aligned}
\tag{3.20}
$$

With these constraints and $(\Delta u_1, \Delta u_2)$ sufficiently small (both electrically, and by comparison to the path curvature (along ψ) of the duct), and with changes in Δu_1 and Δu_2 small over a wavelength (at the highest frequencies of interest), and with small changes in Δu_1 and Δu_2 with respect to changes in u_3, then a duct can be considered an ideal TEM transmission line. Note that with the assumed boundary conditions in (3.20) then

$$
\begin{aligned}
u_1 \text{ constant surfaces } &\equiv \text{ ideal electric boundaries} \\
&\quad \text{(perfectly electric conducting surfaces)} \\
u_2 \text{ constant surfaces } &\equiv \text{ ideal magnetic boundaries} \\
&\quad \text{(perfectly magnetic conducting boundaries)} \, .
\end{aligned}
\tag{3.21}
$$

Viewed another way we have

$$
\begin{aligned}
[\vec{E} \times \vec{H}] \cdot \vec{1}_1 = 0 &\equiv \text{ Poynting vector normal to } u_1 \text{ boundaries} \\
[\vec{E} \times \vec{H}] \cdot \vec{1}_2 = 0 &\equiv \text{ Poynting vector normal to } u_2 \text{ boundaries}
\end{aligned} \tag{3.22}
$$

so that energy flow through duct boundaries is zero. Let us refer to these duct boundaries as electric and magnetic walls.

The region between S_1 and S_2 we call a duct space. The cross section of a duct of sides Δu_1 and Δu_2 is to move from S_1 to S_2 is such a way that the impedance and total transit time is preserved. However, within the constraint of (3.11) one can shrink both Δu_1 and Δu_2 as u_3 increases, thereby separating the ducts from one another. Given that no electromagnetic energy can flow through the electric and magnetic boundaries of each duct, then consider the region between S_1 and S_2 as in Fig. 3.2 which we might call a duct space. Part of this region (between the ducts) has no electromagnetic energy. A wave in one duct does not affect a wave in another duct except via S_1 and S_2. This allows a considerable variation in the routing of ducts in the duct space without interfering one with another.

In a more general sense S_1 and S_2 need not be surfaces of constant u_3. What is important is that

(a) on S_1 and S_2 differential impedance be matched between external waves and ducts, and

(b) the transit time from the wave before S_1 (say on some u_3 before the lens) be matched to the wave after S_2 (say on some u_3 after the lens.

This requires that one consider the possibility that S_1 and S_2 are not in general orthogonal to $\vec{1}_3$ on the various sides of the two lens boundaries. Examples of this are found in [3], [4], [5]. From (a) above this requires that the differential voltage and current relationships in

(3.4) allow for the projection of $\vec{1}_1$ for the electric field and $\vec{1}_2$ for the magnetic field on the lens boundaries be matched across the boundaries since it is the tangential components of E_1 and H_2 that must be matched through the lens boundaries.

For this purpose one can define a tangential dyad on S_1 and S_2 as

$$\begin{aligned} \overleftrightarrow{1}_{t_{\frac{1}{2}}} &\equiv \overleftrightarrow{1} - \vec{1}_{S_{\frac{1}{2}}} \vec{1}_{S_{\frac{1}{2}}} \\ \overleftrightarrow{1} &\equiv \vec{1}_x \vec{1}_x + \vec{1}_y \vec{1}_y + \vec{1}_z \vec{1}_z \\ &= \vec{1}_1 \vec{1}_1 + \vec{1}_2 \vec{1}_2 + \vec{1}_3 \vec{1}_3 \\ &\equiv \text{identity dyad} . \end{aligned} \tag{3.23}$$

If the u_i are all defined so as to be continuous through the lens boundaries, then in general the h_1 and h_2 are discontinuous through these boundaries. Basically the h_1 and h_2 are discontinuous so that E_1' and H_2' are matched, but E_1 and H_2 are not matched (in general). So basically we require that

$$\left. \begin{array}{l} E_1 \vec{1}_1 \cdot \vec{1}_t \\ H_2 \vec{1}_2 \cdot \vec{1}_t \end{array} \right\} \text{are continuous through } S_1 \text{ and } S_2 . \tag{3.24}$$

In a more general sense if the duct size is allowed to abruptly change at the lens boundaries (with electric and/or magnetic boundaries on S_1 and S_2 as required), then one can interpret (3.4) as requiring both ΔV and ΔI to be conserved so that

$$\left. \begin{array}{l} E_1 h_1 \vec{1}_1 \cdot \overleftrightarrow{1}_t \\ H_2 h_2 \vec{1}_2 \cdot \vec{1}_t \end{array} \right\} \text{are continuous through } S_1 \text{ and } S_2 . \tag{3.25}$$

The transit-time matching is a macroscopic requirement (except outside the lens being on a local or differential basis). This must be considered in the overall geometry. In particular, (3.17) through (3.19) can be modified so that $u_3^{(1)}$ and $u_3^{(2)}$ can be considered as functions of u_1, u_2, and $T_{n,m}$ can be modified to allow for the required transit time through the lens as

$$T_{n,m} = t\Big|_{\substack{\text{required departure} \\ \text{on } S_1}} - t\Big|_{\substack{\text{required arrival} \\ \text{on } S_2}} . \tag{3.26}$$

4 Use of Ducts to Reorder Position on Wavefront

If as in Figs. 3.2 and 3.3 we have a duct starting on S_1 and ending on S_2 there are certain requirements which must be met. Impedances must be matched, transit times conserved, and the wave must have the same polarization on the S_2 as on entering S_1 to synthesize the wave at the S_2 end. There can moreover be no power flow through a duct boundary (per (3.22)).

Are there other limitations which must be considered? Could we, for example, tie a duct in a knot, or could we reorder positions on wave fronts as in Fig. 4.1. In this figure we have ducts labeled A, B, C, D for entering waves on S_1 and a reordering in which A, B, C, D are mapped to respectively positions C, A, B, D on S_2. Thus within the limits of our assumptions, what strange things can happen? Note that duct cross sections are small compared to wavelength, that $T_{n,m}$ is the same for all ducts (or as in (3.26)), and Z_d is invariant to position along a duct. Furthermore the duct cross sections are shrunk for positions in the duct space away from S_1 and S_2 so that each duct may pass between the others.

This is a strange beast as can be seen by the following gedankenexperiment. Consider sufficiently low frequencies that the fields are quasi-static. We start out as in Fig. 4.2 by imagining a test electric charge (say an electron) starting out at a point P_1 with potential u_{P_1} and cross S_1 to travel down a particular duct. Say as in Fig. 4.1 this were duct C. On crossing S_2 the test charge is at a new potential, say u_{P_2}, appropriate to the second transmission line. As in Fig. 4.2 this corresponds to an increase in potential u_1. Now go from P_2 to P_3 (in a direction out of the page) to a position to enter duct A on S_2 in Fig. 4.1. The potential here, u_{P_3}, is the same as u_{P_2}. Next enter duct A, crossing S_2, and leave crossing S_1 to P_4 with potential u_{P_4} as appropriate to the first transmission line. As in Fig. 4.2, back in the first transmission line the test charge has risen in potential. If the

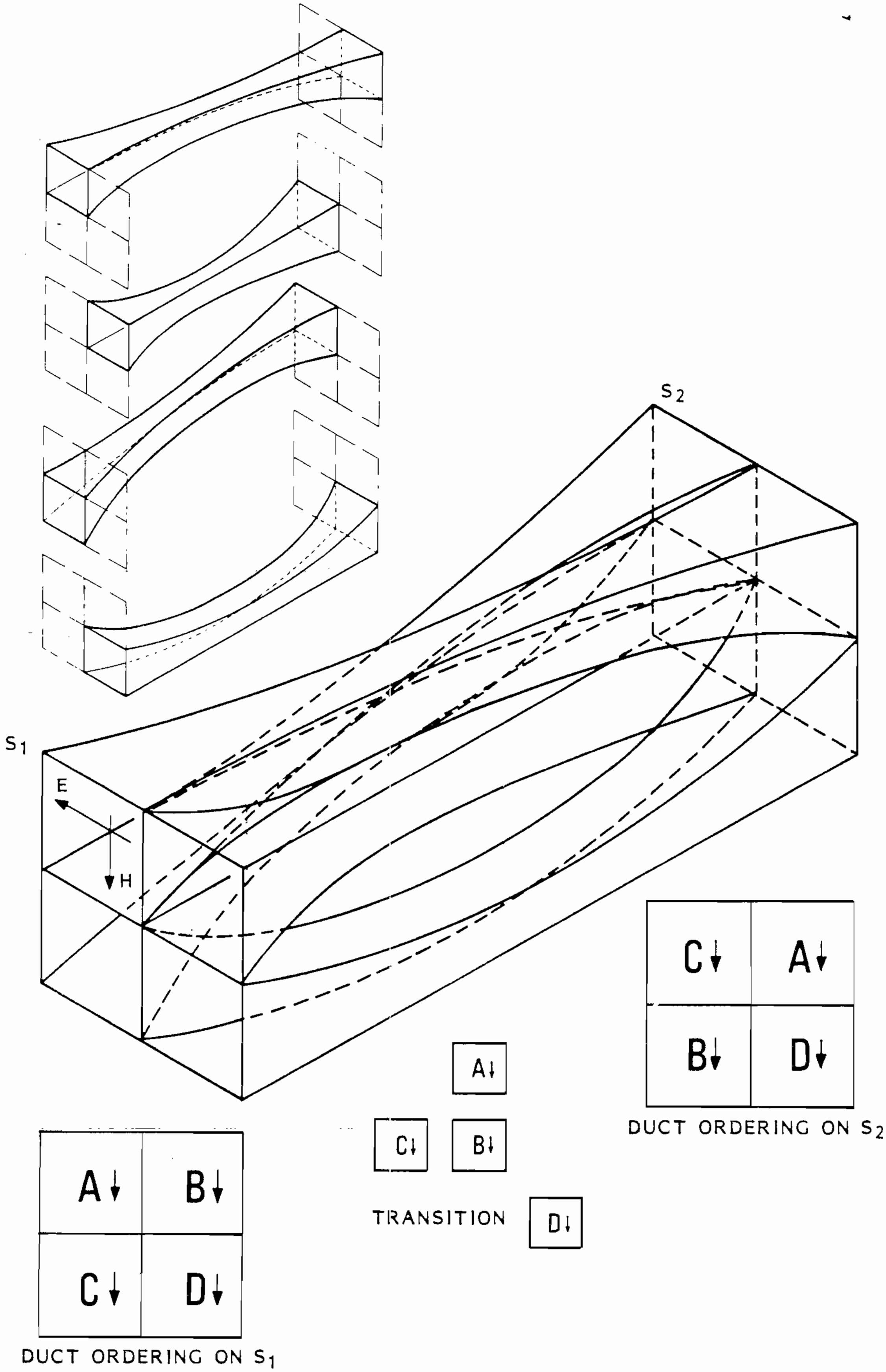


Figure 4.1: Reordering Ducts in Passing Through Duct Space

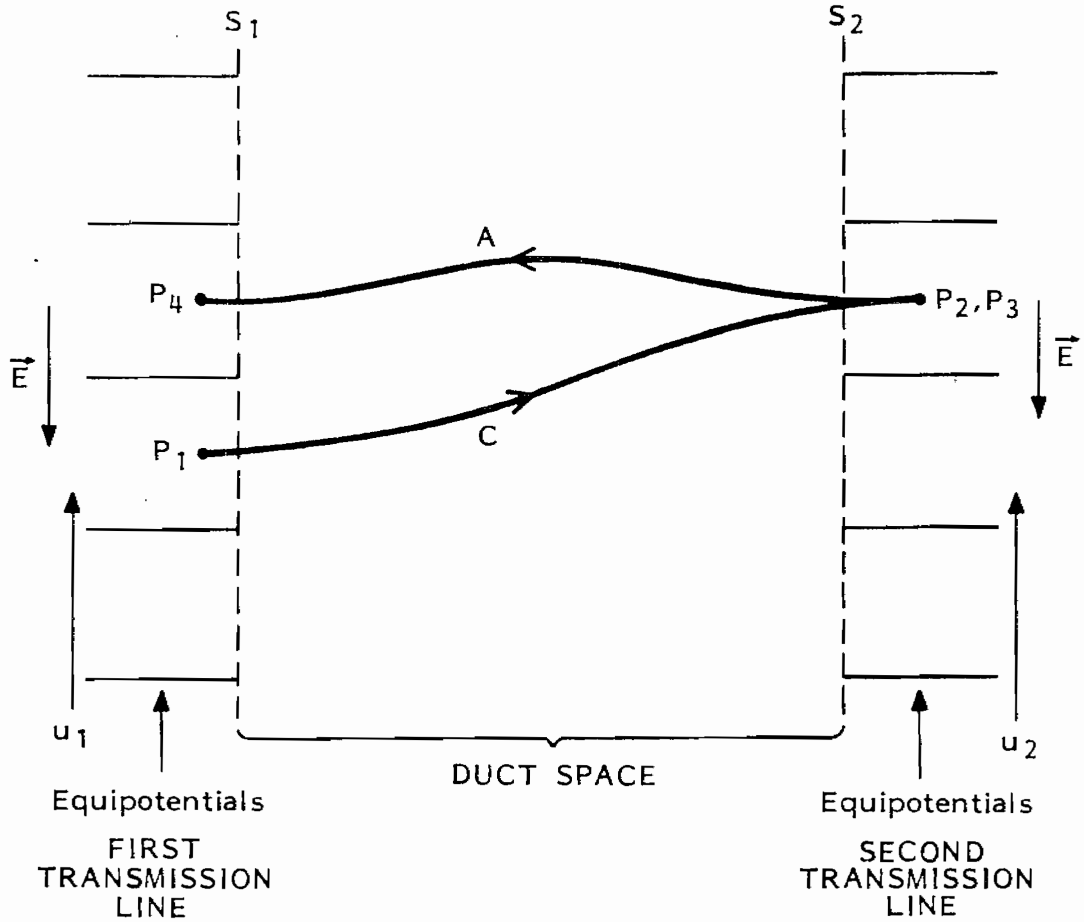


Figure 4.2: Test Charge Meandering Through Ducts

test charge is moved directly from P_4 to P_1 energy will be given to or taken from the field.

