Sensor and Simulation Notes

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AN INVESTIGATION OF THE DETECTION OF THE SURFACE FIELDS OF A RIGHT-ANGLE CORNER REFLECTOR BY SOME ELECTRICALLY SMALL SENSORS

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ABSTRACT

This report deals with the surface fields or the surface charge and current densities on a right-angle corner reflector induced by a polarized uniform plane wave. Equivalent circuit parameters of a short monopole and a small semi-loop mounted on the wedge are derived and explicit correlations between measurable quantities and local surface fields are established.

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1. INTRODUCTION

The class of problems related to the measurements of electromagnetic field quantities have attracted the attention of many engineers engaged in sensor research. The major difficulty encountered in the measuring process of any energy related physical quantity is the interaction of the measuring device(s) with the physical field that always produces a perturbation of the field. Therefore it is essential to have an apriori estimate of the amount of the energy extracted by the sensor and the extent of the perturbation. In this report we will consider a canonical problem of this class, namely, the measurement of surface fields on a right-angle corner reflector. The characteristics of many sensors are discussed in [1], and some related problems are treated in [2].

The geometry of the problem is shown in Figure 1.1. The walls of the wedge are perfectly conducting and the medium of propagation of the waves is air with parameters (ϵ, μ, $\sigma = 0$). We will assume that the illuminating polarized uniform plane wave is propagating in a plane normal to the axis of the wedge. Furthermore only linearly polarized waves will be considered with polarization of the E-field either perpendicular or parallel to the axis of the wedge.

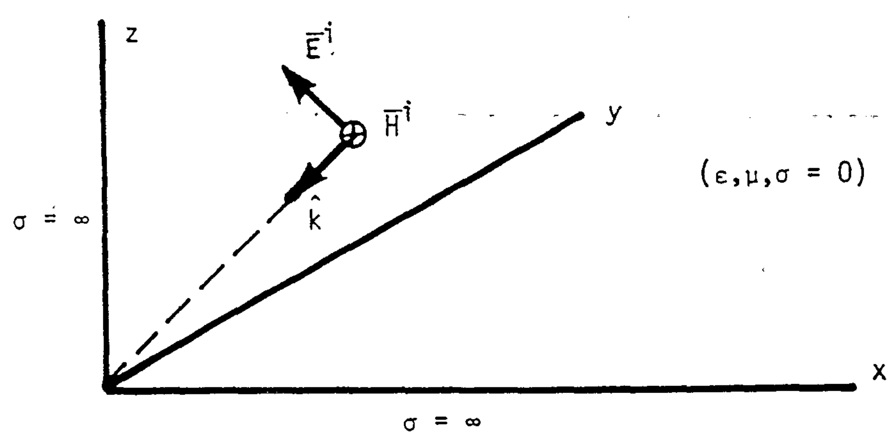

Figure 1.1. Plane wave illumination of a corner reflector.

2. SURFACE CURRENT AND CHARGE DENSITIES FOR POLARIZATION PERPENDICULAR TO THE AXIS OF THE WEDGE

To obtain expressions for surface current density $\overline{K}$ and surface charge density ρ_s, let us replace the above problem with an equivalent problem as shown in Figure 2.1. The time dependence $e^{j\omega t}$ will be understood throughout the report. Then

$$\overline{E}^i(\overline{R},t) = \mathrm{Re}\left[\overline{E}^i(\overline{R})e^{j\omega t}\right] = \mathrm{Re}\left[\overline{E}_0{}^i e^{j(\omega t - \overline{k}\cdot\overline{R})}\right] \tag{2.1}$$

with:

$$\overline{k} = k(-\sin\theta\ \hat{x} + -\cos\theta\ \hat{z}),\ k = \frac{\omega}{c},\quad c = \frac{1}{\sqrt{\mu\varepsilon}}$$

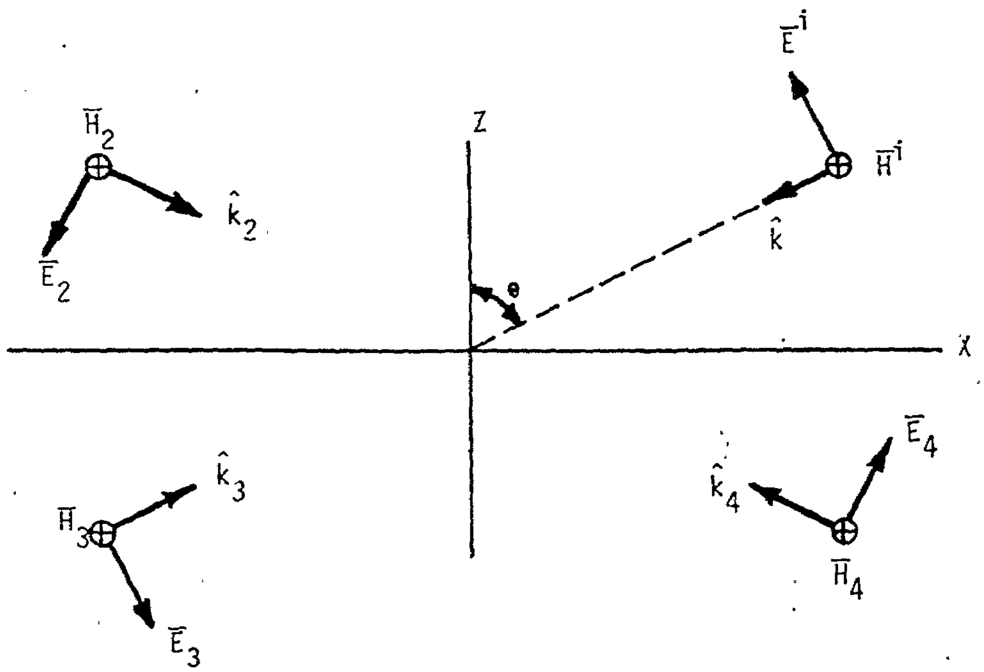

Figure 2.1. Equivalent problem obtained using image fields

$$
\left.
\begin{aligned}
|\overline{k}_\alpha| &= k \\
|\overline{E}_{\alpha o}| &= E_o{}^i \qquad \alpha = 2, 3, 4 \\
\hat{k}_2 &= \sin\theta\ \hat{x} - \cos\theta\ \hat{z} \\
\hat{k}_3 &= \sin\theta\ \hat{x} + \cos\theta\ \hat{z} \\
\hat{k}_4 &= -\sin\theta\ \hat{x} + \cos\theta\ \hat{z} \\
\overline{E}_o{}^i &= E_o{}^i\ (-\cos\theta\ \hat{x} + \sin\theta\ \hat{z}) \\
\overline{E}_{2o} &= E_{2o}\ (-\cos\theta\ \hat{x} - \sin\theta\ \hat{z}) \\
\overline{E}_{3o} &= E_{3o}\ (\cos\theta\ \hat{x} - \sin\theta\ \hat{z}) \\
\overline{E}_{4o} &= E_{4o}\ (\cos\theta\ \hat{x} + \sin\theta\ \hat{z})
\end{aligned}
\right\} \qquad (2.2)
$$

