Electromagnetics, 1985, pp. 37-62

Sensor and Simulation Notes

Note 285

June 1984

AN ANISOTROPIC LENS FOR LAUNCHING TEM WAVES ON A CONDUCTING CIRCULAR CONICAL SYSTEM

A. P. Stone *
Dept. of Mathematics and Statistics
University of New Mexico
Albuquerque, New Mexico 87131
and
C. E. Baum
Air Force Weapons Laboratory
Kirtland AFB
Albuquerque, New Mexico 87117.

ABSTRACT

A differential impedance and transit-time matching approach is used in the design of an anisotropic lens for launching TEM waves from a small source, through the lens, and onto a conducting circular conical system. This approach leads to a system of ordinary differential equations which may be solved exactly to obtain the lens parameters. An approximate solution, which would be applicable to a design procedure, is also given.

* Work of this author was supported in part by AFOSR under grant number AFOSR-83-0040.

1. INTRODUCTION: Remarks on EM Lens Design

A technique commonly used in the solution of electromagnetic boundary value problems is that of expressing Maxwell's equations in orthogonal curvilinear coordinates and then solving for quantities which combine the physical components of fields with the scale factors of the coordinate transformation. The constitutive parameters of the medium are also combined with the scale factors with the hope that the coordinate transformation simplifies the problem. An extension of this technique was made by Baum in his research on EM lens design [1]. In his work he found many cases of isotropic inhomogeneous media for which certain types of electromagnetic wave propagation could be simply expressed. His differential geometric scaling method represented a design procedure for a certain kind of electromagnetic lens. Particular conductor geometries and media inhomogeneities were utilized to transition waves between conical and/or cylindrical transmission lines, and the transition was accomplished with neither reflection nor distortion of the wave. These lenses can then transmit arbitrary pulse waveforms without distortion. We note that the properties of such lenses, when combined with perfect conductors, are independent of frequency if it is assumed that the permittivity and permeability of the medium are real and frequency independent and that its conductivity is zero. This result is in contrast to lenses based on a geometrical optics approximation, such as the Luneberg lens which relies on the frequency being sufficiently high. Of course in practical applications the range (maximum and minimum values) of the permittivity and permeability, the spatial extent of the lens, and range of frequencies of interest would be important considerations.

Further research on this differential geometric approach to electromagnetic lens design was reported on by Stone in [2]. The main result obtained, for the case of transitioning TEM waves between cylindrical and conical transmission lines and isotropic inhomogeneous media, was that a class of solutions to the design problem exists. This class of solutions can be obtained from rotational coordinate systems arising from complex analytic transformations in the plane.

An important alternative way of viewing this differential geometric method in lens design is through a differential impedance matching and transit time matching approach. If a lens is to be inserted between two transmission lines it is clear that not only do impedances have to be matched but that the travel time for waves following paths of different lengths have to be equal if we want no reflection and no distortion of the propagated wave front. This impedance matching approach, which is described in section 2 of this paper, will be used in a specific lens design problem. This problem involves the design of a certain anisotropic lens suitable for launching TEM waves on a conducting circular conical system. Following some Brewster angle considerations in section 3, the statement of this problem is given in section 4, and an analytic solution is described in section 5. An approximate solution which would be applicable to a design procedure appears in section 6, and conclusions and summary appear in section 7.

This section is concluded with some comments regarding applications of EM lenses. Such transient lenses have application to EMP simulators [4] and energy transport in pulse power equipment. Thus included in these applications are highly directional high frequency antennae in which the spatial geometry and medium inhomogeneity are used to launch approximate TEM waves over a cross section with dimensions much larger than a wave length. In particular, in studying TEM waves on circular conical systems one major application of EM lenses would be their use in EMP simulators. For example, a possible way to simulate an electromagnetic pulse on a system in flight is to radiate a large amplitude pulse from an antenna to the systems. Hence one approach to such a simulator is to discharge a fast, high-voltage, capacitive generator into a large antenna which then radiates a narrow, large-amplitude pulse in the direction of the system under test. This type of simulator, which is of the dipole type, is useful in cases where the simulator is far from the system under test compared to the size of the dipole structure [3],[4]. Another realized type of dipole simulator is a resistively loaded circular cone mounted on a ground plane. Such vertically polarized electric dipoles have been constructed at various locations [4].

II. REVIEW OF RESULTS FOR A LENS BETWEEN COAXIAL CIRCULAR CYLINDERS

An excellent illustration of the use of the impedance matching approach to a lens design problem appears in [5]. In this reference the authors consider two cylindrical coaxial waveguides of different size as illustrated in figure 2.1. The waveguide section of the left, which is region I, has inner and outer cylindrical radii denoted by $\rho = A$ and $\rho = B \equiv \lambda_t A$ respectively, while the waveguide section on the right, which is region III has its respective inner and outer radii given by $\rho = A'$ and $\rho = B' \equiv \lambda_t' B'$ with $A' > A$. Both waveguides are filled with the same simple uniform medium of constant ϵ and μ, and $\sigma \equiv 0$. Physical considerations will require that the transverse ratios λ_t, λ'_t be equal. The problem considered in [5] was that of finding a perfect matching section II between regions I and III such that a TEM wave incident from the left side could propagate into a TEM wave in region III without reflection and without distortion. In the lens region II a variable $\epsilon(\theta)$ and anisotropic conductivity σ was permitted, but the lens was to have the same fixed μ as in the cylindrical regions. One way of obtaining a perfect matching for regions I, II, and III is to insert coaxial conducting layers in all the regions with spacing d and thickness Δ of each sheath satisfying $\Delta << d$ (for negligible reduction of impedance by conductor thickness) and also satisfying $d << \lambda$ (for propagation of only TEM mode), where λ is the wavelength of the TEM wave. This condition is needed if there is to be no distortion. Moreover, for negligible loss to be introduced by the conductors one would also need

$$\sqrt{\frac{\omega\epsilon}{\sigma}}\, L << d \quad \text{if} \quad \frac{1}{\sqrt{\omega\mu\sigma}} << \Delta \tag{2.1}$$

and

$$\frac{L}{\sigma\Delta}\sqrt{\frac{\epsilon}{\mu}} << d \quad \text{if} \quad \Delta << \frac{1}{\sqrt{\omega\mu\sigma}} \tag{2.2}$$

where L is the longitudinal dimension in region II. Finally, for a TEM wave to propagate from I, through II, and into III without distortion a plane wave front in I should appear as a plane wave front in III: that is, the traveling time of waves along paths of different radii should be equal. Thus, (see figure 2.2) we require the travel times along $MM'\ M''$ and its infinitesimally changed version $OO'\ O''$ to be equal, and so we obtain

$$\begin{aligned} &\sqrt{\mu\epsilon(\theta)}\ [r_2(\theta) - r_1(\theta)] + \sqrt{\mu\epsilon}\ \Delta_1 \\ &= [r_2(\theta+d\theta) - r_1(\theta+d\theta)]\sqrt{\mu\epsilon(\theta+d\theta)} + \sqrt{\mu\epsilon}\ \Delta_2 \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} \Delta_1 &= r_1(\theta)\cos(\theta) - r_1(\theta+d\theta)\cos(\theta+d\theta) \\ \Delta_2 &= r_2(\theta)\cos(\theta) - r_2(\theta+d\theta)\cos(\theta+d\theta)\,. \end{aligned} \tag{2.4}$$

Some elementary calculus coupled with the fact that the $r_1(\theta)$ which describes the boundary between region I and II, and the $r_2(\theta)$ which describes the boundary between regions II and III,

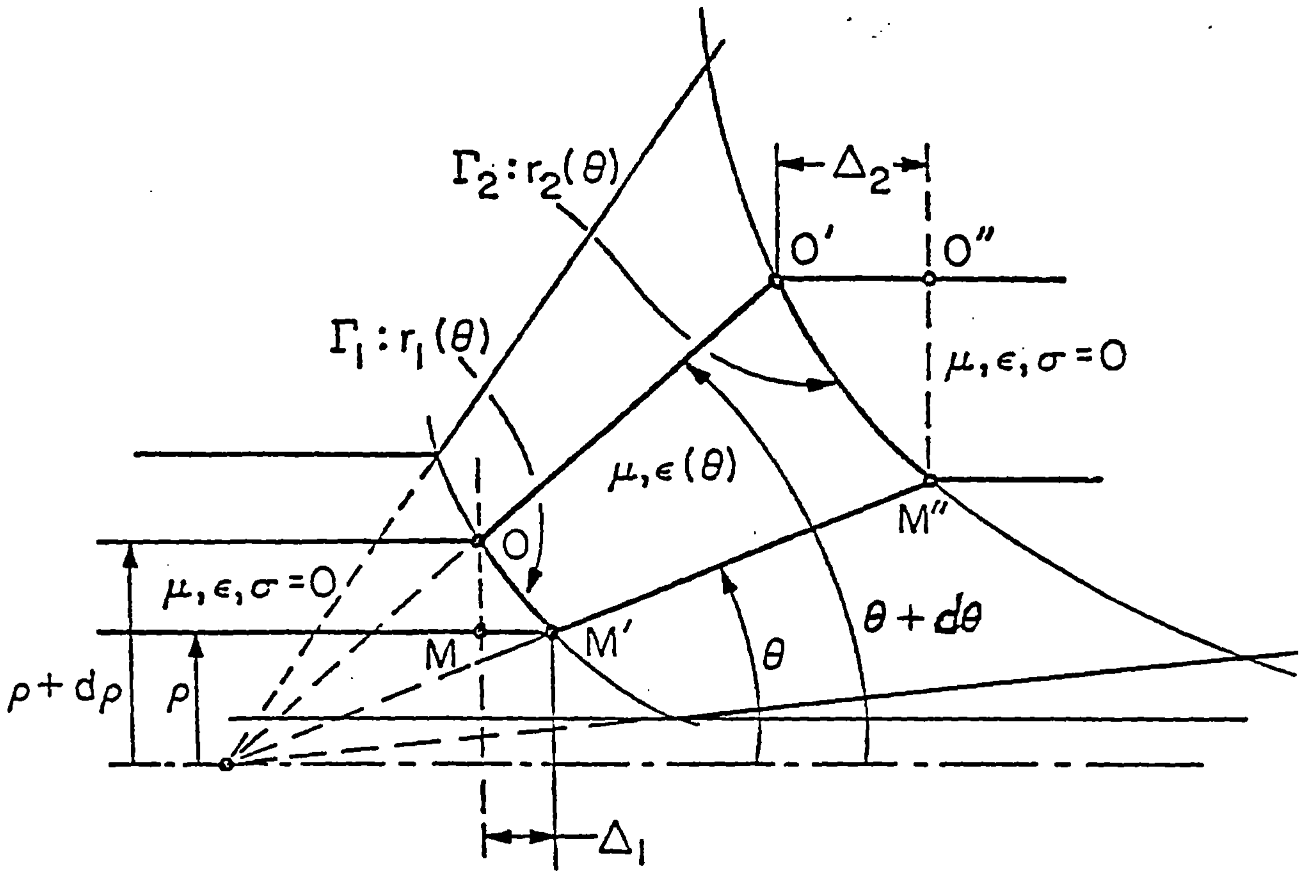

Figure 2.2: Travel Time Geometry (SSN #169)

considered in the impedance matching approach. That this matching obtained indeed matches a reflectionless and distortionless TEM wave from region I to III can be verified by a fields approach. One may consider, for example, TEM solutions to Maxwell's equations in regions I and II with appropriate boundary conditions and find an $\epsilon(\theta)$ which is consistent with that given by (2.11). A similar result is obtained by considering regions II and III, and hence the solutions to the system (2.5) and (2.9) do indeed give yield to perfect matchings. We note that this particular matching problem was also solved in [5] by the more general procedure of differential geometric scaling.

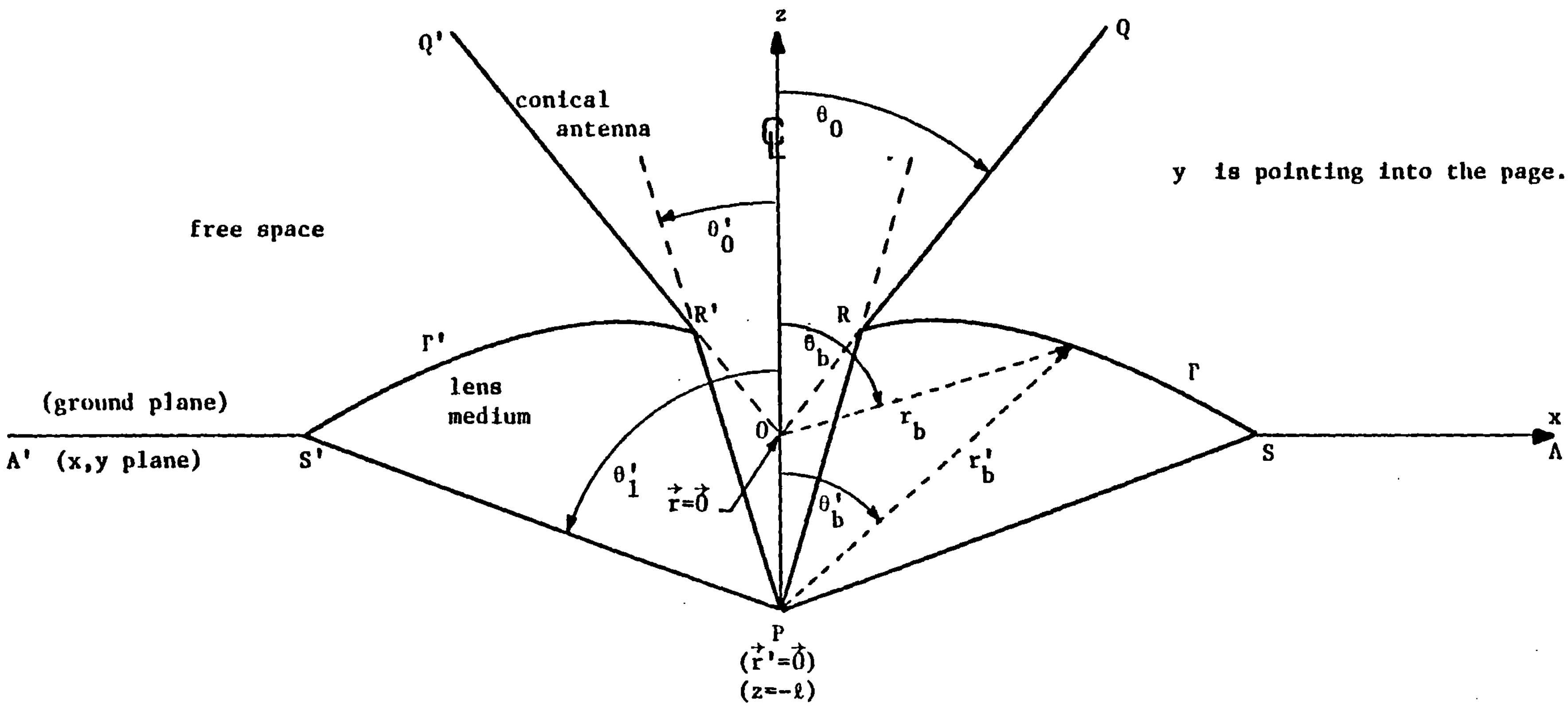

Figure 3.1. Geometry of Circular Conical Lens Feeding Circular Conical Antenna with Ground Plane.

$$\Psi = r\sin(\theta) = r'\sin(\theta') \qquad (3.5)$$

$$z = r\cos(\theta) = r'\cos(\theta') - l.$$

The coordinates described by (3.5) are taken on the boundary curve Γ and so they should all appear at Ψ_b, z_b, etc., as they do in figure 3.2. Again, this is not done in the sequel in order to simplify the notation.