To explain where this comes from we must reexamine our hypotheses and decide what constraints need to be imposed to avoid this problem. Clearly electric potential was not conserved, even quasi-statically, in the (lens) region because of magnetic barriers. That is, our path went around magnetic currents. Put otherwise,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} - \vec{J}_m \neq \vec{0} \, . \tag{4.1}$$

The resolution to our apparent contradiction lies in how we viewed the ducts. We have magnetic (and also electric) currents induced in the duct walls. Hence we must insist that no path through the lens (i.e., the region between surfaces S_1 and S_2) will allow one to come back to a different potential. That is, energy must be conserved. Hence ducts cannot be switched "vertically". These considerations will lead us in the next section to the idea of removing magnetic boundaries in ducts by removing the side walls of ducts.

We could, of course, consider a dual problem in which we imagine a test magnetic charge following a path from S_1 to S_2 (say via duct A as in Fig. 4.1) and back to S_1 (say via duct D) again. As in the previous case, energy is gained or lost and there is a difference in magnetic potential. In this case we will insist that ducts cannot be switched "horizontally". Thus in a later section we consider the removal of electric boundaries in ducts by removing the tops and bottoms of ducts.

5 Restriction to the Case of No Magnetic Currents on Magnetic Boundaries

In order to keep waves confined to ducts we have made the walls either electrically or magnetically perfectly conducting (as in (3.22)). Since magnetically perfectly conducting boundaries are hard to come by, one may wish to do without them. As indicated in Fig. 5.1, the presence of magnetic walls allows ducts to be separated by providing for a discontinuity in the tangential electric field, the electric field being zero outside the duct (but still in the duct space).

5.1 Removal of Magnetic Boundaries to Produce Jackets

Now if we wish to eliminate the magnetic currents there are various ways one can look at this. As indicated in Fig. 5.1 one can consider one of the Maxwell equations in either differential or integral forms as

$$\begin{aligned} \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} - \vec{J}_m \\ \oint_{C'_m} \vec{E} \cdot d\vec{\ell} &= -\int_{S'_m} \left[\frac{\partial \vec{B}}{\partial t} + \vec{J}_m \right] \cdot d\vec{S} \end{aligned} \tag{5.1}$$

where S'_m and its boundary contour C'_m are illustrated in Fig. 5.1 which shows the effect of magnetic currents on walls in separating waves in ducts.

By hypothesis the magnetic field is zero outside of the ducts. Letting C'_m be parallel to the electric field just inside the two ducts illustrated, then the difference in the tangential electric field between two adjacent ducts is just given by

$$[\vec{E}_1 - \vec{E}_2] \cdot \vec{1}_1 = -[\vec{J}_{s_{m_1}} - \vec{J}_{s_{m_2}}] \cdot \vec{1}_3 \tag{5.2}$$

where the unit vectors correspond to the (u_1, u_2, u_3) coordinate system as in Fig. 5.1.

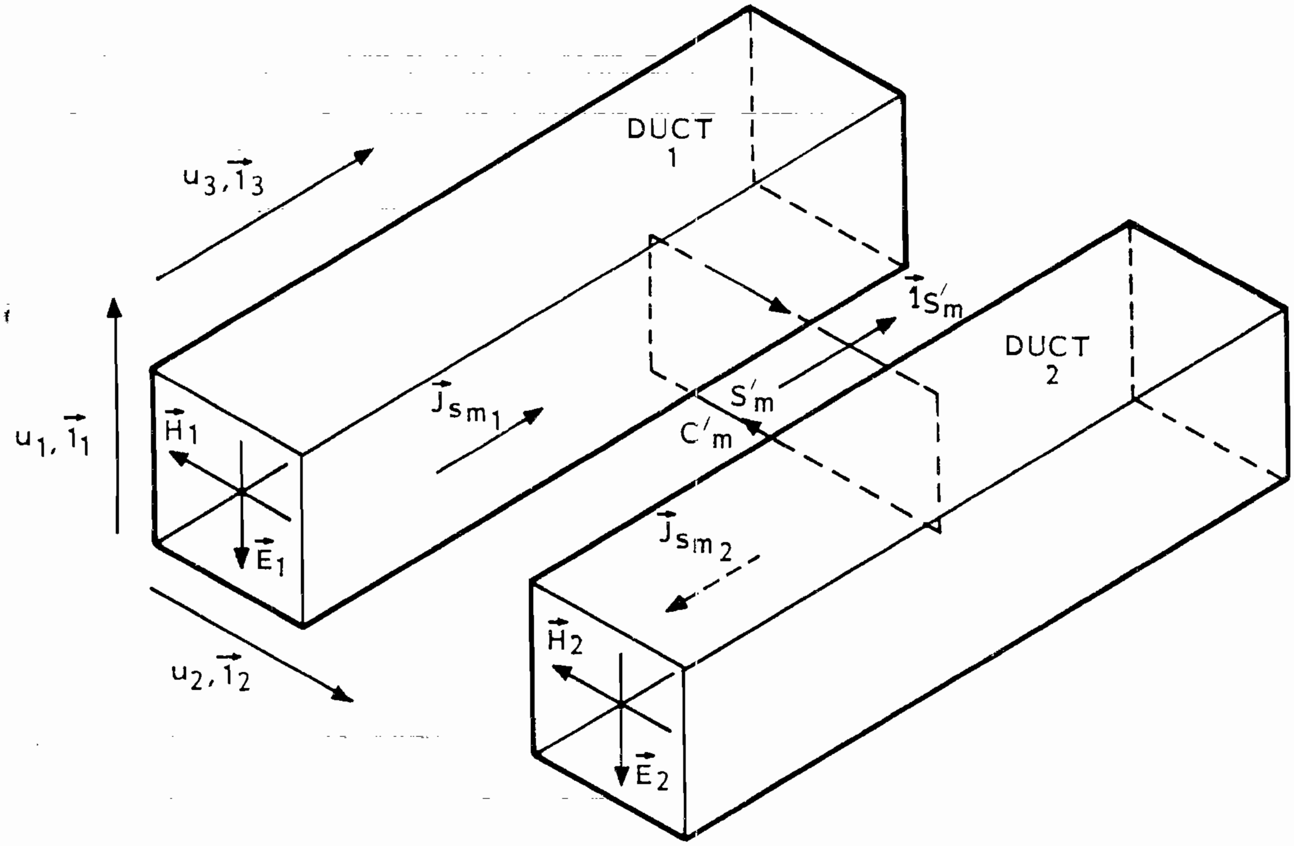


Figure 5.1: Effect of Magnetic Currents on Walls in Separating Waves in Ducts

In this form let the distance between the two ducts shrink to zero with the walls containing $\vec{J}_{s_{m_1}}$ and $\vec{J}_{s_{m_2}}$ becoming coincident. Then (5.2) represents the discontinuity in the tangential electric field between the two ducts across the common wall. If the surface magnetic currents are made to be zero (effectively merging the two ducts) then we have

$$\vec{E}_1 = \vec{E}_2 \tag{5.3}$$

and, of course, consequently (from (3.6))

$$\vec{H}_1 = \vec{H}_2 \tag{5.4}$$

noting the vanishingly small cross section of a duct.

Let us now consider the impedance-matching and transit-time-conservation requirements. Certainly within a duct we must have

$$Z_d = \frac{h_1 \Delta u_1}{h_2 \Delta u_2}\sqrt{\frac{\mu}{\varepsilon}} \neq \text{ function of } u_3 \tag{5.5}$$

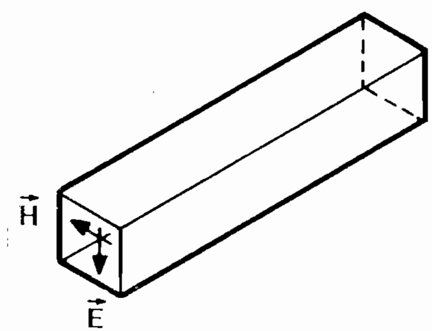
since we have propagation in the u_3 direction, and differential impedances are required to be matched. The same condition should then hold in a jacket which we describe below. Moreover, starting at $u_3 = 0$ and following a path in a duct to $u_3 \neq 0$, the transit time T is

$$T = \int_0^{u_3} \sqrt{\mu\varepsilon} h_3 du_3 \neq \text{ function of } u_2 \ . \tag{5.6}$$

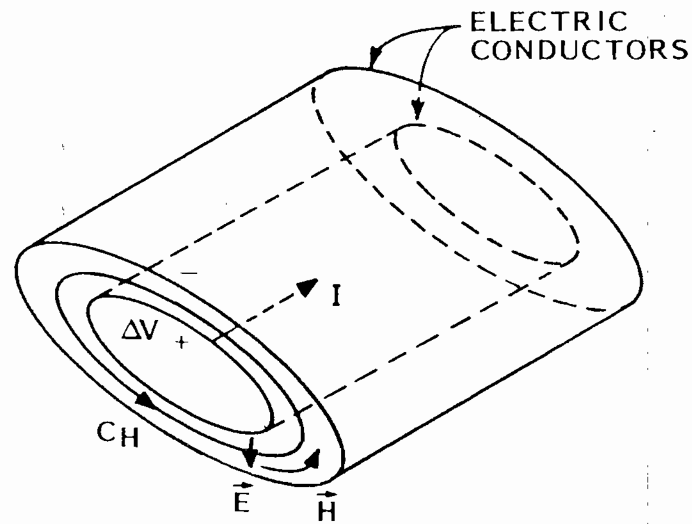
Thus transit time conservation in a duct, and hence a jacket also requires

$$\mu\varepsilon h_3^2 \neq \text{ function of } u_2 \ . \tag{5.7}$$

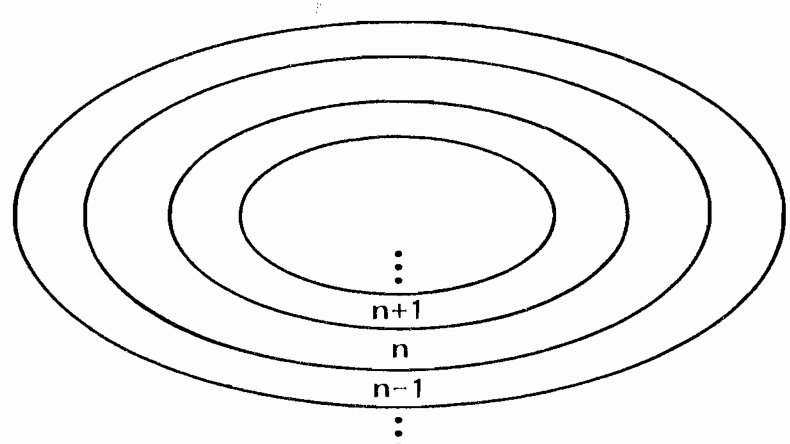
Thus, if magnetic boundaries (i.e., "side" boundaries of a duct) are removed, the duct ensemble becomes a "jacket" (or E hyperduct). This may be thought of as something resembling "parallel" (or curved) plates. The evolution is described as follows as in Fig. 5.2. The jacket is referred to as an E 2-lens, which can be regarded as a two-dimensional space.



a) Duct



b) Jacket (Electric Hyperduct, E 2-lens)



c) Hyperjacket (E 3-lens)

Figure 5.2: Evolution of Duct by Removing Magnetic Boundaries

The totality of jackets, which can be regarded as a composite E 3-lens (or hyperjacket) is an E lens and is composed (in the limit) of an infinite number of such two-dimensional spaces. Following the same convention as in Fig. 5.2 jacket n encloses jacket $n+m$, where $m \geq 1$. The ensemble of these jackets is what we might term an E 3-lens or a hyperjacket.

Note the presence of perfectly conducting boundaries on surfaces of constant u_1 which prevent propagation in the $\vec{1}_1$ direction. While these surfaces can be of zero thickness to allow no space between one jacket and the next, the converse is also possible. Except on S_1 and S_2 where the electric conductors from jacket n must connect to those from jackets $n-1$ and $n+1$ to keep electromagnetic energy from the space between the jackets, these surfaces can in principal be separate. By adjustment at h_1 in the jackets, this spacing can be allowed for.