For surface current density we have

$$
\overline{K} = \hat{n} \times \overline{H}_{total} = \hat{n} \times \frac{1}{\eta}(\hat{k} \times \overline{E}^i + \hat{k}_2 \times \overline{E}_2 + \hat{k}_3 \times \overline{E}_3 + \hat{k}_4 \times \overline{E}_4)
$$

where

$$
\eta = \sqrt{(\mu/\varepsilon)}
$$

is the intrinsic impedance of air. On the surface $x > 0$, $z = 0$ we have

$$
\left.
\begin{aligned}
\overline{K}(\overline{R}) &= \hat{z} \times \frac{1}{\eta}\{(\hat{k} \times \overline{E}_o{}^i + \hat{k}_4 \times \overline{E}_{4o})e^{jkx\sin\theta} + (\hat{k}_2 \times \overline{E}_{2o} + \hat{k}_3 \times \overline{E}_{3o})e^{-jkx\sin\theta}\} \\
&= -\hat{x}\ 2H_o{}^i(e^{jkx\sin\theta} + e^{-jkx\sin\theta}) \\
&= -\hat{x}\ 4H_o{}^i\cos(kx\sin\theta);
\end{aligned}
\right\} (2.3)
$$

on the surface $x = 0$, $z > 0$ we have

$$
\overline{K}(\overline{R}) = \hat{x} \times \hat{y}\ H_o{}^i(2e^{jkz\cos\theta} + 2e^{-jkz\cos\theta}) = \hat{z}\ 4H_o{}^i\cos(kz\cos\theta) \qquad (2.4)
$$

To obtain the surface charge density ρ_s we will make use of a boundary condition derived from the continuity equation. When one of the media is a perfectly conducting surface this boundary condition reads:

$$\nabla \cdot \overline{K} = -j\omega\rho_s$$

Therefore, on the surface $x > 0$, $z = 0$, we have

$$\begin{aligned} \rho_s(\overline{R}) &= \frac{j}{\omega} 4H_0{}^i k \sin\theta \sin(kx\sin\theta) \\ \rho_s(\overline{\overline{R}}) &= j4\varepsilon E_0{}^i \sin\theta \sin(kx\sin\theta) \ ; \end{aligned} \tag{2.5}$$

similarly on the surface $x = 0$, $z > 0$ we have

$$\rho_s(\overline{R}) = -j4\varepsilon E_0{}^i \cos\theta \sin(kz\cos\theta). \tag{2.6}$$

2.1. Surface current and charge densities for polarizations parallel to the wedge axis

For this polarization as it is apparent from Figure 2.2 the following changes should be made in the formulation of the previous section:

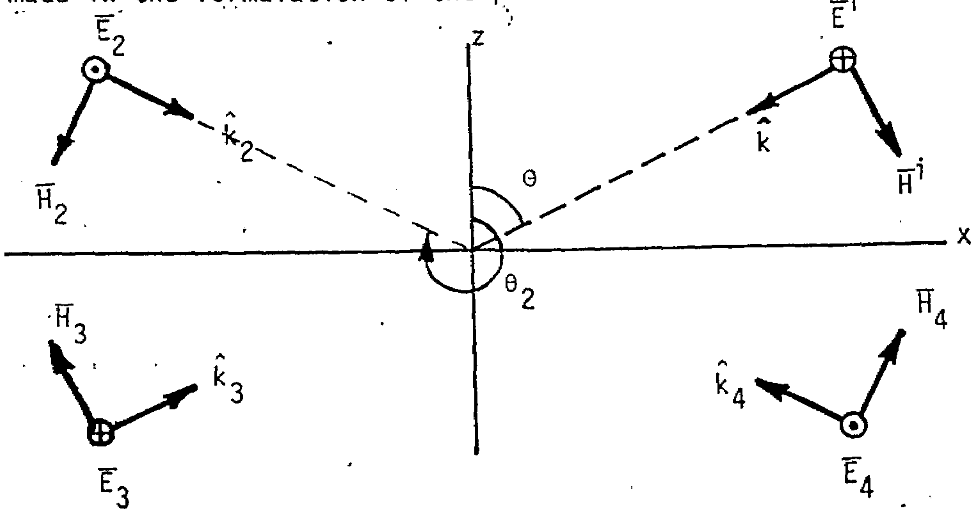

Figure 2.2. Equivalent problem obtained by using image field.

$$\overline{H}_o^i = H_o^i(\cos\theta\ \hat{x} - \sin\theta\ \hat{z})$$

$$\overline{H}_{2o} = H_o^i(-\cos\theta\ \hat{x} - \sin\theta\ \hat{z})$$

$$\overline{H}_{3o} = H_o^i(-\cos\theta\ \hat{x} + \sin\theta\ \hat{z})$$

$$\overline{H}_{4o} = H_o^i(\cos\theta\ \hat{x} + \sin\theta\ \hat{z})$$

$$\overline{K}(\overline{R}) = \hat{y}j4H_o^i\cos\theta\ \sin(kx\sin\theta) \qquad x > 0,\ z = 0 \tag{2.7}$$

$$\overline{K}(\overline{R}) = \hat{y}j4H_o^i\sin\theta\ \sin(kz\cos\theta) \qquad x = 0,\ z > 0 \tag{2.8}$$

$$\rho_s(\overline{R}) = 0 \quad \text{for either } x > 0,\ z = 0 \text{ or } x = 0,\ z > 0\ . \tag{2.9}$$

3. OPEN CIRCUIT VOLTAGE OF A SHORT MONOPOLE MOUNTED ON THE RIGHT-ANGLE CORNER REFLECTOR

The geometry of the problem and the significant parameters are shown in Figure 3.1 We assume that the probe is electrically small. We will

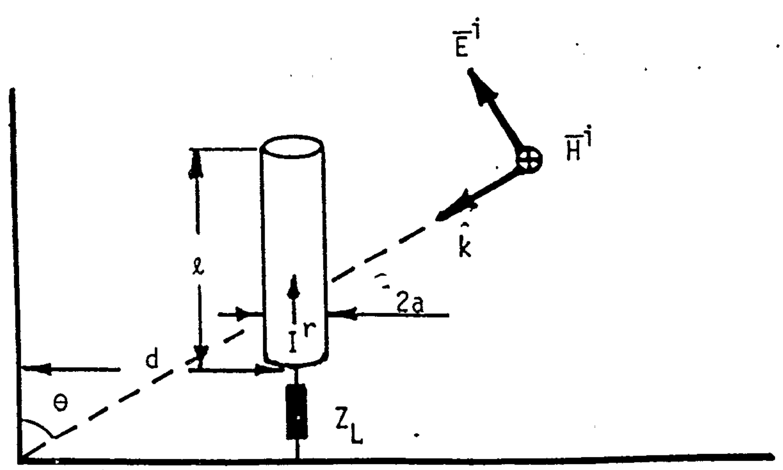

Figure 3.1. Short monopole mounted on a wedge.

consider only the non-trivial polarization of the incident field. As a result of the image theorem, the induced current in the receiving antenna is related to the open circuit voltage by

$$I^r = \frac{V_{oc}^r}{Z_{in}^t/2 + Z_L} \tag{3.1}$$

where

I^r = current at the base of the receiving antenna

Z_{in}^t = input impedance of two transmitting parallel dipole antennas obtained by removing both conducting planes and using the image probes

V_{oc}^r = open circuit voltage of the receiving monopole.