The law of sines then yields the equations

$$\frac{\sin(\theta-\theta')}{l} = \frac{\sin(\theta')}{r} = \frac{\sin(\theta)}{r'} \qquad (3.6)$$

from which we obtain

$$\frac{r}{l} = \frac{\sin(\theta')}{\sin(\theta-\theta')} \qquad (3.7)$$

$$\frac{r'}{l} = \frac{\sin(\theta)}{\sin(\theta-\theta')}.$$

Note that the trigonometric equation

$$r^2 = (r')^2 + l^2 - 2r'l\cos(\theta') \qquad (3.8)$$

is just the law of cosines.

The ranges of θ and θ' will be $\theta_0 \leq \theta \leq \pi/2$ and $\theta_0' \leq \theta' \leq \theta_1'$, and we take $\epsilon_r = \epsilon_{r_0}$ at $\theta' = \theta_0'$ on the boundary. The value of ϵ_{r_0} should be a minimum of ϵ_r in the range of interest. Then for various choices of θ_0 our problem is to find θ_0', θ_1', $\frac{l}{r_0}$, θ in terms of θ', and $\epsilon_r(\theta')$. Thus in seeking solutions to the system of ordinary differential equations, obtained by infinitesimal impedance matching and transit time considerations,

$$\frac{d\theta}{d\theta'} = \frac{1}{\sqrt{\epsilon_r(\theta')}}\,\frac{\sin(\theta)}{\sin(\theta')} \qquad (3.9)$$

$$\frac{d}{d\theta'}\left[r'\sqrt{\epsilon_r(\theta')} - r\right] = 0$$

with geometric relations

$$\frac{2\sqrt{\epsilon_{r_0}}}{1 + \epsilon_{r_0}} = cos(\theta_0 - \theta_0') , \tag{3.14}$$

which may be rewritten in the form

$$cos(\theta_0 - \theta_0') = sech[\frac{1}{2} ln(\epsilon_{r_0})] \tag{3.15}$$

Thus θ_0 , θ_0' and ϵ_{r_0} , are now known, and so the geometric relations (3.10) yield the values of r_0'/r_0 and l/r_0 . Hence, since the system (3.9) yields

$$r' \sqrt{\epsilon_r(\theta')} - r = L \tag{3.16}$$

where L is a constant of integration, we obtain

$$L = r_0' \sqrt{\epsilon_{r_0}} - r_0 . \tag{3.17}$$

If we substitute (3.10) into (3.17) we find that

$$L = r_0[\frac{\sqrt{\epsilon_{r_0}} sin(\theta_0) - sin(\theta_0')}{sin(\theta_0')}]$$

and in terms of l we find that

$$\frac{L}{l} = [\sqrt{\epsilon_{r_0}} sin(\theta_0) - sin(\theta_0')]/sin(\theta_0 - \theta_0') . \tag{3.18}$$

Since the quantity (L/l) appears in several places in our analysis, it is convenient to also express L/l in terms of the impedance Z_c given in (3.11). If (3.11) is rewritten as

$$tan(\frac{\theta_0}{2}) = exp\left[\frac{-2\pi Z_c}{Z_0}\right] \tag{3.19}$$

one finds, using standard trigonometric and hyperbolic relations, that

$$\begin{aligned} cos(\theta_0) &= tanh(\frac{2\pi Z_c}{Z_0}) \\ sin(\theta_0) &= sech(\frac{2\pi Z_c}{Z_0}) . \end{aligned} \tag{3.20}$$

One then finds from (3.14) coupled with some trigonometric relations that

IV. BREWSTER-ANGLE CONSIDERATIONS

One of the implications of (3.12) is that $\epsilon_r(\theta')$ is approximately uniform (or better asymptotically uniform) near $\theta' = \theta_0'$, the inner conical boundary ($\theta' = \theta_0'$ in the lens, $\theta = \theta_0$ outside it). With this insight let us consider what can be locally approximated as a plane wave propagating in a uniform dielectric medium. Referring back to the lens geometry in figure 3.1 let us expand the picture near the meeting point of the lens/free-space boundary Γ with the perfectly conducting circular conical boundaries of free space ($\theta = \theta_0$) and the lens ($\theta' = \theta_0'$). Figure 4.1 gives a picture showing a ray passing through Γ. Along this ray a TEM wave propagates with a TM polarization, i.e., the magnetic field is parallel to boundaries discussed above. In this polarization there exists the well-known Brewster angle ψ_B at which the wave passes through Γ with no distortion [6] given by

$$tan(\psi_B) = \epsilon_{r_0}^{-1/2} \tag{4.1}$$

where ψ_B is measured between Γ and the ray on the free-space side. On the lens side similarly ψ_B' is given by

$$tan(\psi_B') = \epsilon_{r_0}^{1/2} \tag{4.2}$$

which exhibits the fact that in going from one medium to the other, and then reversing ones path, the ratio of one dielectric constant to the other is inverted on path reversal, a rather symmetrical situation. Hence we have

$$\begin{gathered} tan(\psi_B)tan(\psi_B') = 1 \\ \psi_B + \psi_B' = \frac{\pi}{2} \end{gathered} \tag{4.3}$$

which is easily found by geometrical construction with a right triangle.

Referring to figure 4.1, consider the extension of the boundary $\theta' = \theta_0'$ which makes an angle of $\theta_0 - \theta_0'$ with respect to the boundary $\theta = \theta_0$. Furthermore, the extension of the incident ray direction makes an angle of $\psi_B' - \psi_B$ with respect to the transmitted ray direction. Since the two boundaries (on a plane of constant ϕ) are parallel to the local ray directions in the respective media, we conclude that the free-space boundary $\theta = \theta_0$ makes an angle of ψ_B with respect to Γ and that

$$\begin{aligned} \theta_0 - \theta_0' &= \psi_B' - \psi_B \\ &= arctan(\epsilon_{r_0}^{1/2}) - arctan(\epsilon_{r_0}^{-1/2}) \\ &= arctan(\frac{1}{2}[\epsilon_{r_0}^{1/2} - \epsilon_{r_0}^{-1/2}]) \\ &= function\ only\ of\ \epsilon_{r_0} \end{aligned} \tag{4.4}$$

using standard relations for trigonometric functions.

This immediately implies a practical limit on our lens geometry. Define

$$\theta_{0_{min}} \equiv arctan(\frac{1}{2}[\epsilon_{r_0}^{1/2} - \epsilon_{r_0}^{-1/2}]) . \tag{4.5}$$

Now, θ_0' is geometrically constrained as

$$\theta_0' > 0 \tag{4.6}$$

since for $\theta_0' \rightarrow 0+$ then $l \rightarrow \infty$ and negative θ_0' violates the Brewster angle condition near the boundary junction in Figure 4.1. Thus $\theta_{0_{min}}$ actually represents the minimum allowable value of θ_0, and fixing the value of θ_1 as $\pi/2$ the maximum value of the characteristic impedance of our conical system is then

$$Z_{c_{max}} = \frac{Z_0}{2\pi} \, ln(cot(\frac{\theta_{0_{min}}}{2})) \tag{4.7}$$

We thus have $Z_{c_{max}}$ expressed in geometric terms (i.e., in terms of $\theta_{0_{min}}$). It is also possible to express $Z_{c_{max}}$ in physical terms (i.e., in terms of ϵ_{r_0}) as follows. Since the identities

$$cot\left(\frac{x}{2}\right) = \frac{1+cos(x)}{sin(x)}$$

$$\tag{4.8}$$

$$arctan(y) = arcsin\left(\frac{y}{\sqrt{1+y^2}}\right) = arccos\left(\frac{1}{\sqrt{1+y^2}}\right)$$

hold for appropriate x and y, we find that

$$Z_{c_{max}} = \frac{Z_0}{2\pi} \, ln\left(\frac{1+cos(\theta_{0_{min}})}{sin(\theta_{0_{min}})}\right) \tag{4.9}$$

where

$$\begin{aligned} \theta_{0_{min}} &= arctan \; \frac{1}{2}(\epsilon_{r_0}^{1/2}-\epsilon_{r_0}^{-1/2}) \\ &= arcsin\left(\frac{\epsilon_{r_0}^{1/2}-\epsilon_{r_0}^{-1/2}}{\epsilon_{r_0}^{1/2}+\epsilon_{r_0}^{-1/2}}\right) \\ &= arccos\left(\frac{2}{\epsilon_{r_0}^{1/2}+\epsilon_{r_0}^{-1/2}}\right) \end{aligned} \tag{4.10}$$