5.2 TEM Waves in a Jacket

If we now assume, as in Section 2.5, but now in some nth jacket, that we have field components in the form

$$
\begin{aligned}
E_1' &= h_1 E_1 = E_0' f\left(t - u_3/c'\right) \\
E_2' &= 0\ ,\ E_3' = 0 \\
H_2' &= h_2 H_2 = \frac{E_0'}{Z_0'} f\left(t - u_3/c'\right) \\
H_1' &= H_3' = 0 \\
\mu' &= \text{constant in } n\text{the jacket}\ ,\ \varepsilon' = \text{ constant in } n\text{th jacket} \\
Z_0' &= \sqrt{\frac{\mu'}{\varepsilon'}} = \text{ constant in } n\text{th jacket} \\
c' &= \frac{1}{\sqrt{\mu'\varepsilon'}} = \text{ constant in } n\text{th jacket}
\end{aligned}
\tag{5.8}
$$

so that our TEM wave propagates in the u_3 direction, then the Maxwell equations assume the form

$$\begin{aligned}\frac{\partial E_1'}{\partial u_3} &= -\mu' \frac{\partial H_2'}{\partial t} \\ \frac{\partial H_2'}{\partial u_3} &= -\varepsilon' \frac{\partial E_1'}{\partial t}\ .\end{aligned} \tag{5.9}$$

Moreover the formal medium is described, as in (2.41) by,

$$\begin{aligned}\varepsilon' &= \frac{h_2 h_3}{h_1}\varepsilon_1 \\ \mu' &= \frac{h_1 h_3}{h_2}\mu_2\ .\end{aligned} \tag{5.10}$$

Now one jacket (the nth) is characterized by

$$\begin{aligned}&u_1^{(n-1)} < u_1 < u_1^{(n)} \\ &u_1^{(n)} - u_1^{(n-1)} = \Delta u_1\ .\end{aligned} \tag{5.11}$$

Essentially in (3.3) we have united all ducts for the same n and removed m from consideration. We have an incremental voltage (jacket voltage)

$$\Delta V = -E_1' \Delta u_1 = -E_1 h_1 \Delta u_1 \tag{5.12}$$

as before. However, the jacket current is the sum of all the duct currents in the nth jacket as

$$\begin{aligned}I = \sum_{m=1}^{M} \Delta I &= -\int_{u_2^{(0)}}^{u_2^{(M)}} H_2' du_2 = -H_2' u_2^{(M)} = -\oint_{C_H} \vec{H} \cdot d\vec{\ell} \\ u_2^{(M)} &= \text{total change in } u_2 \text{ around jacket}\end{aligned} \tag{5.13}$$

where the limit as $\Delta u_2 \to 0$ ($M \to \infty$) is taken. Note as in Fig. 5.2 the contour C_H is contained on a surface of constant u_3 in the nth jacket. Here both I and ΔV are functions of u_3, but not u_2. The convention here is that increasing potential is in the direction of increasing u_1, and the current on the high-potential electrode is in the direction of increasing u_3.

With this we have the jacket impedance

$$\Delta Z = \frac{\Delta V}{I} = \frac{E_1' \Delta u_1}{H_2' \int_{u_2^{(0)}}^{u_2^{(M)}} du_2} = Z_0' \frac{\Delta u_1}{u_2^{(M)}} \ . \tag{5.14}$$

Note that ΔZ is independent of u_3 (as Z_d is independent of u_3) and also independent of u_2 (over which (5.13) integrates and ΔV is independent). ΔZ is only a function of u_1, or better n concerning the nth jacket. So we have

$$Z_0' = \sqrt{\frac{\mu'}{\varepsilon'}} = \frac{h_1}{h_2}\sqrt{\frac{\mu_2}{\varepsilon_1}} = \text{ constant in } n\text{th jacket} \ . \tag{5.15}$$

In addition we have from the velocity

$$\frac{1}{c'} = \sqrt{\mu'\varepsilon'} = h_3\sqrt{\mu_2\varepsilon_1} = \text{ constant in } n\text{th jacket} \ . \tag{5.16}$$

Note that (5.15) and (5.16) correspond to (2.42) in Section 2.5, except that their status as constants is only in each jacket separately.

5.3 Case of μ and ε Singly or Both Independent of u_2 and u_3 in a Jacket

In any jacket only the u_2 and u_3 coordinates are of importance, and consequently our first fundamental form is

$$(d\ell)^2 = h_2^2(du_2)^2 + h_3^2(du_3)^2 \ . \tag{5.17}$$

We, of course, have the added geometrical property of closure in one dimension (u_2).

If we now assume μ_2 is a constant, then in any jacket we have

$$\begin{aligned} \frac{h_2}{h_1}\sqrt{\varepsilon_1} &= \sqrt{\varepsilon'\mu_2/\mu'} = \text{ constant} \\ h_3\sqrt{\varepsilon_1} &= \sqrt{\mu'\varepsilon'/\mu_2} = \text{ constant} \end{aligned} \tag{5.18}$$

from (5.15) and (5.16). Particular examples of the above situation appear in [1] and [5], where h_2 is a constant.

Therefore if h_1 is a constant (corresponding to uniformly spaced electric boundaries) we would have

$$\frac{h_2}{h_3} = h_1 \frac{\mu_2}{\mu'} = \text{ constant} \tag{5.19}$$

and (5.17) assumes the form ($\tilde{u}_3 =$ constant times u_3)

$$(d\ell)^2 = h_2^2[(du_2)^2 + (d\tilde{u}_3)^2] \tag{5.20}$$

with associated Gaussian curvature (see Appendix A, Section A.2)

$$\mathcal{K} = -\frac{1}{h_2^2}\left[\frac{\partial^2 \ell n(h_2)}{\partial u_2^2} + \frac{\partial^2 \ell n(h_2)}{\partial \tilde{u}_3^2}\right] . \tag{5.21}$$

Thus for constant curvature $\mathcal{K}$, the sign of the Laplacian of $\ell n(h_2)$ determines whether or not we are in a positively or negatively curved space. If $\mathcal{K} = 0$, our space is Euclidean. In any event, for constant h_1, we have

$$\varepsilon_1 = \frac{1}{h_3^2}\frac{\mu'\varepsilon'}{\mu_2} = \frac{1}{h_2^2}\left[h_1^2 \frac{\varepsilon'\mu_2}{\mu'}\right] \tag{5.22}$$

and the medium is nonuniform. See [1,5] for examples of such nonuniform media.

If, on the other hand, we assume ε_1 is a constant, then (5.15) and (5.16) yield

$$\begin{aligned} \frac{h_1}{h_2}\sqrt{\mu_2} &= \sqrt{\frac{\mu'\varepsilon_1}{\varepsilon'}} = \text{ constant} \\ h_3\sqrt{\mu_2} &= \sqrt{\frac{\mu'\varepsilon'}{\varepsilon_1}} = \text{ constant} \end{aligned} \tag{5.23}$$

and the assumption that h_1 is a constant would imply

$$h_2 h_3 = h_1 \frac{\varepsilon'}{\varepsilon_1} = \text{ constant} . \tag{5.24}$$

Next, if both ε_1 and μ_2 are taken to be constant, then

$$\begin{aligned} h_3 &= \text{ constant} \\ h_2 &= \left(\sqrt{\frac{\varepsilon'\mu_2}{\varepsilon_1\mu'}}\right) h_1 \end{aligned} \tag{5.25}$$

and (5.17) assumes the form (A.31).

Finally, our surface is Euclidean when ε_1, μ_2, and h_1 are constant as in this case (5.17) assumes the form (A.17). Noting that Euclidean surfaces involve only planes, cylinders, cones, and various combinations of these surfaces, particular examples of these lenses are given in references [3] and [4]. In these examples, the jackets are cones.

6 Restriction to Case of No Additional Electric Boundaries

If now electric boundaries of a duct (i.e., top and bottom faces of a duct) are removed and continuity in u_1 enforced, objects which we refer to as "slices" are obtained. Slices are in effect H 2-lenses. As in the case of E 2-lenses, an H 2-lens is a two-dimensional Riemannian space and the totality of these slices yields a composite H-lens, or an infinite set of Riemannian spaces.

6.1 Removal of Electric Boundaries to Produce Slices

Fig. 6.1 shows the effects of electric currents on surfaces in separating waves in ducts. The presence of electric boundaries allows ducts to be separated by providing for a discontinuity in the tangential magnetic field, the magnetic field being zero outside of the duct space.

Thus, if we wish to eliminate electric currents we consider the equation

$$\begin{aligned} \nabla \times \vec{H} &= \vec{J} + \frac{\partial \vec{D}}{\partial t} \\ \oint_{C_e'} \vec{H} \cdot d\vec{\ell} &= \int_{S_e'} \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot d\vec{S} \end{aligned} \tag{6.1}$$

where C_e' is the boundary curve for surface S_e' as shown in Fig. 6.1.

We assume the electric field vanishes outside of the ducts, and so if C_e' is parallel to the magnetic field inside the ducts shown in Fig. 6.1, then the difference of the tangential components of magnetic fields between two adjacent ducts is clearly

$$(\vec{H}_1 - \vec{H}_2) \cdot \vec{1}_2 = [\vec{J}_{s_1} - \vec{J}_{s_2}] \cdot \vec{1}_3 \tag{6.2}$$

where, as usual, the unit vectors correspond to the (u_1, u_2, u_3) coordinate system of Fig. 6.1.

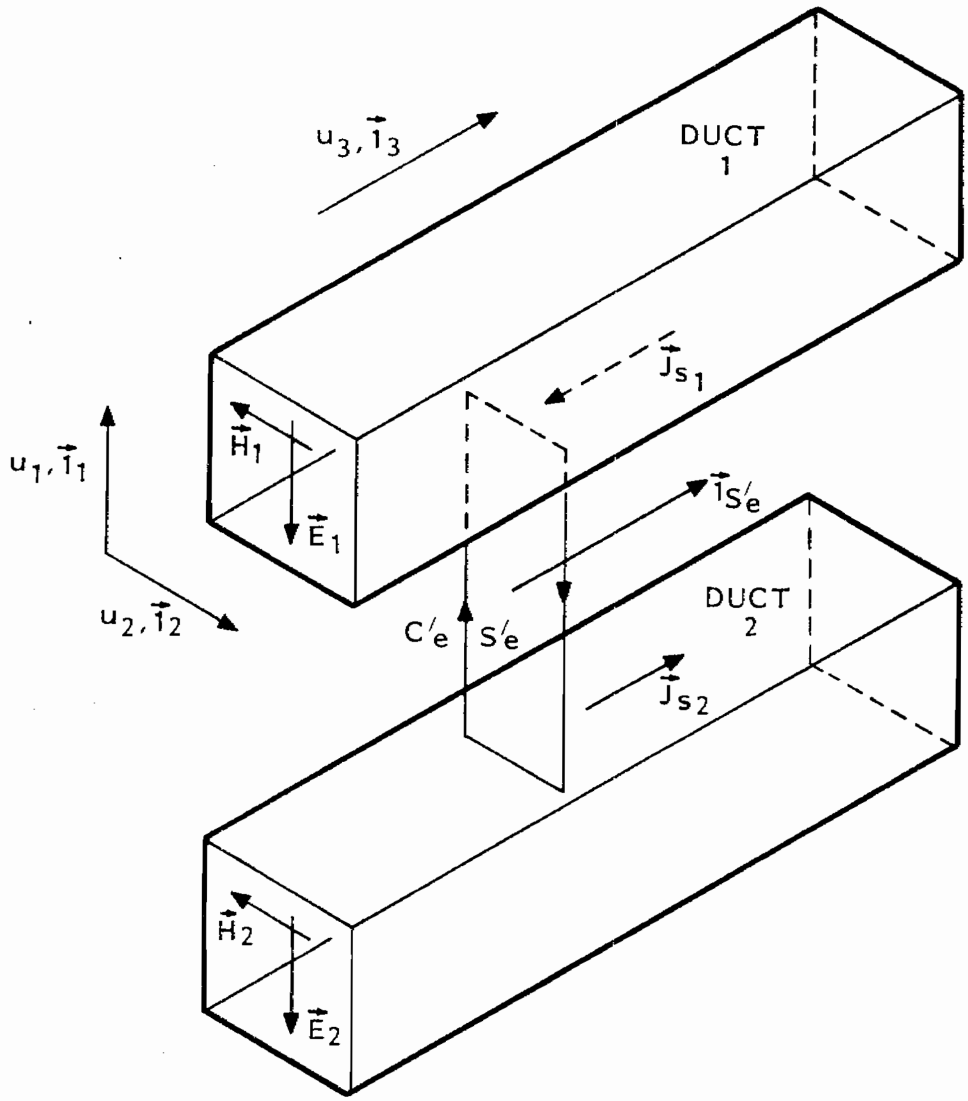


Figure 6.1: Effect of Electric Currents on Walls in Separating Waves in Ducts

If now the distance between ducts shrinks down to zero with the walls containing $\vec{J}_{s_1}, \vec{J}_{s_2}$ becoming coincident, then (6.2) is just the discontinuity in the tangential magnetic field. We then obtain

$$\vec{H}_1 = \vec{H}_2 \tag{6.3}$$

and hence

$$\vec{E}_1 = \vec{E}_2 \tag{6.4}$$

if the currents are zero.

As with the jackets we now consider the impedance-matching and transit-time conservation. Within a duct we have

$$Z_d = \frac{h_1 \Delta u_1}{h_2 \Delta u_2}\sqrt{\frac{\mu}{\varepsilon}} \neq \text{ function of } u_3 \tag{6.5}$$

matching differential impedances. This same condition holds in a slice which is an ensemble of ducts. For transit times, starting at $u_3 = 0$ and following a path in a duct to $u_3 \neq 0$ the transit time T is

$$T = \int_0^{u_3} \sqrt{\mu\varepsilon} h_3 du_3 \neq \text{ function of } u_1 \, . \tag{6.6}$$

So requiring this to be the case for all u_1 gives

$$\mu\varepsilon h_3^2 \neq \text{ function of } u_2 \tag{6.7}$$

as the differential transit-time matching requirement in a slice.