In this section we will deal with open circuit voltage only. The input impedance problem will be discussed in a separate section. Using the vector effective height $\overline{h}^t$ of an antenna, we can write

$$V_{oc}^r = \overline{E}^i \cdot \overline{h}^t. \tag{3.2}$$

The vector effective height of a short cylindrical dipole (or equivalently a monopole on a ground plane) is given by

$$\overline{h}^t(\overline{R}) = -\ell \sin\theta \, \hat{\theta},$$

which corresponds to a linear current distribution

$$I^t(z) = I_o(1 - |z|/\ell) \quad |z| \leq \ell, \text{ with } k\ell << 1.$$

For the problem at hand we have

$$V_{oc}^r = \bar{E}^i(\bar{R}') \cdot \bar{h}^t(\bar{R})\Big|_{(R' = d,\ \theta' = \frac{\pi}{2},\ \phi' = 0)}$$

$$= (-\ell sin\theta\ \hat{\theta}) \cdot \bar{E}_o^{\,i} e^{jkd sin\theta} + \ell sin\theta\ \hat{\theta}_2 \cdot \bar{E}_{2o} e^{-jkd sin\theta}$$

or

$$V_{oc}^r = j2E_o^{\,i}\ell sin\theta\ sin(kd sin\theta) \tag{3.3}$$

3.1. Open circuit voltage of a semi-loop probe whose axis is parallel to the wedge axis

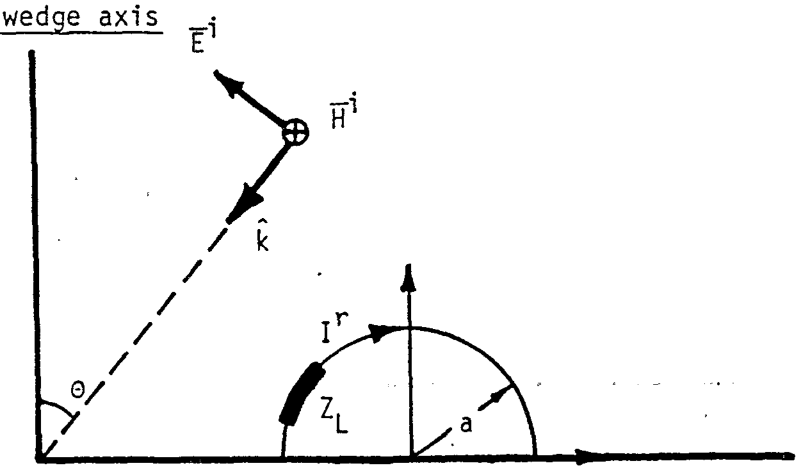

Figure 3.2. Small semi-loop mounted on the wedge.

The open circuit voltage for the probe shown in Figure 3.2 is defined as

$$V_{oc}^{r} = -\sum \overline{E}^{i} \cdot \overline{h}^{t}$$

For a small loop with constant current distribution $\overline{h}^{t}$ is given by

$$\overline{h}^{t}(\overline{R}) = j\,\frac{\pi}{k}(ka)^2\, sin\theta\, \hat{\phi}$$

$$\begin{aligned}
V_{oc}^{r} &= -\sum \overline{E}^{i}(\overline{R}') \cdot \overline{h}^{t}(\overline{R})\Big|_{(R' = d,\ \theta' = \frac{\pi}{2},\ \phi' = 0)} \\
&= -\frac{1}{k}\{j\pi(ka)^2 sin\,\frac{\pi}{2}(-\hat{\theta}) \cdot \overline{E}_{0}^{i}\, e^{jkd\,sin\theta} \\
&\qquad + j\pi(ka)^2 sin\,\frac{\pi}{2}(-\hat{\theta}_2) \cdot \overline{E}_{2o}\, e^{-jkd\,sin\theta}\} \\
&= -j2\pi(ka)^2\,\frac{E_{0}^{i}}{k}\, cos(kd\,sin\theta) \\
&= -j2\omega(\mu\,\pi a^2)H_{0}^{i} cos\,(kd\,sin\theta). \qquad (3.4)
\end{aligned}$$

3.2. Open circuit voltage of a semi-loop probe whose axis is perpendicular to the wedge axis.

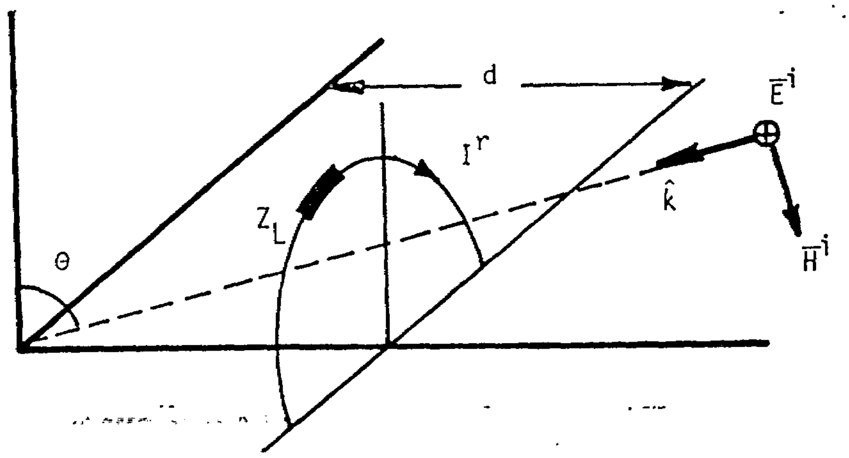

Figure 3.3 Small semi-loop probe with axis perpendicular to the wedge axis.

The pertinant polarization of the incident field for this configuration of probe is depicted in Figure 3.3. Following the same procedures of the previous section and considering the direction of the current density of the probe we can write:

$$V_{oc}^{r} = -\sum \bar{E}^{i}\, \bar{h}^{t}\Big|_{(R' = d,\ \theta' = \frac{\pi}{2},\ \phi' = 0)}$$

i.e.
$$V_{oc}^{r} = -\, 2\omega(\mu\pi a^{2})\, H_{o}^{i}\, \cos\theta\, \sin(kd\sin\theta) \tag{3.5}$$

4. INPUT IMPEDANCE OF THE PROBES

Equation (3.1) of Section 3 indicates that for completion of the equivalent circuit parameters we need to evaluate Z_{in}^{t}, which will be simply referred to as Z_{in} from now on, in each of the probing configurations considered previously. Let us once more recall that Z_{in} is the input impedance of the transmitting antenna in the presence of its images. With this in mind we will begin by deriving analytical formulas for the input impedance functions involved in the problems at hand under the assumptions imposed on the electrical sizes of the probes. The problem of determining the impedance for the cases considered here has been extensively explored previously; however, most often in the form of tables and curves. We will include here the complete expressions for these functions.

4.1. Impedance parameters of two identical, parallel, and short transmitting dipoles.

The problem arising from the application of image theory to two antisymmetrically-driven antennas is shown in Figure 4.1. Since the probe is assumed to be thin and short ($a \ll \ell$, $k\ell \ll 1$), a linear current distribution is a suitable approximation and the induced EMF method can be applied successfully to determine the input impedance.

For convenience we will proceed by assuming a sinusoidal current distribution. At the final stages the results will be simplified by using the conditions imposed on a, ℓ, and ρ.

