Sensor and Simulation Notes
Note 303
13 May 1987

Positioning Loops with Parallel Magnetic Dipole Moments to Avoid Mutual Inductance

J. D. Quinn[1] and C. E. Baum

Air Force Weapons Laboratory
Kirtland AFB, New Mexico

Abstract

A scheme is described for arranging a number of circular loops on a spherical surface to produce magnetic dipole moments, while cancelling higher multipole moments. The ability to arrange loops so as to behave as dipoles then gives rise to the possibility of arranging a set of such dipoles in space so that there is zero mutual inductance. The relative location of parallel magnetic dipoles for zero mutual inductance is given by a particular angle between their dipole axes and a line between the dipoles, and so is valid regardless of the radius of the magnetic dipoles. An earlier paper investigated the feasibility of adding the magnetic dipole moments of a number of dipole loops to produce an electromagnetic field with magnetic dipole moments covering a wide frequency range. The combined results of the two associated papers indicate that it is possible to construct a log-periodic magnetic dipole array to produce a controllable electromagnetic field for some interaction measurement purposes.

[1]Materials Research Laboratories, Maribyrnong, Australia. Stationed temporarily at Air Force Weapons Laboratory.

WL-EMP-SSN-303

Sensor and Simulation Notes
Note 303
13 May 1987

SEP 1 1 36 PM '87

Positioning Loops with Parallel Magnetic Dipole Moments to Avoid Mutual Inductance

J. D. Quinn[1] and C. E. Baum

Air Force Weapons Laboratory
Kirtland AFB, New Mexico

Abstract

A scheme is described for arranging a number of circular loops on a spherical surface to produce magnetic dipole moments, while cancelling higher multipole moments. The ability to arrange loops so as to behave as dipoles then gives rise to the possibility of arranging a set of such dipoles in space so that there is zero mutual inductance. The relative location of parallel magnetic dipoles for zero mutual inductance is given by a particular angle between their dipole axes and a line between the dipoles, and so is valid regardless of the radius of magnetic dipoles. An earlier paper investigated the feasibility of adding the magnetic dipole moments of a number of dipole loops to produce an electromagnetic field with magnetic dipole moments covering a wide frequency range. The combined results of the two associated papers indicate that it is possible to construct a log-periodic magnetic dipole array to produce a controllable electromagnetic field for some interaction measurement purposes.

dipole antennas, loop antennas

[1]Materials Research Laboratories, Maribyrnong, Australia. Stationed temporarily at Air Force Weapons Laboratory.

1 Introduction

As part of the interest in producing electromagnetic environments a local illuminator has been proposed which is based on adding parallel magnetic dipole moments from an array of loops. Models [1] have been developed to describe the effective magnetic dipole moment as a function of frequency for such an array of loops, whose self-inductance is included in a cascaded constant-resistance network. Successful implementation of the concept for this illuminator is dependent upon the ability to effectively produce magnetic dipole moments while cancelling higher moments and as well there should be no mutual inductance between dipoles. Methods for achieving such a controlled electromagnetic environment by loops are described and a configuration of loops forming a log-periodic magnetic dipole array is developed.

The production of accurate magnetic fields inside a volume enclosed by a number of loops has received considerable attention, particularly for generating very uniform magnetic fields. The process usually involves accurately winding loops around symmetrical shapes such as spheres and cylinders. Analysis of magnetic fields in this form involve terms in a series expansion. It is generally possible to eliminate individual terms by virtue of symmetry considerations. The analysis for parallel loops on a spherical shape can be divided into two separate solutions corresponding respectively to an interior volume and an exterior volume. Each of these contains, amongst other terms, a common Legendre polynomial expansion term. Elimination of terms other than the initial term, $(n = 1)$ produces a uniform magnetic field within the volume, while for the exterior it eliminates all higher moments and leaves only a dipole term.

In considering magnetic fields produced by loops and the interaction between loops, it is useful to relate both fields and generating sources to a common origin. The analyses are done in two sections: one for mutual induction between two parallel circular loops and the other for the fields produced by loops symmetrically wound on the surface of a sphere. In both cases an angular dependence about the common origin is obtained. The angles can be chosen to eliminate mutual induction between the dipole component of loops in one case, and for the elimination of higher order components in the exterior magnetic field produced by loops wound on the surface of a sphere. Both analyses produce results that correspond to some form of Legendre function in the angle measured from the parallel dipole axes.

2 Mutual Inductance of Circular Coils with Parallel Axes

The mutual induction between two circular loops of unequal radius and parallel axes is examined. In accordance with the stipulation that the origin of the field and generating systems should be the same, the origin is chosen to be at the center of loop A (Fig. 2.1). The two loops are separated by a distance D between centers, and the angle between the axis of loop A, and the line joining centers is θ. The radius of loop A is a, and the radius of loop B is b. The mutual inductance M_{BA} of coil B due to a unit current on coil A can be written as [2]

$$M_{BA} = \oint_{C_B} \vec{A}_A . d\vec{\ell}_B \tag{2.1}$$

where $\vec{A}_A$ is the magnetic vector potential due to a unit current in loop A and $d\vec{\ell}_B$ is an incremental element of loop B. The magnetic vector potential can be expressed as

$$\vec{A}_A = \frac{\mu}{4\pi} \oint_{C_A} \frac{d\vec{\ell}_A}{r} \tag{2.2}$$

where $d\vec{\ell}_A$ is an incremental element of loop A and r is the distance from $d\vec{\ell}_A$ to the observer. On substituting this expression for $\vec{A}_A$ in (2.1), the expression for the mutual inductance, M_{BA}, becomes

$$M_{BA} = \frac{\mu}{4\pi} \oint_{C_A} \oint_{C_B} \frac{d\vec{\ell}_A . d\vec{\ell}_B}{r} . \tag{2.3}$$

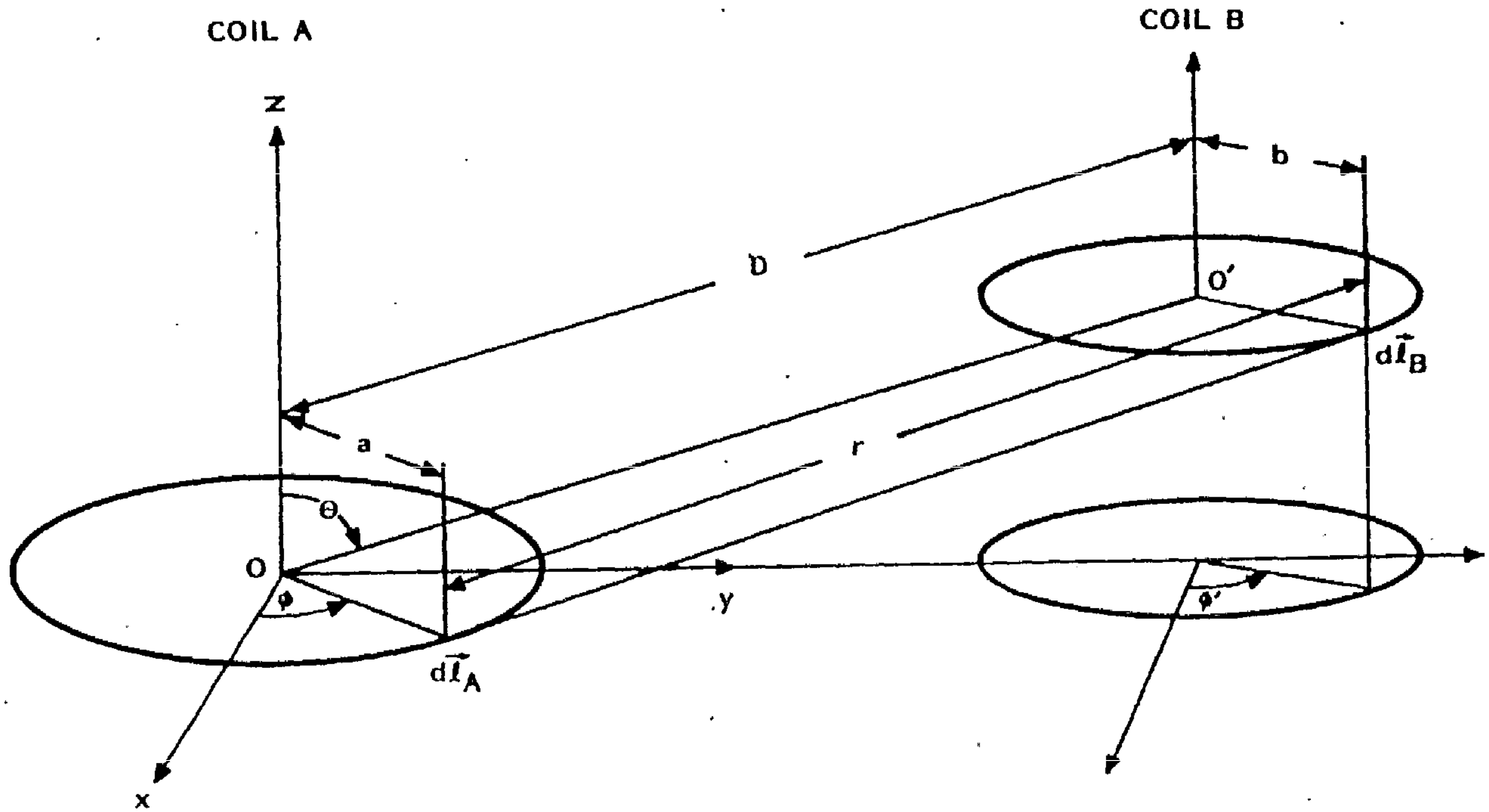

Figure 2.1: Two Parallel Loops with Separation D between Centers

Due to symmetry considerations the magnetic vector potential has only a ϕ component, hence (2.3) can be written in terms of angles ϕ and ϕ'. It can be seen by referring to Fig. 2.2 that the angle between $d\vec{\ell}_A . d\vec{\ell}_B$ is $\phi' - \phi$, and so (2.3) can be rewritten as

$$\begin{aligned} M_{BA} &= \frac{\mu ab}{4\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{\cos(\phi' - \phi)}{r} d\phi\, d\phi' \qquad (2.4) \\ a d\phi &= |d\vec{\ell}_A| \qquad \text{and} \qquad b d\phi' = |d\vec{\ell}_B|\,, \\ d\vec{\ell}_A . d\vec{\ell}_B &= |d\vec{\ell}_A|\,|d\vec{\ell}_B| \cos(\phi' - \phi) \end{aligned}$$

and r is the magnitude of the distance between the incremental elements $d\vec{\ell}_A$ and $d\vec{\ell}_B$.

The value r can be expressed in terms of D, ϕ, ϕ', and θ with three components r_x, r_y, and r_z as

$$\begin{aligned} r_x &= b\cos(\phi') - a\cos(\phi) \qquad (2.5) \\ r_y &= D\sin(\theta) - a\sin(\phi) + b\sin(\phi') \\ r_z &= D\cos(\theta) \\ r^2 &= r_x^2 + r_y^2 + r_z^2 \\ &= D^2 + a^2 + b^2 - 2ab\cos(\phi' - \phi) + 2D\sin(\theta)(b\sin(\phi') - a\sin(\phi)) \qquad (2.6) \end{aligned}$$

or

$$\frac{1}{r} = \frac{1}{D}\left(1 + \frac{2\sin(\theta)(b\sin(\phi') - a\sin(\phi))}{D} + \frac{a^2 + b^2 - 2ab\cos(\phi' - \phi)}{D^2}\right)^{-\frac{1}{2}} . \qquad (2.7)$$

Using the binomial expansion and limiting the analysis to the case where $a + b < D$ this expression can be written as a series with powers of D. The expansion is carried out in detail in Appendix A, and when the result (A.4) is substituted into (2.4) the mutual induction M_{BA} can be written as:

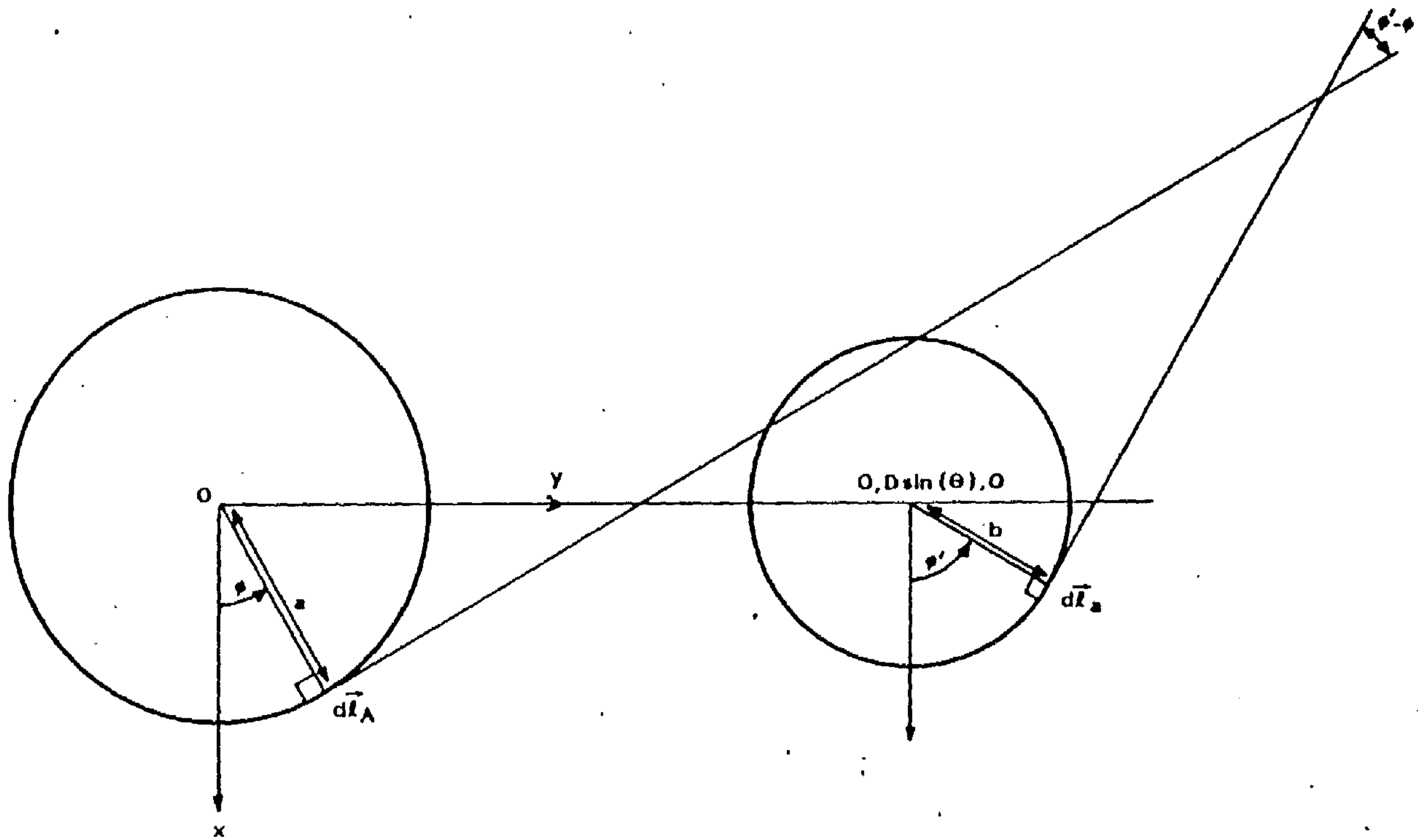

Figure 2.2: Angular Relationships Between Loop Elements

$$
\begin{aligned}
M_{BA} \;=\; & \frac{\mu ab}{4\pi}\Big\{\frac{1}{D}\int_0^{2\pi}\int_0^{2\pi}\cos(\phi'-\phi)d\phi\,d\phi' \\
& -\frac{\sin\theta}{D^2}\int_0^{2\pi}\int_0^{2\pi}\cos(\phi'-\phi)(b\sin(\phi')-a\sin(\phi))d\phi\,d\phi' \\
& +\frac{1}{2D^3}\Big[3\sin^2(\theta)\int_0^{2\pi}\int_0^{2\pi}\cos(\phi'-\phi)(b\sin(\phi')-a\sin(\phi))^2 d\phi\,d\phi' \\
& -\int_0^{2\pi}\int_0^{2\pi}\cos(\phi'-\phi)(a^2+b^2-2ab\cos(\phi'-\phi))d\phi\,d\phi'\Big] \\
& +\frac{1}{2D^4}\Big[3\sin(\theta)\int_0^{2\pi}\int_0^{2\pi}\cos(\phi'-\phi)(a^2+b^2-2ab\cos(\phi'-\phi)) \\
& (b\sin(\phi')-a\sin(\phi))d\phi\,d\phi' \\
& -5\sin^3(\theta)\int_0^{2\pi}\int_0^{2\pi}\cos(\phi'-\phi)(b\sin(\phi')-a\sin(\phi))^3 d\phi\,d\phi'\Big] \\
& +\frac{1}{8}\frac{1}{D^5}\Big[3\int_0^{2\pi}\int_0^{2\pi}\cos(\phi'-\phi)(a^2+b^2-2ab\cos(\phi'-\phi))^2 d\phi\,d\phi' \\
& -30\sin^2(\theta)\int_0^{2\pi}\int_0^{2\pi}\cos(\phi'-\phi)(b\sin(\phi')-a\sin(\phi))^2 \\
& (a^2+b^2-2ab\cos(\phi'-\phi))d\phi\,d\phi' \\
& +35\sin^4(\theta)\int_0^{2\pi}\int_0^{2\pi}\cos(\phi'-\phi)(b\sin(\phi')-a\sin(\phi))^4 d\phi\,d\phi'\Big] \\
& +\frac{1}{D^6}\cdots\,.
\end{aligned}
\tag{2.8}
$$

The evaluation of the integrals in (2.8) is described in Appendices B to E. Appendix B has the integral evaluation for terms in D^{-1} and D^{-2}. Appendices C, D, and E contain the detailed evaluation of terms in D^{-3}, D^{-4}, and D^{-5}, respectively. The results (B.1), (B.2), (C.8), (D.7), (E.12) when substituted into (2.8) yield

$$
M_{BA}=\frac{\mu ab}{4\pi}\left\{\frac{1}{D^3}\pi^2 ab(3\cos^2(\theta)-1)-\frac{1}{D^5}\frac{3}{8}\pi^2 ab(a^2+b^2)(35\sin^4(\theta)-40\sin^2(\theta)+8)\cdots\right\}\,. \tag{2.9}
$$

The integrals related to $1/D$ and all even powers of $1/D$ are equal to zero. Collecting variables and expressing the angle dependence of the second term as $\cos(\theta)$ we can write

$$
M_{BA}=\frac{\mu a^2 b^2\pi}{4}\left\{\frac{1}{D^3}(3\cos^2(\theta)-1)-\frac{3}{8}\frac{(a^2+b^2)}{D^5}(35\cos^4(\theta)-30\cos^2(\theta)+3)\cdots\right\}\,. \tag{2.10}
$$

The factors of the terms in $\cos(\theta)$ correspond to those for Legendre polynomials for $P_1^{(0)}(\cos\theta)$ and $P_4^{(0)}(\cos\theta)$. The mutual inductance between the loops is then given for the first two terms corresponding to the dipole and quadrupole components. For no mutual inductance for the dipole component the result of $\theta = \theta_{0_d}$ can be obtained where

$$\cos(\theta_{0_d}) = \frac{1}{\sqrt{3}}\ , \ \theta_{0_d} \simeq 54.74^\circ\ . \tag{2.11}$$

For no quadrupole mutual inductance there are two angles of $\theta_{0_{q1}}$ and $\theta_{0_{q2}}$ where

$$\begin{aligned} \sin^2(\theta_{0_{q1}}) &= \frac{4+2\sqrt{\frac{6}{5}}}{7}\ , \quad \theta_{0_{q1}} = 70.12^\circ \\ \sin^2(\theta_{0_{q2}}) &= \frac{4-2\sqrt{\frac{6}{5}}}{7}\ , \quad \theta_{0_{q2}} = 30.56^\circ\ . \end{aligned} \tag{2.12}$$

From (2.11) it can be seen that zero mutual inductance occurs at the nulls for the Legendre coefficients for each individual dipole, quadrupole, etc., and can be achieved by placing the loops at the appropriate angle θ, independent of D provided $a + b < D$.

Mutual inductance between loops can be eliminated on an individual multipole basis, at least for magnetic dipoles and quadrupoles. The angles of zero mutual inductance for these cases cannot be met simultaneously, and hence the need exists for eliminating the production of higher moments at the source, when the loops are to be placed close together. On the assumption that this can be done, a satisfactory distribution of loops in space is possible which produces zero mutual inductance for magnetic dipoles and possibly keeping multipole interaction to a minimum. The aim is to keep, at least for neighboring loops, the angle θ_{0_d} between the line between the central points of each loop and the parallel magnetic-dipole vectors.

The simplest array of such nonmutually inducting loops is a linear array, all with their centers situated on a line chosen through an arbitrary origin at an angle θ_{0_d} with their magnetic dipole axes. This fulfills the angle requirement but may not be practical from the point of view of efficient use of space. A more compact array may be made, where the angle criterion is released for loops other than the nearest neighbors. In this case the greater rate of falloff in intensity for higher moment magnetic field components with distance (as evident in (2.10)) can be utilized to reduce these contributions of mutual induction.

3 Magnetic Fields From Currents

3.1 Magnetic Fields Due to Parallel Loops on the Surface of a Sphere

If parallel loops carrying a current are constrained to lie on the surface of a sphere, then an analysis in spherical coordinates can readily be carried out. Consider a sphere, radius r, with current limited to the ϕ direction as in Fig. 3.1. The magnetic field intensity $\vec{H}$ is related to the scalar magnetic potential Φ as

$$\vec{H} = \nabla\Phi \; . \tag{3.1}$$

Also

$$\nabla \cdot \vec{B} = \mu\nabla \cdot \vec{H} = 0 \; . \tag{3.2}$$

Hence

$$\nabla \cdot \vec{H} = \nabla \cdot \nabla\Phi = \nabla^2\Phi = 0 \; . \tag{3.3}$$

A solution for this equation, ignoring the ϕ component because of symmetry produced by constraining currents to constant values of θ, can be written in terms of Legendre polynomials as

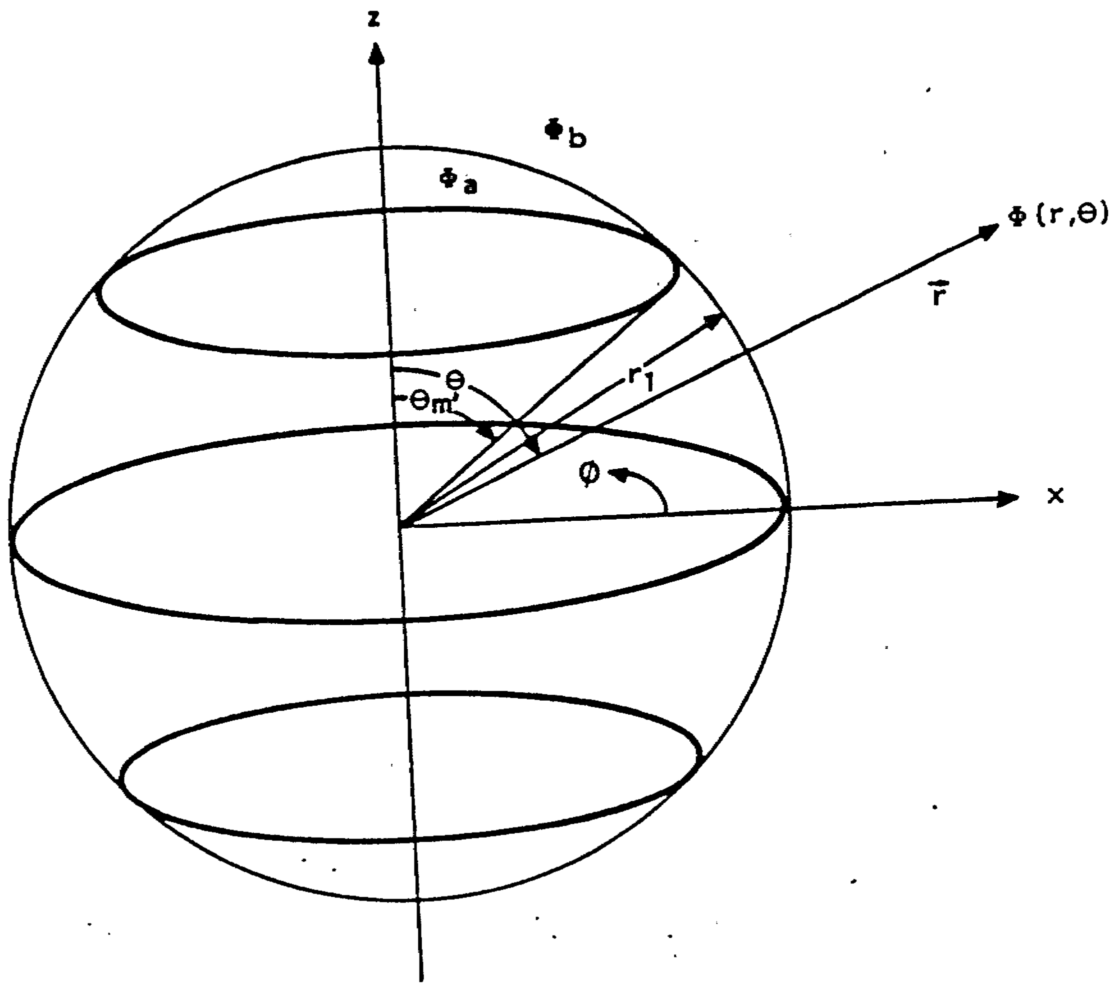

Figure 3.1: Sphere with Loops Wound in ϕ Direction. Scalar magnetic field is described for two regions, with Φ_a for the region inside the sphere, and Φ_b for the region outside the sphere.

$$\Phi = \sum_{n=0}^{\infty} \left[a_n \left(\frac{r}{r_1} \right)^n + b_n \left(\frac{r_1}{r} \right)^{n+1} \right] P_n^{(0)}(\cos(\theta)) \ . \tag{3.4}$$

Since the scalar magnetic potential must be finite at the origin, and tend to zero for r equal to infinity the solution must be in two parts. One corresponds to Φ_a for which $r < r_1$ and the other corresponds to Φ_b for which $r > r_1$. Thus normalizing coordinates by putting R for r/r_1 (3.4) can be expressed as two separate equations

$$\begin{aligned} \Phi_a &= \sum_{n=0}^{\infty} a_n R^n P_n^{(0)}(\cos(\theta)) \\ \Phi_b &= \sum_{n=0}^{\infty} b_n R^{-(n+1)} P_n^{(0)}(\cos(\theta)) \end{aligned} \tag{3.5}$$

The magnetic field intensity $\vec{H}$ for the external case with $r > r_1$ can be found using (3.1) together with the vector relation in spherical coordinates, i.e.,

$$\nabla\Phi = \frac{\partial \Phi}{\partial r}\vec{1}_r + \frac{1}{r}\frac{\partial \Phi}{\partial \theta}\vec{1}_\theta + \frac{1}{r\sin(\theta)}\frac{\partial \Phi}{\partial \phi}\vec{1}_\phi \ . \tag{3.6}$$

Thus we can write for the two non-zero magnetic field intensity components outside the sphere

$$\begin{aligned} H_{b_r} &= -\frac{1}{r_1}\sum_{n=1}^{\infty} b_n(n+1)R^{-(n+2)} P_n^{(0)}(\cos(\theta)) \\ H_{b_\theta} &= \frac{1}{r_1}\sum_{n=1}^{\infty} b_n R^{-(n+2)} P_n^{(1)}(\cos(\theta)) \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} P_n^{(m)}(x) &= (-1)^m(1-x^2)^{\frac{1}{2}m}\frac{d^m P_n(x)}{dx^m} \\ P_n^{(1)}(x) &= -(1-x^2)^{\frac{1}{2}}\frac{d(P_n(x))}{dx} \ . \end{aligned} \tag{3.8}$$

In a similar way expression for the magnetic field intensity inside the sphere become

$$\begin{aligned} H_{a_r} &= \frac{1}{r_1}\sum_{n=1}^{\infty} n a_n R^{n-1} P_n^{(0)}(\cos(\theta)) \\ H_{a_\theta} &= \frac{1}{r_1}\sum_{n=1}^{\infty} a_n R^{n-1} P_n^{(1)}(\cos(\theta)) \ . \end{aligned} \tag{3.9}$$

There is no ϕ component for either the internal or external case.