and hence

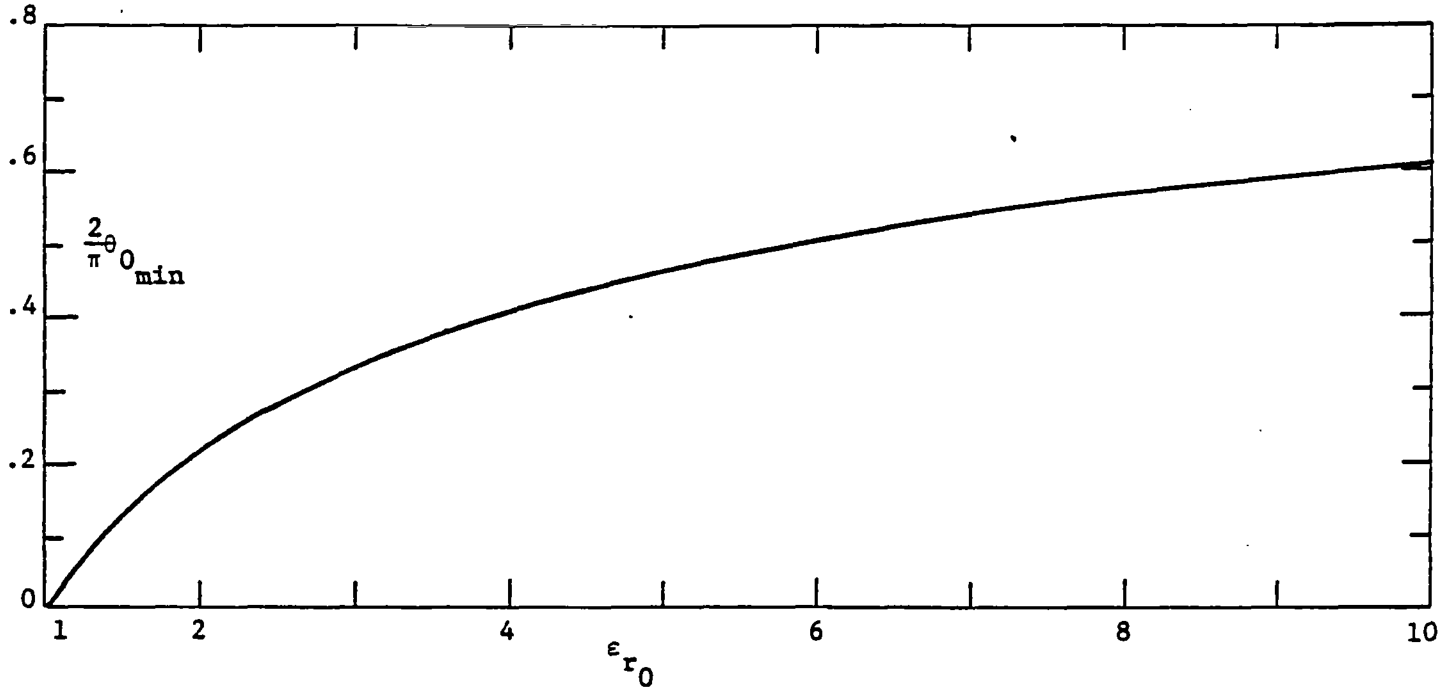

Figure 4.2. Minimum Cone Angle for Given Initial Dielectric Constant

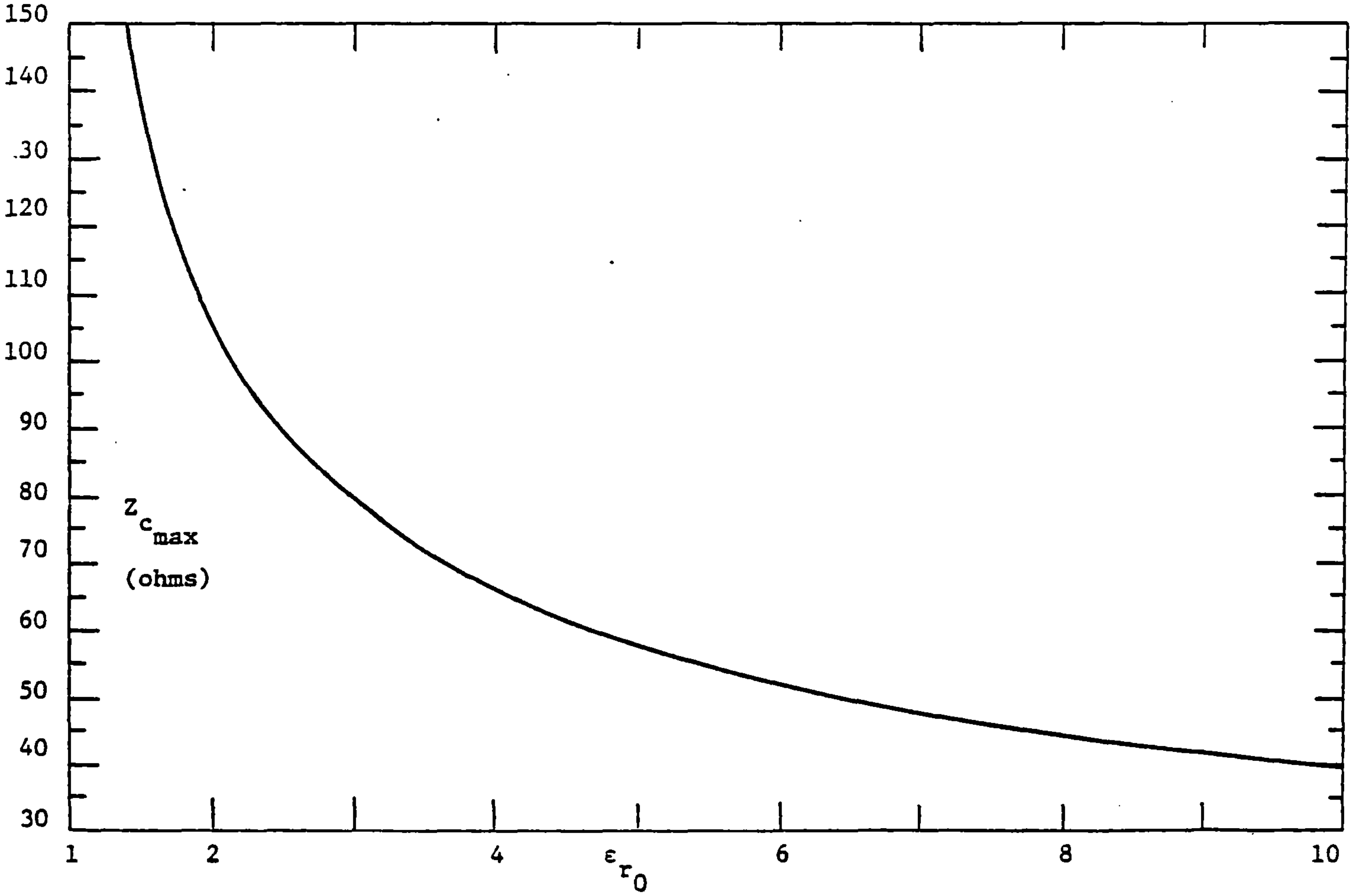

Figure 4.3. Maximum Cone Impedance for Given Initial Relative Dielectric Constant

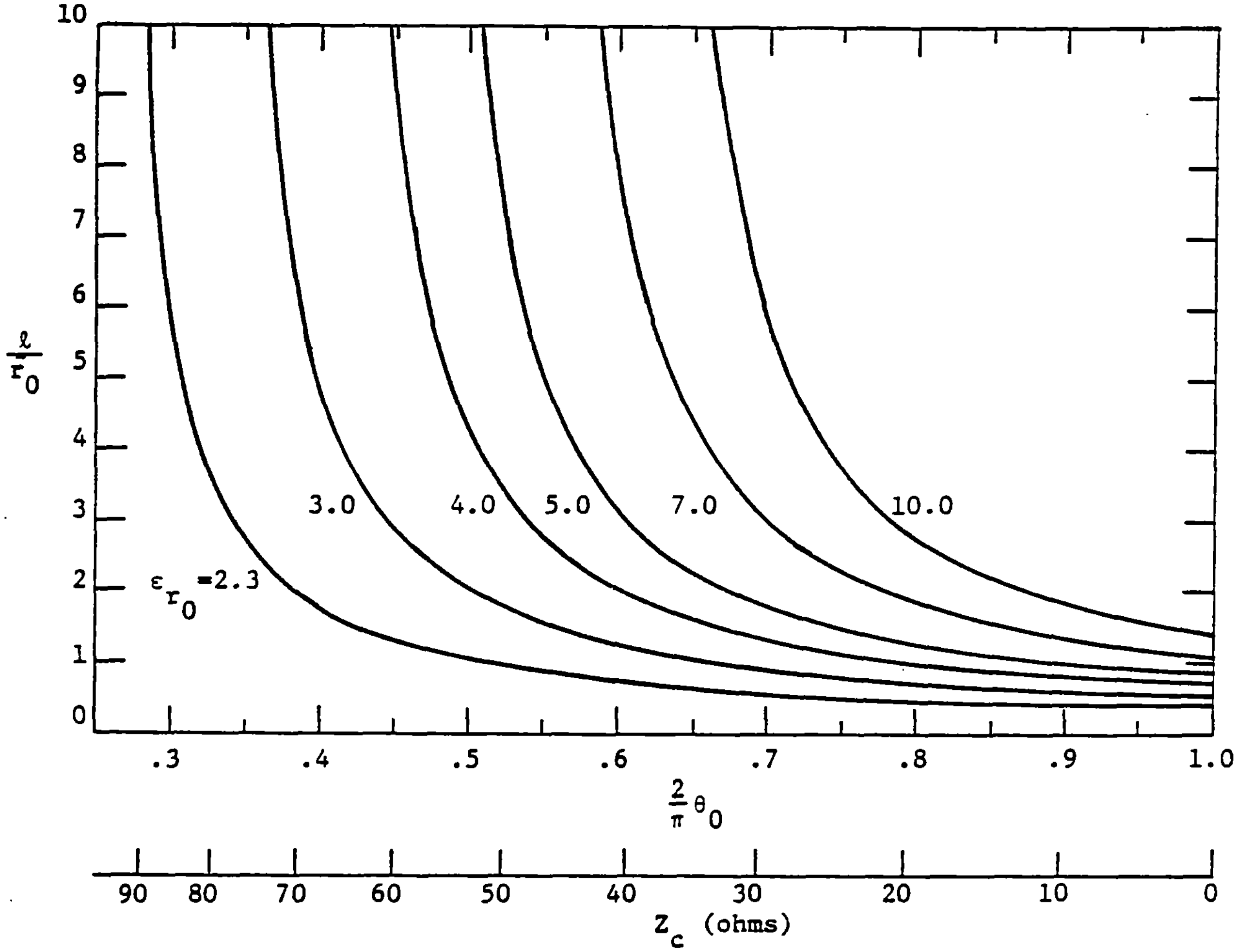

Figure 4.4. Apex of Conical Lens as a Function of Cone Angle with Initial Relative Dielectric Constant as a Parameter

$$\frac{dT}{dT'} = \frac{l\,cosh(T')}{l\,cosh(T) + L(sinh(T') - sinh(T))} \,. \tag{5.9}$$

Actually T represents the well known spherical TEM wave potential function with boundary values

$$T(\frac{\pi}{2}) = 0 \tag{5.10}$$

$$T(\theta_0) = ln(cot(\frac{\theta_0}{2})) \,. \tag{5.11}$$

Since we have from (3.22)

$$\frac{L}{l} = \sqrt{\epsilon_{r_0}}\,sech(\frac{2\pi Z_c}{Z_0}) + tanh(\frac{2\pi Z_c}{Z_0}) \,, \tag{5.12}$$

we can restrict Z_c so that $\frac{L}{l} \geq 1$ and hence rewrite (5.9) as

$$\frac{dT}{dT'} = \frac{l\,cosh(T')}{L\,sinh(T') - \sqrt{L^2-l^2}\,sinh(T-a)} \tag{5.13}$$

where $a = arccosh(\frac{L}{\sqrt{L^2-l^2}})$. We then transform (5.12) into a linear, inhomogeneous differential equation by the substitutions

$$x = sinh(T-a)$$

$$\tag{5.14}$$

$$y = sinh(T') \,.$$

The resulting equation is then

$$\frac{dy}{dx} - \frac{L}{l}\frac{y}{\sqrt{1+x^2}} = -\frac{\sqrt{L^2-l^2}}{l}\frac{x}{\sqrt{1+x^2}} \tag{5.15}$$

which can also be written in the form

$$\frac{d}{dx}[y(x+\sqrt{1+x^2})^{\alpha}] = -\frac{\sqrt{L^2-l^2}}{l}\frac{x}{\sqrt{1+x^2}}[x + \sqrt{1+x^2}]^{\alpha} \tag{5.16}$$

where $\alpha = -L/l$. The integral of the right hand side of (5.16) reduces to an obvious standard form by a substitution $x = sinh(u)$, with the result that the solution to (5.5) is

The preceding equations (5.21) and (5.22) will be useful in determining the lens profile in our subsequent numerical work.

B. *Constraints on Z_c and ϵ_r* .

In sections 3 and 4 it was noted that the condition (3.12), i.e.,

$$\frac{d\epsilon_r(\theta')}{d\theta'}\Big|_{\theta'=\theta_0'} = 0 \tag{5.23}$$

enabled us to specify θ_0' by specifying ϵ_{r_0}. Since $\epsilon_r(\theta')$, given by equation (5.4), is

$$\sqrt{\epsilon_r(\theta')} = \frac{L\,sin(\theta-\theta') + l\,sin(\theta')}{l\,sin(\theta)} \tag{5.24}$$

condition (5.23) implies $\epsilon_r(\theta')$ will be an increasing function of θ' for $\theta_0' \le \theta' \le \theta_1'$ provided we have

$$\frac{d\epsilon_r(\theta')}{d\theta'} \ge 0 \tag{5.25}$$

in the range $\theta_0' \le \theta' \le \theta_1'$. In view of differential equation (3.13), i.e.,

$$\frac{d\epsilon_r}{d\theta'} = \frac{2}{sin(\theta-\theta')}[2\sqrt{\epsilon_r}-(1+\epsilon_r)cos(\theta-\theta')] \, , \tag{5.26}$$

the condition (5.25) means that we must have

$$\frac{2\sqrt{\epsilon_r}}{1+\epsilon_r} \ge cos(\theta-\theta') \tag{5.27}$$

in the range of interest. The latter inequality however will in general not hold throughout this range, and consequently we impose a less restrictive condition, namely that

$$\epsilon_r(\theta') \ge \epsilon_{r_0} \tag{5.28}$$

for $\theta_0' \le \theta' \le \theta_1'$. This condition can be realized, for a given choice of ϵ_{r_0} , by restricting the range of the impedance Z_c so that

$$Z_{c_{min}} \le Z_c \le Z_{c_{max}} \tag{5.29}$$

where the upper and lower limits, Z_{c_1} and Z_{c_0} , are determined by the following considerations.

From the formulas for Z_c and $cos(\theta_0-\theta_0')$, given in (3.11) and (3.14), we note that θ_0 decreases as Z_c increases and that $(\theta_0-\theta_0')$ increases as ϵ_{r_0} increases. The geometry of our problem dictates that θ_0 , θ_0' , and $(\theta_0-\theta_0')$ be positive. Hence, for fixed ϵ_{r_0} , the quantity $(1-\epsilon_{r_0})\tanh(\frac{2\pi Z_c}{Z_0}) + 2\sqrt{\epsilon_{r_0}}sech(\frac{2\pi Z_c}{Z_0})$, which appears in the formulas for $sin(\theta_0')$ and

$$a = (\epsilon_{r_0}-1)(cosh(\frac{2\pi Z_c}{Z_0})) \{ \exp\left[\frac{2\pi Z_c}{Z_0} \right] \alpha \}$$

and

$$b = (1-\epsilon_{r_0}) tanh(\frac{2\pi Z_c}{Z_0}) + 2\sqrt{\epsilon_{r_0}} sech(\frac{2\pi Z_c}{Z_0})$$

so that $cot(\theta') = a/b$ and rewrite (5.35) in the form

$$\sqrt{\epsilon_{r_0}}\ \sqrt{a^2+b^2} = [\sqrt{\epsilon_{r_0}} sech(\frac{2\pi Z_c}{Z_0}) + tanh(\frac{2\pi Z_c}{Z_0})] a + b \ , \tag{5.36}$$

then the first solution Z_c strictly less than $Z_{c_{max}}$ we will denote by $Z_{c_{min}}$. Thus we may obtain a range of Z_c, for fixed ϵ_{r_0}, for which the inequality

$$Z_{c_{min}} \leq Z_c \leq Z_{c_{max}} \tag{5.37}$$

is satisfied. Numerical values of $Z_{c_{min}}$ and $Z_{c_{max}}$, obtained by computer calculations, are given in table 5.1. We note that for this range of Z_c the constant L/l satisfies the condition $\frac{L}{l} \geq 1$ which is implicit in the result finally obtained in (5.18). Values of L/l for the range $Z_{c_{min}} \leq Z_c \leq Z_{c_{max}}$ with ϵ_{r_0} as a parameter appear in table 5.2.

C. *Numerical Results*

In this section we present numerical results based on the analytical results of section 5A. Equation (5.18) gives θ' as a function of θ with parameters ϵ_{r_0} and Z_c. Thus if we take $\epsilon_{r_0} = 2.3$ and take Z_c as a parameter with values ranging between 58Ω and 95Ω, then we obtain graphs of θ' versus θ as in figure 5.1. In this figure, as well as other figures, the scales for θ and θ' have been normalized by a factor of $\frac{2}{\pi}$. The corresponding numerical data appears in table 5.3. The range of impedances chosen corresponds to the inequality (5.37). For the same value of ϵ_{r_0}, namely 2.3, and the same range of impedances, graphs of ϵ_r versus θ' are given in figure 5.2 with the corresponding numerical data in table 5.4. These results are obtained from (5.20) which gives ϵ_r as a function of θ'.

If one takes $\theta_1 = \pi/2$, then the corresponding values of θ_1' and ϵ_r are given in (5.21) and (5.22). This value of ϵ_r is denoted by ϵ_{r_1}. Thus if we choose a particular ϵ_{r_0}, then θ_1' and ϵ_{r_1} are functions of the impedance Z_c, whose range is restricted by the choice of ϵ_{r_0}, and we may obtain plots of ϵ_{r_1} versus impedance with ϵ_{r_0} as parameter. These graphs appear in figure 5.3 with values of ϵ_{r_0} given by 2.3, 3.0, 4.0, 5.0, 7.0, and 10.0. The corresponding numerical data is in table 5.5.