Based on filamentary current distribution we have for the magnetic vector potential $\overline{A}(\overline{R})$:

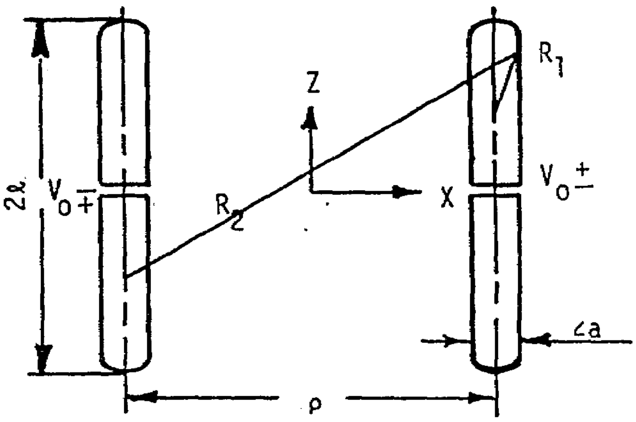

Figure 4.1

Two identical,parallel transmitting antennas.

$$\overline{A}(\overline{R}) = \frac{\mu}{4\pi}\int_{V'} G\,(\overline{R}|\overline{R}')\overline{J}(\overline{R}')\,dv'$$

$$= \hat{z}\,\frac{\mu}{4\pi}\int_{-\ell}^{\ell} I(z')\left\{\frac{e^{-jkR_1}}{R_1} - \frac{e^{-jkR_2}}{R_2}\right\} dz'.$$

R_1 and R_2 are the distances from the observation point to the source points on the axes of the dipoles. For $\overline{R}$ on the surface of one of the antennas we have

$$R_1 = \sqrt{[(z - z')^2 + a^2]},\; R_2 \simeq \sqrt{[(z - z')^2 + \rho^2]} \text{ for } \rho >> a. \quad (4.1)$$

The electric field on the surface of the right hand side antenna is given by

$$\overline{E} = -\,j\omega(1 + \frac{1}{k^2}\nabla\nabla\cdot)\,\overline{A} = -\,\hat{z}\,j\omega(1 + \frac{1}{k^2}\frac{\partial^2}{\partial z^2})\,A_z(z)$$

$$= -\,\hat{z}\,V_0\,\delta(z) \quad (4.2)$$

where as usual a slice generator has been assumed. Multiplying (4.2) by I(z) and integrating over the source region we obtain

$$V_o\, I(0) = \frac{j\omega\mu}{4\pi}\int_{-\ell}^{\ell}\int_{-\ell}^{\ell} I(z)I(z')\left(1 + \frac{1}{k^2}\frac{\partial^2}{\partial z^2}\right)(G_1 - G_2)\, dz dz'$$

where $$G_1(z,z') = \frac{e^{-jkR_1}}{R_1},\quad G_2(z,z') = \frac{e^{-jkR_2}}{R_2}$$

$$G(z,z') = G_1(z,z') - G_2(z,z').$$

The first integration with respect to z amounts to the evaluation of the z-component of the near zone electric field due to a current distribution of the form

$$I(z) = I_m\, \sin k(\ell - |z|) \qquad\qquad |z| \le \ell$$

Therefore it is given by [3]

$$\int_{-\ell}^{\ell} I(z)\left(1 + \frac{1}{k^2}\frac{\partial^2}{\partial z^2}\right)G(z,z')\, dz = \frac{I_m}{k}\ \{G(\ell,z') +$$

$$+ G(-\ell,z') - 2\cos k\ell\ G(0,z')\}.$$

We then have

$$Z_{in}\, I^2(0) = \frac{j\omega\mu I_m}{4\pi k}\int_{-\ell}^{\ell}\{G(\ell,z') + G(-\ell,z')$$

$$- 2\cos k\ell\ G(0,z')\}\ I(z')dz',$$

$$Z_{in} = \frac{j\eta}{2\pi \sin^2 k\ell}\int_{0}^{\ell}\{G(\ell,z) + G(-\ell,z) - 2\cos k\ell\ G(0,z)\} \times$$

$$\times \sin k(\ell - z)\, dz. \qquad (4.3)$$

Let us note that from the circuit relations for these antennas

$$V_1 = Z_{11}I_1 + Z_{12}I_2$$

$$V_2 = Z_{21}I_1 + Z_{22}I_2 \cdot$$

It follows that for antisymmetrically driven antennas we have

$$Z_{in} = Z_{11} - Z_{12} \cdot \tag{4.4}$$

Let us define

$$\zeta(a) = \int_0^{\ell} \{G_1(\ell,z) + G_1(-\ell,z) - 2\cos k\ell\, G_1(0,z)\}\sin k(\ell - z)\, dz. \tag{4.5}$$

Combining (4.3) - (4.5) we obtain

$$Z_{11} = \frac{j\eta}{2\pi \sin^2 k\ell}\ \zeta(a),\quad Z_{12} = \frac{j\eta}{2\pi \sin^2 k\ell}\ \zeta(\rho). \tag{4.6}$$

Therefore the problem of finding the input impedance reduces to the evaluation of $\zeta(a)$ and $\zeta(\rho)$.

$\zeta(a)$ has been previously evaluated by expressing it in terms of sine and cosine-integrals [4] We will choose another approach which enables us to obtain a series expansion for $\zeta(a)$ and $\zeta(\rho)$, and in particular to reduce the results to simplified forms under certain assumptions on the parameters. We have

$$\int_0^{\ell} G_1(\ell,z)\, \sin k\,(\ell - z)dz = \int_0^{\ell} \frac{e^{-jk\sqrt{(x^2 + a^2)}}}{\sqrt{(x^2 + a^2)}} \sin kx dx$$

$$\int_0^{\ell} G_1(-\ell,z) \sin k(\ell - z)dz = \int_{\ell}^{2\ell} \frac{e^{-jk\sqrt{(x^2 + a^2)}}}{\sqrt{(x^2 + a^2)}} \sin k(2\ell - x)dx$$

$$\int_0^{\ell} G_1(0,z) \sin k(\ell - z)dz = \int_0^{\ell} \frac{e^{-jk\sqrt{(x^2 + a^2)}}}{\sqrt{(x^2 + a^2)}} \sin k(\ell - x)dx$$

Therefore we are led to define

$$s(\xi,\alpha) = \int_0^{\ell} \frac{e^{-jk\sqrt{(x^2 + a^2)}}}{\sqrt{(x^2 + a^2)}} \sin kx dx \tag{4.7}$$

$$c(\xi,\alpha) = \int_0^{\ell} \frac{e^{-jk\sqrt{(x^2 + a^2)}}}{\sqrt{(x^2 + a^2)}} \cos kx dx \tag{4.8}$$

Then in terms of $s(\ell,a)$ and $c(\ell,a)$ we can write

$$\begin{aligned}\zeta(a) &= s(\ell,a) + \sin 2k\ell(c(2\ell,a) - c(\ell,a)) + \\ &\quad - \cos 2k\ell(s(2\ell,a) - s(\ell,a)) - 2\sin k\ell \cos k\ell\, c(\ell,a) + 2\cos^2 k\ell\ s(\ell,a) \\ \zeta(a) &= 2\ s(\ell,a) + (2\ s(\ell,a) - s(2\ell,a))\cos 2k\ell - (2c(\ell,a) - c(2\ell,a))\sin 2k\ell\end{aligned} \tag{4.9}$$

In order to evaluate $s(\ell,a)$ and $c(\ell,a)$ let us introduce the following dimensionless parameters:

$$\xi = k\ell\ ,\ \alpha = \frac{a}{\ell}\ ,\ \beta = \sqrt{(1 + \alpha^2)} \tag{4.10}$$