So removing electric boundaries (i.e., "top" and "bottom" boundaries of a duct), except for the ending electric boundaries (the transmission-line conductors), the duct ensemble becomes a slice. As in Fig. 6.2 the evolution is described. The slice is referred to as an H 2-lens (or H hyperduct), which is a kind of two-dimensional space. The totality of slices, which can be referred to as an composite H 3-lens (or H hyperslice) which is an H lens and is composed of an infinite number of such two-dimensional spaces. Now slices are not closed in the sense of jackets, beginning and ending as they do on electric conductors.

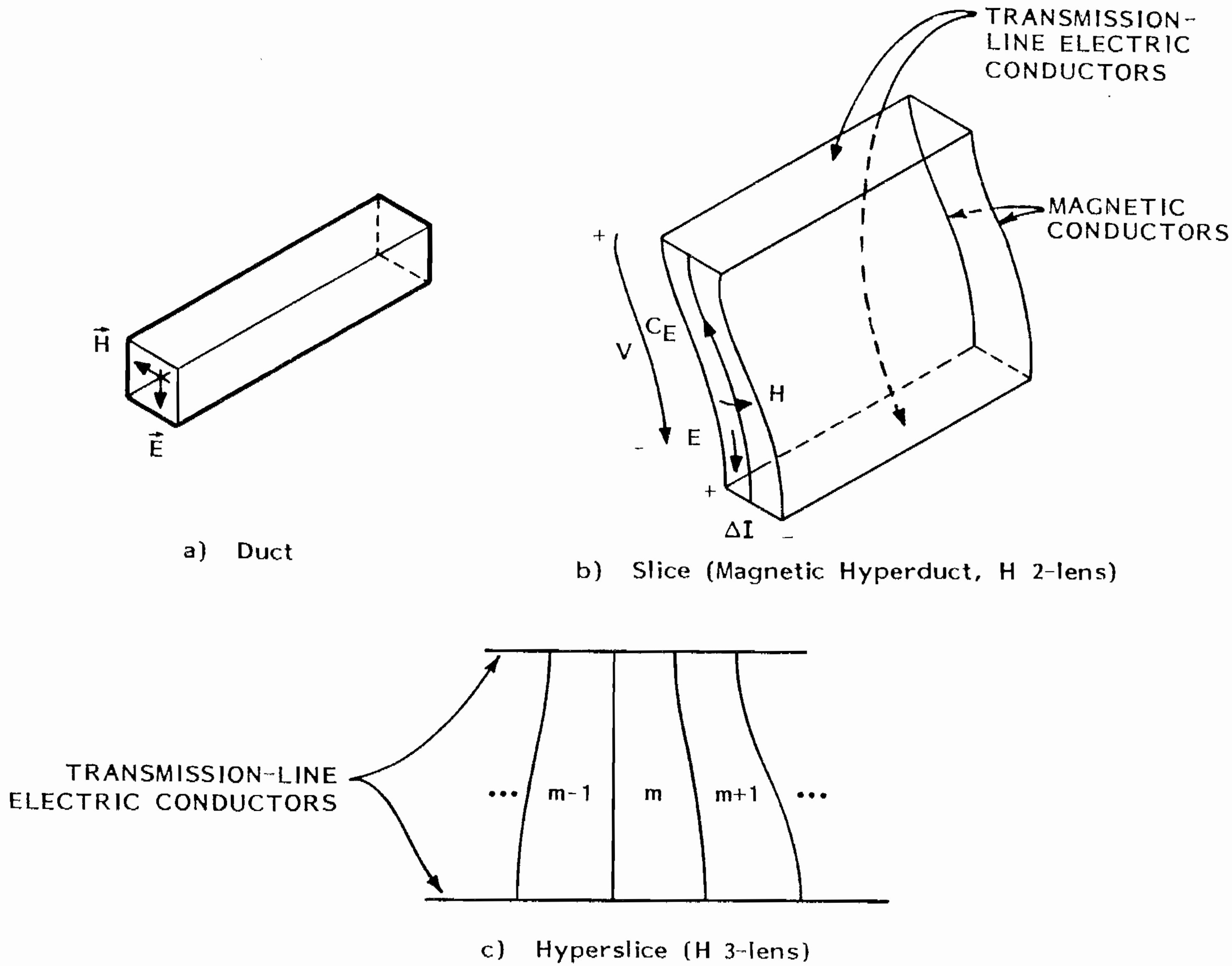


Figure 6.2: Evolution of Duct by Removing Electric Boundaries

Jackets, as noted earlier, have the additional geometrical property of being closed in one dimension.

Note the presence of perfectly magnetically conducting boundaries on surfaces of constant u_2 which prevent propagation in the $\vec{1}_2$ direction. Again these surfaces can be of zero thickness, but space between them can also be allowed. Except on S_1 and S_2 where the magnetic conductors from slice m must connect to those from slices $m-1$ and $m+1$ to keep the electromagnetic energy from the space between the slices, these surfaces can in principle be separate. By adjustment of h_2 in the jackets this spacing can be allowed for.

6.2 TEM Waves in a Slice

We now assume, as in Section 2.5, but now in some mth slice, that we have field components in the form

$$\begin{aligned}
E_1' &= h_1 E_1 = E_0' f\left(t - u_3/c'\right) \\
E_2' &= 0\,,\ E_3' = 0 \\
H_2' &= h_2 H_2 = \frac{E_0'}{Z_0'} f\left(t - u_3/c'\right) \\
H_1' &= H_3' = 0 \\
\mu' &= \text{constant in } m\text{th slice}\,,\ \varepsilon' = \text{ constant in } m\text{th slice} \\
Z_0' &= \sqrt{\frac{\mu'}{\varepsilon'}} = \text{constant in } m\text{th slice} \\
c' &= \frac{1}{\sqrt{\mu'\varepsilon'}} = \text{ constant in } m\text{th slice}
\end{aligned} \tag{6.8}$$

so that our TEM wave propagates in the u_3 direction, then the Maxwell equations assume the form

$$\begin{aligned}
\frac{\partial E_1'}{\partial u_3} &= -\mu' \frac{\partial H_2'}{\partial t} \\
\frac{\partial H_2'}{\partial u_3} &= -\varepsilon' \frac{\partial E_1'}{\partial t}\,.
\end{aligned} \tag{6.9}$$

Moreover the formal medium is described in (2.41) by

$$\begin{aligned} \varepsilon' &= \frac{h_2 h_3}{h_1}\varepsilon_1 \\ \mu' &= \frac{h_1 h_3}{h_2}\mu_2 \,. \end{aligned} \tag{6.10}$$

Now one slice (the mth) is characterized by

$$\begin{gathered} u_2^{(m-1)} < u_2 < u_2^{(m)} \\ u_2^{(m)} - u_1^{(m+1)} = \Delta u_2 \,. \end{gathered} \tag{6.11}$$

Essentially in (3.3) we have united all ducts for the same m and removed n from consideration. We have an incremental current (slice current)

$$\Delta I = -H_2' \Delta u_2 = -H_2 h_2 \Delta u_2 \tag{6.12}$$

as before. However, the slice voltage is the sum of all the duct voltages in the mth slice as

$$\begin{aligned} V = \sum_{n=1}^{N} \Delta V &= -\int_{u_1^{(0)}}^{u_1^{(N)}} E_1' du_1 = -E_1' u_1^{(N)} = -\int_{C_E} \vec{E}\cdot d\vec{\ell} \\ u_1^{(N)} &= \text{ total change in } u_1 \text{ across slice} \end{aligned} \tag{6.13}$$

where the limit as $\Delta u_1 \to 0$ $(N \to \infty)$ is taken. Note as in Fig. 6.2 the contour C_E is contained on a surface of constant u_3 in the mth slice. Here both ΔI and V are functions of u_3, but not u_1. Again the convention is that increasing (electric) potential is in the direction of increasing u_1, and the current on the high-potential electrode is in the direction of increasing u_3.

With this we have the slice admittance

$$\Delta Y = \frac{\Delta I}{V} = \frac{H_2' \Delta u_2}{E_1' \int_{u_1^{(0)}}^{u_1^{(M)}} du_1} = \frac{1}{Z_0'} \frac{\Delta u_2}{u_1^{(M)}} \,. \tag{6.14}$$

Note that ΔY is independent of u_3 (as Z_d is independent of u_3) and also independent of u_1 (over which (6.13) integrates and ΔI is independent). ΔY is only a function of u_2, or better m concerning the mth slice. So we have

$$Z_0' = \sqrt{\frac{\mu'}{\varepsilon'}} = \frac{h_1}{h_2}\sqrt{\frac{\mu_2}{\varepsilon_1}} = \text{ constant in } m\text{th slice .} \tag{6.15}$$

In addition we have from the velocity

$$\frac{1}{c'} = \sqrt{\mu'\varepsilon'} = h_3\sqrt{\mu_2\varepsilon_1} = \text{ constant in } m\text{th slice .} \tag{6.16}$$

Note that (6.15) and (6.16) correspond to (2.42) in Section 2.5, except that their status as constants is only in each slice separately.

6.3 Case of μ and ε Singly or Both Independent of u_1, u_3 in a Slice

The results in this section parallel those in Section 5.3 because of the duality between the roles of the electric and magnetic fields. For a slice, our fundamental form will be

$$(d\ell)^2 = h_1^2(du_1)^2 + h_3^2(du_3)^2 \tag{6.17}$$

so that only the u_1 and u_3 directions are under consideration. Thus in any slice if we assume ε_1 to be a constant, the equations (6.15) and (6.16) yield

$$\begin{aligned} \frac{h_1}{h_2}\sqrt{\mu_2} &= \sqrt{\frac{\mu'\varepsilon_1}{\varepsilon'}} = \text{ constant} \\ h_3\sqrt{\mu_2} &= \sqrt{\frac{\mu'\varepsilon'}{\varepsilon_1}} = \text{ constant .} \end{aligned} \tag{6.18}$$

Hence if we take h_2 to be a constant (corresponding to uniformly spaced magnetic boundaries) we obtain

$$\frac{h_1}{h_3} = \frac{h_2\varepsilon_1}{\varepsilon'} = \text{ constant} \tag{6.19}$$

and therefore we can rewrite (6.17) as

$$(d\ell)^2 = h_3^2[(d\tilde{u}_1)^2 + (du_3)^2] \tag{6.20}$$

where $\tilde{u}_1$ constant times u_1. The Gaussian curvature (see Appendix A, Section A.2) associated with (6.20) is

$$K = -\frac{1}{h_3^2}\left[\frac{\partial^2 \ell n(h_3)}{\partial \tilde{u}_1^2} + \frac{\partial^2 \ell n(h_3)}{\partial u_3^2}\right] \tag{6.21}$$

and thus for surfaces of constant Gaussian curvature the sign of the Laplacian of $\ell n(h_3)$ determines whether surfaces are positively or negatively curved, corresponding to spheres or pseudospheres. If $K = 0$, the surface is Euclidean. In any case, when h_2 is a constant

$$\mu_2 = \frac{1}{h_3^2}\left[\frac{\mu'\varepsilon'}{\varepsilon_1}\right] = \frac{1}{h_1^2}\left[\frac{h_2^2\mu'\varepsilon_1}{\varepsilon'}\right] \tag{6.22}$$

and we have a nonuniform medium.

If instead of taking ε_1 to be a constant, we take μ_2 to be a constant, then equations (6.15) and (6.16) yield

$$\begin{aligned} \frac{h_2}{h_1}\sqrt{\varepsilon_1} &= \sqrt{\frac{\varepsilon'\mu_2}{\mu'}} = \text{ constant} \\ h_3\sqrt{\varepsilon_1} &= \sqrt{\frac{\varepsilon'\mu'}{\mu_2}} = \text{ constant}\,. \end{aligned} \tag{6.23}$$

Hence the assumption that h_2 is a constant leads to the conclusion that

$$h_1 h_3 = h_2\frac{\mu'}{\mu_2} = \text{ constant}\,. \tag{6.24}$$

Finally, if both ε_1 and μ_2 are constants (corresponding to a homogeneous medium), then

$$\begin{aligned} h_3 &= \sqrt{\frac{\varepsilon'\mu'}{\varepsilon_1\mu_2}} = \text{ constant} \\ h_1 &= \left(\sqrt{\frac{\mu'\varepsilon_1}{\mu_2\varepsilon'}}\right) h_2\,. \end{aligned} \tag{6.25}$$

Therefore when ε_1, μ_2 and h_2 are constant our surface is Euclidean. This result is the analog of that obtained in Section 5 for jackets.

7 Summary

In this paper we have investigated the relations connecting the differential geometry method and the transit-time and differential impedance-matching approach in designing lenses for inhomogeneous TEM plane waves. In Section 2 of this paper, the differential geometry scaling method was shown to lead to the transit-time and impedance-matching method. Various cases were considered (e.g., inhomogeneous isotropic media, inhomogeneous TEM waves, homogeneous TEM waves) and the results appear in a summary given in Table 2.1.

In the remaining sections, 3 through 6, of this paper we show that the physical assumptions of transit-time conservation and differential impedance matching lead to the differential geometric scaling approach. This is accomplished by showing how a duct can evolve into an EM 3-lens in one of two ways. First, if all magnetic walls are removed, we obtain an E 2-lens which is called a jacket in this paper. The totality of all jackets, called a hyperjacket, is an E 3-lens. At this stage if all intermediate electric boundaries are removed, the end result is an EM 3-lens. This process is described in Sections 4 and 5.