Next, the determination of the curved boundary of the lens is made in the following analysis. In figure 3.2, we let z denote the height above the ground plane and Ψ the cylindrical radius. Thus we have

can be simplified with the result that

$$x \equiv \left(\frac{\Psi}{sin\ \theta_0}\right)^{l/L} \tan(\frac{\theta_0}{2}) \ . \tag{5.46}$$

Now (5.38) yields the result that

$$tan(\theta) = \frac{\Psi}{z} \tag{5.47}$$

and so (5.43) through (5.46) may be combined and we obtain

$$z = \Psi\left(\frac{1-x^2}{2x}\right) \tag{5.48}$$

where x is given by (5.45). Note that

$$\frac{1-x^2}{2x} = \frac{x^{-1}-x}{2} = \frac{e^{-lnx}-e^{lnx}}{2}$$

and since

$$\tan\left(\frac{\theta_0}{2}\right) = exp\left(\frac{-2\pi Z_c}{Z_0}\right)$$

$$sin\ \theta_0 = sech\left(\frac{2\pi Z_c}{Z_0}\right)$$

from (3.19) and (3.20), one obtains

$$z = \Psi\ sinh\left[\frac{l}{L}\ sech\left(\frac{2\pi Z_c}{Z_0}\right) - \frac{l}{L}\ ln\Psi + \left(\frac{2\pi Z_c}{Z_0}\right)\right] \tag{5.49}$$

where $\frac{L}{l} = \sqrt{\epsilon_{r_0}} sech\left(\frac{2\pi Z_c}{Z_0}\right) + tanh\left(\frac{2\pi Z_c}{Z_0}\right)$. We can now use (5.49) to obtain the graphs of figures 5.4 and 5.5 and the corresponding numerical results in tables 5.6 and 5.7. In figure 5.4 we take $\epsilon_{r_0} = 2.3$ and take Z_c as a parameter to obtain plots of the height z versus the cylindrical radius Ψ. We can also fix the value of Z_c at 60Ω and take ϵ_{r_0} as a parameter to obtain similar plots in figure 5.5. In each case $r_0 = 1$ and we obtain the lens profile as indicated in these figures.

One may also attempt to solve numerically the system of ordinary differential equations, given in (3.23) and (3.13), as

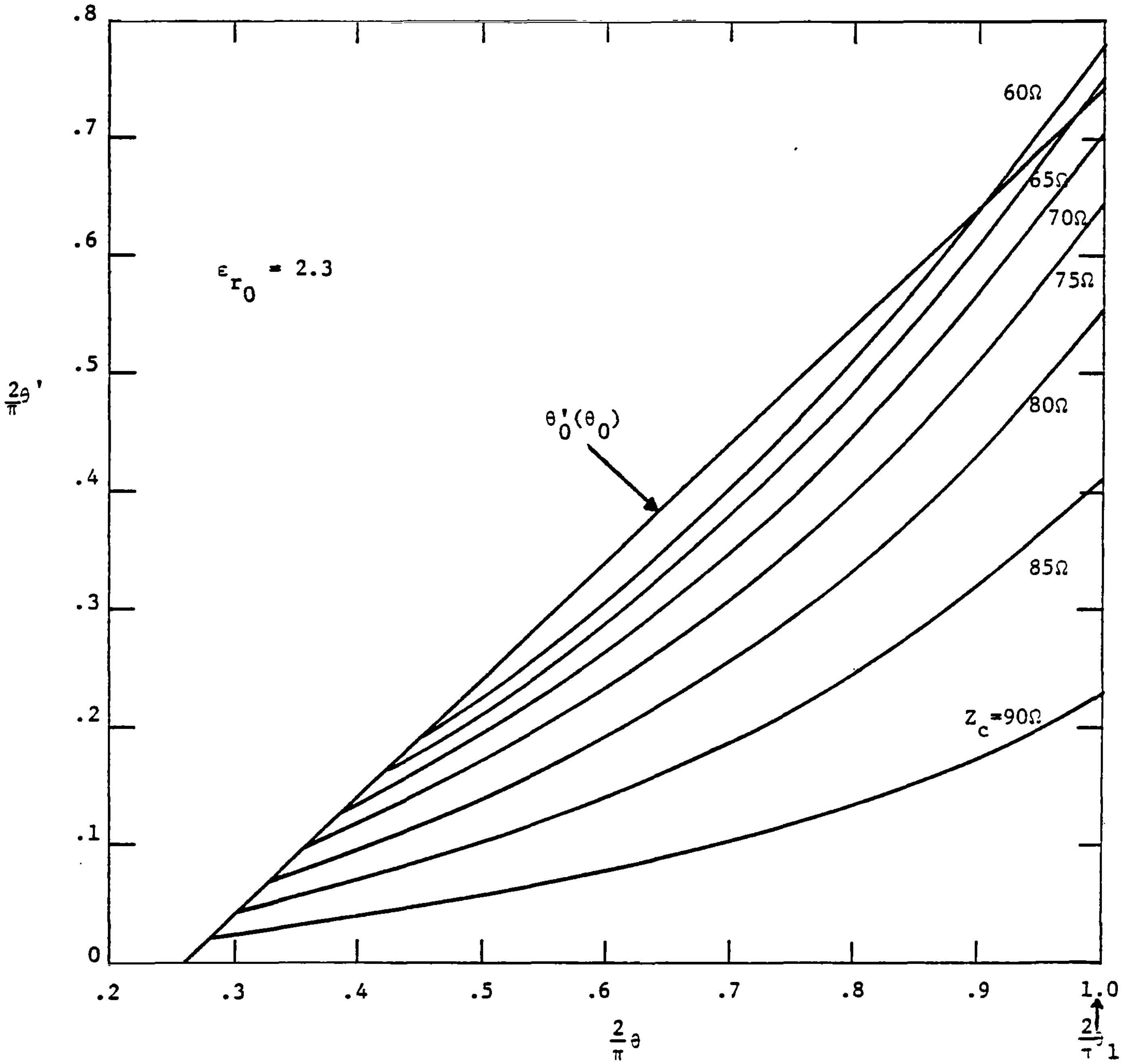

Figure 5.1. Relation Between Lens Angle θ' and Conical Antenna Angle θ for $\varepsilon_{r_0} = 2.3$ with Cone Impedance as a Parameter

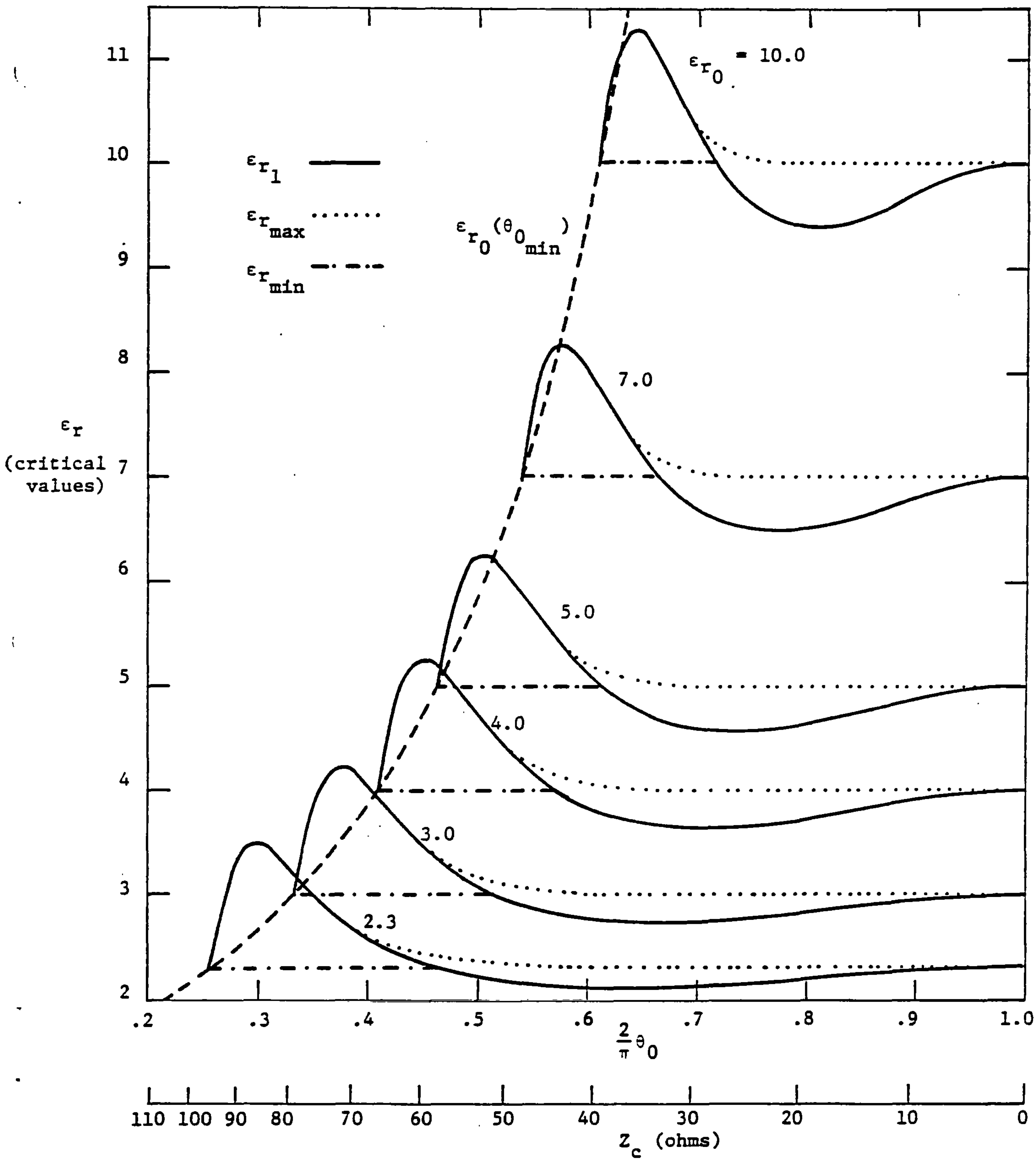

Figure 5.3. Maxima, Minima, and Boundary Values of the Relative Dielectric Constant as a Function of the Conical Antenna Angle θ_0 with Initial Dielectric Constant as a Parameter

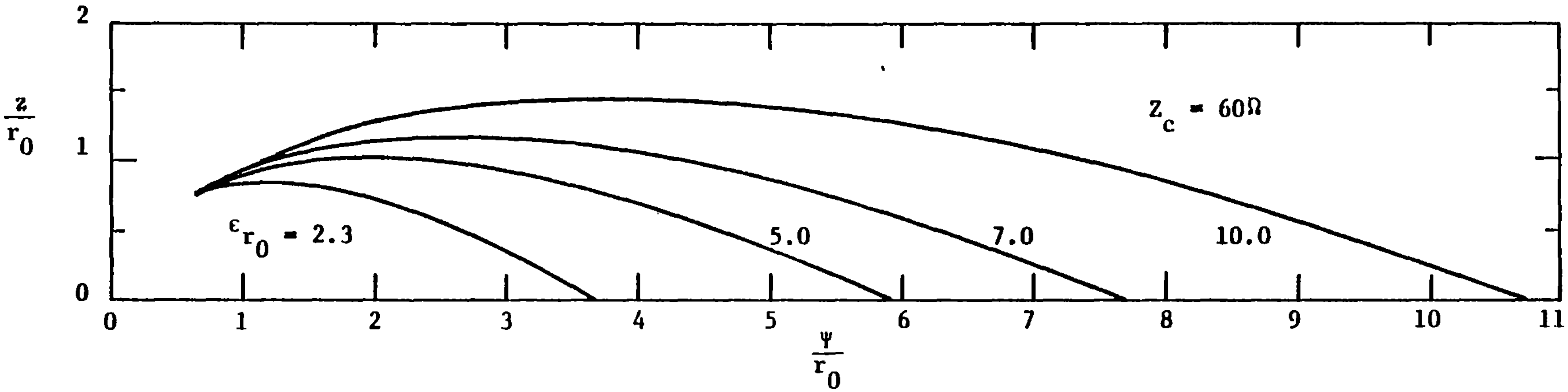

Figure 5.5. Shape of Lens/Free-Space Boundary for Z_c = 60Ω with Initial Relative Dielectric Constant as a Parameter

$\varepsilon_{r_0} = 2.30$

Z_c	L/ℓ	ℓ/r_0	L/r_0
58.11000	1.754518	1.249413	2.192117
59.00000	1.749824	1.287407	2.252737
60.00000	1.744417	1.332549	2.324522
61.00000	1.738876	1.380505	2.400528
62.00000	1.733209	1.431529	2.481139
63.00000	1.727423	1.485908	2.566793
64.00000	1.721527	1.543965	2.657978
65.00000	1.715529	1.606065	2.755250
66.00000	1.709434	1.672625	2.859242
67.00000	1.703251	1.744121	2.970676
68.00000	1.696986	1.821101	3.090383
69.00000	1.690646	1.904194	3.219319
70.00000	1.684237	1.994134	3.358595
71.00000	1.677766	2.091773	3.509505
72.00000	1.671238	2.198115	3.673573
73.00000	1.664660	2.314342	3.852593
74.00000	1.658037	2.441866	4.048703
75.00000	1.651374	2.582377	4.264470
76.00000	1.644677	2.737925	4.503002
77.00000	1.637951	2.911012	4.768096
78.00000	1.631201	3.104729	5.064438
79.00000	1.624431	3.322943	5.397894
80.00000	1.617647	3.570549	5.775887
81.00000	1.610851	3.853834	6.207953
82.00000	1.604049	4.181032	6.706580
83.00000	1.597244	4.563110	7.288402
84.00000	1.590441	5.015022	7.976096
85.00000	1.583642	5.557707	8.801418
86.00000	1.576852	6.221390	9.810210
87.00000	1.570073	7.051428	11.07126
88.00000	1.563309	8.119020	12.69254
89.00000	1.556563	9.542828	14.85402
90.00000	1.549838	11.53633	17.87945
91.00000	1.543137	14.52616	22.41584
92.00000	1.536461	19.50660	29.97113
93.00000	1.529814	29.45454	45.05996
94.00000	1.523198	59.18390	90.14879
95.00000	1.516615	9980.497	15136.57
95.00600	1.516575	3355443.	5088782.