Then

$$s(\ell,a) = \int_0^{1} \frac{e^{-j\xi\sqrt{(x^2 + \alpha^2)}}}{\sqrt{(x^2 + \alpha^2)}} \sin \xi x dx \equiv s(\xi,\alpha) \tag{4.7'}$$

$$c(\ell,a) = \int_0^{1} \frac{e^{-j\xi\sqrt{(x^2 + \alpha^2)}}}{\sqrt{(x^2 + \alpha^2)}} \cos \xi x dx \equiv c(\xi,\alpha) \tag{4.8'}$$

with

$$\frac{\partial s}{\partial \xi} = - j\int_0^1 e^{-j\xi\sqrt{(x^2+\alpha^2)}} sin\xi x dx + \int_0^1 \frac{xe^{-j\xi\sqrt{(x^2+\alpha^2)}}}{\sqrt{(x^2+\alpha^2)}} cos\xi x dx$$

$$\int_0^1 \frac{xe^{-j\xi\sqrt{(x^2+\alpha^2)}}}{(x^2+\alpha^2)} cos\xi x dx = - \frac{1}{j\xi} e^{-j\xi\sqrt{(x^2+\alpha^2)}} cos\xi x \Big|_0^1 -$$

$$- \frac{1}{j} \int_0^1 e^{-j\xi\sqrt{(x^2+\alpha^2)}} sin\xi x dx$$

Hence

$$\frac{\partial s}{\partial \xi} = - \frac{1}{j\xi} e^{-j\xi\sqrt{(x^2+\alpha^2)}} cos\xi x \Big|_0^1 = - \frac{1}{j\xi} (e^{-j\beta\xi} cos\xi - e^{-j\alpha\xi}) \qquad (4.11)$$

Similarly

$$\frac{\partial c}{\partial \xi} = - j \int_0^1 e^{-j\xi\sqrt{(x^2+\alpha^2)}} cos\xi x dx - \int_0^1 \frac{xe^{-j\xi\sqrt{(x^2+\alpha^2)}}}{\sqrt{(x^2+\alpha^2)}} sin\xi x dx$$

$$\frac{\partial c}{\partial \xi} = \frac{1}{j\xi} e^{-j\xi\sqrt{(x^2+\alpha^2)}} sin\xi x \Big|_0^1 = \frac{1}{j\xi} e^{-j\xi\beta} sin\xi \qquad (4.12)$$

We will proceed by finding a series expansion for $\frac{\partial s}{\partial \xi}$ and $\frac{\partial c}{\partial \xi}$. For the sake of numerical compution we will develop two different series expansions depending on the relative values of $\alpha\xi = ka$.

I) $\alpha\xi << 1$

$$\frac{\partial s}{\partial \xi} = - \frac{1}{j\xi} \{\frac{1}{2} (e^{-j\xi(\beta-1)} + e^{-j\xi(\beta+1)}) - e^{-j\xi\alpha}\}$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-j\xi)^{n-1}}{n!} \{(\beta-1)^n + (\beta+1)^n - 2\alpha^n\}$$

$$s(\xi,\alpha) = - \frac{1}{j2} \sum_{n=1}^{\infty} \frac{(-j\xi)^n}{n!n} \{ (\beta-1)^n + (\beta+1)^n - 2\alpha^n\} \qquad (4.13)$$

$$2s(\xi,\alpha) - s(2\xi,\tfrac{\alpha}{2}) = -\frac{1}{j2} \sum_{n=1}^{\infty} \frac{(-j2\xi)^n}{n!n} s_n$$

$$s_n = 2^{1-n}\{(\beta-1)^n + (\beta+1)^n\} - \{(\beta(\tfrac{\alpha}{2})-1)^n + (\beta(\tfrac{\alpha}{2})+1)^n\} - 2^{1-n}\alpha^n \,. \tag{4.14}$$

Similarly we have

$$\frac{\partial c}{\partial \xi} = \frac{1}{j\xi} e^{-j\xi\beta} \sin\xi = \frac{j}{2} \sum_{n=1}^{\infty} \frac{(-j\xi)^{n-1}}{n!} \{(\beta-1)^n - (\beta+1)^n\}$$

$$c(\xi,\alpha) - c(0,\alpha) = -\frac{1}{2} \sum_{n=1}^{\infty} \frac{(-j\xi)^n}{n!n} \{(\beta-1)^n - (\beta+1)^n\}$$

$$c(0,\alpha) = \int_0^1 \frac{d\alpha}{\sqrt{(x^2+\alpha^2)}} = \ln\,(x + \sqrt{(x^2 + \alpha^2)})\,\Big|_0^1 = \ln \frac{\beta+1}{\alpha}$$

$$c(\xi,\alpha) = \ln \frac{\beta+1}{\alpha} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-j\xi)^n}{n!n} \{(\beta-1)^n - (\beta+1)^n\} \tag{4.15}$$

$$2c(\xi,\alpha) - c(2\xi,\tfrac{\alpha}{2}) = -\ln \frac{2[\beta(\frac{\alpha}{2})+1]}{(\beta+1)^2} - \ln\alpha - \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-j2\xi)^n}{n!n} c_n$$

$$c_n = 2^{1-n}[(\beta-1)^n - (\beta+1)^n] - [(\beta(\tfrac{\alpha}{2})-1)^n - (\beta(\tfrac{\alpha}{2})+1)^n] \,. \tag{4.16}$$

The previous expressions are so far exact. Let us introduce approximations under the assumptions

$$\alpha = \frac{a}{\ell} \ll 1 \qquad \xi = k\ell \ll 1$$

Then

$$\beta(\alpha) = 1 + \frac{1}{2}\alpha^2 + 0(\alpha^4)$$

$$\ln \frac{2[\beta(\frac{\alpha}{2})+1]}{(\beta(\alpha)+1)^2} = \ln 4(1 + \frac{\alpha^2}{16} + \ldots) - 2\ln 2(1 + \frac{\alpha^2}{4} + \ldots) = 0(\alpha^2)$$

$$s_n = 2^{1-n} \sum_{m=0} \binom{n}{m} \beta^m [1+(-1)^{n-m}] - \sum_{m=0}^{n} \binom{n}{m} \beta^m(\tfrac{\alpha}{2}) \times$$

$$\times [1 + (-1)^{n-m}] - 2^{1-n}\alpha^n$$

$$2^{1-n}\,\beta^{m}(\alpha) - \beta^{m}(\tfrac{\alpha}{2}) = 2^{1-n}(1 + \alpha^{2})^{\frac{m}{2}} - (1 + \tfrac{\alpha^{2}}{4})^{\frac{m}{2}} =$$

$$= (2^{1-n} - 1) + 0(\alpha^{2})$$

$$s_{n} = \sum_{m=0}^{n} (2^{1-n} - 1)\binom{n}{m}[1 + (-1)^{n-m}] - \alpha\delta_{n1} + 0(\alpha^{2})$$

$$= (2^{1-n} - 1)\ 2^{n} - \alpha\delta_{n1} + 0(\alpha^{2}) = (2 - 2^{n}) - \alpha\delta_{n1} + 0(\alpha^{2})$$

where δ_{n1} is the Kronecker delta.