At the boundary of the sphere ($R = 1$) the surface current density is constrained to have only a ϕ component and is related to the magnetic field intensities $\vec{H}_a$ and $\vec{H}_b$, so using

$$\nabla \times \vec{H} = \vec{J} \tag{3.10}$$

and

$$H_{r_a} = H_{r_b} \qquad \text{at the boundary} \tag{3.11}$$

we can write

$$J_{s_\phi} = H_{\theta_b} - H_{\theta_a} \,. \tag{3.12}$$

Consider that the surface current is not allowed to vary continuously with θ over the surface of the sphere but is contained in an infinitesimally narrow band at $\theta_{m'}$ as a current $I_{m'}$. A Dirac delta function can relate the current h surface current density $J_{s_{\phi_{m'}}}$ as

$$\int_0^{\pi r_1} J_{s_{\phi_{m'}}} d(r_1\theta) = \int_0^{\pi r_1} I_{m'}\delta(r_1\theta - r_1\theta_{m'})d(r_1\theta) \tag{3.13}$$

and as $d\ell = dr_1(\theta)$ for this case

$$J_{s_{\phi_{m'}}} = I_{m'}\delta(r_1\theta - r_1\theta_{m'}) = I_{m'}\delta[r_1(\theta - \theta_{m'})] \,. \tag{3.14}$$

From (3.9) and (3.12) this can be written as

$$J_{s_{\phi_{m'}}} = \frac{I_{m'}}{r_1}\sum_{n=1}^{\infty} c_{n,m'}P_n^{(1)}(\cos(\theta)) \,. \tag{3.15}$$

A solution for the $c_{n,m'}$ can be formed by multiplying (3.13) and (3.14) by $P_n^{(1)}(\cos\theta)\sin(\theta)$ and integrating over the respective arcs. Thus

$$\begin{aligned} &\frac{I_{m'}}{r_1}\int_0^{\pi r_1} \delta[r_1(\theta - \theta_{m'})]P_n^{(1)}(\cos(\theta))\sin(\theta)d(r_1\theta) \\ &= \frac{I_{m'}}{r_1}c_{n,m'}\int_0^{\pi r_1} P_n^{(1)}(\cos(\theta))P_n^{(1)}(\cos(\theta))\sin(\theta)d(r_1\theta) \end{aligned} \tag{3.16}$$

where the orthogonality of the Legendre functions leaves only the nth term in the sum. But

$$\int_{-\infty}^{\infty} f(x)\delta(x-x')dx = f(x') \tag{3.17}$$

and from [3]

$$\int_{-1}^{1} [P_n^{(m)}(x)]^2 dx = \left(n+\frac{1}{2}\right)^{-1} (n+m)! \Big/ (n-m)! \, . \tag{3.18}$$

Hence

$$c_{n,m'} = \frac{2n+1}{2n(n+1)} P_n^{(1)}(\cos(\theta_{m'}))\sin(\theta_{m'}) \, . \tag{3.19}$$

Substituting (3.19) into (3.15) gives

$$J_{s_{\phi_{m'}}} = \frac{I_{m'}}{r_1}\sum_{n=1}^{\infty} \frac{2n+1}{2n(n+1)} P_n^{(1)}(\cos(\theta_{m'}))\sin(\theta_{m'})P_n^{(1)}(\cos\theta) \, . \tag{3.20}$$

A new quantity can be defined which relates the accumulated effect of a number of loops M', at discrete values of $\theta_{m'}$, as

$$\begin{aligned} A'_n &\equiv \frac{2n+1}{2n(n+1)} \sum_{m'=1}^{M'} P_n^{(1)}(\cos(\theta_{m'}))\sin(\theta_{m'})p_{m'} \\ I_{m'} &\equiv I_0 p_{m'} \, . \end{aligned} \tag{3.21}$$

Then

$$J_{s_\phi} = \sum_{m'=1}^{M'} J_{s_{\phi_{m'}}} = +\frac{I_0}{r_1}\sum_{n=1}^{\infty} A'_n P_n^{(1)}(\cos(\theta)) \, . \tag{3.22}$$

The boundary conditions at r_1 (i.e. (3.11)) are known for the r components of magnetic field, hence the r components of (3.7) and (3.9) can be equated to give

$$-b_n(n+1) = na_n \, . \tag{3.23}$$

And from (3.12) and (3.22)

$$\begin{aligned} J_{s_\phi} &= H_{\theta b} - H_{\theta a} = \sum_{n=1}^{\infty} b_n P_n^{(1)}(\cos(\theta)) - \sum_{n=1}^{\infty} a_n P_n^{(1)}(\cos(\theta)) \\ &= \frac{I_0}{r_1}\sum_{n=1}^{\infty} A'_n P_n^{(1)}(\cos(\theta)) \end{aligned} \tag{3.24}$$

giving

$$b_n - a_n = \frac{I_0}{r_1} A'_n \, . \tag{3.25}$$

Eliminating a_n in (3.23) and (3.25) gives'

$$b_n = \frac{n}{(2n+1)} \frac{I_0}{r_1} A'_n \, . \tag{3.26}$$

The magnetic field $\vec{H}$ outside the sphere is obtained by substituting (3.26) into (3.7) giving

$$\begin{aligned} H_{b_r} &= -\sum_{n=1}^{\infty} b_n (n+1) R^{-(n+2)} P_n^{(0)}(\cos(\theta)) \\ &= -\frac{I_0}{r_1} \sum_{n=1}^{\infty} \frac{n(n+1)}{2n+1} A'_n R^{-(n+2)} P_n^{(0)}(\cos(\theta)) \\ H_{b_\theta} &= \sum_{n=1}^{\infty} b_n R^{-(n+2)} P_n^{(1)}(\cos(\theta)) \\ &= \frac{I_0}{r_1} \sum_{n=1}^{\infty} \frac{n}{2n+1} A'_n R^{-(n+2)} P_n^{(1)}(\cos(\theta)) \, . \end{aligned} \tag{3.27}$$

These equations describe the fields external to the sphere on which the parallel loops are wound, with a common origin for both field and generating source. Furthermore, the field can be calculated for each position R, θ for any combination of loops at angles $\theta_{m'}$ with contribution $p_{m'}$ for $m' = 1$ to M' using (3.27).

The magnetic field $\vec{H}$ inside the sphere can be expressed in the form of a similar set of equations using a_n, which is obtained from (3.23) and (3.25) as

$$a_n = -\frac{I_0}{r_1} A'_n \frac{n+1}{2n+1} \, . \tag{3.28}$$

On substituting (3.28) into (3.9) the magnetic field can be expressed fully. Writing in terms of its two components in spherical coordinates

$$\begin{aligned} H_{a_r} &= \sum_{n=1}^{\infty} n a_n R^{n-1} P_n^{(0)}(\cos(\theta)) \\ &= -\frac{I_0}{r_1} \sum_{n=1}^{\infty} \frac{n(n+1)}{2n+1} A'_n R^{n-1} P_n^{(0)}(\cos(\theta)) \end{aligned} \tag{3.29}$$

$$
\begin{aligned}
H_{a_\phi} &= \sum_{n=1}^{\infty} a_n R^{n-1} P_n^{(1)}(\cos(\theta)) \\
&= -\frac{I_0}{r_1} \sum_{n=1}^{\infty} \frac{n+1}{2n+1} A'_n R^{n-1} P_n^{(1)}(\cos(\theta))
\end{aligned}
$$

3.2 Use of Loop Symmetry for Eliminating Magnetic Moments with Even Values of n

Examination of (3.27) and (3.29) shows that the field components are determined by A_n, which is itself a function of $P_n^{(1)}(\cos(\theta_{m'}))$. In seeking to eliminate various components of the field, nulls and symmetry associated with each degree (n) of $P_n^{(1)}(\cos(\theta_{m'}))$ can be utilized. Symmetrical considerations can be utilized by choosing the distribution of $\theta_{m'}$ in a way such that the field is composed of pairs of contributions with magnitudes that are equal in magnitude, but vary in sign dependent upon degree. The definition of an associated Legendre function with real variable on examination shows that its parity is determined by n and m, i.e.

$$
P_n^{(m)}(\xi) \equiv (-1)^m (1-\xi^2)^{\frac{1}{2}m} \frac{d^m P_n(\xi)}{d\xi^m} \tag{3.30}
$$

and

$$
P_n(\xi) \equiv P_n^{(0)}(\xi) = \frac{1}{2^n n!} \frac{d^n[(\xi^2-1)^n]}{d\xi^n} .
$$

As ξ is changed to $(-\xi)$ the sign will be exactly dependent on the m-fold differentiation of $P_n(\xi)$ whose sign is dependent on the power of n, and so

$$
P_n^{(m)}(-\xi) = (-1)^{n+m} P_n^{(m)}(\xi) . \tag{3.31}
$$

This results in $P_n^{(1)}(\cos(\xi))$ being an odd function for n even and an even function for n odd, i.e.;

$$
P_n^{(1)}(-\cos(\theta_{m'})) = -P_n^{(1)}(\cos(\theta_{m'})) \qquad \text{for} \quad n \text{ even}
$$

and

$$P_n^{(1)}(-\cos(\theta_{m'})) = P_n^{(1)}(\cos(\theta_{m'})) \qquad \text{for} \qquad n \text{ odd} \tag{3.32}$$
$$\mu_{m'} = \cos(\theta_{m'}) .$$

An even arrangement for loops on a unit sphere is shown in Fig. 3.2. An evenly spaced set of loops from m'equals 1 to M' is symmetrically placed about $\cos(\theta_{m'})$ equals 0 with M' even. This means that there is no loop at $\cos(\theta_{m'})$ equals 0. For such a configuration, the relationship (3.32) applies. Further, the function is also odd when forming a product with $\sin(\theta)$, i.e.,

$$\begin{aligned} \sin(\theta)P_n^{(1)}(\cos(\theta)) &= (1-\cos^2(\theta))^{1/2}P_n^{(1)}(\cos(\theta)) \\ &= -(1-(-\cos^2(\theta))^{1/2}P_n^{(1)}(-\cos(\theta)) \qquad \text{for} \quad n \text{ even} . \end{aligned} \tag{3.33}$$

Setting $\cos(\theta_{m'}) = \mu_{m'}$ and setting the constraints

$$i_{m'} = i_{M'-m+1} \tag{3.34}$$

and

$$\mu_{m'} = -\mu_{M'-m+1} . \tag{3.35}$$

Then from (3.21)

$$\begin{aligned} A'_n &= \frac{2n+1}{2n(n+1)} \sum_{m'=1}^{M'} P_n^{(1)}(\mu_{m'})(1-\mu_{m'}^2)^{1/2} i_{m'} \\ &= 0 \qquad \text{for } M' \text{ even and } n \text{ even.} \end{aligned} \tag{3.36}$$

Thus only odd values of n need be considered for the magnetic fields represented by (3.27) and (3.29) in such an arrangement of loops. A'_n exists only for odd values of n.