Table 5.2: L/ℓ , ℓ/r_0 , L/r_0 versus cone impedance Z_c with ε_{r_0} as a parameter

Table 5.2

(continuation)

$\varepsilon_{r_0} = 4.00$

Z_c	L/ℓ	ℓ/r_0	L/r_0
43.84000	2.187163	2.386710	5.220124
44.00000	2.186189	2.406013	5.260000
45.00000	2.179928	2.533545	5.522945
46.00000	2.173374	2.674242	5.812129
47.00000	2.166539	2.830190	6.131717
48.00000	2.159437	3.003936	6.486809
49.00000	2.152079	3.198635	6.883716
50.00000	2.144478	3.418231	7.330321
51.00000	2.136646	3.667728	7.836638
52.00000	2.128596	3.953563	8.415538
53.00000	2.120339	4.284173	9.083899
54.00000	2.111887	4.670824	9.864253
55.00000	2.103252	5.128909	10.78739
56.00000	2.094444	5.680033	11.89651
57.00000	2.085474	6.355490	13.25421
58.00000	2.076354	7.202386	14.95470
59.00000	2.067093	8.295081	17.14671
60.00000	2.057703	9.758166	20.07941
61.00000	2.048192	11.81765	24.20482
62.00000	2.038571	14.93032	30.43652
63.00000	2.028849	20.17943	40.94101
64.00000	2.019035	30.90882	62.40600
65.00000	2.009138	65.05243	130.6993

$\varepsilon_{r_0} = 5.00$

Z_c	L/ℓ	ℓ/r_0	L/r_0
39.16300	2.405302	2.920503	7.024690
40.00000	2.399874	3.069721	7.366943
41.00000	2.393045	3.267981	7.820426
42.00000	2.385855	3.491914	8.331202
43.00000	2.378319	3.746737	8.910937
44.00000	2.370450	4.039188	9.574695
45.00000	2.362265	4.378114	10.34226
46.00000	2.353776	4.775383	11.24018
47.00000	2.344998	5.247278	12.30485
48.00000	2.335945	5.816756	13.58762
49.00000	2.326631	6.517292	15.16333
50.00000	2.317069	7.399636	17.14547
51.00000	2.307273	8.544646	19.71483
52.00000	2.297257	10.08943	23.17803
53.00000	2.287033	12.28674	28.10019
54.00000	2.276614	15.65921	35.64998
55.00000	2.266013	21.49093	48.69872
56.00000	2.255241	34.00879	76.69802
57.00000	2.244311	80.12195	179.8186
57.74500	2.236073	145888.8	326218.0

$Z_c = 60\Omega$

$\frac{2}{\pi}\theta$	θ	$\frac{2}{\pi}\theta'$	θ'	$\frac{2}{\pi}\theta$	θ	$\frac{2}{\pi}\theta'$	θ'
0.440	0.691	0.185	0.291	0.730	1.147	0.437	0.686
0.450	0.707	0.192	0.301	0.740	1.162	0.447	0.703
0.460	0.723	0.199	0.312	0.750	1.178	0.458	0.720
0.470	0.738	0.206	0.324	0.760	1.194	0.469	0.737
0.480	0.754	0.213	0.335	0.770	1.210	0.481	0.755
0.490	0.770	0.221	0.346	0.780	1.225	0.492	0.773
0.500	0.785	0.228	0.358	0.790	1.241	0.504	0.791
0.510	0.801	0.236	0.370	0.800	1.257	0.516	0.810
0.520	0.817	0.243	0.382	0.810	1.272	0.527	0.829
0.530	0.833	0.251	0.394	0.820	1.288	0.540	0.848
0.540	0.848	0.259	0.407	0.830	1.304	0.552	0.867
0.550	0.864	0.267	0.420	0.840	1.319	0.564	0.886
0.560	0.880	0.275	0.433	0.850	1.335	0.577	0.906
0.570	0.895	0.284	0.446	0.860	1.351	0.589	0.926
0.580	0.911	0.292	0.459	0.870	1.367	0.602	0.946
0.590	0.927	0.301	0.473	0.880	1.382	0.615	0.966
0.600	0.942	0.310	0.486	0.890	1.398	0.628	0.987
0.610	0.958	0.319	0.500	0.900	1.414	0.641	1.008
0.620	0.974	0.328	0.514	0.910	1.429	0.655	1.029
0.630	0.990	0.337	0.529	0.920	1.445	0.668	1.050
0.640	1.005	0.346	0.544	0.930	1.461	0.682	1.071
0.650	1.021	0.355	0.558	0.940	1.477	0.696	1.093
0.660	1.037	0.365	0.573	0.950	1.492	0.710	1.115
0.670	1.052	0.375	0.589	0.960	1.508	0.723	1.136
0.680	1.068	0.385	0.604	0.970	1.524	0.738	1.159
0.690	1.084	0.395	0.620	0.980	1.539	0.752	1.181
0.700	1.100	0.405	0.636	0.990	1.555	0.766	1.203
0.710	1.115	0.415	0.652	1.000	1.571	0.780	1.226
0.720	1.131	0.426	0.669				

Table 5.3: Lens angle θ' versus conical antenna angle θ for $\epsilon_{r_0} = 2.30$, with cone impedance as a parameter

Table 5.3

(continuation)

$Z_c = 70\Omega$

$\frac{2}{\pi}\theta$	θ	$\frac{2}{\pi}\theta'$	θ'
0.400	0.628	0.135	0.212
0.410	0.644	0.140	0.221
0.420	0.660	0.146	0.229
0.430	0.675	0.152	0.238
0.440	0.691	0.157	0.247
0.450	0.707	0.163	0.257
0.460	0.723	0.169	0.266
0.470	0.738	0.175	0.276
0.480	0.754	0.182	0.285
0.490	0.770	0.188	0.295
0.500	0.785	0.194	0.305
0.510	0.801	0.201	0.316
0.520	0.817	0.208	0.326
0.530	0.833	0.214	0.337
0.540	0.848	0.221	0.348
0.550	0.864	0.228	0.359
0.560	0.880	0.235	0.370
0.570	0.895	0.243	0.381
0.580	0.911	0.250	0.393
0.590	0.927	0.258	0.405
0.600	0.942	0.265	0.417
0.610	0.958	0.273	0.429
0.620	0.974	0.281	0.441
0.630	0.990	0.289	0.454
0.640	1.005	0.297	0.467
0.650	1.021	0.306	0.480
0.660	1.037	0.314	0.494
0.670	1.052	0.323	0.507
0.680	1.068	0.332	0.521
0.690	1.084	0.341	0.535

$\frac{2}{\pi}\theta$	θ	$\frac{2}{\pi}\theta'$	θ'
0.700	1.100	0.350	0.550
0.710	1.115	0.359	0.564
0.720	1.131	0.369	0.579
0.730	1.147	0.378	0.594
0.740	1.162	0.388	0.610
0.750	1.178	0.398	0.625
0.760	1.194	0.408	0.641
0.770	1.210	0.419	0.658
0.780	1.225	0.429	0.674
0.790	1.241	0.440	0.691
0.800	1.257	0.451	0.708
0.810	1.272	0.462	0.725
0.820	1.288	0.473	0.743
0.830	1.304	0.485	0.761
0.840	1.319	0.496	0.779
0.850	1.335	0.508	0.798
0.860	1.351	0.520	0.817
0.870	1.367	0.532	0.836
0.880	1.382	0.545	0.856
0.890	1.398	0.557	0.876
0.900	1.414	0.570	0.896
0.910	1.429	0.583	0.916
0.920	1.445	0.596	0.937
0.930	1.461	0.610	0.958
0.940	1.477	0.623	0.979
0.950	1.492	0.637	1.001
0.960	1.508	0.651	1.023
0.970	1.524	0.665	1.045
0.980	1.539	0.680	1.068
0.990	1.555	0.694	1.090
1.000	1.571	0.709	1.113

Table 5.3

(continuation)

$Z_c = 80\Omega$

$\frac{2}{\pi}\theta$	θ	$\frac{2}{\pi}\theta'$	θ'	$\frac{2}{\pi}\theta$	θ	$\frac{2}{\pi}\theta'$	θ'
0.330	0.518	0.071	0.112	0.670	1.052	0.235	0.369
0.340	0.534	0.074	0.117	0.680	1.068	0.241	0.379
0.350	0.550	0.078	0.122	0.690	1.084	0.248	0.390
0.360	0.565	0.082	0.128	0.700	1.100	0.255	0.401
0.370	0.581	0.085	0.134	0.710	1.115	0.262	0.412
0.380	0.597	0.089	0.140	0.720	1.131	0.270	0.423
0.390	0.613	0.093	0.146	0.730	1.147	0.277	0.435
0.400	0.628	0.097	0.152	0.740	1.162	0.285	0.447
0.410	0.644	0.101	0.158	0.750	1.178	0.292	0.459
0.420	0.660	0.105	0.164	0.760	1.194	0.300	0.472
0.430	0.675	0.109	0.171	0.770	1.210	0.308	0.484
0.440	0.691	0.113	0.177	0.780	1.225	0.317	0.497
0.450	0.707	0.117	0.184	0.790	1.241	0.325	0.511
0.460	0.723	0.121	0.191	0.800	1.257	0.334	0.524
0.470	0.738	0.126	0.198	0.810	1.272	0.343	0.538
0.480	0.754	0.130	0.205	0.820	1.288	0.352	0.552
0.490	0.770	0.135	0.212	0.830	1.304	0.361	0.567
0.500	0.785	0.140	0.219	0.840	1.319	0.370	0.582
0.510	0.801	0.144	0.227	0.850	1.335	0.380	0.597
0.520	0.817	0.149	0.234	0.860	1.351	0.390	0.613
0.530	0.833	0.154	0.242	0.870	1.367	0.400	0.629
0.540	0.848	0.159	0.250	0.880	1.382	0.411	0.645
0.550	0.864	0.164	0.258	0.890	1.398	0.421	0.662
0.560	0.880	0.170	0.266	0.900	1.414	0.432	0.679
0.570	0.895	0.175	0.275	0.910	1.429	0.443	0.696
0.580	0.911	0.180	0.283	0.920	1.445	0.455	0.714
0.590	0.927	0.186	0.292	0.930	1.461	0.466	0.732
0.600	0.942	0.192	0.301	0.940	1.477	0.478	0.751
0.610	0.958	0.197	0.310	0.950	1.492	0.490	0.770
0.620	0.974	0.203	0.319	0.960	1.508	0.503	0.790
0.630	0.990	0.209	0.329	0.970	1.524	0.516	0.810
0.640	1.005	0.216	0.339	0.980	1.539	0.529	0.831
0.650	1.021	0.222	0.348	0.990	1.555	0.542	0.852
0.660	1.037	0.228	0.358	1.000	1.571	0.556	0.873

Table 5.3

(continuation)

$Z_c = 90\Omega$

$\frac{2}{\pi}\theta$	θ	$\frac{2}{\pi}\theta'$	θ'	$\frac{2}{\pi}\theta$	θ	$\frac{2}{\pi}\theta'$	θ'
0.270	0.424	0.021	0.032	0.640	1.005	0.086	0.135
0.280	0.440	0.022	0.034	0.650	1.021	0.089	0.139
0.290	0.456	0.023	0.036	0.660	1.037	0.091	0.143
0.300	0.471	0.024	0.038	0.670	1.052	0.094	0.147
0.310	0.487	0.026	0.040	0.680	1.068	0.097	0.152
0.320	0.503	0.027	0.042	0.690	1.084	0.099	0.156
0.330	0.518	0.028	0.044	0.700	1.100	0.102	0.160
0.340	0.534	0.030	0.047	0.710	1.115	0.105	0.165
0.350	0.550	0.031	0.049	0.720	1.131	0.108	0.170
0.360	0.565	0.032	0.051	0.730	1.147	0.111	0.174
0.370	0.581	0.034	0.053	0.740	1.162	0.114	0.179
0.380	0.597	0.035	0.056	0.750	1.178	0.117	0.184
0.390	0.613	0.037	0.058	0.760	1.194	0.120	0.189
0.400	0.628	0.038	0.060	0.770	1.210	0.124	0.194
0.410	0.644	0.040	0.063	0.780	1.225	0.127	0.200
0.420	0.660	0.042	0.065	0.790	1.241	0.130	0.205
0.430	0.675	0.043	0.068	0.800	1.257	0.134	0.211
0.440	0.691	0.045	0.071	0.810	1.272	0.138	0.216
0.450	0.707	0.047	0.073	0.820	1.288	0.141	0.222
0.460	0.723	0.048	0.076	0.830	1.304	0.145	0.228
0.470	0.738	0.050	0.079	0.840	1.319	0.149	0.234
0.480	0.754	0.052	0.082	0.850	1.335	0.153	0.241
0.490	0.770	0.054	0.085	0.860	1.351	0.157	0.247
0.500	0.785	0.056	0.087	0.870	1.367	0.161	0.254
0.510	0.801	0.058	0.090	0.880	1.382	0.166	0.260
0.520	0.817	0.060	0.094	0.890	1.398	0.170	0.268
0.530	0.833	0.062	0.097	0.900	1.414	0.175	0.275
0.540	0.848	0.064	0.100	0.910	1.429	0.180	0.282
0.550	0.864	0.066	0.103	0.920	1.445	0.185	0.290
0.560	0.880	0.068	0.106	0.930	1.461	0.190	0.298
0.570	0.895	0.070	0.110	0.940	1.477	0.195	0.306
0.580	0.911	0.072	0.113	0.950	1.492	0.200	0.314
0.590	0.927	0.074	0.117	0.960	1.508	0.206	0.323
0.600	0.942	0.077	0.120	0.970	1.524	0.211	0.332
0.610	0.958	0.079	0.124	0.980	1.539	0.217	0.341
0.620	0.974	0.081	0.128	0.990	1.555	0.223	0.350
0.630	0.990	0.084	0.131	1.000	1.571	0.229	0.360

Table 5.4

(continuation)