$$c_{n} = -2^{1-n} \sum_{m=0}^{n} \binom{n}{m}\ \beta^{m}[1 - (-1)^{n-m}] + \sum_{m=0}^{n} \binom{n}{m}\ \beta^{m}(\tfrac{\alpha}{2})[1 - (-1)^{n-m}]$$

$$= (2^{n} - 2) + 0(\alpha^{2})\ .$$

Thus we have the following approximations:

$$s(\xi,\alpha) = \frac{1}{j2}\{\sum_{n=1}^{\infty} \frac{(-j2\xi)^{n}}{n!n} + j2\alpha\xi\} + 0(\alpha^{2}\xi) \tag{4.13'}$$

$$2s(\xi,\alpha) - s(2\xi, \tfrac{\alpha}{2}) = \frac{1}{2j}\{\sum_{n=1}^{\infty} \frac{(-j2\xi)^{n}}{n!n}(2^{n} - 2) - j2\alpha\xi\} + 0(\alpha^{2}\xi) \tag{4.14'}$$

$$2c(\xi,\alpha) - c(2\xi, \tfrac{\alpha}{2}) = -\ln\alpha - \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-j2\xi)^{n}}{n!n}(2^{n} - 2) + 0(\alpha^{2}\xi)\ . \tag{4.16'}$$

Finally for $\zeta(a)$ we obtain

$$\zeta(a) = \sin 2\xi \ln\alpha + (\sin 2\xi - j\cos 2\xi)\sum_{n=1}^{\infty} \frac{(-j2\xi)^{n}}{n!n} \times$$

$$\times (2^{n-1} - 1) + j\sum_{n=1}^{\infty} \frac{(-j2\xi)^{n}}{n!n} - (2 + \cos 2\xi)\alpha\xi + 0(\alpha^{2}\xi)$$

$$\zeta(a) = 2\xi - \frac{5}{36}(2\xi)^{3} - 3\alpha\xi + \sin 2\xi \ln\alpha - \frac{j}{48}(2\xi)^{4} +$$

$$+ 0\{\max(\xi^{5}, \alpha^{2}\xi)\}. \tag{4.17}$$

ii) $\alpha\xi >> 1$

For this case we have $\xi=k\ell$, $\alpha=\frac{\rho}{\ell}$. Following the above formulations we set:

$$s(\xi,\alpha) = s(0,\alpha) + e^{-j\xi\alpha} \sum_{n=0}^{\infty} s_n \xi^n \tag{4.18}$$

$$c(\xi,\alpha) = c(0,\alpha) + e^{-j\xi\alpha} \sum_{n=0}^{\infty} c_n \xi^n . \tag{4.19}$$

Then

$$\frac{\partial s}{\partial \xi} = e^{-j\xi\alpha} \sum_{n=1}^{\infty} n s_n \xi^{n-1} - j\alpha e^{-j\xi\alpha} \sum_{n=0}^{\infty} s_n \xi^n$$

$$= e^{-j\xi\alpha} \sum_{n=0}^{\infty} [-j\alpha s_n + (n+1) s_{n+1}] \xi^n$$

$$= - \frac{1}{j\xi} (e^{-j\xi\beta} cos\xi - e^{-j\xi\alpha}) = - \frac{e^{-j\xi\alpha}}{j2\xi} \sum_{n=1}^{\infty} \frac{(-j\xi)^n}{n!} \times$$

$$\times \{(\beta - \alpha - 1)^n + (\beta - \alpha + 1)^n\}$$

$$= e^{-j\xi\alpha} \sum_{n=1}^{\infty} \frac{(-j\xi)^{n-1}}{n!} \sigma_n = e^{-j\xi\alpha} \sum_{n=0}^{\infty} \frac{(-j\xi)^n}{(n+1)!} \cdot \sigma_{n+1} ,$$

where we have defined

$$\sigma_n = \frac{1}{2} \{(\beta - \alpha - 1)^n + (\beta - \alpha + 1)^n\}; \; n=0,1,2,... \tag{4.20}$$

Therefore we have

$$-j\alpha s_n + (n+1) s_{n+1} = (-j)^n \frac{\sigma_{n+1}}{(n+1)!}; \; n=0,1,2,3,... \tag{4.21}$$

Recurrsion relation (4.21) starts off with $s_0 = 0$ which is an immediate consequence of (4.18):

$$s_0 = 0$$

$$s_1 = \sigma_1$$

$$s_2 = - \frac{j}{2!}(\frac{\sigma_2}{2} - \frac{\alpha\sigma_1}{1})$$

$$s_3 = \frac{(-j)2}{3!}\left(\frac{\sigma_3}{3} - \frac{\alpha\,\sigma_2}{2} + \alpha^2\sigma_1\right)$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$s_n = \frac{(-j)^{n-1}}{n!}\left(\frac{\sigma_n}{n} - \frac{\alpha\sigma_{n-1}}{n-1} + \frac{\alpha^2\sigma_{n-2}}{n-2} - - - - + (-1)^{n-1}\alpha^{n-1}\sigma_1\right)$$

$$n=1,2,3,\ldots \qquad (4.22)$$

Similarly for $\frac{\partial c}{\partial \xi}$ we have

$$\frac{\partial c}{\partial \xi} = e^{-j\xi\alpha}\sum_{n=0}^{\infty}(-j\alpha c_n + (n+1)c_{n+1})\xi^n$$

$$= \frac{1}{j\xi}e^{-j\xi\beta}\sin\xi = -\frac{1}{2\xi}e^{-j\xi\alpha}\sum_{n=1}^{\infty}\frac{(-j\xi)^n}{n!}[(\beta-\alpha-1)^n - (\beta-\alpha+1)^n]$$

$$= e^{-j\xi\alpha}\sum_{n=0}^{\infty}\frac{(-j\xi)^n}{(n+1)!}\gamma_{n+1} ,$$

with

$$\gamma_n = \frac{1}{2j}\{(\beta-\alpha+1)^n - (\beta-\alpha-1)^n\} . \qquad (4.23)$$

Finally

$$-j\alpha\ c_n + (n+1)\ c_{n+1} = \frac{(-j)^n}{(n+1)!}\ \gamma_{n+1};\ n=0,1,2,\ldots \qquad (4.24)$$

$$c_0 = 0$$

$$c_1 = \gamma_1$$

$$c_2 = -\frac{j}{2!}\left(\frac{\gamma_2}{2} - \frac{\alpha\gamma_1}{1}\right)$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$c_n = \frac{(-j)^{n-1}}{n!}\left(\frac{\gamma_n}{n} - \frac{\alpha\,\gamma_{n-1}}{n-1} + \ldots . + (-1)^{n-1}\alpha^{n-1}\gamma_1\right) \qquad (4.25)$$