3.3 Relationship Between Spherical Harmonics and Multipoles

The potential or field due to circulating currents can be expressed in series of two distinct forms, which are essentially relatable term by term. The concept of dipoles, quadrupoles

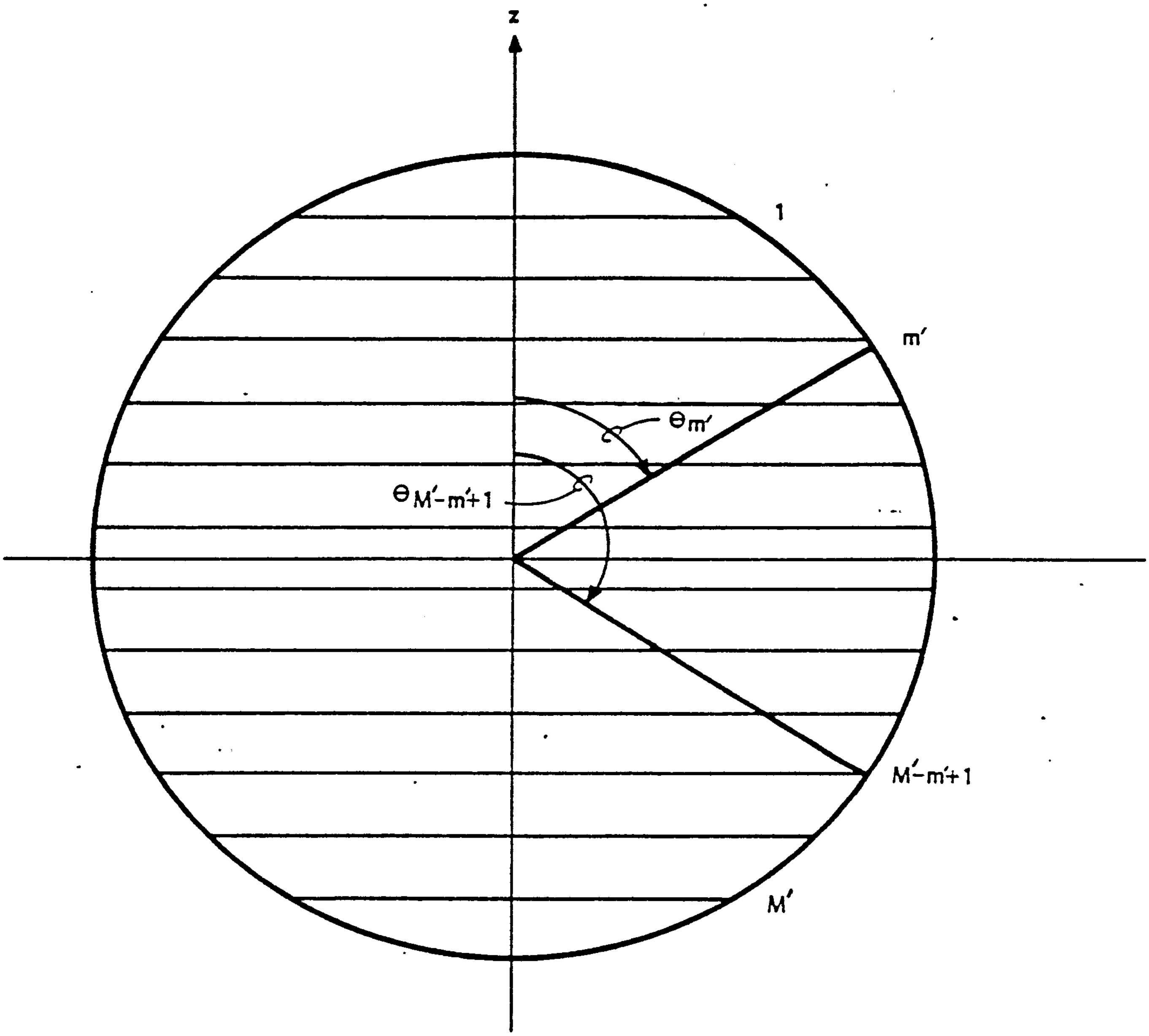

Figure 3.2: An Arrangement of an Even Number of Loops on a Unit Sphere for which $\cos(\theta_{m'}) = -\cos(\theta_{M'-m'+1})$. The current in each loop is constrained by the relationship $I_{m'} = I_{M'-m'+1}$.

and multipoles follow directly from one of these which is based on a Taylor's series. It applies to both magnetic and electric poles, although easier to understand in terms of the electric field and point charges [4]. The electric potential from an arbitrary distribution of charge density contained within the volume of a sphere can be expressed as

$$\psi = \frac{\rho}{4\pi\varepsilon_0} \int \int_V \int \frac{\rho(x_0, y_0, z_0)}{R_1} dx_0\, dy_0\, dz_0 \tag{3.37}$$

where $\rho(x_0, y_0, z_0)$ is the distribution of charge density and

$$R_1^2 = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 \,. \tag{3.38}$$

The expansion of this becomes (outside the sphere)

$$\begin{aligned} \psi &= \frac{\rho}{4\pi\varepsilon_0} \sum_{n=0}^{\infty} \sum_{\ell=0}^{n} \sum_{k=0}^{n-\ell} \frac{(-\ell)^n}{\ell! k! (n-\ell-k)!} \left\{ \int \int \int x_0^\ell y_o^k z_0^{n-\ell-k} \rho(x_0, y_0, z_0)\, dx_0\, dy_0\, dz_0 \right\} \\ & \frac{\partial^n}{\partial x^\ell \partial y^k \partial z^{n-\ell-k}} \left[\frac{1}{r} \right] \\ r &= x^2 + y^2 + z^2 \,. \end{aligned} \tag{3.39}$$

A series of multipoles with appropriate coefficients can be postulated that can produce a potential distribution outside the sphere equal to that from the arbitrary charge density. The multipoles are visualized as a number of equal and opposite point charges separated by some distance and each term is related in the series by an order n. The numerical value of charges, and the associated value of the multipole is 2^n in the expansion of (3.40). This series contains some redundant terms, i.e., ones which do not contribute to the external potential. When these are removed the remaining terms are equal to the series expression for potential based on spherical harmonics, again with n terms, as

$$\begin{aligned} \psi &= \frac{\rho}{4\pi\varepsilon_0} \sum_{n=0}^{\infty} \sum_{m=0}^{n} [A_{n,m} Y_{m,n,e}(\theta, \phi) \\ & + B_{n,m} Y_{m,n,o}(\theta, \phi)] \frac{1}{r^{n+1}} \\ A_{n,m} &= (2 - 1_{o,m}) \frac{(n-m)!}{(n+m)!} \end{aligned} \tag{3.40}$$

$$\int_0^{2\pi} \cos(m\phi_0)d\phi_0 \int_0^{\pi} P_n^{(m)}(\cos\theta_0)\sin\theta_0 d\theta_0$$
$$\int_0^{a} \rho(r_0,\theta_0,\phi_0) r_0^{n+2} dr_0$$
$$Y_{m,n,e} = P_n^{(m)}(\cos(\theta))\cos(m\phi) \qquad \text{(even form)}$$
$$Y_{m,n,o} = P_n^{(m)}(\cos(\theta))\sin(m\phi) \qquad \text{(odd form)}$$
$$\int_0^{2\pi} d\phi \int_0^{\pi} [Y_{m,n,o}(\theta,\phi)]^2 \sin\theta d\theta = \frac{4\pi}{(2-1_{0,m})(2n+1)} \frac{(n+m)!}{(n-m)!}$$
$$1_{m,m''} = \begin{cases} 1 & \text{if } m = m'' \\ 0 & \text{otherwise} \end{cases}$$

The coefficient $B_{n,m}$ is a similar volume integral to $A_{n,m}$ with $\sin(m\phi_0)$ substituted for $\cos(m\phi_0)$ in the integrand. The magnetic case can be developed by analogy with the electric case by replacing the electric multipole by a magnetic multipole. A similar expansion results, with terms exhibiting the same behavior with n. In the case of magnetic moments, the magnetic monopole ($n = 0$) equals zero. Nomenclature for the individual terms and their relationship to n for the static case are shown in Table 3.1. The traditional schemes do not cater readily for higher terms as is shown. A scheme based on the index n, as the term 2^n-pole was used in [5]. Others avoid the problem altogether, and simply state the relations in terms of n. If a scheme is desired to refer to the poles for larger values of n, it could be based on the fact that they correspond to the integer values of the power of two, or the integer binary orders. One scheme might be to call them n-binary order poles. In this case the first binary order pole would correspond to the dipole, the second binary order pole the quadrupole, etc.

Multipole Expansion for Static Electric and Magnetic Case					
Index n	Usual Name	2^n	Relation to Distance		Morse & Feshbach
			Potential	Field	
0	Monopole	1	r^{-1}	r^{-2}	Monopole
1	Dipole	2	r^{-2}	r^{-3}	Dipole
2	Quadrupole	4	r^{-3}	r^{-4}	Dipole of Dipoles
3	Octupole	8	r^{-4}	r^{-5}	Dipole of Quadrupoles?
4	Hexadecimapole?	16	r^{-5}	r^{-6}	—
5	—	32	r^{-6}	r^{-7}	—

Table 3.1. Relationship of Terms in Multipole Expansion

4 Spherical Coils

4.1 Helmholtz Coil—Two Loops Wound Symmetrically on Sphere

Two loops can be wound on the surface of a sphere in such a way that they are symmetrically located with respect to $\theta = \pi/2$, thus complying with (3.35) and (3.36). Thus a particular form of (3.37) can be written as

$$A'_n = \frac{(2n+1)}{n(n+1)} P_n^{(1)}(\mu_1)(1-\mu_1^2)^{1/2} p_1 \tag{4.1}$$

with $A'_n = 0$ for all n even.

The individual components of A'_n can be expanded for values of $n = 1, 3, 5$ as

$$\begin{aligned} A'_1 &= \frac{3}{2} P_1^{(1)}(\mu_1)(1-\mu_1^2)^{1/2} p_1 & (4.2) \\ &= -\frac{3}{2}(1-\mu_1^2) p_1 = -\frac{3}{2}\sin^2(\theta_1) p_1 \\ A'_3 &= \frac{7}{12} P_3^{(1)}(\mu_1)(1-\mu_1^2)^{1/2} p_1 & (4.3) \\ &= -\frac{7}{8}(5\mu_1^2 - 1)(1-\mu_1^2) p_1 . \end{aligned}$$

If we want A_3' to be zero (to maximize uniformity near the origin)

$$\begin{aligned} 5\mu_1^2 - 1 &= 0 \quad , \quad \theta_1 \simeq 63.43^\circ \\ \mu_1 = \cos(\theta_1) &= \frac{1}{\sqrt{5}} \quad , \quad \theta_2 = 180^\circ - \theta_1 \simeq 116.57^\circ \end{aligned} \tag{4.4}$$

$$\begin{aligned} A_5' &= \frac{11}{30} P_5^{(1)}(\mu_1)(1-\mu_1^2)p_1 \\ &= -\frac{11}{16}(21\mu_1^4 - 14\mu_1^2 + 1)(1-\mu_1^2)p_1 \,. \end{aligned} \tag{4.5}$$

It can be seen that A_3' can be made zero by locating the loops on the sphere at angles in (4.4). This corresponds to the configuration of a Helmholtz coil where the radius of its two constituent loops is equal to their separation. By setting A_3' equal to zero, the octupole term in the expansion for the magnetic field intensity is cancelled and the field is fully described by the dipole term and the odd multipole terms from the 5-binary order pole to infinty. The magnetic field intensity components for points outside the spherical volume, from (3.27) and (4.1) are

$$\begin{aligned} H_{b_r} &= -\frac{I_0 p_1}{r_1} \sum_{n=1}^{\infty} P_n^{(1)}(\mu_1)(1-\mu_1^2)^{1/2} R^{-(n+2)} P_n^{(0)}(\cos(\theta)) \\ &= \frac{I_0 p_1}{r_1} \left\{ \frac{4}{5} R^{-3} \cos(\theta) - \frac{2}{\sqrt{5}} \sum_{n=5}^{\infty\, (n \text{ odd})} P_n^{(1)} \left(\frac{1}{\sqrt{5}} \right) R^{-(n+2)} P_n^{(0)}(\cos(\theta)) \right\} \\ H_{b_\theta} &= \frac{I_0 p_1}{r_1} (1-\mu_1^2)^{1/2} \sum_{n=1}^{\infty} \frac{1}{n+1} P_n^{(1)}(\mu_1) R^{-(n+2)} P_n^{(1)}(\cos(\theta)) \\ &= \frac{I_0 p_1}{r_1} \left\{ \frac{2}{5} R^{-3} \sin(\theta) + \frac{2}{\sqrt{5}} \sum_{n=5}^{\infty\, (n \text{ odd})} \frac{1}{n+1} P_n^{(1)} \left(\frac{1}{\sqrt{5}} \right) R^{-(n+2)} P_n^{(1)}(\cos(\theta)) \right\} . \end{aligned} \tag{4.6}$$

For inside the volume of the sphere, the magnetic field intensity for the Helmholtz coil pair can be written from (3.29) and (4.1) as

$$H_{a_r} = -\frac{I_0 p_1}{r_1} (1-\mu_1^2)^{1/2} \sum_{n=1}^{\infty} P_n^{(1)}(\mu_1) R^{n-1} P_n^{(0)}(\cos(\theta)) \tag{4.7}$$

$$= \frac{I_0 p_1}{r_1}\left\{\frac{4}{5}\cos(\theta) - \frac{2}{\sqrt{5}}\sum_{n=5}^{\infty \; n\,\mathrm{odd}} P_n^{(1)}\left(\frac{1}{\sqrt{5}}\right) R^{n-1} P_n^{(0)}(\cos(\theta))\right\}$$

$$H_{a_\theta} = -\frac{I_0 p_1}{r_1}(1-\mu_1^2)^{1/2}\sum_{n=1}^{\infty}\frac{1}{n}P_n^{(1)}(\mu_1)R^{n-1}P_n^{(1)}(\cos(\theta))$$

$$= -\frac{I_0 p_1}{r_1}\left\{\frac{4}{5}\sin(\theta) + \frac{2}{\sqrt{5}}\sum_{n=5}^{\infty \; n\,\mathrm{odd}}\frac{1}{n}P_n^{(1)}\left(\frac{1}{\sqrt{5}}\right) R^{n-1} P_n^{(1)}(\cos(\theta))\right\} .$$

The magnetic field intensity inside the volume due to the $n = 1$ term, is independent of distance from the origin, and has a direction that is parallel to the axis of rotation. This can be appreciated readily from the relation that converts a unit vector $\vec{1}_z$ parallel to the z axis in Cartesian coordinates to spherical coordinates, i.e.,

$$\vec{1}_z = \vec{1}_r \cos(\theta) - \vec{1}_\theta \sin(\theta) . \tag{4.8}$$

4.2 Uniform Field and Magnetic Dipole Fields from Loops on a Sphere

It was shown in (4.6) and (4.7) that the first terms in the magnetic field expansions correspond to a magnetic dipole outside the sphere and a uniform field inside the sphere. By use of symmetry it is possible to eliminate all values of A'_n for even values of n. By choosing the location of two loops it is possible to eliminate one odd order, in this case corresponding to the octupole or 3-binary order pole (4.4). By adding extra pairs of symmetrical loops more A'_n terms, for n odd, can be eliminated. By eliminating all terms from 3 to infinity for A'_n in (3.37), there results a uniform magnetic field for the interior case and a dipole magnetic field for the exterior case.