$Z_c = 65\Omega$

$\frac{2}{\pi}\theta'$	θ'	ε_r	$\frac{2}{\pi}\theta'$	θ'	ε_r
0.158	0.248	2.300	0.500	0.786	2.497
0.160	0.252	2.300	0.510	0.800	2.501
0.170	0.267	2.301	0.520	0.816	2.506
0.180	0.283	2.302	0.530	0.832	2.509
0.190	0.299	2.305	0.540	0.848	2.513
0.200	0.314	2.308	0.550	0.864	2.516
0.210	0.329	2.311	0.560	0.880	2.519
0.220	0.345	2.316	0.570	0.895	2.522
0.230	0.361	2.321	0.580	0.911	2.524
0.240	0.377	2.326	0.590	0.926	2.525
0.250	0.393	2.332	0.600	0.942	2.527
0.260	0.408	2.338	0.610	0.958	2.527
0.270	0.424	2.344	0.620	0.973	2.528
0.280	0.440	2.351	0.630	0.989	2.528
0.290	0.455	2.357	0.640	1.005	2.527
0.300	0.472	2.365	0.650	1.022	2.526
0.310	0.487	2.372	0.660	1.037	2.525
0.320	0.503	2.379	0.670	1.052	2.523
0.330	0.519	2.386	0.680	1.069	2.521
0.340	0.534	2.393	0.690	1.084	2.518
0.350	0.551	2.401	0.700	1.100	2.515
0.360	0.566	2.408	0.710	1.116	2.511
0.370	0.581	2.415	0.720	1.131	2.507
0.380	0.597	2.422	0.730	1.147	2.503
0.390	0.612	2.430	0.740	1.163	2.497
0.400	0.629	2.437	0.750	1.178	2.492
0.410	0.644	2.444	0.760	1.194	2.486
0.420	0.659	2.450	0.770	1.209	2.479
0.430	0.676	2.457	0.780	1.225	2.472
0.440	0.691	2.463	0.790	1.240	2.465
0.450	0.707	2.469	0.800	1.257	2.457
0.460	0.722	2.475	0.810	1.272	2.449
0.470	0.738	2.481	0.820	1.289	2.439
0.480	0.753	2.487	0.827	1.299	2.434
0.490	0.769	2.492			

Table 5.4

(continuation)

$Z_c = 75\Omega$

$\frac{2}{\pi}\theta'$	θ'	ε_r	$\frac{2}{\pi}\theta'$	θ'	ε_r
0.097	0.153	2.300	0.420	0.660	2.714
0.100	0.157	2.300	0.430	0.676	2.730
0.110	0.173	2.302	0.440	0.691	2.744
0.120	0.188	2.305	0.450	0.707	2.759
0.130	0.205	2.310	0.460	0.722	2.773
0.140	0.220	2.316	0.470	0.739	2.788
0.150	0.236	2.324	0.480	0.753	2.800
0.160	0.252	2.332	0.490	0.770	2.814
0.170	0.267	2.341	0.500	0.786	2.827
0.180	0.283	2.352	0.510	0.801	2.839
0.190	0.299	2.363	0.520	0.816	2.850
0.200	0.314	2.375	0.530	0.833	2.862
0.210	0.329	2.387	0.540	0.848	2.873
0.220	0.345	2.400	0.550	0.864	2.884
0.230	0.362	2.415	0.560	0.880	2.894
0.240	0.377	2.428	0.570	0.895	2.903
0.250	0.392	2.443	0.580	0.911	2.912
0.260	0.408	2.458	0.590	0.927	2.921
0.270	0.424	2.473	0.600	0.943	2.928
0.280	0.440	2.489	0.610	0.959	2.936
0.290	0.456	2.505	0.620	0.974	2.942
0.300	0.472	2.521	0.630	0.990	2.948
0.310	0.486	2.536	0.640	1.006	2.953
0.320	0.503	2.553	0.650	1.021	2.958
0.330	0.518	2.569	0.660	1.037	2.962
0.340	0.534	2.585	0.670	1.052	2.965
0.350	0.550	2.602	0.680	1.068	2.967
0.360	0.566	2.619	0.690	1.084	2.969
0.370	0.580	2.634	0.700	1.100	2.971
0.380	0.597	2.651	0.710	1.115	2.971
0.390	0.613	2.667	0.720	1.131	2.971
0.400	0.628	2.682	0.730	1.147	2.970
0.410	0.644	2.698	0.732	1.149	2.970

Table 5.4

(continuation)

$Z_c = 85\Omega$

$\frac{2}{\pi}\theta'$	θ'	ε_r	$\frac{2}{\pi}\theta'$	θ'	ε_r
0.045	0.071	2.300	0.240	0.378	2.767
0.050	0.079	2.301	0.250	0.392	2.801
0.060	0.095	2.305	0.260	0.409	2.840
0.070	0.110	2.314	0.270	0.424	2.878
0.080	0.126	2.326	0.280	0.440	2.915
0.090	0.142	2.341	0.290	0.456	2.955
0.100	0.157	2.357	0.300	0.471	2.994
0.110	0.174	2.378	0.310	0.487	3.035
0.120	0.189	2.399	0.320	0.502	3.072
0.130	0.205	2.422	0.330	0.518	3.112
0.140	0.220	2.446	0.340	0.533	3.151
0.150	0.236	2.474	0.350	0.550	3.193
0.160	0.251	2.500	0.360	0.565	3.232
0.170	0.267	2.530	0.370	0.581	3.275
0.180	0.283	2.561	0.380	0.597	3.315
0.190	0.299	2.594	0.390	0.613	3.355
0.200	0.315	2.626	0.400	0.628	3.393
0.210	0.329	2.658	0.410	0.644	3.434
0.220	0.346	2.695	0.420	0.660	3.475
0.230	0.361	2.728	0.424	0.667	3.490

Table 5.5

(continuation)

$\varepsilon_{r_0} = 3.00$

$\frac{2}{\pi}\theta_0$	θ_0	Z_c	ε_{r_1}	$\frac{2}{\pi}\theta_0$	θ_0	Z_c	ε_{r_1}
0.330	0.518	79.65	3.153	0.670	1.052	32.60	2.728
0.340	0.534	77.77	3.367	0.680	1.068	31.52	2.729
0.350	0.550	75.94	3.800	0.690	1.084	30.45	2.732
0.360	0.565	74.16	4.070	0.700	1.100	29.38	2.736
0.370	0.581	72.43	4.192	0.710	1.115	28.33	2.741
0.380	0.597	70.73	4.204	0.720	1.131	27.28	2.747
0.390	0.613	69.07	4.144	0.730	1.147	26.25	2.754
0.400	0.628	67.45	4.044	0.740	1.162	25.22	2.762
0.410	0.644	65.86	3.926	0.750	1.178	24.19	2.770
0.420	0.660	64.31	3.803	0.760	1.194	23.18	2.780
0.430	0.675	62.79	3.684	0.770	1.210	22.16	2.790
0.440	0.691	61.30	3.571	0.780	1.225	21.16	2.801
0.450	0.707	59.83	3.467	0.790	1.241	20.16	2.812
0.460	0.723	58.39	3.373	0.800	1.257	19.17	2.823
0.470	0.738	56.98	3.288	0.810	1.272	18.18	2.835
0.480	0.754	55.59	3.212	0.820	1.288	17.20	2.847
0.490	0.770	54.23	3.144	0.830	1.304	16.22	2.859
0.500	0.785	52.88	3.084	0.840	1.319	15.24	2.871
0.510	0.801	51.56	3.030	0.850	1.335	14.27	2.883
0.520	0.817	50.26	2.983	0.860	1.351	13.30	2.895
0.530	0.833	48.97	2.942	0.870	1.367	12.34	2.907
0.540	0.848	47.71	2.905	0.880	1.382	11.38	2.919
0.550	0.864	46.46	2.873	0.890	1.398	10.42	2.930
0.560	0.880	45.23	2.845	0.900	1.414	9.46	2.940
0.570	0.895	44.01	2.821	0.910	1.429	8.51	2.951
0.580	0.911	42.81	2.801	0.920	1.445	7.56	2.960
0.590	0.927	41.63	2.783	0.930	1.461	6.61	2.969
0.600	0.942	40.46	2.769	0.940	1.477	5.66	2.976
0.610	0.958	39.30	2.756	0.950	1.492	4.72	2.983
0.620	0.974	38.15	2.747	0.960	1.508	3.77	2.989
0.630	0.990	37.02	2.739	0.970	1.524	2.83	2.994
0.640	1.005	35.90	2.734	0.980	1.539	1.89	2.997
0.650	1.021	34.79	2.730	0.990	1.555	0.94	2.999
0.660	1.037	33.69	2.728	1.000	1.571	0.00	3.000

Table 5.5

(continuation)

$\varepsilon_{r_0} = 5.00$

$\frac{2}{\pi}\theta_0$	θ_0	Z_c	ε_{r_1}	$\frac{2}{\pi}\theta_0$	θ_0	Z_c	ε_{r_1}
0.460	0.723	58.39	5.229	0.740	1.162	25.22	4.577
0.470	0.738	56.98	5.341	0.750	1.178	24.19	4.581
0.480	0.754	55.59	5.822	0.760	1.194	23.18	4.589
0.490	0.770	54.23	6.111	0.770	1.210	22.16	4.600
0.500	0.785	52.88	6.236	0.780	1.225	21.16	4.613
0.510	0.801	51.56	6.239	0.790	1.241	20.16	4.628
0.520	0.817	50.26	6.162	0.800	1.257	19.17	4.646
0.530	0.833	48.97	6.039	0.810	1.272	18.18	4.665
0.540	0.848	47.71	5.894	0.820	1.288	17.20	4.686
0.550	0.864	46.46	5.742	0.830	1.304	16.22	4.708
0.560	0.880	45.23	5.594	0.840	1.319	15.24	4.730
0.570	0.895	44.01	5.453	0.850	1.335	14.27	4.754
0.580	0.911	42.81	5.324	0.860	1.351	13.30	4.777
0.590	0.927	41.63	5.207	0.870	1.367	12.34	4.801
0.600	0.942	40.46	5.103	0.880	1.382	11.38	4.825
0.610	0.958	39.30	5.010	0.890	1.398	10.42	4.848
0.620	0.974	38.15	4.929	0.900	1.414	9.46	4.870
0.630	0.990	37.02	4.859	0.910	1.429	8.51	4.892
0.640	1.005	35.90	4.798	0.920	1.445	7.56	4.912
0.650	1.021	34.79	4.746	0.930	1.461	6.61	4.931
0.660	1.037	33.69	4.703	0.940	1.477	5.66	4.948
0.670	1.052	32.60	4.667	0.950	1.492	4.72	4.963
0.680	1.068	31.52	4.638	0.960	1.508	3.77	4.976
0.690	1.084	30.45	4.615	0.970	1.524	2.83	4.986
0.700	1.100	29.38	4.598	0.980	1.539	1.89	4.994
0.710	1.115	28.33	4.586	0.990	1.555	0.94	4.998
0.720	1.131	27.28	4.579	1.000	1.571	0.00	5.000
0.730	1.147	26.25	4.576				

Table 5.5

(continuation)

$\varepsilon_{r_0} = 10.00$

$\frac{2}{\pi}\theta_0$	θ_0	Z_c	ε_{r_1}
0.610	0.958	39.30	10.002
0.620	0.974	38.15	10.717
0.630	0.990	37.02	11.122
0.640	1.005	35.90	11.274
0.650	1.021	34.79	11.251
0.660	1.037	33.69	11.118
0.670	1.052	32.60	10.928
0.680	1.068	31.52	10.714
0.690	1.084	30.45	10.499
).700	1.100	29.38	10.296
0.710	1.115	28.33	10.112
0.720	1.131	27.28	9.950
0.730	1.147	26.25	9.810
0.740	1.162	25.22	9.693
0.750	1.178	24.19	9.597
0.760	1.194	23.18	9.522
0.770	1.210	22.16	9.465
0.780	1.225	21.16	9.424
0.790	1.241	20.16	9.399
0.800	1.257	19.17	9.387

$\frac{2}{\pi}\theta_0$	θ_0	Z_c	ε_{r_1}
0.810	1.272	18.18	9.387
0.820	1.288	17.20	9.397
0.830	1.304	16.22	9.417
0.840	1.319	15.24	9.444
0.850	1.335	14.27	9.477
0.860	1.351	13.30	9.516
0.870	1.367	12.34	9.558
0.880	1.382	11.38	9.603
0.890	1.398	10.42	9.650
0.900	1.414	9.46	9.698
0.910	1.429	8.51	9.745
0.920	1.445	7.56	9.791
0.930	1.461	6.61	9.834
0.940	1.477	5.66	9.874
0.950	1.492	4.72	9.910
0.960	1.508	3.77	9.941
0.970	1.524	2.83	9.966
0.980	1.539	1.89	9.984
0.990	1.555	0.94	9.996
1.000	1.571	0.00	10.000

Table 5.6

(continuation)

$Z_c = 80\Omega$

Ψ/r_0	z/r_0
0.5000	0.8736
0.6000	0.9186
0.7000	0.9544
0.8000	0.9827
0.9000	1.0046
1.0000	1.0209
1.1000	1.0322
1.2000	1.0391
1.3000	1.0420
1.4000	1.0411
1.5000	1.0367
1.6000	1.0291
1.7000	1.0184
1.8000	1.0048
1.9000	0.9885
2.0000	0.9696
2.1000	0.9481
2.2000	0.9242
2.3000	0.8981
2.4000	0.8696
2.5000	0.8391
2.6000	0.8064
2.7000	0.7717
2.8000	0.7350
2.9000	0.6963
3.0000	0.6559
3.1000	0.6135
3.2000	0.5694
3.3000	0.5236
3.4000	0.4760
3.5000	0.4267
3.6000	0.3758
3.7000	0.3233
3.8000	0.2692
3.9000	0.2135
4.0000	0.1563
4.1000	0.0976

$Z_c = 90\Omega$

Ψ/r_0	z/r_0
0.5000	0.9471
0.6000	0.9928
0.7000	1.0292
0.8000	1.0579
0.9000	1.0801
1.0000	1.0966
1.1000	1.1081
1.2000	1.1150
1.3000	1.1179
1.4000	1.1169
1.5000	1.1124
1.6000	1.1046
1.7000	1.0938
1.8000	1.0799
1.9000	1.0633
2.0000	1.0440
2.1000	1.0222
2.2000	0.9979
2.3000	0.9712
2.4000	0.9423
2.5000	0.9111
2.6000	0.8778
2.7000	0.8425
2.8000	0.8051
2.9000	0.7657
3.0000	0.7244
3.1000	0.6813
3.2000	0.6363
3.3000	0.5895
3.4000	0.5409
3.5000	0.4906
3.6000	0.4387
3.7000	0.3850
3.8000	0.3297
3.9000	0.2729
4.0000	0.2144
4.1000	0.1543
4.2000	0.0928
4.3000	0.0297