In general (4.18) - (4.20) together with (4.22), (4.23), and (4.25) are compact enough to permit numerical computation of $s(\xi,\alpha)$ and $c(\xi,\alpha)$. However we would like to discuss the approximation and further simplification under the conditions:

$$\alpha = \frac{\rho}{\ell} >> 1, \quad \xi << 1.$$

We have

$$\beta - \alpha = \surd(1+\alpha^2) - \alpha = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(2m-3)!!}{2^m m!} \alpha^{-(2m-1)}$$

with

$$(2m-3)!! = (2m-3)(2m-5) - - - 3.1 \quad m=2,3,...$$

$$(-1)!! = 1$$

$$\sigma_n = \frac{1}{2} \{(\beta-\alpha+1)^n + (\beta-\alpha-1)^n\} = \frac{1}{2} \sum_{m=0}^{n} \binom{n}{m} (\beta-\alpha)^m [1+(-1)^{n-m}]$$

$$\gamma_n = \frac{1}{2j} \{(\beta-\alpha+1)^n - (\beta-\alpha-1)^n\} = \frac{1}{2j} \sum_{m=0}^{n} \binom{n}{m} (\beta-\alpha)^m [1-(-1)^{n-m}]$$

$$(\beta-\alpha)^m = (\frac{1}{2} \alpha^{-1})^m (1 + \sum_{r=0}^{\infty} (-1)^{r+1} \frac{(2r+1)!!}{2^{r+1}(r+2)!} \alpha^{-(2r+2)})^m$$

$$= (\frac{1}{2} \alpha^{-1})^m (1 + m \sum_{r=0}^{\infty} (-1)^{r+1} \frac{(2r+1)!!}{2^{r+1}(r+2)!} \alpha^{-(2r+2)} +$$

$$+ \frac{m(m-1)}{2} \frac{\alpha^{-4}}{16} (1 + \sum_{r=0}^{\infty} (-1)^{r+1} \frac{(2r+3)!!}{2^r(r+3)!} \alpha^{-(2r+2)})^2 + - - -)$$

$$\beta-\alpha = \frac{1}{2} \alpha^{-1} (1 - \frac{1}{4} \alpha^{-2} + \frac{1}{8} \alpha^{-4} - \frac{5}{64} \alpha^{-6}) + 0(\alpha^{-9})$$

$$(\beta-\alpha)^2 = \frac{1}{4} \alpha^{-2} (1 - \frac{1}{2} \alpha^{-2} + \frac{5}{16} \alpha^{-4}) + 0(\alpha^{-8})$$

$$(\beta-\alpha)^3 = \frac{1}{8} \alpha^{-3} (1 - \frac{3}{4} \alpha^{-2}) + 0(\alpha^{-7})$$

$$(\beta-\alpha)^4 = \frac{1}{16} \alpha^{-4} + 0(\alpha^{-6})$$

$$\left.\begin{aligned}
\sigma_1 &= (\beta-\alpha) = \frac{1}{2} \alpha^{-1} - \frac{1}{8} \alpha^{-3} + \frac{1}{16} \alpha^{-5} - \frac{5}{128} \alpha^{-7} + 0(\alpha^{-9}) \\
\sigma_2 &= 1 + (\beta-\alpha)^2 = 1 + \frac{1}{4} \alpha^{-2} - \frac{1}{8} \alpha^{-4} + \frac{5}{64} \alpha^{-6} + 0(\alpha^{-8}) \\
\sigma_3 &= 3(\beta-\alpha) + (\beta-\alpha)^3 = \frac{3}{2} \alpha^{-1} - \frac{1}{4} \alpha^{-3} + \frac{3}{32} \alpha^{-5} + 0(\alpha^{-7}) \\
\sigma_4 &= 1 + 6(\beta-\alpha)^2 + (\beta-\alpha)^4 = 1 + \frac{3}{2} \alpha^{-2} - \frac{11}{16} \alpha^{-4} + 0(\alpha^{-6})
\end{aligned}\right\} \quad (4.26)$$

. . . .

$$
\left.\begin{aligned}
\gamma_1 &= \frac{1}{j} \\
\gamma_2 &= \frac{2}{j}(\beta-\alpha) = \frac{1}{j}(\alpha^{-1} - \frac{1}{4}\alpha^{-3} + \frac{1}{8}\alpha^{-5} - \frac{5}{64}\alpha^{-7}) + 0(\alpha^{-9}) \\
\gamma_3 &= \frac{1}{j}[1 + 3(\beta-\alpha)^2] = \frac{1}{j}(1 + \frac{3}{4}\alpha^{-2} - \frac{3}{8}\alpha^{-4}) + 0(\alpha^{-6}) \\
\gamma_4 &= \frac{1}{j}[4(\beta-\alpha) + 4(\beta-\alpha)^3] = \frac{1}{j}(2\alpha^{-1}) + 0(\alpha^{-5}) \\
&\dots\dots\dots\dots\dots\dots
\end{aligned}\right\} \quad (4.26')
$$

From (4.22), (4.25) and (4.26) it follows that

$$
\left.\begin{aligned}
s_1(\alpha) &= (2\alpha)^{-1} - (2\alpha)^{-3} + 0(\alpha^{-5}) \\
s_2(\alpha) &= -\frac{j}{2!}\,[(2\alpha)^{-2} - 2(2\alpha)^{-4}] + 0(\alpha^{-5}) \\
s_3(\alpha) &= -\frac{1}{3!}\,[\frac{1}{2}(2\alpha)^{-1} + \frac{1}{3}(2\alpha)^{-3}\} + 0(\alpha^{-5}) \\
s_4(\alpha) &= \frac{j}{4!}\,[\frac{4}{3}(2\alpha)^{-2} - 2(2\alpha)^{-4}] + 0(\alpha^{-5}) \\
&\dots\dots\dots\dots\dots\dots \\
c_1(\alpha) &= -j \\
c_2(\alpha) &= -\frac{1}{2!}(-\alpha + (2\alpha)^{-1} - (2\alpha)^{-3}) + 0(\alpha^{-5}) \\
c_3(\alpha) &= \frac{j}{3!}(\alpha^2 - \frac{1}{6} + \frac{3}{2}(2\alpha)^{-2} - 3(2\alpha)^{-4} + 0(\alpha^{-5}) \\
c_4(\alpha) &= \frac{1}{4!}(-\alpha^3 + \frac{\alpha}{6} + \frac{1}{4}(2\alpha)^{-1} + \frac{3}{2}(2\alpha)^{-3} + 0(\alpha^{-5}) \\
&\dots\dots\dots\dots\dots\dots
\end{aligned}\right\} \quad (4.27)
$$

Also, let us note that

$$
c(0,\alpha) = \ln(\alpha^{-1} + \sqrt{(1+\alpha^{-2})}) = \ln(1 + \alpha^{-1} + \sum_{m=1}^{\infty}(-1)^{m+1} \times
$$

$$
\times \frac{(2m-3)!!}{2^m m!}\alpha^{-2m})
$$

$$
c(0,\alpha) = \alpha^{-1} + \sum_{m=1}^{\infty}(-1)^{m+1}\frac{(2m-3)!!}{2^m m\ !}\alpha^{-2m} \ .
$$

$$- \frac{1}{2} \{\alpha^{-1} + \sum_{m=1}^{\infty} (-1)^{m+2} \frac{(2m-3)!!}{2^m m!} \alpha^{-2m}\}^2 + \ldots$$

$$= \alpha^{-1} - \frac{1}{6} \alpha^{-3} + 0(\alpha^{-5})$$

$$2c(0,\alpha) - c(0, \frac{\alpha}{2}) = \alpha^{-3} + 0(\alpha^{-5}) \tag{4.28}$$

Now we can proceed to evaluate $\zeta(\rho)$.