Alternatively, one could remove intermediate electric walls from a duct and obtain an H 2-lens which is called a slice in this paper. The totality of slices, called a hyperslice, is a H 3-lens, and removal of all magnetic walls leads once more to an EM 3-lens. Slices are investigated in Section 6 of this paper.

These remarks are summarized in Table 7.1, which describes the evolution of a duct to an EM 3-lens.

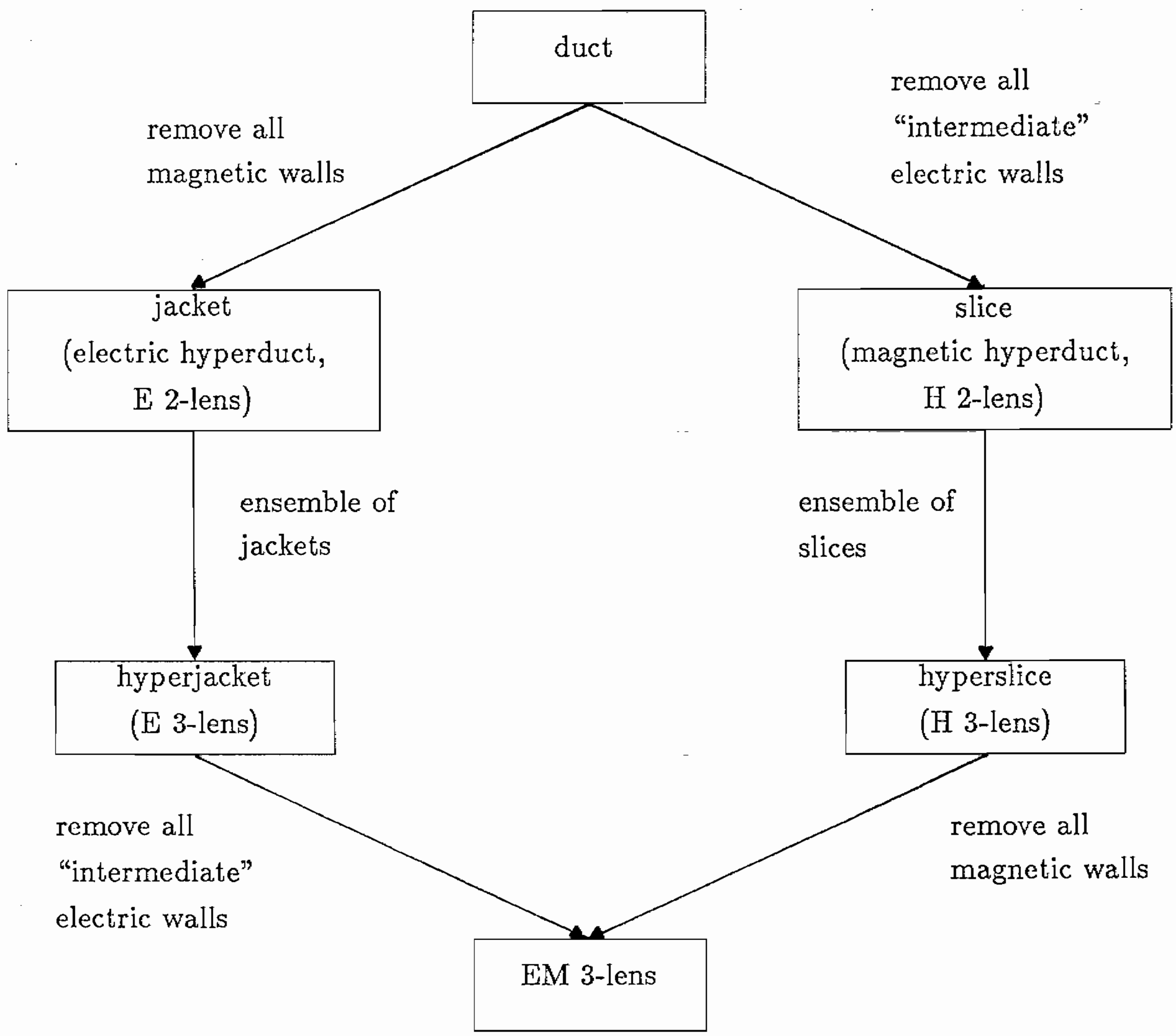


Table 7.1. Evolution of Duct to EM 3-lens

Epilogue

(Hexagon) "Can I learn a little more?" I asked when we took leave of each other. "Couldn't you lift a tip of the veil for me, so that I can know in what direction to look for the problem?"

(Puncto) "All right," he said. "Tell me what the sum of the angles of a triangle is."

(Hexagon) That unexpected question did surprise me, I must admit, but I answered: "180°, of course."

(Puncto) "Always?" he asked, and he left.

⋮

⋮

(Hexagon) I was looking forward in high spirits to my next meeting with Mr. Puncto. From the moment we greeted each other he knew there had to be a reason for my good mood.

(Puncto) "Did you find the solution?" he asked.

(Hexagon) "No," I replied, "I can't shout Eureka yet, but do believe that I am very close to discovering the basis of the problem. I think that our curious phenomenon—the sum of the angles of a triangle exceeding 180°—has to be explained by assuming that the sides of the triangle are curved but that this curvature is not visible, I mean: *not visible to us.* It occurs in a direction perpendicular to our world. A three-dimensional creature must be able to see the curvature; we cannot."

from Sphereland (1965)
by Dionys Burger
translated by C. J. Rheinboldt

Appendix A. Differential Geometry of Surfaces

A.1 Fundamental Forms

In this appendix some differential geometric results which are relevant to electromagnetic lens design are discussed. Our starting point in this discussion is the introduction of a metric form (also called the first fundamental form) for a surface. It is given by the expression

$$(d\ell)^2 = h_a^2(du_a)^2 + h_b^2(du_b)^2 \ , \tag{A.1}$$

where we are dealing with an orthogonal curvilinear coordinate system (u_a, u_b, u_n) with line element

$$(d\ell)^2 = h_a^2(du_a)^2 + h_b^2(du_b)^2 + h_n^2(du_n)^2 \ . \tag{A.2}$$

Then, corresponding to surfaces u_n = constant, the form (A.1) is obtained. The coefficients appearing in (A.2) are computed by taking

$$h_i^2 = \left(\frac{\partial x}{\partial u_i}\right)^2 + \left(\frac{\partial y}{\partial u_i}\right)^2 + \left(\frac{\partial z}{\partial u_i}\right)^2 \tag{A.3}$$

where $x = x(u_a, u_b, u_n)$, $y = y(u_a, u_b, u_n)$, $z = z(u_a, u_b, u_n)$ are rectangular Cartesian coordinates, and the index i can be equal to any one of the indices a, b, or n.

The first fundamental form for a surface can also be computed from

$$(d\ell)^2 = (d\vec{r}_s) \cdot (d\vec{r}_s) \tag{A.4}$$

where the position vector $\vec{r}_s(u_a, u_b)$ describes the surface with parameters u_a and u_b. Thus

$$\vec{r}_s(u_a, u_b) = x(u_a, u_b)\vec{1}_x + y(u_a, u_b)\vec{1}_y + z(u_a, u_b)\vec{1}_z \ . \tag{A.5}$$

Another quadratic differential form which is of importance in the differential geometry of curves and surfaces is the second fundamental form, denoted by most authors as II, and defined by

$$II = -(d\vec{r}_s) \cdot (d\vec{1}_n(\vec{r}_s)) \tag{A.6}$$

where

$$\vec{1}_n(\vec{r}_s) = \left(\frac{\partial \vec{r}_s}{\partial u_a} \times \frac{\partial \vec{r}_s}{\partial u_b}\right) \Big/ \left|\frac{\partial \vec{r}_s}{\partial u_a} \times \frac{\partial \vec{r}}{\partial u_b}\right| . \tag{A.7}$$

The vector $\vec{1}_n(\vec{r}_s)$ can also be expressed in the form

$$\vec{1}_n(\vec{r}_s) = \vec{1}_a \times \vec{1}_b \tag{A.8}$$

where $\vec{1}_a$ and $\vec{1}_b$ are unit vectors in the coordinate directions u_a and u_b.

The geometrical significance of the fundamental forms (A.4) and (A.6) lies in the fact that the first fundamental form is useful in studying such things as lengths of curves and angles between curves on surfaces while the second fundamental form is useful in studying curvature of a surface. The second fundamental form (A.6) may also be written in the form (analogous to (A.1))

$$II = \ell_a (du_a)^2 + \ell_b (du_b)^2 \tag{A.9}$$

where

$$\begin{aligned} \ell_a &= \frac{1}{h_a h_b}\left\{\frac{\partial^2 \vec{r}_s}{\partial u_a^2} \cdot \left(\frac{\partial \vec{r}_s}{\partial u_a} \times \frac{\partial \vec{r}_s}{\partial u_b}\right)\right\} \\ \ell_b &= \frac{1}{h_a h_b}\left\{\frac{\partial^2 \vec{r}_s}{\partial u_b^2} \cdot \left(\frac{\partial \vec{r}_s}{\partial u_a} \times \frac{\partial \vec{r}_s}{\partial u_b}\right)\right\} . \end{aligned} \tag{A.10}$$

The derivation of formula (A.9) from (A.6) is straightforward though lengthy (see [11(p. 75)]).

A.2 Surface Curvature

Since the Cartesian coordinates x_i of a surface are functions of u_a and u_b, one finds that certain compatibility conditions (like $\frac{\partial^3 \vec{r}_s}{\partial u_a^2 \partial u_b} = \frac{\partial^3 \vec{r}_s}{\partial u_b \partial u_a^2}$) must be satisfied and hence that certain differential equations are obtained. One result of all of this is Gauss' "Theorema egreguim" which asserts that the Gaussian curvature K is a bending invariant. This means that K is unchanged by deformations of the surface which do not involve stretching, shrinking, or tearing. The Gaussian curvature K, associated with (A.1), has several different formulations. One of the most useful for our purposes is given by

$$K = -\frac{1}{h_a h_b}\left[\frac{\partial}{\partial u_a}\left(\frac{1}{h_a}\frac{\partial h_b}{\partial u_a}\right) + \frac{\partial}{\partial u_b}\left(\frac{1}{h_b}\frac{\partial h_a}{\partial u_b}\right)\right] . \qquad \text{(A.11)}$$

The compatibility conditions mentioned above also lead to a set of differential equations which are sometimes referred to as the Codazzi equations [11]. For the fundamental forms (A.1) and (A.9) these equations are

$$\begin{aligned} \frac{\partial \ell_a}{\partial u_b} &= \Gamma^a_{a,b}\ell_a - \Gamma^b_{a,a}\ell_b \\ \frac{\partial \ell_b}{\partial u_a} &= \Gamma^a_{b,b}\ell_a - \Gamma^b_{a,b}\ell_b \end{aligned} \qquad \text{(A.12)}$$

where the $\Gamma^i_{j,k}$ are Christoffel symbols for our surface. The complete set of these symbols are given by

$$\begin{aligned} \Gamma^a_{a,a} &= \frac{1}{h_a}\frac{\partial h_a}{\partial u_a} & \Gamma^b_{a,a} &= -\frac{h_a}{h_b^2}\frac{\partial h_a}{\partial u_b} \\ \Gamma^a_{a,b} &= \frac{1}{h_a}\frac{\partial h_a}{\partial u_b} & \Gamma^b_{a,b} &= \frac{1}{h_b}\frac{\partial h_b}{\partial u_a} \\ \Gamma^a_{b,b} &= -\frac{h_b}{h_a^2}\frac{\partial h_b}{\partial u_a} & \Gamma^b_{b,b} &= \frac{1}{h_b}\frac{\partial h_b}{\partial u_b} \end{aligned} \quad , \qquad \text{(A.13)}$$

We note that these symbols are symmetric in the lower subscripts, that is, $\Gamma^i_{j,k} = \Gamma^i_{k,j}$.

The Codazzi equations take a particularly simple form in the case that the lines of curvature are coordinate lines. The principal curvatures, which are normal curvatures in the curvature directions, are denoted by K_a and K_b and are given by

$$K_a = \frac{\ell_a}{h_a^2} \qquad \text{and} \qquad K_b = \frac{\ell_b}{h_b^2} . \qquad \text{(A.14)}$$

We also note that the Gaussian curvature may be expressed as

$$K = K_a K_b = \frac{\ell_a}{h_a^2}\frac{\ell_b}{h_b^2} . \qquad \text{(A.15)}$$

The Codazzi equations may then be reexpressed in the form

$$\begin{aligned} \frac{\partial K_a}{\partial u_b} &= \frac{1}{h_a}\frac{\partial h_a}{\partial u_b}(K_b - K_a) = \frac{\partial \ell n(h_a)}{\partial u_b}(K_b - K_a) \\ \frac{\partial K_b}{\partial u_a} &= \frac{1}{h_b}\frac{\partial h_b}{\partial u_a}(K_a - K_b) = \frac{\partial \ell n(h_b)}{\partial u_a}(K_a - K_b) . \end{aligned} \qquad \text{(A.16)}$$

A.3 Riemannian Surfaces

We next note that any surface which admits a form (A.1) is called a Riemannian surface. A particular instance of such a surface is a Euclidean surface which admits a first fundamental form

$$(d\ell)^2 = (dx')^2 + (dy')^2 \tag{A.17}$$

for some coordinates (x', y'). Thus a basic test for Euclideaness is the existence of a real coordinate transformation

$$(u_a, u_b) \longrightarrow (x', y') \tag{A.18}$$

which transforms (A.1) to (A.17). If no such transformation exists, then (A.1) defines a non-Euclidean surface. A necessary and sufficient condition for a surface to be Euclidean is the existence of such a transformation (A.18). The reader should see [12] for details. The simplest examples of Euclidean surfaces include planes, cylinders, and cones and more complex examples would involve combinations of these (as discussed later in Section A.6). We now look at some of these simpler cases.