Table 5.7

(continuation)

$\varepsilon_{r_0} = 3.00$

Ψ/r_0	z/r_0	Ψ/r_0	z/r_0
0.6500	0.7623	2.5000	0.7181
0.7000	0.7791	2.5500	0.7046
0.7500	0.7942	2.6000	0.6907
0.8000	0.8077	2.6500	0.6762
0.8500	0.8198	2.7000	0.6614
0.9000	0.8305	2.7500	0.6461
0.9500	0.8399	2.8000	0.6304
1.0000	0.8481	2.8500	0.6142
1.0500	0.8551	2.9000	0.5976
1.1000	0.8611	2.9500	0.5806
1.1500	0.8660	3.0000	0.5632
1.2000	0.8699	3.0500	0.5454
1.2500	0.8730	3.1000	0.5272
1.3000	0.8751	3.1500	0.5086
1.3500	0.8763	3.2000	0.4896
1.4000	0.8767	3.2500	0.4702
1.4500	0.8763	3.3000	0.4504
1.5000	0.8751	3.3500	0.4303
1.5500	0.8732	3.4000	0.4097
1.6000	0.8705	3.4500	0.3888
1.6500	0.8672	3.5000	0.3676
1.7000	0.8632	3.5500	0.3460
1.7500	0.8584	3.6000	0.3240
1.8000	0.8531	3.6500	0.3017
1.8500	0.8471	3.7000	0.2790
1.9000	0.8405	3.7500	0.2560
1.9500	0.8333	3.8000	0.2326
2.0000	0.8255	3.8500	0.2090
2.0500	0.8172	3.9000	0.1849
2.1000	0.8083	3.9500	0.1606
2.1500	0.7988	4.0000	0.1359
2.2000	0.7888	4.0500	0.1108
2.2500	0.7783	4.1000	0.0855
2.3000	0.7672	4.1500	0.0598
2.3500	0.7557	4.2000	0.0339
2.4000	0.7436	4.2500	0.0076
2.4500	0.7311		

Table 5.7

(continuation)

$\varepsilon_{r_0} = 5.00$

Ψ/r_0	z/r_0	Ψ/r_0	z/r_0
0.6500	0.7625	3.3000	0.8804
0.7000	0.7855	3.3500	0.8701
0.7500	0.8068	3.4000	0.8596
0.8000	0.8266	3.4500	0.8488
0.8500	0.8451	3.5000	0.8376
0.9000	0.8623	3.5500	0.8262
0.9500	0.8783	3.6000	0.8145
1.0000	0.8932	3.6500	0.8024
1.0500	0.9070	3.7000	0.7901
1.1000	0.9198	3.7500	0.7775
1.1500	0.9316	3.8000	0.7647
1.2000	0.9426	3.8500	0.7515
1.2500	0.9527	3.9000	0.7381
1.3000	0.9620	3.9500	0.7244
1.3500	0.9704	4.0000	0.7105
1.4000	0.9781	4.0500	0.6962
1.4500	0.9851	4.1000	0.6817
1.5000	0.9914	4.1500	0.6670
1.5500	0.9970	4.2000	0.6520
1.6000	1.0019	4.2500	0.6368
1.6500	1.0063	4.3000	0.6212
1.7000	1.0100	4.3500	0.6055
1.7500	1.0131	4.4000	0.5895
1.8000	1.0156	4.4500	0.5733
1.8500	1.0175	4.5000	0.5568
1.9000	1.0190	4.5500	0.5401
1.9500	1.0199	4.6000	0.5231
2.0000	1.0202	4.6500	0.5059
2.0500	1.0201	4.7000	0.4885
2.1000	1.0195	4.7500	0.4708
2.1500	1.0184	4.8000	0.4529
2.2000	1.0168	4.8500	0.4348
2.2500	1.0147	4.9000	0.4165
2.3000	1.0123	4.9500	0.3979
2.3500	1.0093	5.0000	0.3791
2.4000	1.0060	5.0500	0.3601
2.4500	1.0022	5.1000	0.3409
2.5000	0.9980	5.1500	0.3215
2.5500	0.9934	5.2000	0.3018
2.6000	0.9884	5.2500	0.2820
2.6500	0.9830	5.3000	0.2619
2.7000	0.9773	5.3500	0.2416
2.7500	0.9711	5.4000	0.2211
2.8000	0.9646	5.4500	0.2005
2.8500	0.9577	5.5000	0.1796
2.9000	0.9505	5.5500	0.1585
2.9500	0.9429	5.6000	0.1372
3.0000	0.9350	5.6500	0.1157
3.0500	0.9267	5.7000	0.0940
3.1000	0.9181	5.7500	0.0721
3.1500	0.9091	5.8000	0.0500
3.2000	0.8999	5.8500	0.0277
3.2500	0.8903	5.9000	0.0052

Table 5.7

(continuation)

$\varepsilon_{r_0} = 10.00$

Ψ/r_0	z/r_0	Ψ/r_0	z/r_0	Ψ/r_0	z/r_0	Ψ/r_0	z/r_0
0.6500	0.7628	3.2000	1.4254	5.7500	1.2952	8.3000	0.7714
0.7000	0.7933	3.2500	1.4281	5.8000	1.2882	8.3500	0.7582
0.7500	0.8224	3.3000	1.4305	5.8500	1.2810	8.4000	0.7448
0.8000	0.8502	3.3500	1.4326	5.9000	1.2737	8.4500	0.7313
0.8500	0.8768	3.4000	1.4345	5.9500	1.2662	8.5000	0.7177
0.9000	0.9023	3.4500	1.4362	6.0000	1.2586	8.5500	0.7041
0.9500	0.9267	3.5000	1.4376	6.0500	1.2508	8.6000	0.6903
1.0000	0.9501	3.5500	1.4388	6.1000	1.2429	8.6500	0.6764
1.0500	0.9727	3.6000	1.4398	6.1500	1.2349	8.7000	0.6624
1.1000	0.9943	3.6500	1.4405	6.2000	1.2267	8.7500	0.6483
1.1500	1.0151	3.7000	1.4410	6.2500	1.2184	8.8000	0.6342
1.2000	1.0351	3.7500	1.4413	6.3000	1.2100	8.8500	0.6199
1.2500	1.0544	3.8000	1.4413	6.3500	1.2014	8.9000	0.6055
1.3000	1.0730	3.8500	1.4412	6.4000	1.1927	8.9500	0.5910
1.3500	1.0908	3.9000	1.4408	6.4500	1.1838	9.0000	0.5764
1.4000	1.1080	3.9500	1.4402	6.5000	1.1749	9.0500	0.5618
1.4500	1.1246	4.0000	1.4394	6.5500	1.1657	9.1000	0.5470
1.5000	1.1406	4.0500	1.4384	6.6000	1.1565	9.1500	0.5321
1.5500	1.1560	4.1000	1.4371	6.6500	1.1472	9.2000	0.5172
1.6000	1.1708	4.1500	1.4357	6.7000	1.1377	9.2500	0.5021
1.6500	1.1851	4.2000	1.4341	6.7500	1.1281	9.3000	0.4870
1.7000	1.1988	4.2500	1.4323	6.8000	1.1183	9.3500	0.4717
1.7500	1.2121	4.3000	1.4303	6.8500	1.1084	9.4000	0.4564
1.8000	1.2248	4.3500	1.4280	6.9000	1.0985	9.4500	0.4410
1.8500	1.2371	4.4000	1.4256	6.9500	1.0883	9.5000	0.4255
1.9000	1.2489	4.4500	1.4230	7.0000	1.0781	9.5500	0.4099
1.9500	1.2602	4.5000	1.4203	7.0500	1.0677	9.6000	0.3941
2.0000	1.2711	4.5500	1.4173	7.1000	1.0573	9.6500	0.3784
2.0500	1.2816	4.6000	1.4142	7.1500	1.0467	9.7000	0.3625
2.1000	1.2917	4.6500	1.4108	7.2000	1.0360	9.7500	0.3465
2.1500	1.3014	4.7000	1.4073	7.2500	1.0251	9.8000	0.3304
2.2000	1.3107	4.7500	1.4036	7.3000	1.0142	9.8500	0.3143
2.2500	1.3196	4.8000	1.3998	7.3500	1.0031	9.9000	0.2980
2.3000	1.3281	4.8500	1.3957	7.4000	0.9919	9.9500	0.2817
2.3500	1.3362	4.9000	1.3915	7.4500	0.9806	10.0000	0.2653
2.4000	1.3440	4.9500	1.3872	7.5000	0.9692	10.0500	0.2487
2.4500	1.3514	5.0000	1.3826	7.5500	0.9577	10.1000	0.2321
2.5000	1.3585	5.0500	1.3779	7.6000	0.9460	10.1500	0.2155
2.5500	1.3653	5.1000	1.3730	7.6500	0.9343	10.2000	0.1987
2.6000	1.3717	5.1500	1.3680	7.7000	0.9224	10.2500	0.1818
2.6500	1.3778	5.2000	1.3628	7.7500	0.9104	10.3000	0.1649
2.7000	1.3836	5.2500	1.3574	7.8000	0.8983	10.3500	0.1478
2.7500	1.3891	5.3000	1.3519	7.8500	0.8861	10.4000	0.1307
2.8000	1.3943	5.3500	1.3462	7.9000	0.8738	10.4500	0.1135
2.8500	1.3992	5.4000	1.3404	7.9500	0.8614	10.5000	0.0962
2.9000	1.4038	5.4500	1.3344	8.0000	0.8488	10.5500	0.0789
2.9500	1.4081	5.5000	1.3282	8.0500	0.8362	10.6000	0.0614
3.0000	1.4121	5.5500	1.3219	8.1000	0.8235	10.6500	0.0439
3.0500	1.4158	5.6000	1.3155	8.1500	0.8106	10.7000	0.0262
3.1000	1.4193	5.6500	1.3089	8.2000	0.7977	10.7500	0.0085
3.1500	1.4225	5.7000	1.3021	8.2500	0.7846		

D. *Length of Boundary Curve*

We conclude this section with a derivation of an exact formula for the length of the curved boundary of the lens. The length is given by a rather complicated definite integral which can be evaluated numerically. We proceed with our derivation. From the geometry indicated in figure 3.2 we have:

$$\Psi = r\,cos(\frac{\pi}{2} - \theta) = r\,sin(\theta)$$

$$z = r\,sin(\frac{\pi}{2} - \theta) = r\,cos(\theta) \tag{5.51}$$

Hence, since $(ds)^2 = (d\Psi)^2 + (dz)^2$, we obtain

$$(ds)^2 = r^2(d\theta)^2 + (dr)^2 \tag{5.52}$$

and so if r can be expressed as a function of θ along the boundary curve we can determine its length. Since (5.18) can be reexpressed in the form

$$cot(\theta) - cot(\theta') = \frac{sin(\theta-\theta')}{sin(\theta)\cdot sin(\theta')} = cot(\theta_1')\cdot \tan^{\alpha}(\frac{\theta}{2}) \tag{5.53}$$

the law of sines (see(3.25)) leads us to

$$r = \frac{l\,cot^{\alpha}(\theta/2)}{cot(\theta_1')\cdot sin(\theta)} \tag{5.54}$$

where $\alpha = -L/l$ and θ_1' is the angle corresponding to $\theta_1 = \frac{\pi}{2}$. The preceding result can also be written as

$$r = \frac{l}{cot(\theta_1')}\left(\frac{1+cos(\theta)}{sin(\theta)}\right)^{\alpha}\frac{1}{sin(\theta)} . \tag{5.55}$$

Since

$$\frac{dr}{d\theta} = \frac{1}{cot(\theta_1')}\left(\frac{1+cos(\theta)}{sin(\theta)}\right)^{\alpha}\frac{[L-l\,cos(\theta)]}{\sin^2(\theta)} \tag{5.56}$$

we then find by a straightforward calculation that

$$(ds)^2 = \frac{[L^2+l^2-2lL\,cos(\theta)]}{cot^2(\theta_1')\cdot\sin^4(\theta)}\left(\frac{1+\cos(\theta)}{sin(\theta)}\right)^{2\alpha}(d\theta)^2 . \tag{5.57}$$

Thus the total length of the boundary Γ , denoted by $||\Gamma||$, is given by integration of (5.57) over the range $\theta_0 \leq \theta \leq \pi/2$. The result is

VI. APPROXIMATION OF EXACT CASE BY UNIFORM ISOTROPIC ϵ_r

Now that we have found the exact solution to our lens problem involving an $\epsilon_r(\theta')$, let us recall that our constraint of zero electric field in the r' direction requires an anisotropic medium which is in effect perfectly conducting (or has an infinite dielectric constant) in the r' direction. Practically, this can be realized by a set of (thin) conducting sheets in the form of circular cones which are surfaces of constant θ' ; these sheets should be closely spaced in terms of wavelength, particularly in regions where the required $\frac{d\epsilon_r(\theta')}{d\theta'}$ is not small. By extending ridges of thin metal from these surfaces in the $\pm\theta'$ direction (surfaces of constant r') the electric field can be loaded to increase the effective ϵ_r by what are known as artificial-dielectric techniques. No conducting sheets should be placed on surfaces of constant ϕ or otherwise to interfere with H_ϕ because such would decrease the effective permeability which we have assumed to be a constant.