Most interest in symmetrical loops has been in producing uniform magnetic fields in the interior region. Schemes have been reported for eliminating various of the higher moments using up to eight loops, which can eliminate A'_n from $n = 2$ up to 14 [6]. All moments higher than 1 can be eliminated using an infinite number of evenly spaced loops wound on the surface of a sphere with the current $i_{m'}$ in each loop equalling $I_0 \sin(\theta_{m'})$ [7] and [2]. The solutions in both these cases involve irrational ratios between the currents in each loop and so can limit the practicability of the various solutions found. A systematic discussion of axially symmetric magnetic fields is given in [8].

5 Log-Periodic Magnetic Dipole Array

The preceding analyses have succeeded in describing completely the magnetic field intensity due to currents in a series of loops wound symmetrically on the surface of a sphere. The magnetic field intensity is described as a sum of elements in a series expansion of multipole terms. Furthermore, all of the terms higher than one in the series can be cancelled, so it is theoretically possible by manipulations of loop position and fractional currents to generate the desired dipole only. An approximation to this condition can be obtained for small numbers of loop pairs, and if the cancellation of the quadrupole, octupole, and hexadecimapole terms is sufficient then a Helmholtz coil will suffice. In general, loops wound on the surface of a sphere provide a method whereby a predominantly dipole magnetic field can be generated.

The possibility of generating pure dipole magnetic fields can be combined with the result for mutual induction between loops, which also resolves into a multipole series. For parallel magnetic dipoles the mutual inductance is zero, so long as their centers lie on a line making the angle θ_{0_4} with the dipole axes. The value of θ_{0_4} is equal to 54.74°, from (2.11). A minimum separation can be set for pure dipoles, based on the non-intersection of their forming spheres. This is based on the fact that the magnetic dipole-moment field exists outside the sphere only, and the angle of zero mutual inductance applies only for dipoles. For the situation that some higher order terms are still remaining, the separation between spheres will need to be extended to meet the desired accuracy requirements.

A scheme for utilizing a series of parallel magnetic dipoles to produce a combined dipole-moment field has been established in an associated paper [1]. An analysis in that paper developed a network model for producing a controllable electromagnetic field made up of composite magnetic dipole moments which cover a wide frequency range. A successful configuration of magnetic dipoles was a series of loops which is increased successively by a factor β^2 in area and associatedly by a factor β in inductance in the network. Successful implementation of this model was dependent upon two conditions, addressed in this paper, being met. These conditions are firstly that an efficient means of producing magnetic dipole moments be found, and secondly a means for locating these dipoles with zero mutual inductance be configured.

Implementation of the loop-geometry conditions developed in this paper can be used in conjunction with the model in the associated paper to produce a suitable magnitude of magnetic dipole moment over a wide frequency range. The spatial arrangement of the loops is shown in Fig. 5.1, and can be visualized as a set of spheres located with their centers on a line making angle θ_{0_ℓ} with respect to the z axis when Cartesian coordinates are used, as with the axis of rotation in spherical coordinates. In the case of pure dipole generation it is sufficient that the separations of the spheres are such that their surfaces do not overlap or touch. When this condition is approximated by a few loops, such as for a Helmholtz coil, a greater wavelength dependent separation may be required depending on the accuracy required. The locations for two coils on the surface of each appropriate sphere are shown in Fig. 5.1 as two parallel lines above and below the sphere center and parallel to the x, y plane. The packing illustrated shows the spheres associated with each dipole touching, which is a limit for no interaction between perfect dipoles. It can be seen that the spheres fit into a circular cone, whose apex half-angle α_ℓ can, after a little

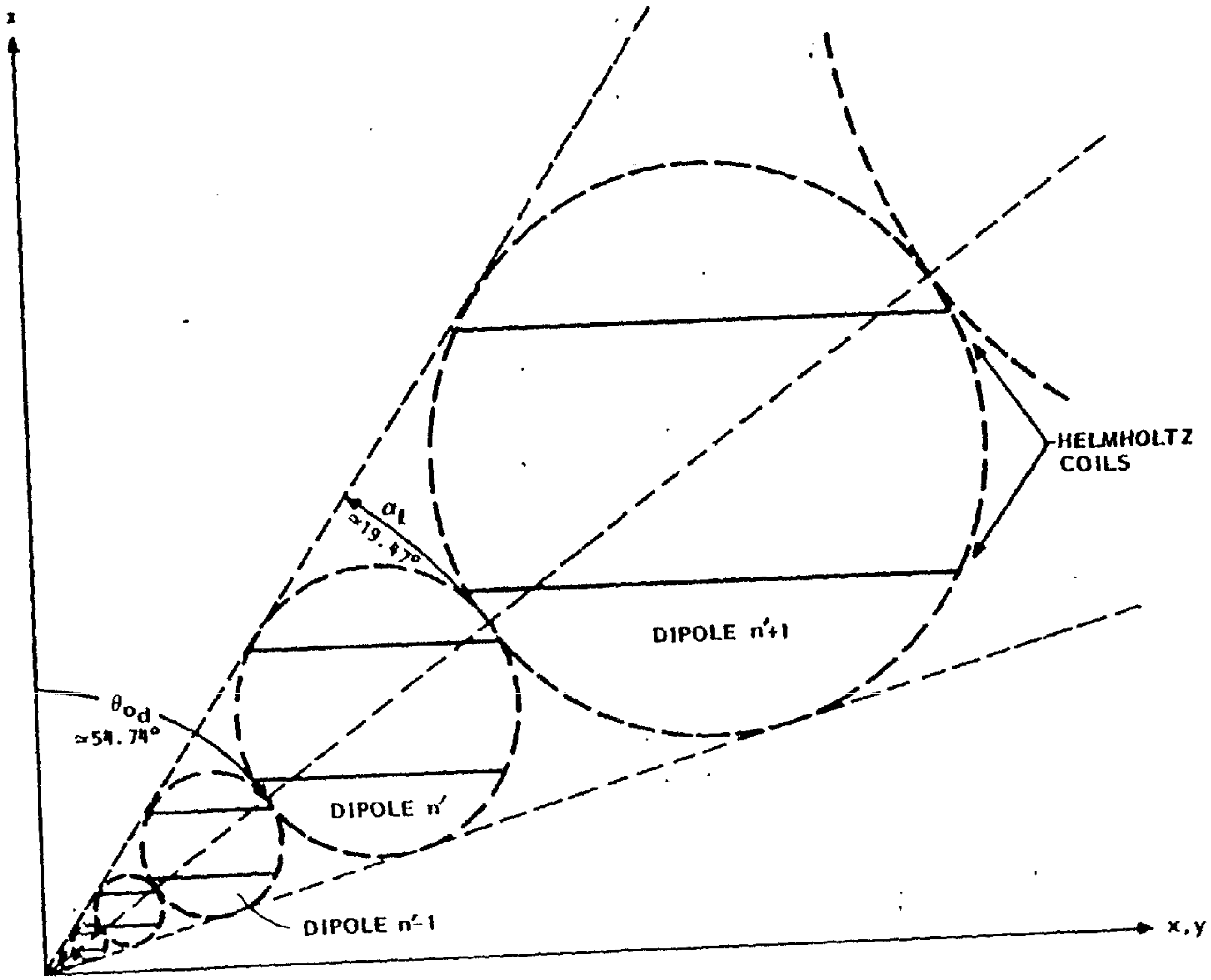

Figure 5.1: A Log-Periodic Array of Magnetic Dipoles with No Mutual Inductance. Each dipole occupies the volume of a sphere, and each successive radius increases by a factor of $\beta = 2$. The configuration shown is for maximum packing efficiency of perfect dipoles.

analytical geometry, be expressed as

$$\sin(\alpha_\ell) = \frac{\beta - 1}{\beta + 1} \,. \tag{5.1}$$

Note for our example (here and [1])

$$\beta = 2 \,,\ \sin(\alpha_\ell) = \frac{1}{3} \,,\ \alpha_\ell \simeq 19.47 \,. \tag{5.2}$$

The relationship between the area of successive loops is based on a constant ratio (β^2), which simultaneously influences the related inductance in the associated constant-resistance network by a ratio β. When the conditions of efficient dipole generation and zero mutual inductance are complied with, a large magnitude of magnetic dipole moment can be achieved over a wide frequency range. The configuration shown in Fig. 5.1 is consistent with that for $\beta = 2$, and the characteristic curves for such an array of dipole loops for this value can be found in [1] for 1, 2 and 5 dipole arrays. The general performance of this type of array and the relationship between the successive loop elements is consistent with the principles of the log-periodic class of antennas [9]. The network/loop combination then can be considered to be a magnetic dipole log-periodic array.

This new type of magnetic-dipole antenna can be referred to in brief as SCYLLA (scaled constant-resistance yet log-periodic loop array).

6 Summary

A relationship was derived which describes the field produced by a series of parallel loops wound on the surface of a sphere. The relationship has a common origin for the field generating systems and the resultant field, and this provides a convenient form to show their natural symmetries. The symmetries are manipulated to eliminate the contributions of higher order multipoles. Another relationship was derived to determine the mutual inductance between two parallel circular loops. It is shown, that at least for the dipole and quadrupole static case, mutual inductance can be eliminated separately, but not simultaneously. A configuration of loops based on these two concepts gave rise to a proposed scheme which is effectively a log-periodic magnetic dipole array antenna.

The log-periodic magnetic dipole array is associated with a cascaded constant-resistance network model described in a previous paper investigating networks for producing composite magnetic dipole moments from a number of loops [1]. The resultant system, with proper choice of a proportional factor β (a recurrence relation between successive stages) can produce a constant magnitude magnetic dipole moment over a wide frequency range. The response for this type of antenna has been given in the previous paper for selected values of β and number of dipoles. This new class of antenna can find application in providing accurately known electromagnetic environments over limited volumes. One such application is to provide the electromagnetic field, in a confined space say, to measure interaction behavior in accordance with the concept of PARTES [10].

Appendix A. Binomial Expansion of $\frac{1}{|r|}$

From (2.6) we can use the binomial expansion in the form

$$(1+x)^{-1/2} = 1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5x^3}{16} + \frac{35x^4}{128} \cdots \tag{A.1}$$

where

$$x = \frac{2\sin(\theta)(b\sin(\phi') - a\sin(\phi))}{D} + \frac{a^2 + b^2 - 2ab\cos(\phi' - \phi)}{D^2}\,. \tag{A.2}$$

Substituting the expanded forms of x from (A.2) into (A.1) we get the expansion for the first five terms

$$\begin{aligned}
&(1+x)^{-1/2} = 1\Big\}\ \text{term 1} \\
&\left.\begin{aligned}&-\tfrac{1}{D}\sin(\theta)(b\sin(\phi') - a\sin(\phi)) \\ &-\tfrac{1}{2D^2}(a^2 + b^2 - 2ab\cos(\phi' - \phi))\end{aligned}\right\}\ \text{term 2}
\end{aligned} \tag{A.3}$$

$$
\left.\begin{aligned}
&+ \tfrac{3}{8}\tfrac{2^2}{1}\tfrac{1}{D^2}\sin^2(\theta)(b\sin(\phi') - a\sin(\phi))^2 \\
&+ \tfrac{3}{8}\tfrac{2^2}{1}\tfrac{1}{D^3}\sin(\theta)(b\sin(\phi') - a\sin(\phi))(a^2+b^2-2ab\cos(\phi'-\phi)) \\
&+ \tfrac{3}{8}\tfrac{1}{D^4}(a^2+b^2-2ab\cos(\phi'-\phi))^2
\end{aligned}\right\}\ \text{term 3}
$$

$$
\left.\begin{aligned}
&- \tfrac{5}{16}\tfrac{2^3}{1}\tfrac{1}{D^3}\sin^3(\theta)(b\sin(\phi') - a\sin(\phi))^3 \\
&- \tfrac{5}{16}\tfrac{3}{1}\tfrac{2^2}{1}\tfrac{1}{D^4}\sin^2(\theta)(b\sin(\phi') - a\sin(\phi))^2(a^2+b^2-2ab\cos(\phi'-\phi)) \\
&- \tfrac{5}{16}\tfrac{3}{1}\tfrac{2}{1}\tfrac{1}{D^5}\sin(\theta)(b\sin(\phi') - a\sin(\phi))(a^2+b^2-2ab\cos(\phi'-\phi))^2 \\
&- \tfrac{5}{16}\tfrac{1}{D^6}(a^2+b^2-2ab\cos(\phi'-\phi))^3
\end{aligned}\right\}\ \text{term 4}
$$

$$
\left.\begin{aligned}
&+ \tfrac{35}{128}\tfrac{2^4}{1}\tfrac{1}{D^4}\sin^4(\theta)(b\sin(\phi') - a\sin(\phi))^4 \\
&+ \tfrac{35}{128}\tfrac{4}{1}\tfrac{2^3}{1}\tfrac{1}{D^5}\sin^3(\theta)(b\sin(\phi') - a\sin(\phi))^3(a^2+b^2-2ab\cos(\phi'-\phi)) \\
&+ \tfrac{35}{128}\tfrac{6}{1}\tfrac{2^2}{1}\tfrac{1}{D^6}\sin^2(\theta)(b\sin(\phi') - a\sin(\phi))^2(a^2+b^2-2ab\cos(\phi'-\phi))^2 \\
&+ \tfrac{35}{128}\tfrac{4}{1}\tfrac{2}{1}\tfrac{1}{D^7}\sin(\theta)(b\sin(\phi') - a\sin(\phi))(a^2+b^2-2ab\cos(\phi'-\phi))^3 \\
&+ \tfrac{35}{128}\tfrac{1}{D^8}(a^2+b^2-2ab\cos(\phi'-\phi))^4
\end{aligned}\right\}\ \text{term 5}\ .
$$