Thus, for a cylinder of radius r_0, the transformation

$$\begin{aligned} x &= \Psi\cos(\phi) \\ y &= \Psi\sin(\phi) \\ z &= z \end{aligned} \tag{A.19}$$

will yield the first fundamental form for a circular cylinder if we take $\Psi = r_0$. We thus obtain the first fundamental form

$$(d\ell)^2 = r_0^2(d\phi)^2 + (dz)^2 \,. \tag{A.20}$$

The form (A.19) may be put into the Euclidean form (A.16) by the transformation

$$\begin{aligned} x' &= r_0\phi \\ y' &= z \end{aligned} \tag{A.21}$$

and hence we obtain

$$(d\ell)^2 = (dx')^2 + (dy')^2 \,. \tag{A.22}$$

Thus a circular cylinder of radius r_0 is our first example of a Euclidean surface.

A second example of a Euclidean surface is a circular cone. For the transformation

$$\begin{aligned} x &= r\sin(\theta)\cos(\phi) = \Psi\cos(\phi) \\ y &= r\sin(\theta)\sin(\phi) = \Psi\sin(\phi) \\ z &= r\cos(\theta) \end{aligned} \tag{A.23}$$

we can obtain the first fundamental form for a circular cone with generating angle $\theta = \theta_0$. The fundamental form for such a cone is

$$(d\ell)^2 = (dr)^2 + (r^2\sin^2(\theta_0))(d\phi)^2 \,. \tag{A.24}$$

The transformation

$$\begin{aligned} x' &= r\cos(\phi\sin(\theta_0)) \\ y' &= r\sin(\phi\sin(\theta_0)) \end{aligned} \tag{A.25}$$

will transform (A.24) to the Euclidean form $(d\ell)^2 = (dx')^2 + (dy')^2$ thereby demonstrating that a circular cone is also a Euclidean surface.

Our last example is that of a sphere of radius r_0. The surface of such a sphere is a simple example of a non-Euclidean surface. The first fundamental form for this surface is

$$(d\ell)^2 = r_0^2(d\theta)^2 + (r_0^2\sin^2\theta)(d\phi)^2 \,. \tag{A.26}$$

One may show that the surface of this sphere is non-Euclidean by showing that there is no real coordinate transformation (A.18) which transforms (A.26) to the Euclidean form (A.17). Alternatively we may make use of a fundamental result which states (see [12]) that the Gaussian curvature K vanishes identically on a surface if and only if there is a real coordinate transformation (A.18) which transforms (A.1) to (A.17). Thus, for the surface of a sphere of radius r_0, we have

$$\begin{aligned} h_a &= r_0 \\ h_b &= r_0 \sin(\theta) \end{aligned} \tag{A.27}$$

and hence

$$K = -\frac{1}{r_0^2 \sin(\theta)} \frac{\partial}{\partial \theta} \left[\frac{1}{r_0} \frac{\partial (r_0 \sin(\theta))}{\partial \theta} \right] = \frac{1}{r_0^2} \,. \tag{A.28}$$

Therefore, since $K \neq 0$ at all points on the surface of a sphere, this surface is a non-Euclidean surface. We note that for the previous examples of a circular cylinder and a circular cone that K vanishes identically for these surfaces thereby establishing once more the Euclidean nature of these surfaces.

Finally, we note that a surface is Euclidean if and only if at least one of the principal curvatures, K_a or K_b, vanishes identically on the surface. This assertion is an immediate consequence of (A.14).

A.4 Surfaces of Constant Gaussian Curvature

Surfaces of constant Gaussian curvature are of considerable importance. For example, if $K = 1/r_0^2 > 0$, then the surface associated with (A.1) is "essentially" a sphere of radius r_0. The technical statement is that the surface is "isometric" to a sphere of radius r_0. This means that there is a mapping of the given surface, say S_1, into the surface of the sphere, say S_2, which preserves lengths of curves (i.e., the length of the image of an arc in S_1 equals the length of the arc in S_2). Thus (see [13]) we note that a surface of positive constant Gaussian curvature $K = 1/r_0^2$ is isometric to the surface of a sphere of radius r_0, a surface of Gaussian curvature $K = 0$ is isometric to a plane, and a surface of negative constant Gaussian curvature $K = -1/r_0^2$ is isometric to the pseudosphere. An example of the latter is provided by a surface with fundamental form

$$(d\ell)^2 = r_0^2[\{\cot^2(u_a)\}(du_a)^2 + \{\sin^2(u_a)\}(du_b)^2] \, . \tag{A.29}$$

For this surface it is easily verified that

$$K = -\frac{1}{r_0^2} < 0 \, . \tag{A.30}$$

The surface is a pseudosphere (see Fig. A.1) and it is obtained by revolving a tractrix about its asymptote.

For the special case of Riemannian surfaces with one of the scale factors equal to a constant more can be said. Let us assume for convenience that $h_a = 1$ and investigate further. Our first fundamental form is

$$(d\ell)^2 = (du_a)^2 + h_b^2(du_b)^2 \tag{A.31}$$

with associated Gaussian curvature

$$K = -\frac{1}{h_b}\frac{\partial^2 h_b}{\partial u_a^2} \, . \tag{A.32}$$

Now if K is a constant (positive, negative, or zero corresponding to spheres, pseudo-spheres, or planes) the differential equation (A.32) is integrable and explicit expressions for the scale factor h_b are obtainable. Let us assume that

$$\begin{aligned} h_b(u_a, u_b)\Big|_{u_a=0} &= 1 \\ \frac{\partial h_b}{\partial u_a}(u_a, u_b)\Big|_{u_a=0} &= 0 \end{aligned} \tag{A.33}$$

so that constants of integration may be evaluated. Then, if (a) $K = 1/r_0^2 > 0$, (A.32) can be dealt with as an ordinary differential equation since K does not depend on u_b, and we have

$$\frac{d^2 h_b}{du_a^2} + \frac{1}{r_0^2} h_b = 0 \, . \tag{A.34}$$

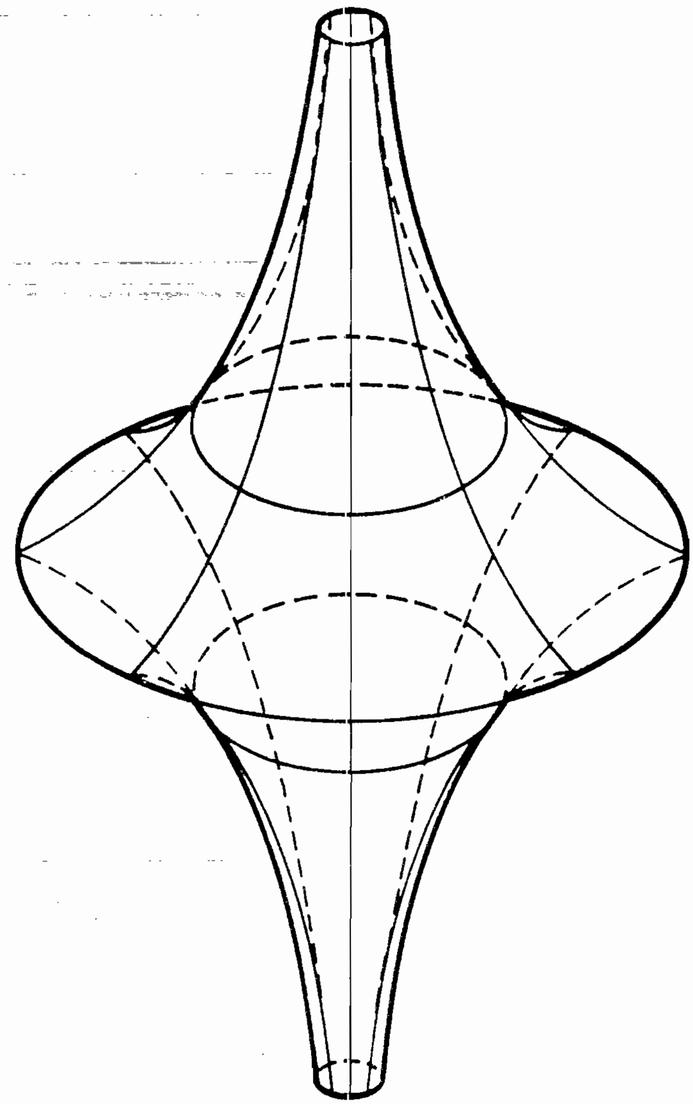


Figure A.1: Pseudosphere

The solution of (A.34) is

$$h_b(u_a, u_b) = \cos\left(\frac{u_a}{r_0}\right) \tag{A.35}$$

while if (b) $K = -1/r_0^2 < 0$, (A.32) has solution

$$h_b(u_a, u_b) = \cosh\left(\frac{u_b}{r_0}\right) \tag{A.36}$$

and if (c), $K = 0$, (A.32) has solution

$$h_b(u_a, u_b) = 1\ . \tag{A.37}$$

Thus we reaffirm our earlier statement that surfaces of constant Gaussian curvature correspond to spheres, pseudospheres, or planes. For Riemannian surfaces with one of the scale factors equal to a constant, the coordinate system (u_a, u_b) is usually called a semi-geodesic coordinate system. The terminology is explained by the fact that one family of coordinate lines consists of geodesic lines and the other family consists of orthogonal trajectories of the first family. Simple examples include Cartesian and polar coordinates in a plane (Euclidean space) and spherical coordinates on a sphere (non-Euclidean space). Surfaces of revolution will also yield semi-geodesic coordinates.

A.5 Equal Scale Factors for Surface Coordinates on a Euclidean Surface

The special case of surfaces whose first fundamental form is

$$(d\ell)^2 = h^2[(du_a)^2 + (du_b)^2] \tag{A.38}$$

(i.e., $h_a = h_b = h$ in (A.1)) is also of interest. If the Gaussian curvature vanishes identically, so that the space is Euclidean, then there is a coordinate transformation

$$\begin{aligned} x' &= x'(u_a, u_b) \\ y' &= y'(u_a, u_b) \end{aligned} \tag{A.39}$$

such that

$$(d\ell)^2 = (dx')^2 + (dy')^2 \ . \tag{A.40}$$

In this case we can show that x' and y' are either the real and imaginary parts of an analytic function of $u_a + ju_b$ or of $u_a - ju_b$. To see this, first note that (A.38) and (A.40) give two distinct expressions for $(d\ell)^2$. Thus, since

$$\begin{aligned} dx' &= \frac{\partial x'}{\partial u_a} du_a + \frac{\partial x'}{\partial u_b} du_b \\ dy' &= \frac{\partial y'}{\partial u_a} du_a + \frac{\partial y'}{\partial u_b} du_b \end{aligned} \tag{A.41}$$

we obtain

$$\begin{aligned} (d\ell)^2 &= (dx')^2 + (dy')^2 \\ &= \left[\left(\frac{\partial x'}{\partial u_a}\right)^2 + \left(\frac{\partial y'}{\partial u_a}\right)^2\right](du_a)^2 + 2\left[\frac{\partial x'}{\partial u_a}\frac{\partial x'}{\partial u_b} + \frac{\partial y'}{\partial u_a}\frac{\partial y'}{\partial u_b}\right](du_a)(du_b) \\ &\quad + \left[\left(\frac{\partial x'}{\partial u_b}\right)^2 + \left(\frac{\partial y'}{\partial u_b}\right)^2\right](du_b)^2 \ . \end{aligned} \tag{A.42}$$

Hence, by comparison of (A.42) with (A.38), we must have

$$\begin{aligned} h^2 = \left(\frac{\partial x'}{\partial u_a}\right)^2 + \left(\frac{\partial y'}{\partial u_a}\right)^2 &= \left(\frac{\partial x'}{\partial u_b}\right)^2 + \left(\frac{\partial y'}{\partial u_b}\right)^2 \\ \left(\frac{\partial x'}{\partial u_a}\right)\left(\frac{\partial x'}{\partial u_b}\right) + \left(\frac{\partial y'}{\partial u_a}\right)\left(\frac{\partial y'}{\partial u_b}\right) &= 0 \ . \end{aligned} \tag{A.43}$$

The equations (A.43) then imply

$$\left(\frac{\partial y'}{\partial u_a}\right)^2\left[1 + \left(\frac{\frac{\partial y'}{\partial u_b}}{\frac{\partial x'}{\partial u_b}}\right)^2\right] = \left(\frac{\partial x'}{\partial u_b}\right)^2\left[1 + \left(\frac{\frac{\partial y'}{\partial u_b}}{\frac{\partial x'}{\partial u_b}}\right)^2\right] \ . \tag{A.44}$$

Thus the only possibility is that

$$\left(\frac{\partial y'}{\partial u_a}\right)^2 = \left(\frac{\partial x'}{\partial u_b}\right)^2 \tag{A.45}$$

as otherwise there is a linear relation between $\frac{\partial y'}{\partial u_b}$ and $\frac{\partial x'}{\partial u_b}$. Hence, either

$$\frac{\partial x'}{\partial u_b} = \frac{\partial y'}{\partial u_a} \tag{A.46}$$

or

$$\frac{\partial x'}{\partial u_b} = -\frac{\partial y'}{\partial u_a} .$$

If the first equation of (A.46) holds, then

$$\frac{\partial x'}{\partial u_a} = -\frac{\partial y'}{\partial u_b} \tag{A.47}$$

while if the second equation of (A.46) holds, then

$$\frac{\partial x'}{\partial u_a} = \frac{\partial y'}{\partial u_b} . \tag{A.48}$$

Thus if

$$\begin{aligned} \frac{\partial x'}{\partial u_a} &= \frac{\partial y'}{\partial u_b} \\ \frac{\partial x'}{\partial u_b} &= -\frac{\partial y'}{\partial u_a} \end{aligned} \tag{A.49}$$

then the Cauchy-Riemann equations are satisfied and $x' + jy'$ is an analytic function of the complex variable $u_a + ju_b$. Similarly if

$$\begin{aligned} \frac{\partial x'}{\partial u_a} &= -\frac{\partial y'}{\partial u_b} \\ \frac{\partial x'}{\partial u_b} &= \frac{\partial y'}{\partial u_a} \end{aligned} \tag{A.50}$$

then $x' - jy'$ is an analytic function of the complex variable $u_a + ju_b$. In either event we can write

$$f(u_a + ju_b) = x' \pm jy' \tag{A.51}$$

which represents a conformal transformation. Thus, if $h_a = h_b = h$ so that our first fundamental form is expressible as (A.38), and if our surface so defined is Euclidean, conformal maps are the only way in which the form given by (A.38) can be obtained.