In this section we address the design problem of obtaining an average value of ϵ_r which approximates in some sense the exact case. For a given set of initial conditions (i.e., given ϵ_{r_0} and Z_c) we will have an inequality

$$\epsilon_{r_{min}} \leq \epsilon_{r_{av}} \leq \epsilon_{r_{max}} \tag{6.1}$$

where typically $\epsilon_{r_{min}} = \epsilon_{r_0}$, and $\epsilon_{r_{max}}$ is determined by the initial data. Let us choose the value of $\epsilon_{r_{av}}$ to be determined by matching the impedances in the lens and free-space conical systems. Thus, if ϵ_{r_0} and Z_c are specified, we then know θ_0 and θ_0' from (3.19) and (3.21). Hence if $\theta_1 = \frac{\pi}{2}$, then the corresponding values of ϵ_{r_1} and θ_1' are known. Thus, we seek a value $\epsilon_{r_{av}}$ which satisfies the relations

$$Z_c = \frac{Z_0}{2\pi}\, ln\left[cot\left(\frac{\theta_0}{2}\right)\right] = \frac{1}{\sqrt{\epsilon_{r_{av}}}}\,\frac{Z_0}{2\pi}\, ln\left[\frac{cot\left(\frac{\theta_0'}{2}\right)}{cot\left(\frac{\theta_1'}{2}\right)}\right] . \tag{6.2}$$

It is important to realize that ϵ_{r_1} and $\epsilon_{r_{max}}$ need not in general be equal unless $\epsilon_r(\theta')$ is a non-decreasing function of θ' for the range of interest. For impedances $Z_c \geq 70\Omega$ and for $\epsilon_{r_0} = 2.3$ we do actually have $\epsilon_{r_1} = \epsilon_{r_{max}}$ (see table 6.1 and figure 5.2). We note that the condition (see (3.13)) that $\epsilon_r(\theta')$ be increasing is the condition that

$$\frac{2\sqrt{\epsilon_r}}{1 + \epsilon_r} \geq cos(\theta - \theta') . \tag{6.3}$$

This condition will in general not be realized over the entire range $\theta_0' \leq \theta' \leq \theta_1'$. For example, when $\epsilon_{r_0} = \epsilon_{r_{min}} = 2.3$ and $Z_c = 60\Omega$, calculations using the data from table 5.8 shows that condition (6.3) holds only for $(.300) \leq \theta' \leq (.910)$, while the range of interest for θ' is $(.300) \leq \theta' \leq 1.226$. For these values of ϵ_{r_0} and Z_c we have $\epsilon_{r_{max}} = 2.42$, while $\epsilon_{r_1} = 2.34$. The value of $\epsilon_{r_{av}}$, from (6.2) is 2.36. This example and an examination of table 6.1 suggest that, since the variation of ϵ_r for some ranges of θ_0 and

$$\sqrt{\epsilon_{r_1}} = \sqrt{1+a_1^2}\left[sin(\theta_1{}' + \gamma_1)\right] \tag{6.7}$$

where $\gamma_1 = arctan(a_1)$. Hence by a easy calculation which uses the obvious inequality $sin(\theta_1{}' + \gamma_1) \leq 1$ and the equations

$$2\sqrt{\epsilon_{r_0}} = (\epsilon_{r_0}+1)cos(\theta_0-\theta_0{}')$$

$$(\epsilon_{r_0}-1) = (\epsilon_{r_0}+1)sin(\theta_0-\theta_0{}') \tag{6.8}$$

we obtain

$$\sqrt{\epsilon_{r_1}} \leq \sqrt{3+\epsilon_{r_0}\sin(\theta_0)cos(\theta_0{}')}\,. \tag{6.9}$$

Thus, we find that

$$\epsilon_{r_1} \leq 3 + \epsilon_{r_0}(sech(\frac{2\pi Z_c}{Z_0})) \tag{6.10}$$

gives a rough upper bound for ϵ_{r_1} for a given ϵ_{r_0} and Z_c.

Table 6.1

(continuation)

$\varepsilon_{r_0} = 3.00$

Z_c	θ_0	θ_0'	θ_1'	ε_{r_1}	$\varepsilon_{r_{max}}$	$\varepsilon_{r_{avg}}$	Percent of r_0 transit time error
50.7350	0.8110	0.2874	1.1315	3.0000	3.0587	3.12	1.43
51.0000	0.8078	0.2842	1.1289	3.0096	3.0620	3.12	1.71
52.0000	0.7959	0.2723	1.1187	3.0476	3.0747	3.15	2.00
53.0000	0.7840	0.2604	1.1076	3.0888	3.0879	3.17	2.56
54.0000	0.7723	0.2487	1.0954	3.1335	3.1015	3.20	2.84
55.0000	0.7608	0.2372	1.0820	3.1816	3.1156	3.23	3.40
56.0000	0.7493	0.2257	1.0675	3.2334	3.1302	3.27	3.67
57.0000	0.7380	0.2144	1.0516	3.2890	3.1450	3.32	4.23
58.0000	0.7269	0.2033	1.0342	3.3484	3.1601	3.36	4.50
59.0000	0.7159	0.1923	1.0153	3.4116	3.1753	3.42	5.05
60.0000	0.7050	0.1814	0.9945	3.4786	3.1905	3.48	5.33
61.0000	0.6943	0.1707	0.9719	3.5493	3.2056	3.55	5.86
62.0000	0.6837	0.1601	0.9471	3.6233	3.2204	3.62	6.13
63.0000	0.6732	0.1496	0.9199	3.7001	3.2346	3.70	6.67
64.0000	0.6629	0.1393	0.8902	3.7788	3.2480	3.78	6.93
65.0000	0.6527	0.1291	0.8577	3.8583	3.2603	3.86	7.20
66.0000	0.6427	0.1191	0.8220	3.9368	3.2710	3.94	7.47
67.0000	0.6327	0.1091	0.7829	4.0122	3.2798	4.01	7.73
68.0000	0.6230	0.0994	0.7402	4.0813	3.2862	4.08	7.99
69.0000	0.6133	0.0897	0.6934	4.1403	3.2894	4.14	7.99
70.0000	0.6038	0.0802	0.6424	4.1843	3.2889	4.18	7.99
71.0000	0.5944	0.0708	0.5870	4.2076	3.2840	4.21	7.73
72.0000	0.5851	0.0615	0.5270	4.2037	3.2738	4.20	7.47
73.0000	0.5760	0.0524	0.4624	4.1656	3.2576	4.17	7.20
74.0000	0.5669	0.0434	0.3933	4.0868	3.2347	4.09	6.67
75.0000	0.5581	0.0345	0.3202	3.9621	3.2042	3.96	5.86
76.0000	0.5493	0.0257	0.2435	3.7887	3.1658	3.79	4.77
77.0000	0.5407	0.0171	0.1642	3.5675	3.1191	3.57	3.40
78.0000	0.5321	0.0085	0.0831	3.3034	3.0641	3.30	1.71
79.0000	0.5237	0.0001	0.0014	3.0054	3.0012	3.00	0.00
79.0175	0.5236	0.0000	0.0000	3.0000	3.0003	3.00	0.00

Table 6.1

(continuation)

$\varepsilon_{r_0} = 7.00$

Z_c	θ_0	θ_0'	θ_1'	ε_{r_1}	$\varepsilon_{r_{max}}$	$\varepsilon_{r_{avg}}$	Percent of r_0 transit time error
33.0500	1.0459	0.1978	0.8120	6.9999	7.0844		1.50
33.0000	1.0466	0.1985	0.8130	6.9945	7.0828		1.50
34.0000	1.0322	0.1842	0.7930	7.1084	7.1151	7.23	2.25
35.0000	1.0180	0.1699	0.7699	7.2348	7.1486	7.30	2.80
36.0000	1.0039	0.1558	0.7433	7.3729	7.1828	7.40	3.36
37.0000	0.9899	0.1418	0.7127	7.5211	7.2168	7.53	4.09
38.0000	0.9760	0.1279	0.6778	7.6761	7.2496	7.68	4.64
39.0000	0.9623	0.1142	0.6378	7.8330	7.2796	7.83	5.19
40.0000	0.9486	0.1006	0.5922	7.9841	7.3049	7.98	5.55
41.0000	0.9352	0.0871	0.5404	8.1182	7.3234	8.12	5.91
42.0000	0.9218	0.0738	0.4816	8.2194	7.3319	8.22	6.09
43.0000	0.9086	0.0606	0.4154	8.2670	7.3272	8.27	6.09
44.0000	0.8955	0.0475	0.3413	8.2356	7.3051	8.24	5.73
45.0000	0.8826	0.0345	0.2592	8.0969	7.2614	8.10	4.82
46.0000	0.8698	0.0217	0.1694	7.8247	7.1916	7.82	3.54
47.0000	0.8571	0.0091	0.0729	7.4012	7.0920	7.40	1.69
47.7219	0.8481	0.0000	0.0000	7.0000	7.0001		0.00

$\varepsilon_{r_0} = 10.00$

Z_c	θ_0	θ_0'	θ_1'	ε_{r_1}	$\varepsilon_{r_{max}}$	$\varepsilon_{r_{avg}}$	Percent of r_0 transit time error
27.6240	1.1258	0.1676	0.6957	10.0000	10.0912		1.42
27.0000	1.1353	0.1770	0.7087	9.9092	10.0653	10.14	1.10
28.0000	1.1202	0.1620	0.6872	10.0585	10.1073	10.21	1.73
29.0000	1.1053	0.1470	0.6617	10.2271	10.1513	10.31	2.35
30.0000	1.0904	0.1322	0.6315	10.4127	10.1961	10.44	3.13
31.0000	1.0757	0.1175	0.5960	10.6102	10.2400	10.61	3.75
32.0000	1.0611	0.1028	0.5545	10.8114	10.2803	10.81	4.37
33.0000	1.0466	0.0884	0.5061	11.0028	10.3138	11.00	4.83
34.0000	1.0322	0.0740	0.4499	11.1638	10.3361	11.16	5.29
35.0000	1.0180	0.0597	0.3852	11.2654	10.3418	11.27	5.29
36.0000	1.0039	0.0456	0.3111	11.2686	10.3241	11.27	4.98
37.0000	0.9899	0.0316	0.2273	11.1259	10.2751	11.13	4.37
38.0000	0.9760	0.0178	0.1338	10.7881	10.1865	10.79	2.98
39.0000	0.9623	0.0040	0.0315	10.2157	10.0502	10.22	0.79
39.2940	0.9582	0.0000	0.0000	10.0000	9.9968		0.00

θ'	θ	ε_r
0.312	0.705027	2.2400
0.322	0.719020	2.2401
0.332	0.732816	2.2405
0.342	0.746423	2.2411
0.352	0.759845	2.2419
0.362	0.773089	2.2429
0.372	0.786160	2.2441
0.382	0.799063	2.2455
0.392	0.811803	2.2470
0.402	0.824385	2.2487
0.412	0.836813	2.2505
0.422	0.849090	2.2524
0.432	0.861222	2.2545
0.442	0.873212	2.2566
0.452	0.885063	2.2588
0.462	0.896779	2.2611
0.472	0.908364	2.2635
0.482	0.919821	2.2659
0.492	0.931153	2.2684
0.502	0.942363	2.2710
0.512	0.953454	2.2736
0.522	0.964429	2.2762
0.532	0.975291	2.2788
0.542	0.986042	2.2815
0.552	0.996685	2.2841
0.562	1.007222	2.2868
0.572	1.017657	2.2895
0.582	1.027990	2.2922
0.592	1.038226	2.2948
0.602	1.048365	2.2975
0.612	1.058410	2.3001
0.622	1.068364	2.3027
0.632	1.078228	2.3052
0.642	1.088004	2.3077
0.652	1.097694	2.3102
0.662	1.107301	2.3126
0.672	1.116826	2.3150
0.682	1.126271	2.3173
0.692	1.135637	2.3195
0.702	1.144927	2.3217
0.712	1.154142	2.3238
0.722	1.163284	2.3258
0.732	1.172355	2.3278
0.742	1.181356	2.3297
0.752	1.190289	2.3314
0.762	1.199154	2.3331
0.772	1.207955	2.3347
0.782	1.216692	2.3362
0.792	1.225366	2.3376
0.802	1.233980	2.3389
0.812	1.242534	2.3401
0.822	1.251031	2.3412
0.832	1.259470	2.3422
0.842	1.267854	2.3430
0.852	1.276184	2.3438
0.862	1.284461	2.3444
0.872	1.292686	2.3449
0.882	1.300860	2.3453
0.892	1.308986	2.3455
0.902	1.317064	2.3457
0.912	1.325094	2.3456
0.922	1.333079	2.3455
0.932	1.341019	2.3452
0.942	1.348916	2.3448
0.952	1.356770	2.3443
0.962	1.364583	2.3436
0.972	1.372356	2.3427
0.982	1.380089	2.3418
0.992	1.387784	2.3406
1.002	1.395442	2.3394
1.012	1.403063	2.3380
1.022	1.410649	2.3364
1.032	1.418201	2.3347
1.042	1.425719	2.3329
1.052	1.433205	2.3308
1.062	1.440660	2.3287
1.072	1.448083	2.3264
1.082	1.455477	2.3239
1.092	1.462842	2.3213
1.102	1.407179	2.3185
1.112	1.477489	2.3156
1.122	1.484773	2.3125
1.132	1.492031	2.3093
1.142	1.499264	2.3059
1.152	1.506474	2.3024
1.162	1.513660	2.2987
1.172	1.520825	2.2949
1.182	1.527968	2.2909
1.192	1.535090	2.2867
1.202	1.542192	2.2824
1.212	1.549275	2.2779
1.222	1.556340	2.2733
1.232	1.563387	2.2686
1.242	1.570418	2.2636
1.252	1.577432	2.2586

Table 6.3. Numerical solution to the system (5.50) with $\varepsilon_{r_0} = 2.24$ and $Z_c = 60\Omega$

REFERENCES

1. C.E. Baum, "A scaling technique for the design of idealized electromagnetic lens", Sensor and Simulation Note 64, August 1968, Kirtland AFB, AF Weapons Laboratory, Albuquerque, NM 87117.

2. A.P. Stone, "A differential geometric approach to electromagnetic lens design", Sensor and Simulation Note 282, April 1983, Kirtland AFB, AF Weapons Laboratory, Albuquerque, NM 87117. (also to appear under same title in Electromagnetics).

3. C.E. Baum, "A circular conical antenna simulator", Sensor and Simulation Note 36, March 1967, Kirtland AFB, AF Weapons Laboratory, Albuquerque, NM 87117.

4. C.E. Baum, "EMP simulators for various types of nuclear EMP environments: An interim categorization", Sensor and Simulation Note 240 (also in IEEE Transactions on Antennas and Propagation, Vol. AP-26, No. 1, Jan. 1978, pp. 35-53, and in IEEE Transactions on EM Compatibility, Feb. 1978).

5. T.C. Mo, C.H. Papas, and C.E. Baum, "Differential geometry scaling method for electromagnetic field and its applications to coaxial waveguide functions", Sensor and Simulation Note 169, March 1973 (also as a paper by the same authors under the title of "General scaling method for electromagnetic field with application to a matching problem", J. Math. Physics, Vol. 14, pp. 479-483, April 1973.

6. C.E. Baum, "The Brewster angle wave matcher", Sensor and Simulation Note 37, March 1967, Kirtland AFB, AF Weapons Laboratory, Albuquerque, NM 87117.