The terms of equal powers of D can be collected giving

$$
\begin{aligned}
\frac{1}{|r|} &= \frac{1}{D}(1+x)^{-1/2} \\
&= \frac{1}{D} - \frac{1}{D^2}\sin(\theta)(b\sin(\phi') - a\sin(\phi)) \\
&\quad + \frac{1}{D^3}\frac{1}{2}[3\sin^2(\theta)(b\sin(\phi') - a\sin(\phi))^2 - (a^2+b^2-2ab\cos(\phi'-\phi))] \\
&\quad + \frac{1}{D^4}\frac{1}{2}[3\sin(\theta)(b\sin(\phi') - a\sin(\phi))(a^2+b^2-2ab\cos(\phi'-\phi)) \\
&\qquad - 5\sin^3(\theta)(b\sin(\phi') - a\sin(\phi))^3] \\
&\quad + \frac{1}{D^5}\frac{1}{8}[3(a^2+b^2-2ab\cos(\phi'-\phi))^2 \\
&\qquad - 30\sin^2(\theta)(b\sin(\phi') - a\sin(\phi))^2(a^2+b^2-2ab\cos(\phi'-\phi)) \\
&\qquad + 35\sin^4(\theta)(b\sin(\phi') - a\sin(\phi))^4] \\
&\quad + O\left(\frac{1}{D^6}\right) .
\end{aligned}
\tag{A.4}
$$

Appendix B. Evaluation of Integrals for Terms in D^{-1} and D^{-2}

From (2.7) the first term can be evaluated as

$$\begin{aligned}
\frac{1}{D}\int_0^{2\pi}\int_0^{2\pi}\cos(\phi'-\phi)d\phi\,d\phi' &= \frac{1}{D}\int_0^{2\pi}\int_0^{2\pi}\cos(\phi)\cos(\phi')+\sin(\phi)\sin(\phi')d\phi\,d\phi' \\
&= \frac{1}{D}[\sin(\phi)]\Big|_0^{2\pi}[\sin(\phi')]\Big|_0^{2\pi}+[-\cos(\phi)]\Big|_0^{2\pi}[-\cos(\phi')]\Big|_0^{2\pi} \\
&= 0\,. \qquad \text{(B.1)}
\end{aligned}$$

The second term of (2.7) evaluates as

$$\begin{aligned}
&-\frac{\sin(\theta)}{D^2}\int_0^{2\pi}\int_0^{2\pi}\cos(\phi'-\phi)(b\sin(\phi')-a\sin(\phi))d\phi\,d\phi' \qquad \text{(B.2)} \\
&= -\frac{\sin(\theta)}{D^2}\int_0^{2\pi}\int_0^{2\pi}[b\cos(\phi)\cos(\phi')\sin(\phi')+b\sin^2(\phi')\sin(\phi) \\
&\quad -a\cos(\phi)\cos(\phi')\sin(\phi)-a\sin^2(\phi)\sin(\phi')]d\phi\,d\phi' \\
&= -\frac{\sin(\theta)}{D^2}\Big\{b\left[\frac{\sin^2(\phi')}{2}\right]\Big|_0^{2\pi}[\sin(\phi)]\Big|_0^{2\pi}+b\left[\frac{\phi'}{2}-\frac{\sin(2\phi')}{4}\right]\Big|_0^{2\pi}[\cos(\phi)]\Big|_0^{2\pi} \\
&\quad -a\left[\frac{\sin^2(\phi)}{2}\right]\Big|_0^{2\pi}[\sin(\phi')]\Big|_0^{2\pi}-a\left[\frac{\phi}{2}-\frac{\sin(2\phi)}{4}\right]\Big|_0^{2\pi}[-\cos(\phi')]\Big|_0^{2\pi}\Big\} \\
&= 0\,.
\end{aligned}$$

Appendix C. Evaluation of Integrals in D^{-3}

The third term of (2.7) can be evaluated in terms of two integrals Λ_1, Λ_2 as

$$\begin{aligned}
&\frac{1}{2D^3}[3\sin^2(\theta)\int_0^{2\pi}\int_0^{2\pi}\cos(\phi'-\phi)(b\sin(\phi')-a\sin(\phi))^2d\phi\,d\phi' \\
&\quad -\int_0^{2\pi}\int_0^{2\pi}\cos(\phi'-\phi)(a^2+b^2-2ab\cos(\phi'-\phi))d\phi\,d\phi'] \\
&= \frac{1}{2D^3}(3\sin^2(\theta)\Lambda_1-\Lambda_2)\,. \qquad \text{(C.1)}
\end{aligned}$$

A_1 can be evaluated as

$$\begin{aligned}
A_1 &= \int_0^{2\pi}\int_0^{2\pi}\cos(\phi'-\phi)(b\sin(\phi')-a\sin(\phi))^2 d\phi\, d\phi' \\
&= \int_0^{2\pi}\int_0^{2\pi} b^2\cos(\phi')\cos(\phi)\sin^2(\phi')d\phi\, d\phi' + \int_0^{2\pi}\int_0^{2\pi} b^2\sin(\phi)\sin^3(\phi')d\phi\, d\phi' \\
&\quad + \int_0^{2\pi}\int_0^{2\pi} a^2\cos(\phi')\cos(\phi)\sin^2(\phi)d\phi\, d\phi' + \int_0^{2\pi}\int_0^{2\pi} a^2\sin^3(\phi)\sin(\phi')d\phi\, d\phi' \\
&\quad - \int_0^{2\pi}\int_0^{2\pi} 2ab\cos(\phi')\cos(\phi)\sin(\phi)\sin(\phi')d\phi\, d\phi' \\
&\quad - \int_0^{2\pi}\int_0^{2\pi} 2ab\sin^2(\phi)\sin^2(\phi')d\phi\, d\phi' \\
&= b^2\left[\frac{\sin^3(\phi')}{3}\right]\Big|_0^{2\pi}[\sin(\phi)]\Big|_0^{2\pi} + b^2\left[\frac{\cos^3(\phi')}{3}-\cos(\phi')\right]\Big|_0^{2\pi}[-\cos(\phi)]\Big|_0^{2\pi} \\
&\quad + a^2[\cos(\phi')]\Big|_0^{2\pi}\left[\frac{\sin^3(\phi')}{3}\right]\Big|_0^{2\pi} + a^2[-\cos(\phi')]\Big|_0^{2\pi}\left[\frac{\cos^3(\phi')}{3}-\cos(\phi')\right]\Big|_0^{2\pi} \\
&\quad - 2ab\left[\frac{\sin^2(\phi')}{2}\right]\Big|_0^{2\pi}\left[\frac{\sin^2(\phi)}{2}\right]\Big|_0^{2\pi} \\
&\quad - 2ab\left[\frac{\phi'}{2}-\frac{\sin(\phi')\cos(\phi')}{2}\right]\Big|_0^{2\pi}\left[\frac{\phi}{2}-\frac{\sin(\phi)\cos(\phi)}{2}\right]\Big|_0^{2\pi} \\
A_1 &= -2ab(\pi^2) \qquad (C.2)
\end{aligned}$$

A_2 can be evaluated as

$$A_2 = \int_0^{2\pi}\int_0^{2\pi}\cos(\phi'-\phi)(a^2+b^2-2ab\cos(\phi'-\phi)d\phi\, d\phi' . \qquad (C.3)$$

But from (B.1)

$$\int_0^{2\pi}\int_0^{2\pi}\cos(\phi'-\phi)d\phi\, d\phi' = 0 . \qquad (C.4)$$

Hence

$$a^2\int_0^{2\pi}\int_0^{2\pi}\cos(\phi'-\phi)d\phi\, d\phi' = b^2\int_0^{2\pi}\int_0^{2\pi}\cos(\phi'-\phi)d\phi\, d\phi' = 0 . \qquad (C.5)$$

Thus

$$\begin{aligned}
A_2 &= -2ab\int_0^{2\pi}\int_0^{2\pi}(\cos^2(\phi'-\phi)d\phi\, d\phi' \qquad (C.6) \\
&= -2ab\int_0^{2\pi}\int_0^{2\pi}(\cos^2(\phi')\cos^2(\phi)+\sin^2(\phi)\sin^2(\phi')
\end{aligned}$$

$$
\begin{aligned}
&+ 2\sin(\phi)\sin(\phi')\cos(\phi)\cos(\phi'))d\phi\, d\phi' \\
= \; & -2ab\Big\{ \left[\frac{\phi'}{2} + \frac{\sin(\phi')\cos(\phi')}{2}\right]\Big|_0^{2\pi} \left[\frac{\phi}{2} + \frac{\sin(\phi)\cos(\phi)}{2}\right]\Big|_0^{2\pi} \\
&+ \left[\frac{\phi'}{2} - \frac{\sin(\phi')\cos(\phi')}{2}\right]\Big|_0^{2\pi} \left[\frac{\phi}{2} - \frac{\sin(\phi)\cos(\phi)}{2}\right]\Big|_0^{2\pi} \\
&+ 2\left[\frac{\sin^2(\phi')}{2}\right]\Big|_0^{2\pi} \left[\frac{\sin^2(\phi)}{2}\right]\Big|_0^{2\pi} \Big\} \\
= \; & -2ab[\pi^2 + \pi^2] = -4ab\pi^2 \,. \qquad \text{(C.7)}
\end{aligned}
$$

Thus referring to (C.1) we can write

$$
\begin{aligned}
\frac{1}{2D^3}(3\sin^2(\theta)\Lambda_1 - \Lambda_2) &= \frac{1}{2D^3}(-6\sin^2(\theta)ab\pi^2 + 4ab\pi^2) \\
&= \frac{ab\pi^2}{D^3}(3\cos^2(\theta) - 3 + 2) \\
&= \frac{ab\pi^2}{D^3}(3\cos^2(\theta) - 1) \qquad \text{(C.8)}
\end{aligned}
$$

Appendix D. Evaluation of Integral in Terms of D^{-4}

The fourth term of (2.7) can be written as

$$
\begin{aligned}
&\frac{1}{2D^4}[3\sin(\theta)\int_0^{2\pi}\int_0^{2\pi} \cos(\phi' - \phi)(a^2 + b^2 - 2ab\cos(\phi' - \phi) \qquad \text{(D.1)} \\
&(b\sin(\phi') - a\sin(\phi))d\phi\, d\phi' \\
&- 5\sin^3(\theta)\int_0^{2\pi}\int_0^{2\pi} \cos(\phi' - \phi)(b\sin(\phi') - a\sin(\phi))^3 d\phi\, d\phi'] \,.
\end{aligned}
$$

This can be rewritten in terms of two integrals Λ_3 and Λ_4 as

$$
\frac{1}{2D^4}[3\sin(\theta)\Lambda_3 - 5\sin^3(\theta)\Lambda_4] \,. \qquad \text{(D.2)}
$$

The first integral can be evaluated as

$$
\begin{aligned}
\Lambda_3 &= \int_0^{2\pi}\int_0^{2\pi} \cos(\phi' - \phi)(a^2 + b^2 - 2ab\cos(\phi' - \phi))(b\sin(\phi') - a\sin(\phi))d\phi\, d\phi' \qquad \text{(D.3)} \\
&= \int_0^{2\pi}\int_0^{2\pi} (\cos(\phi' - \phi)(b\sin(\phi') - a\sin(\phi))(a^2 + b^2 - 2ab\cos(\phi' - \phi))d\phi\, d\phi' \,.
\end{aligned}
$$

Thus, remembering (B.2)

$$
\begin{aligned}
\Lambda_3 &= -2ab\int_0^{2\pi}\int_0^{2\pi}\cos^2(\phi'-\phi)(b\sin(\phi')-a\sin(\phi))d\phi\,d\phi' \\
&= -2ab\int_0^{2\pi}\int_0^{2\pi}(\cos^2(\phi)\cos^2(\phi')+2\cos(\phi)\cos(\phi')\sin(\phi)\sin(\phi') \\
&\quad +\sin^2(\phi)\sin^2(\phi'))(b\sin(\phi')-a\sin(\phi))d\phi\,d\phi' \\
&= -2ab\int_0^{2\pi}\int_0^{2\pi} b\cos^2(\phi)\cos^2(\phi')\sin(\phi')+2b\cos(\phi)\cos(\phi')\sin(\phi)\sin^2(\phi') \\
&\quad +b\sin^2(\phi)\sin^3(\phi')-a\cos^2(\phi)\cos^2(\phi')\sin(\phi)-2a\cos(\phi)\cos(\phi')\sin^2(\phi)\sin(\phi') \\
&\quad -a\sin^3(\phi)\sin^2(\phi')d\phi\,d\phi' \\
&= -2ab\Big\{b\left[\frac{\phi}{2}+\frac{\sin(2\phi)}{4}\right]\Big|_0^{2\pi}\left[-\frac{\cos^3(\phi')}{3}\right]\Big|_0^{2\pi}+2b\left[\frac{\sin^3(\phi')}{3}\right]\Big|_0^{2\pi}\left[\frac{\sin^2(\phi)}{2}\right]\Big|_0^{2\pi} \\
&\quad +b\left[\frac{\cos^3(\phi')}{3}-\cos(\phi')\right]\Big|_0^{2\pi}\left[\frac{\phi}{2}-\frac{\sin(2\phi)}{4}\right]\Big|_0^{2\pi} \\
&\quad -a\left[\frac{\cos^3(\phi)}{3}\right]\Big|_0^{2\pi}\left[\frac{\phi'}{2}+\frac{\sin(2\phi')}{4}\right]\Big|_0^{2\pi}-2a\left[\frac{\sin^2(\phi')}{2}\right]\Big|_0^{2\pi}\left[\frac{\sin^3(\phi)}{3}\right]\Big|_0^{2\pi} \\
&\quad -a\left[\frac{\phi'}{2}-\frac{\sin(2\phi')}{4}\right]\Big|_0^{2\pi}\left[\frac{\cos^3(\phi)}{3}-\cos(\phi)\right]\Big|_0^{2\pi}\Big\} \\
&= 0\,.
\end{aligned}
\tag{D.4}
$$