We note also that the coordinates x' and y' are generalized Cartesian coordinates. If the surface is a plane, they are exactly Cartesian coordinates. However, as discussed at the end of this appendix, x' and y' could be curvilinear. Finally, we observe that the Gaussian curvature $\mathcal{K}$, given in (A.11), can be rewritten in the form

$$\mathcal{K} = -\frac{1}{h^2}\left[\frac{\partial^2(\ell n(h))}{\partial u_a^2} + \frac{\partial^2(\ell n(h))}{\partial u_b^2}\right] \tag{A.52}$$

when $h_a = h_a = h$. Thus our surface is Euclidean if and only if $\ell n(h)$ is a harmonic function of u_a and u_b.

Note, however, that merely having $h_a = h_b$ for a surface will not be enough to guarantee that $\mathcal{K} = 0$ for that surface. Equation (A.15) shows that at least one of the principal curvatures, $\mathcal{K}_a$ or $\mathcal{K}_b$, must vanish in order for the Gaussian curvature to vanish. In the case that $h_a = h_b = h$, equation (A.52) shows that the Gaussian curvature $\mathcal{K}$ vanishes if and only if $\ell n(h)$ is a harmonic function of u_a and u_b. A simple example of a surface for which $h_a = h_b$ with $\mathcal{K} \neq 0$ is given by a sphere on which we choose

$$\begin{aligned} u_a &= 2r_0\left[\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right]\cos(\phi) \\ u_b &= 2r_0\left[\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right]\sin(\phi)\ . \end{aligned} \tag{A.53}$$

The south pole of the sphere ($\theta = \pi$) is tangent to the (u_a, u_b)-plane at the origin of this plane and the north pole of the sphere has coordinates $(r_0, 0, 0)$. In this case the mapping (A.53), which is a conformal mapping, is stereographic projection from $(r_0, 0, 0)$ to the (u_a, u_b)-plane. It is a routine calculation to check that for these coordinates

$$(d\ell)^2 = \frac{16r_0^4}{16r_0^2 + u_a^2 + u_b^2}((du_a)^2 + (du_b)^2)\ . \tag{A.54}$$

A.6 The Scroll: A General Euclidean Surface

Finally, we conclude this appendix with some further comments on Euclidean surfaces. For these surfaces, $K = 0$ and since

$$K = K_a K_b \tag{A.55}$$

at least one of the principal curvatures vanishes. If we suppose that $K_a = 0$, then the Codazzi equations (A.16) imply

$$\begin{aligned} \frac{K_b}{h_a}\frac{\partial h_a}{\partial u_b} &= K_b \frac{\partial \ell n(h_a)}{\partial u_b} = 0 \\ \frac{\partial K_b}{\partial u_a} &= -\frac{K_b}{h_b}\frac{\partial h_b}{\partial u_a} = -K_b \frac{\partial \ell n(h_b)}{\partial u_a} \ . \end{aligned} \tag{A.56}$$

Hence if K_b is nonvanishing, then the scale factors satisfy

$$\begin{aligned} h_a &= f(u_a) \\ h_b K_b &= g(u_b) \end{aligned} \tag{A.57}$$

since the second equation of (A.57) may be rewritten as

$$\frac{\partial (h_b K_b)}{\partial u_a} = 0 \ . \tag{A.58}$$

Hence, for Euclidean spaces with $K_a = 0$, h_a depends only on u_a, at most, and $K_b h_b$ depends only on u_b, at most.

Analogous results hold when $K_b = 0$ and K_a in nonvanishing. In the case where both K_a and K_b vanish, h_a depends only on u_a, and h_b depends only on u_b, and our first fundamental form clearly assumes the Euclidean form, for some (x', y'),

$$(d\ell)^2 = (dx')^2 + (dy')^2 \tag{A.59}$$

which is characteristic of a plane. In any case, when our space is Euclidean, at least one of the principal curvatures vanishes. If, as we have seen in the above analysis, we take

$K_a = 0$ then h_a depends only on u_a. Hence, if we define a new coordinate, say

$$\tilde{u}_a = \int_{u_{a_0}}^{u_a} h_a(u'_a)du'_a \ , \tag{A.60}$$

then $d\tilde{u}_a = h_a du_a$ and

$$(d\ell)^2 = (d\tilde{u}_a)^2 + h_b(du_b) \tag{A.61}$$

which means $(\tilde{u}_a, u_b)$ are semi-geodesic coordinates. Therefore, since $K = 0$, we have

$$\frac{\partial^2 h_b}{\partial \tilde{u}_a^2} = 0 \tag{A.62}$$

and so

$$h_b = A\tilde{u}_a + B \tag{A.63}$$

where A and B are constants of integration. Hence, because of (A.60), h_b is a function of u_a. Therefore for the case where $K_a = 0$ the scale factors h_a and h_b are functions of a single variable. If both principal curvatures vanish then, as noted previously, h_a is a function of u_a (at most) and h_b is a function of u_b (at most).

As we observed in Appendix A.5, the case of equal scale factors for a Euclidean surface led to the statement that general Cartesian coordinates (x', y') exist with the property that

$$f(u_a + ju_b) = x' \pm ju' \tag{A.64}$$

is a conformal map. Thus there are many more ways of producing examples of Euclidean surfaces. The most general Euclidean surface is either a plane or a union of planes, cones, and/or cylinders. Thus one implication of this result is that, for our lens applications, bending of metal sheets in lenses is permitted, since plane sheets and bent sheets have the same (zero) Gaussian curvature. See Figure A.2 for an example of such surfaces, which we choose to call scrolls.

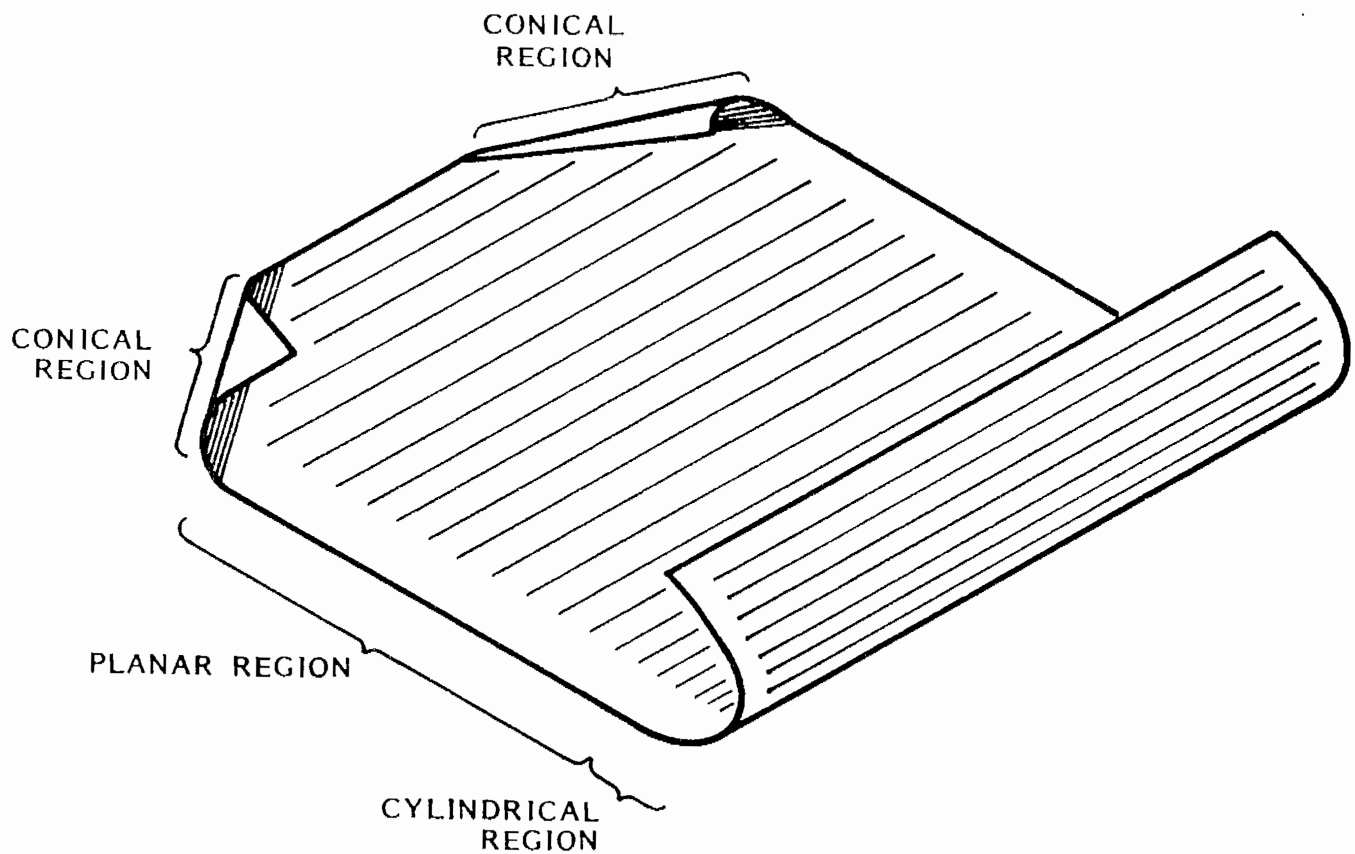


Figure A.2: A Scroll as a General Euclidean Surface

It should also be noted that scrolls can be even more complicated as illustrated in Fig. A.3. In particular, a Euclidean surface can be constructed in such a way that it is multiply connected. A simple example is a cylinder (not necessarily circular) in which the surface closes on itself, i.e., one can leave some point on a geodesic and arrive back at the same point. A similar statement can be made concerning cones. We might call such Euclidean surfaces 1-closed, i.e., they are closed in at least one dimension.

As in Fig. A.3 this closure can be rather convoluted. Consider a cylinder (truncated) which is flattened such that one or more portions are planes (planes having both principal curvatures zero). One of these planar regions can be part of another such flattened cylinder. This process can be extended indefinitely. Similarly conical surfaces can have portions that are planes which can in turn be included in other conical (or cylindrical) surfaces with corresponding planar portions.

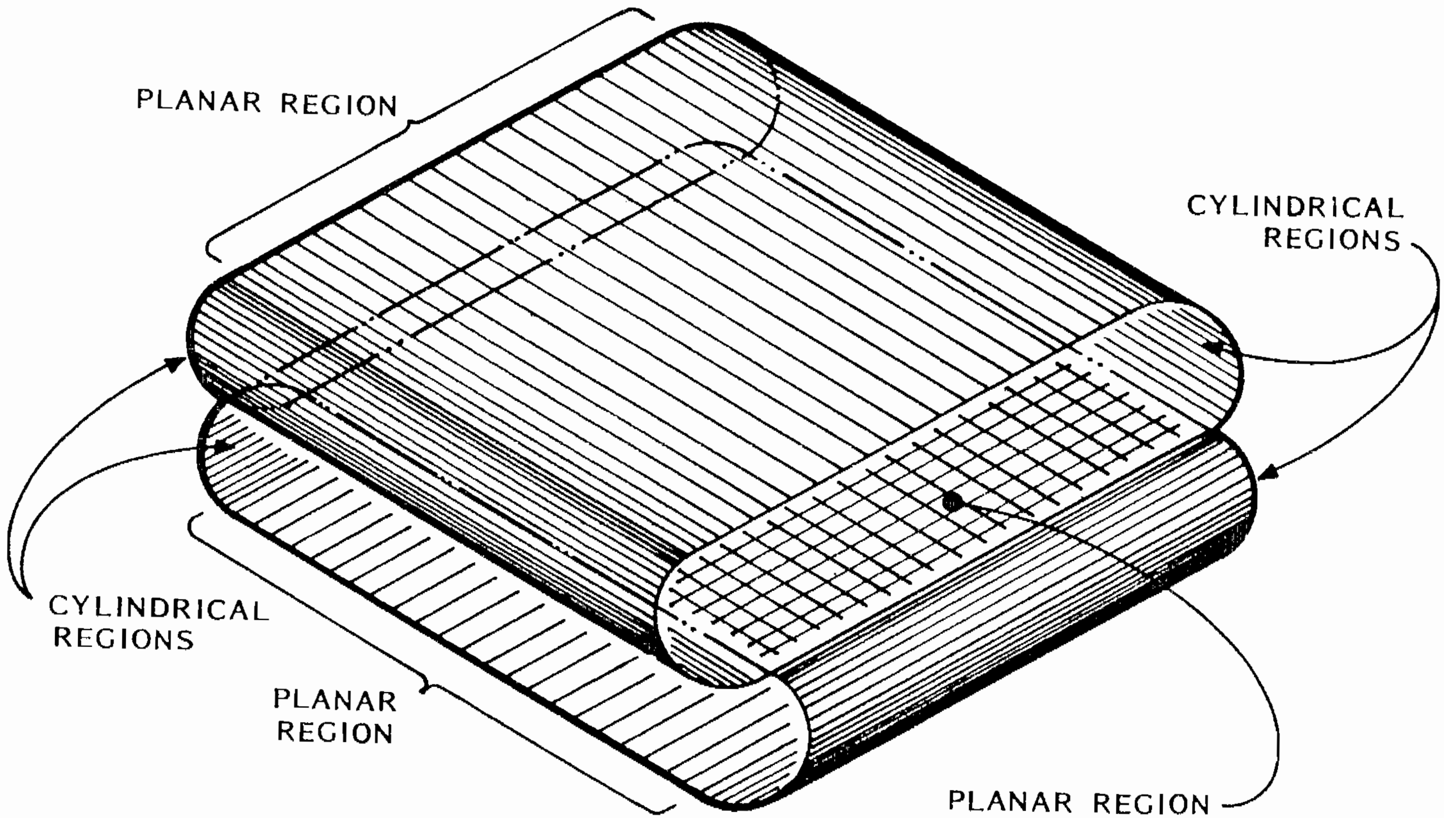


Figure A.3: Multiply Connected Scroll (Euclidean Surface)