The second integral Λ_4 can be evaluated as

$$
\Lambda_4=\int_0^{2\pi}\int_0^{2\pi}\cos(\phi'-\phi)(b\sin(\phi')-a\sin(\phi))^3d\phi\,d\phi'\,. \tag{D.5}
$$

Using

$$
\begin{aligned}
(u-v)^3 &= u^3-v^3-3u^2v+3uv^2 \\
\Lambda_4 &= \int_0^{2\pi}\int_0^{2\pi}[b^3\cos(\phi)\cos(\phi')\sin^3(\phi')-a^3\cos(\phi)\cos(\phi')\sin^3(\phi) \\
&\quad -3ab^2\cos(\phi)\cos(\phi')\sin^2(\phi')\sin(\phi) \\
&\quad +3a^2b\cos(\phi)\cos(\phi')\sin(\phi')\sin^2(\phi)+b^3\sin(\phi)\sin^4(\phi')-a^3\sin^4(\phi)\sin(\phi') \\
&\quad -3ab^2\sin^2(\phi)\sin^3(\phi')+3a^2b\sin^3(\phi)\sin^2(\phi')]d\phi\,d\phi' \\
&= b^3\left[\frac{\sin^4(\phi')}{4}\right]\Big|_0^{2\pi}[\sin(\phi)]\Big|_0^{2\pi}-a^3\left[\frac{\sin^4(\phi)}{4}\right]\Big|_0^{2\pi}[\sin(\phi')]\Big|_0^{2\pi}
\end{aligned}
$$

$$
\begin{aligned}
& - 3ab^2 \left[\frac{\sin^3(\phi')}{3}\right]\Big|_0^{2\pi} \left[\frac{\sin^2(\phi)}{2}\right]\Big|_0^{2\pi} \\
& + 3a^2 b \left[\frac{\sin^3(\phi)}{3}\right]\Big|_0^{2\pi} \left[\frac{\sin^2(\phi')}{2}\right]\Big|_0^{2\pi} \\
& + b^3 \left[-\cos(\phi)\right]\Big|_0^{2\pi} \left[\frac{3\phi'}{8} - \frac{\sin(2\phi')}{4} + \frac{\sin(4\phi')}{32}\right]\Big|_0^{2\pi} \qquad \text{(D.6)} \\
& - a \left[-\cos(\phi')\right]\Big|_0^{2\pi} \left[\frac{3\phi}{8} - \frac{\sin(2\phi')}{4} + \frac{\sin(4\phi')}{32}\right]\Big|_0^{2\pi} \\
& - 3ab^2 \left[\frac{\phi}{2} - \frac{\sin(2\phi)}{4}\right]\Big|_0^{2\pi} \left[\frac{\cos^3(\phi')}{3} - \cos(\phi')\right]\Big|_0^{2\pi} \\
& + 3a^2 b \left[\frac{\cos^3(\phi)}{3} - \cos(\phi)\right]\Big|_0^{2\pi} \cdot \left[\frac{\phi'}{2} - \frac{\sin(2\phi')}{4}\right]\Big|_0^{2\pi} \\
= {} & 0 .
\end{aligned}
$$

Substituting (D.4) and (D.7) into (D.2) evaluate the fourth term as

$$\frac{1}{2D^4}[3\sin(\theta)\Lambda_3 - 5\sin^3(\theta)\Lambda_4] = 0 . \qquad \text{(D.7)}$$

Appendix E. Evaluation of Integrals in D^{-5}

The fifth term from (2.7) can be written as

$$
\begin{aligned}
& + \frac{1}{8}\frac{1}{D^5}[3\int_0^{2\pi}\int_0^{2\pi} \cos(\phi'-\phi)(a^2+b^2-2ab\cos(\phi'-\phi))^2 d\phi\, d\phi' \\
& \quad - 30\sin^2(\theta)\int_0^{2\pi}\int_0^{2\pi}(b\sin(\phi') - a\sin(\phi))^2 \\
& \quad (a^2+b^2-2ab\cos(\phi'-\phi))\cos(\phi'-\phi)d\phi\, d\phi' \\
& \quad + 35\sin^4(\theta)\int_0^{2\pi}\int_0^{2\pi}(b\sin(\phi') - a\sin(\phi))^4\cos(\phi'-\phi)d\phi\, d\phi' \\
& = \frac{1}{8}\frac{1}{D^5}(3\Lambda_5 - 30\sin^2(\theta)\Lambda_6 + 35\sin^4(\theta)\Lambda_7) \\
\Lambda_5 & = \int_0^{2\pi}\int_0^{2\pi}\cos(\phi'-\phi)(a^2+b^2-2ab\cos(\phi'-\phi))^2 d\phi\, d\phi' \qquad \text{(E.1)} \\
& = \int_0^{2\pi}\int_0^{2\pi}[(a^2+b^2)^2 - 4ab(a^2+b^2)\cos(\phi'-\phi)
\end{aligned}
$$

$$+ 4a^2b^2\cos^2(\phi' - \phi)]\cos(\phi' - \phi)d\phi\, d\phi' .$$

This can be broken down into three separate integrals as

$$\Lambda_5 = \Lambda_{5,1} - \Lambda_{5,2} + \Lambda_{5,3} . \tag{E.2}$$

The first of these becomes

$$\Lambda_{5,1} = \int_0^{2\pi}\int_0^{2\pi}(a^2+b^2)^2\cos(\phi'-\phi)d\phi\, d\phi' = (a^2+b^2)^2\int_0^{2\pi}\int_0^{2\pi}\cos(\phi'-\phi)d\phi\, d\phi' . \tag{E.3}$$

This is similar in form to (B.1) and reduces to

$$\begin{aligned}(a^2+b^2)(0) &= 0\\ \Lambda_{5,2} &= \int_0^{2\pi}\int_0^{2\pi}4ab(a^2+b^2)\cos^2(\phi'-\phi)d\phi\, d\phi' .\end{aligned} \tag{E.4}$$

This is similar in form to (C.6) and can be rewritten

$$\begin{aligned}\Lambda_{5,2} &= 4ab(a^2+b^2)(-\Lambda_2\big/2ab)\\ &= 4ab(a^2+b^2)(2\pi^2) = 8\pi^2ab(a^2+b^2)\\ \Lambda_{5,3} &= 4a^2b^2\int_0^{2\pi}\int_0^{2\pi}\cos^3(\phi'-\phi)d\phi\, d\phi'\\ &= 4a^2b^2\int_0^{2\pi}\int_0^{2\pi}(\cos(\phi)\cos(\phi')+\sin(\phi)\sin(\phi'))^3d\phi\, d\phi' .\end{aligned} \tag{E.5}$$

Expand using

$$\begin{aligned}(u+v)^3 &= u^3+v^3+3uv^2+3u^2v\\ u &= \cos(\phi)\cos(\phi')\\ v &= \sin(\phi)\sin(\phi')\\ \Lambda_{5,3} &= 4a^2b^2\int_0^{2\pi}\int_0^{2\pi}(\cos^3(\phi)\cos^3(\phi')+\sin^3(\phi)\sin^3(\phi')\\ &\quad + 3\cos(\phi)\cos(\phi')\sin^2(\phi)\sin^2(\phi')\\ &\quad + 3\cos^2(\phi)\cos^2(\phi')\sin(\phi)\sin(\phi'))d\phi\, d\phi'\end{aligned} \tag{E.6}$$

$$
\begin{aligned}
&= 4a^2b^2\Big\{\left[\sin(\phi) - \frac{\sin^3(\phi)}{3}\right]\Big|_0^{2\pi}\left[\sin(\phi') - \frac{\sin^3(\phi')}{3}\right]\Big|_0^{2\pi} \\
&+ \left[\frac{\cos^3(\phi)}{3} - \cos(\phi)\right]\Big|_0^{2\pi}\left[\frac{\cos^3(\phi')}{3} - \cos(\phi')\right]\Big|_0^{2\pi} \\
&+ 3\left[\frac{\sin^3(\phi)}{3}\right]\Big|_0^{2\pi}\left[\frac{\sin^3(\phi')}{3}\right]\Big|_0^{2\pi} 3\left[-\frac{\cos^3(\phi)}{3}\right]\Big|_0^{2\pi}\left[-\frac{\cos^3(\phi)}{3}\right]\Big|_0^{2\pi}\Big\} = 0\,.
\end{aligned}
$$

From (E.2)

$$
\begin{aligned}
\Delta_5 &= \Delta_{5,1} - \Delta_{5,2} + \Delta_{5,3} = 0 - 8\pi^2 ab(a^2+b^2) + 0 \\
\Delta_5 &= -8\pi^2 ab(a^2+b^2) \\
\Delta_6 &= \int_0^{2\pi}\int_0^{2\pi} \cos(\phi'-\phi)(b\sin(\phi') - a\sin(\phi))^2(a^2+b^2-2ab\cos(\phi'-\phi))\,d\phi\,d\phi' \\
&= \int_0^{2\pi}\int_0^{2\pi} (b\sin(\phi') - a\sin(\phi))^2((a^2+b^2)\cos(\phi'-\phi) - 2ab\cos^2(\phi'-\phi))\,d\phi\,d\phi' \\
&= \int_0^{2\pi}\int_0^{2\pi} (b^2\sin^2(\phi') - 2ab\sin(\phi)\sin(\phi') + a^2\sin^2(\phi))((a^2+b^2)\cos(\phi'-\phi) \\
&\quad - 2ab\cos^2(\phi'-\phi))\,d\phi\,d\phi' \\
&= \Delta_{6,1} - \Delta_{6,2} + \Delta_{6,3} + \Delta_{6,4} + \Delta_{6,5} + \Delta_{6,6} \\
\Delta_{6,1} &= \int_0^{2\pi}\int_0^{2\pi} b^2(a^2+b^2)\cos(\phi'-\phi)\sin^2(\phi')\,d\phi\,d\phi' \\
&= b^2(a^2+b^2)\int_0^{2\pi}\int_0^{2\pi} \cos(\phi'-\phi)\sin^2(\phi')\,d\phi\,d\phi' \\
&= b^2(a^2+b^2)\int_0^{2\pi}\int_0^{2\pi} (\cos(\phi')\cos(\phi)\sin^2(\phi') + \sin(\phi)\sin^3(\phi'))\,d\phi\,d\phi' \\
&= b^2(a^2+b^2)\Big\{[+\sin(\phi)]\Big|_0^{2\pi}\left[\frac{\sin^3(\phi')}{3'}\right]\Big|_0^{2\pi} \\
&\quad + [-\cos(\phi)]\Big|_0^{2\pi}\left[\frac{\cos^3(\phi')}{3} - \cos(\phi')\right]\Big|_0^{2\pi}\Big\} = 0 \\
\Delta_{6,2} &= \int_0^{2\pi}\int_0^{2\pi} (b^2\sin^2(\phi')(-2ab\cos^2(\phi'-\phi))\,d\phi\,d\phi' \qquad \text{(E.7)} \\
\Delta_{6,2} &= -2ab^3\int_0^{2\pi}\int_0^{2\pi} \sin^2(\phi')\cos^2(\phi'-\phi)\,d\phi\,d\phi' \\
&= -2ab^3\int_0^{2\pi}\int_0^{2\pi} (\sin^2(\phi')\cos^2(\phi)\cos^2(\phi') + 2\cos(\phi')\cos(\phi)\sin^3(\phi')\sin(\phi) \\
&\quad + \sin^2(\phi)\sin^4(\phi'))\,d\phi\,d\phi' \\
&= -2ab^2\Big\{\left[\frac{\phi'}{8} - \frac{\sin(4\phi')}{32}\right]\Big|_0^{2\pi}\left[\frac{\phi}{2} + \frac{\sin(\phi)\cos(\phi)}{2}\right]\Big|_0^{2\pi} + 2\left[\frac{\sin^2(\phi)}{2}\right]\Big|_0^{2\pi}\left[\frac{\sin^4(\phi)}{4}\right]\Big|_0^{2\pi}
\end{aligned}
$$

$$
\begin{aligned}
& + \left[\frac{\phi}{2} - \frac{\sin(\phi)\cos(\phi)}{2}\right]\Big|_0^{2\pi} \left[\frac{3\phi'}{8} - \frac{\sin(2\phi')}{4} + \frac{\sin(4\phi')}{32}\right]\Big|_0^{2\pi} \Bigg\} \\
= & -2ab^3 \left\{ \left(\frac{2\pi}{8}\frac{2\pi}{2}\right) + 0 + \left(\frac{2\pi}{2}\right)\left(\frac{6\pi}{8}\right) \right\} \\
= & -2ab^3\pi^2
\end{aligned}
$$

$$
\begin{aligned}
\Lambda_{6,3} &= \int_0^{2\pi}\int_0^{2\pi} a^2 \sin^2(\phi)(a^2+b^2)\cos(\phi'-\phi)\,d\phi\,d\phi' \\
&= a^2(a^2+b^2)\int_0^{2\pi}\int_0^{2\pi} \cos(\phi'-\phi)\sin^2(\phi)\,d\phi\,d\phi' \, .
\end{aligned}
$$