Appendix B. Maxwell's Equations for a Two-Dimensional Space

B.1 Real and Formal Maxwell Equations

The general form of the real and formal Maxwell's equations appear in (2.9) and (2.15) together with the corresponding constitutive relations. If we specialize to a two-dimensional space these equations (for the case $\vec{J} = \vec{0}$) will take the following form if we assume we have formal magnetic field components H'_a, H'_b and a formal electric field E'_n (as in Section 5):

$$\begin{aligned} \frac{\partial E'_n}{\partial u_b} &= -\mu'_a \frac{\partial H'_a}{\partial t} \\ \frac{\partial E'_n}{\partial u_a} &= \mu'_b \frac{\partial H'_b}{\partial t} \\ \frac{\partial H'_b}{\partial u_a} - \frac{\partial H'_a}{\partial u_b} &= \varepsilon'_n \frac{\partial E'_n}{\partial t} . \end{aligned} \tag{B.1}$$

Alternatively we could proceed by interchanging the roles of electric and magnetic fields as in Section 6. The replacement of the formal field equations by the real field equations yields

$$\begin{aligned} \frac{\partial (h_n E_n)}{\partial u_b} &= -h_a h_b \mu_a \frac{\partial H_a}{\partial t} \\ \frac{\partial (h_n E_n)}{\partial u_a} &= h_a h_n \mu_b \frac{\partial H_b}{\partial t} \\ \frac{\partial (h_a H_a)}{\partial u_a} - \frac{\partial (h_b H_b)}{\partial u_b} &= h_a h_b \varepsilon_n \frac{\partial E_n}{\partial t} . \end{aligned} \tag{B.2}$$

Now various assumptions may be made about the real and formal media. Certainly we must have in our present case

$$\begin{aligned} \frac{h_a h_b}{h_n} \varepsilon_n &= \varepsilon'_n \\ \frac{h_b h_n}{h_a} \mu_a &= \mu'_a \\ \frac{h_a h_n}{h_b} \mu_b &= \mu'_b \end{aligned} \tag{B.3}$$

from (2.13). If we then assume the formal medium is isotropic, then $\mu'_a = \mu'_b = \mu'$ and we obtain

$$\frac{h_b}{h_a}\mu_a = \frac{h_a}{h_b}\mu_b \ . \tag{B.4}$$

Thus if the real medium is also isotropic (i.e., $\mu_a = \mu_b = \mu$) we find that

$$\begin{aligned} h_a &= h_b = h \\ \frac{h^2}{h_n}\varepsilon_n &= \varepsilon'_n \\ h_n\mu &= \mu' \ . \end{aligned} \tag{B.5}$$

If our space is Euclidean then this case corresponds to that discussed in Appendix A, equations (A.38) through (A.51), which means the coordinates (u_a, u_b) are expressible as a complex variable $u_a + ju_b$. If the real and formal media have constant permeability, then the scale factor h_n is also a constant (corresponding to a constant spacing of electrical conducting boundaries as discussed in Section 5) and the permittivity satisfies

$$\varepsilon_n = \frac{h_n\varepsilon'_n}{h^2} \ . \tag{B.6}$$

If the real and formal media are both uniform and isotropic (i.e., constant μ, μ', ε_n, ε'_n) then all scale factors are constant and our space is Euclidean. Thus we see that by increasing the rigidity of our assumptions on our real and formal media we arrive at the situation where our space has to be Euclidean for the case of uniform and isotropic media.

On the other hand, let us investigate the consequences of the assumptions that $h_a = h_b \equiv h$, and that our space is Euclidean. Certainly one possibility is the case that h is a constant. A more interesting case occurs when h is not constant. As shown in Appendix A, (A.38) through (A.51), conformal maps yield the complete class of solutions. A particular example corresponds to inversion of coordiantes, where we put

$$\begin{aligned} u_a &= \frac{x'}{(x')^2 + (y')^2} \\ u_b &= \frac{-y'}{(x')^2 + (y')^2} \end{aligned} \tag{B.7}$$

in which case

$$h = \frac{1}{u_a^2 + u_b^2} \tag{B.8}$$

so that

$$(d\ell)^2 = h^2[(du_a)^2 + (du_b)^2] = [(dx')^2 + (dy')^2] \,. \tag{B.9}$$

This example is the analog of the situation discussed in [1 (Appendix A)], characteristics of coordinate systems for field components in all three coordinate directions are investigated (i.e., the case $h_1 = h_2 = h_3$).

B.2 Case of a Uniform TEM Plane Wave in Two-Dimensional Formal Coordinates

Maxwell's equations (in Appendix B.1) may now be further specialized to the case of plane wave propagation in a plane S with propagation in a direction $\vec{1}_0$. We have coordinates (u_a, u_b, u_n) as in Appendix A with u_n constant on S and with corresponding unit vectors $(\vec{1}_a, \vec{1}_b, \vec{1}_n)$ and we assume our formal medium is isotropic and homogeneous. Thus our formal fields are assumed to satisfy

$$\begin{aligned}
\vec{E}' &= E'_n \vec{1}_n \\
\vec{H}' &= \frac{1}{Z'_0}\vec{1}_0 \times \vec{E}' \\
\vec{1}_0 &= (\cos(\eta))\vec{1}_a + (\sin(\eta))\vec{1}_b \\
E'_n &= E'_0 f(t - \vec{1}_0 \cdot \vec{u}/c') \\
\vec{u} &= (u_a, u_b) \\
Z'_0 &= \sqrt{\frac{\mu'}{\varepsilon'}} \,,\; c' = \frac{1}{\sqrt{\mu'\varepsilon'}} \,.
\end{aligned} \tag{B.10}$$

In (B.10), μ', ε', and η are constants and hence Z'_0 and c' are also constants.

We note that as a consequence of (B.10) we must have

$$\begin{aligned} H'_a &= \frac{E'_n}{Z'_0}\sin(\eta) \\ H'_b &= -\frac{E'_n}{Z'_0}\cos(\eta) \end{aligned} \tag{B.11}$$

where η is the constant angle defined in (B.10). We note also that the formal and real media properties are related by the following equations. In these equations, which follow from (2.13), we have taken $\mu' = \mu'_a = \mu'_b$ and $\varepsilon' = \varepsilon'_n$ because of our stated assumptions on the formal medium. Thus we obtain

$$\begin{aligned} \frac{h_b h_n}{h_a}\mu_a &= \frac{h_a}{h_b}h_n\mu_b = \mu' \\ \frac{h_a h_b}{h_n}\varepsilon_n &= \varepsilon' . \end{aligned} \tag{B.12}$$

Since we have formal magnetic field components H'_a, H'_b and a formal electric field component E'_n, the situation for purposes of the present development is the same as that for a jacket, as in Section 5. We note that each of these formal field components satisfies a two-dimensional wave equation

$$\frac{\partial^2 W'_i}{\partial u_a^2} + \frac{\partial^2 W'_i}{\partial u_b^2} = \mu'\varepsilon'\frac{\partial^2 W'_i}{\partial t^2} \tag{B.13}$$

where W'_i can be any one of H'_a, H'_b, or E'_n.

B.3 Specialization of B.2 to the Case of an Isotropic Real Medium

Now the equations (B.12) which give relationships among the scale factors and the real and formal constitutive parameters were based on the assumption that the formal medium is isotropic and homogeneous. If we now impose an added restriction on the real medium, namely, that it is isotropic, then

$$\begin{aligned} \mu &= \mu_a = \mu_b \\ h &= h_a = h_b . \end{aligned} \tag{B.14}$$

Hence the equations (B.12) may now be written as (using $\varepsilon = \varepsilon_n$)

$$\begin{aligned} \varepsilon &= \varepsilon' \frac{h_n}{h^2} \\ \mu &= \frac{\mu'}{h_n} . \end{aligned} \tag{B.15}$$

One consequence of a constant scale factor h_n is that for the real medium the permeability μ is a constant and that the permittivity ε will in general be a function of u_a and u_b since h depends on these coordinates. If our surface is Euclidean, then h is obtainable through conformal mappings of the plane into itself. A further observation is that for constant h_n, equations (B.15) correspond exactly to the case of two-dimensional lenses (with electric field in the direction of the z-axis) discussed in [1 (Section IX)]. The case of constant h_n, as stated earlier, corresponds to a constant spacing of electrical conducting boundaries.

B.4—Further Specialization of B.2 and B.3 to the Case of Constant μ and Constant ε

The equations (B.15) are based on the assumptions that the formal medium is isotropic and homogeneous and that the real medium is also isotropic. If we further specialize these equations to the case of a constant permeability, μ, then the scale factor h_n is also constant. As we saw at the end of the last section these assumptions led to, in the case of Euclidean surfaces, to the case where the complex coordinates and hence h are obtainable by conformal mappings and the permittivity ε is a function of the coordinates u_a and u_b.

If in addition to a constant μ we have a constant ε, then equations (B.15) imply that all scale factors are constant. In this event the surface defined by

$$(d\ell)^2 = h^2[(du_a)^2 + (du_b)^2] \tag{B.16}$$

is obviously Euclidean. Note however that for surfaces which are Euclidean that the permittivity ε need not be a constant.

We note that the results obtained above are applicable in Section 5 (on jackets).

B.5 Dual Cases

The results obtained in Sections B.1 through B.4 can be used in the case where we have formal electric field components E'_a and E'_b and one formal magnetic field component, H'_n. The duality that exists between the roles of the electric and magnetic fields makes this task a simple one. Thus, for example, the formal Maxwell's equations become

$$\begin{aligned} \frac{\partial H'_n}{\partial u_b} &= \varepsilon'_a \frac{\partial E'_a}{\partial t} \\ \frac{\partial H'_n}{\partial u_a} &= -\varepsilon'_b \frac{\partial E'_b}{\partial t} \\ \frac{\partial E'_b}{\partial u_a} - \frac{\partial E'_a}{\partial u_b} &= -\mu'_n \frac{\partial H'_n}{\partial t} . \end{aligned} \tag{B.17}$$

For the case of a uniform TEM plane wave, as discussed in Section B.2, we would take

$$\begin{aligned} \vec{H}' &= H'_n \vec{1}_n = H'_0 g(t - \vec{1}_0 \cdot \vec{u}/c') \vec{1}_n \\ \vec{E}' &= Z'_0 (\vec{1}_0 \times \vec{H}') \end{aligned} \tag{B.18}$$

where $\vec{1}_0, \vec{u}, c'$, and Z'_0 are the same quantities which appear in (B.10).

Thus the assumption that the formal medium is isotropic and homogeneous would lead to the analog of (B.12), namely,

$$\begin{aligned} \frac{h_b h_n}{h_a} \varepsilon_a &= \frac{h_a h_n}{h_b} \varepsilon_b = \varepsilon' \\ \frac{h_a h_b}{h_n} \mu_n &= \mu' . \end{aligned} \tag{B.19}$$

Hence if the further restriction that the real medium be isotropic is made, then we must have

$$\begin{aligned} \varepsilon = \varepsilon_a = \varepsilon_b \\ h = h_a = h_b \end{aligned} \tag{B.20}$$

and consequently (using $\mu_n = \mu$)

$$\varepsilon = \frac{\varepsilon'}{h_n}$$
$$\mu = \frac{\mu' h_n}{h^2} . \tag{B.21}$$

The analogs of the results in Sections B.3 and B.4 then are easy to state. For example, h_n is a constant if and only if the real permittivity ε is a constant. If ε is a constant, then μ will in general be a function of u_a and u_b and the surface can be Euclidean. If both ε and μ are constant then the surface will be Euclidean. Finally, the case of constant h_n corresponds to the case of two-dimensional lenses (with magnetic field in the z-direction) discussed in [1 (Section IX)]. The results in the dual cases discussed above are applicable in Section 6 (on slices).

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