This integral has the same form as (E.6) which equals zero thus

$$
\begin{aligned}
\Lambda_{6,3} &= 0 \\
\Lambda_{6,4} &= \int_0^{2\pi}\int_0^{2\pi} a^2 \sin^2(\phi)(-2ab\cos^2(\phi'-\phi))\,d\phi\,d\phi' \\
&= -2a^3b\int_0^{2\pi}\int_0^{2\pi} \sin^2(\phi)\cos^2(\phi'-\phi))\,d\phi\,d\phi' \, .
\end{aligned} \tag{E.8}
$$

This has an equivalent form in (E.7) by virtue of symmetry of ϕ and ϕ'. Hence

$$
\begin{aligned}
\Lambda_{6,4} &= -2a^3b\pi^2 \, . \\
\Lambda_{6,5} &= \int_0^{2\pi}\int_0^{2\pi} -2ab\sin(\phi)\sin(\phi')(-2ab\cos^2(\phi'-\phi))\,d\phi\,d\phi' \\
&= 4a^2b^2\int_0^{2\pi}\int_0^{2\pi} \cos^2(\phi'-\phi)\sin(\phi)\sin(\phi')\,d\phi\,d\phi' \\
&= 4a^2b^2\int_0^{2\pi}\int_0^{2\pi} (\cos^2(\phi')\cos^2(\phi)\sin(\phi)\sin(\phi') + 2\cos(\phi)\cos(\phi')\sin^2(\phi)\sin^2(\phi') \\
&\quad + \sin^3(\phi)\sin^3(\phi'))\,d\phi\,d\phi' \\
&= 4a^2b^2\Bigg\{ \left[-\frac{\cos^3(\phi')}{3}\right]\Big|_0^{2\pi} \left[-\frac{\cos^3(\phi)}{3}\right]\Big|_0^{2\pi} + 2\left[\frac{\sin^3(\phi')}{3}\right]\Big|_0^{2\pi} \left[\frac{\sin^3(\phi)}{3}\right]\Big|_0^{2\pi} \\
&\quad + \left[\frac{\cos^3(\phi)}{3} - \cos(\phi)\right]\Big|_0^{2\pi} \left[\frac{\cos^3(\phi')}{3} - \cos(\phi')\right]\Big|_0^{2\pi} \Bigg\} \\
&= 4a^2b^2\{0\} \\
&= 0 \\
\Lambda_{6,6} &= \int_0^{2\pi}\int_0^{2\pi} (-2ab\sin(\phi)\sin(\phi'))(a^2+b^2)\cos(\phi'-\phi)\,d\phi\,d\phi' \\
&= -2ab(a^2+b^2)\int_0^{2\pi}\int_0^{2\pi} \cos(\phi'-\phi)\sin(\phi)\sin(\phi')\,d\phi\,d\phi'
\end{aligned}
$$

$$\begin{aligned}
&= -2ab(a^2+b^2)\int_0^{2\pi}\int_0^{2\pi}\cos(\phi)\cos(\phi')\sin(\phi)\sin(\phi')+\sin^2(\phi)\sin^2(\phi')d\phi\,d\phi' \\
&= -2ab(a^2+b^2)\Big\{\left[\frac{\sin^2(\phi)}{2}\right]\Big|_0^{2\pi}\left[\frac{\sin^2(\phi')}{2}\right]\Big|_0^{2\pi} \qquad (E.9)\\
&\quad+\left[\frac{\phi}{2}-\frac{\sin(\phi)\cos(\phi)}{2}\right]\Big|_0^{2\pi}\left[\frac{\phi'}{2}-\frac{\sin(\phi')\cos(\phi')}{2}\right]\Big|_0^{2\pi}\Big\} \\
&= -2ab(a^2+b^2)(\pi^2) \\
&= -2\pi^2 ab(a^2+b^2)
\end{aligned}$$

$$\begin{aligned}
\Lambda_6 &= \Lambda_{6,1}+\Lambda_{6,2}+\Lambda_{6,3}+\Lambda_{6,4}+\Lambda_{6,5}+\Lambda_{6,6} \\
&= 0-2ab^3\pi^2+0-2a^3b\pi^2-2ab(a^2+b^2)\pi^2 \\
&= -2\pi^2ab(b^2+a^2+a^2+b^2) = -4\pi^2ab(b^2+a^2) \\
\Lambda_7 &= \int_0^{2\pi}\int_0^{2\pi}(b\sin(\phi')-a\sin(\phi))^4\cos(\phi'-\phi)d\phi\,d\phi' . \\
(u-v)^4 &= u_1^4-4u_1^3v_1-4u_1v_1^3+6u_1^2v_1^2+v_1^4 \\
\Lambda_7 &= \int_0^{2\pi}\int_0^{2\pi}(b^4\sin^4(\phi')-4ab^3\sin^3(\phi')\sin(\phi)-4a^3b\sin^3(\phi)\sin(\phi') \\
&\quad+6a^2b^2\sin^2(\phi)\sin^2(\phi')+a^4\sin^4(\phi)) \\
&\quad(\cos(\phi)\cos(\phi')+\sin(\phi)\sin(\phi'))d\phi\,d\phi' .
\end{aligned}$$

This can be separated into 10 individual integrals as

$$\begin{aligned}
\Lambda_7 &= \Lambda_{7,1}+\Lambda_{7,2}+\Lambda_{7,3}+\Lambda_{7,4}+\cdots+\Lambda_{7,10} \\
\Lambda_{7,1} &= \int_0^{2\pi}\int_0^{2\pi} b^4\sin^4(\phi')\cos(\phi)\cos(\phi')d\phi\,d\phi' \\
&= b^4\left[\sin(\phi)\right]\Big|_0^{2\pi}\left[\frac{\sin^5(\phi')}{5}\right]\Big|_0^{2\pi} = 0 \\
\Lambda_{7,2} &= \int_0^{2\pi}\int_0^{2\pi} -4ab^3\sin^3(\phi')\sin(\phi)\cos(\phi)\cos(\phi')d\phi\,d\phi' \\
&= -4ab^3\int_0^{2\pi}\int_0^{2\pi}\sin^3(\phi')\cos(\phi')\sin(\phi)\cos(\phi)d\phi\,d\phi' \\
&= -4ab^3\left[\frac{\sin^2(\phi)}{2}\right]\Big|_0^{2\pi}\left[\frac{\sin^4(\phi')}{4}\right]\Big|_0^{2\pi} = 0 \\
\Lambda_{7,3} &= -4a^3b\int_0^{2\pi}\int_0^{2\pi}\sin^3(\phi)\cos(\phi)\sin(\phi')\cos(\phi')d\phi\,d\phi' \qquad (E.10)
\end{aligned}$$

$$
\begin{aligned}
&= -4a^3b\left[\frac{\sin^2(\phi')}{2}\right]\Big|_0^{2\pi}\left[\frac{\sin^4(\phi)}{4}\right]\Big|_0^{2\pi} = 0\\
\Lambda_{7,4} &= 6a^2b^2\int_0^{2\pi}\int_0^{2\pi}\sin^2(\phi)\sin^2(\phi')\cos(\phi)\cos(\phi')d\phi\, d\phi'\\
&= 6a^2b^2\left[\frac{\sin^3(\phi)}{3}\right]\Big|_0^{2\pi}\left[\frac{\sin^3(\phi')}{3}\right]\Big|_0^{2\pi} = 0\\
\Lambda_{7,5} &= a^4\int_0^{2\pi}\int_0^{2\pi}\sin^4(\phi)\cos(\phi)\cos(\phi')d\phi\, d\phi'\\
&= a^4\left[\sin(\phi)\right]\Big|_0^{2\pi}\left[\frac{\sin^5(\phi)}{5}\right]\Big|_0^{2\pi} = 0\\
\Lambda_{7,6} &= b^4\int_0^{2\pi}\int_0^{2\pi}\sin^5(\phi')\sin(\phi)d\phi\, d\phi'\\
&= b^4\left[-\cos(\phi)\right]\Big|_0^{2\pi}\left[-\frac{5\cos(\phi')}{8}+\frac{5\cos(3\phi')}{48}-\frac{\cos(5\phi')}{80}\right]\Big|_0^{2\pi} = 0\\
\Lambda_{7,7} &= -4ab^2\int_0^{2\pi}\int_0^{2\pi}\sin^4(\phi')\sin^2(\phi)d\phi\, d\phi'\\
&= -4ab^3\left[\frac{3\phi'}{8}-\frac{\sin(2\phi)}{4}+\frac{\sin(4\phi)}{32}\right]\Big|_0^{2\pi}\left[\frac{\phi}{2}-\frac{\sin(\phi)\cos(\phi)}{2}\right]\Big|_0^{2\pi}\\
&= -4ab^3\left(\frac{6\pi}{8}\right)\left(\frac{2\pi}{2}\right)\\
&= -3ab^3\pi^2\\
\Lambda_{7,8} &= -4a^3b\int_0^{2\pi}\int_0^{2\pi}\sin^3(\phi)\sin(\phi')\sin(\phi)\sin(\phi')d\phi\, d\phi'\\
&= -4a^3b\int_0^{2\pi}\int_0^{2\pi}\sin^4(\phi)\sin^2(\phi')d\phi\, d\phi'\\
&= -4a^3b\left[\frac{3\phi}{8}-\frac{\sin(2\phi)}{4}+\frac{\sin(4\phi)}{32}\right]\Big|_0^{2\pi}\left[\frac{\phi'}{2}-\frac{\sin(\phi')\cos(\phi')}{2}\right]\Big|_0^{2\pi}\\
&= -3a^3b\pi^2\\
\Lambda_{7,9} &= 6a^2b^2\int_0^{2\pi}\int_0^{2\pi}\sin^3(\phi)\sin^3(\phi')d\phi\, d\phi'\\
&= 6a^2b^2\left[\frac{\cos^3(\phi)}{3}-\cos(\phi)\right]\Big|_0^{2\pi}\left[\frac{\cos^2(\phi')}{3}-\cos(\phi')\right]\Big|_0^{2\pi} = 0\\
\Lambda_{7,10} &= a^4\int_0^{2\pi}\int_0^{2\pi}\sin^5(\phi)\sin(\phi')d\phi\, d\phi'\\
&= a^4\left[-\cos(\phi')\right]\Big|_0^{2\pi}\left[-\frac{5\cos(\phi)}{8}+\frac{5\cos(3\phi)}{48}-\frac{\cos(5\phi)}{80}\right]\Big|_0^{2\pi} = 0\\
\Lambda_7 &= \Lambda_{7,1}+\Lambda_{7,2}+\Lambda_{7,3}+\Lambda_{7,4}+\Lambda_{7,5}+\Lambda_{7,6}+\Lambda_{7,7}+\Lambda_{7,8}+\Lambda_{7,9}+\Lambda_{7,10}
\end{aligned}
$$

$$= 0+0+0+0+0+0-3ab^3\pi^2 - 3a^3b\pi^2 + 0 + 0$$
$$= -3\pi^2 ab(a^2+b^2)$$

The desired term in D^{-5} in the expansion then becomes from (E.1)

$$\frac{1}{8}\frac{1}{D^5}(3\Lambda_5 - 30\sin^2(\theta)\Lambda_6 + 35\sin^4(\theta)\Lambda_7) \ .$$

$$\begin{aligned} \Lambda_5 &= -8\pi^2 ab(a^2+b^2) \\ \Lambda_6 &= -4\pi^2 ab(b^2+a^2) \\ \Lambda_7 &= -3\pi^2 ab(a^2+b^2) \end{aligned} \tag{E.11}$$

and substituting these into the integral for the term in D^{-5} we get

$$\begin{aligned} &\frac{1}{8}\frac{1}{D^5}ab(a^2+b^2)\pi^2\{-24+120\sin^2(\theta) - 105\sin^4(\theta)\} \\ &= -\frac{1}{D^5}\frac{3}{8}ab(a^2+b^2)\pi^2\{35\sin^4(\theta) - 40\sin^2(\theta) + 8\} \ . \end{aligned} \tag{E.12}$$

References

[1] J. D. Quinn and C. E. Baum, "Networks for Producing Composite Magnetic Dipole Moments from Various Loops," Sensor and Simulation Note 298, January 1987.

[2] W. R. Smythe, *Static and Dynamic Electricity*, Third Edition, McGraw-Hill Book Company (1968).

[3] M. Abramowitz and I. E. Stegun, *Handbook of Mathematical Functions*, NBS (1964).

[4] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics, Part II*, McGraw-Hill Book Company (1953).

[5] H. Jeffreys and B. S. Jeffreys, *Methods of Mathematical Physics*, Third Edition, Cambridge University Press (1966).

[6] M. E. Pittman and D. L. Wardelich, "Three and Four Coil Systems for Homogeneous Magnetic Fields," IEEE Trans. Aerospace, Vol. AS-2, February 1964, pp. 36-45.

[7] J. E. Everett and J. E. Osemeikhian, "Spherical Coils for Uniform Magnetic Fields," J. Sci. Instrum., Vol. 42, 1966, pp. 470-474.

[8] M. W. Garrett, "Axially Symmetric Systems for Generating and Measuring Magnetic Fields," Part 1, J. Appl. Phys., Vol. 22, No. 9, September 1951, pp. 1091-1107.

[9] E. C. Jordan et al., "Developments in Broadband Antennas," IEEE Spectrum, April 1964, pp. 58-71.

[10] C. E. Baum, "The PARTES Concept in EMP Simulation," Sensor and Simulation Note 260, December 1979, and Electromagnetics, Vol. 3, No. 1, 1983, pp. 1-19.