$$\zeta(\rho) = 2s(\xi,\alpha) + \cos 2\xi(2s(\xi,\alpha) - s(2\xi, \frac{\alpha}{2}))$$

$$- \sin 2\xi(2c(\xi,\alpha) - c(2\xi, \frac{\alpha}{2}))$$

$$= e^{-j\xi\alpha} \sum_{n=0}^{\infty} (2\xi)^n \{2^{1-n} s_n(\alpha) + \cos 2\xi \times$$

$$\times (2^{1-n} s_n(\alpha) - s_n(\frac{\alpha}{2})) - \sin 2\xi(2^{1-n} c_n(\alpha) -$$

$$- c_n(\frac{\alpha}{2}))\} - \sin 2\xi(2c(0,\alpha) - c(0,\frac{\alpha}{2}))$$

$$= e^{-j\xi\alpha} \sum_{n=0}^{\infty} (2\xi)^n \{2^{2-n} s_n(\alpha) - s_n(\frac{\alpha}{2}) +$$

$$(-\frac{(2\xi)^2}{2} + \frac{(2\xi)^4}{24} - - - -)(2^{1-n} s_n(\alpha) - s_n(\frac{\alpha}{2}))$$

$$- (2\xi - \frac{(2\xi)^3}{6} + - - - -)(2^{1-n} c_n(\alpha) - c_n(\frac{\alpha}{2}))\}$$

$$- \sin 2\xi \ (2c(0,\alpha) - c(0,\frac{\alpha}{2}))$$

$$\zeta(\rho) = e^{-j\xi\alpha} \{\frac{3}{4} \alpha^{-3}(2\xi) + j\frac{3}{8} \alpha^{-2}(2\xi)^2 - (2\xi)^3(\frac{1}{16}\alpha^{-1}) +$$

$$+ 0 \{max(\xi\alpha^{-5}, \xi^3\alpha^{-3})\}\} -$$

$$- \alpha^{-3}(2\xi) + 0(max(\alpha^{-3}\xi^3, \alpha^{-5}\xi)). \tag{4.29}$$

Noting that

$$\alpha = \frac{\rho}{\ell} = \frac{k\rho}{\xi},$$

we can write

$$\zeta(\rho) = \frac{1}{32} e^{-jk\rho}(2\xi)^4\{-(k\rho)^{-1} + j3(k\rho)^{-2} + 3(k\rho)^{-3}\}$$

$$- \frac{1}{8}(2\xi)^4(k\rho)^{-3} + O\{max(\xi\alpha^{-5}, \xi^3\alpha^{-3})\}. \tag{4.30}$$

Using (4.6) together with (4.17) and (4.30) we finally obtain expressions for Z_{11} and Z_{12}:

$$Z_{11} = \frac{jn}{2\pi sin^2\xi} \zeta(a) = \frac{jn}{2\pi} \xi^{-2}(1 - \frac{\xi^2}{6} + \frac{\xi^4}{24} + . . .)^{-2} \times \zeta(a)$$

$$= \frac{jn}{2\pi} (\xi^{-2} + \frac{1}{3} + O(\xi^4)) \zeta(a)$$

$$= \frac{jn}{2\pi} \{(4 - 6\alpha)(2\xi)^{-1} - (\frac{2}{9} + \frac{\alpha}{2})(2\xi) +$$

$$2cot\xi \, \ell n\alpha - \frac{j}{12} (2\xi)^2\} + O\{max(\xi^3, \alpha^2\xi^{-1})\} \tag{4.31}$$

with

$$\alpha = \frac{a}{\ell}, \; \xi = k\ell;$$

$$Z_{12} = \frac{jn}{2\pi sin^2\xi} \zeta(\rho) = \frac{jn}{2\pi} (\xi^{-2} + \frac{1}{3} + O(\xi^4))\zeta(\rho)$$

$$= \frac{jn}{16\pi} e^{-jk\rho}(2\xi)^2\{-(k\rho)^{-1} + j3(k\rho)^{-2} + 3(k\rho)^{-3}\}$$

$$- \frac{jn}{4\pi}(2\xi)^2(k\rho)^{-3} + O\{\xi^4 max[(k\rho)^{-3}, (k\rho)^{-5}]\}. \tag{4.32}$$

Then the input impedance for the problem at hand is given by

$$Z_{in} = Z_{11} - Z_{12} . \tag{4.4}$$

Let us note that in (4.32), as in previous sections, we have ρ=2d.

4.2 <u>Input impedance of two coaxial small circular loops in transmitting mode</u>.

For an electrically small loop we assume that its current distribution is uniform when driven by a localized voltage. The conventional induced EMF method will be applied to determine its impedance. However, the formation is much more complicated than the case for linear antennas. Therefore assumptions imposed on the geometry of the probing antennas justify adoption of simpler approximate methods in ad hoc bases for evaluation of self and mutual impedances.

As can be seen from the application of the EMF method to the antennas of the previous section the self impedance can be obtained

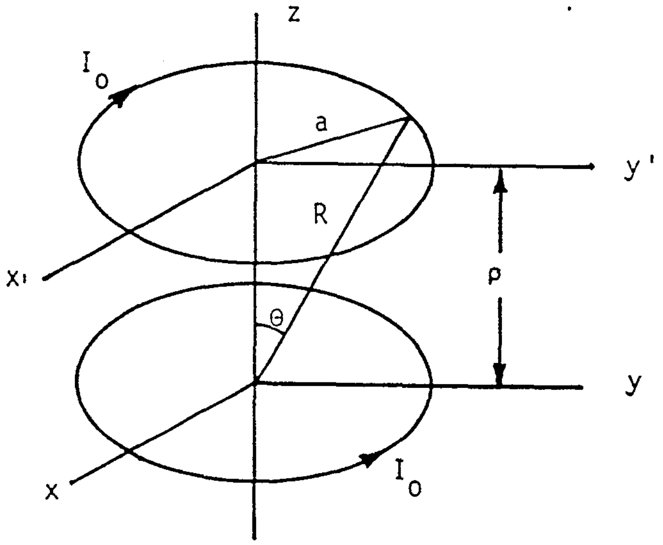

Figure 4.2 Coaxial transmitting loop antennas

by removing the image of one antenna and evaluating the input impedance of the isolated loop.

The input resistance of a constant current loop may be simply evaluated by an application of Poynting's theorem [5]. The result is

$$R_{in} \simeq \frac{\eta}{6\pi} [\pi(ka)^2]^2. \tag{4.33}$$

A rather simple way of obtaining a compact formula for the input reactance of the loop is to make use of the reactance of the loop based on the circuit theory. According to [6] the reactance of the loop is given by:

$$X_{in} \simeq \omega(L_i + L_0) \simeq \omega\mu\{\frac{1}{8\pi} + a(\ln \frac{8a}{b} - 2)\} \qquad (4.34)$$

where L_i is the internal inductance of the wire, L_0 is the so-called selected mutual inductance, and b is the radius of the wire of the loop antenna. This result compares very well with the leading term of the formula obtained for X_{in} by application of the wave theory as discussed in [7]. Combining (4.33) and (4.34) we obtain the self impedance of the loop:

$$Z_{11} \simeq R_{in} + j\ X_{in} \simeq \frac{\eta}{6\pi}[\pi(ka)^2]^2 + $$
$$+ j\omega\ \{\frac{1}{8\pi} + a[\ln(\frac{8a}{b}) - 2]\} \qquad (4.35)$$

4.3 Mutual impedance of the two coaxial loop antennas

As for the case of monopole on a corner reflector, we will present a simple and compact formulation for Z_{12} of two small coaxial loop antennas. We will use the EMF method again and therefore the Fresnel field of a constant current loop is needed.

The Fresnel field of a constant current loop antenna has been previously obtained in the form of a rapidly converging power series in [8], [9], [10], and [11]. We will obtain another series expansion which closely follows the ones given in [9] and will prove more suitable in application of the EMF method.

Using the addition theorem for Legendre and spherical Bessel

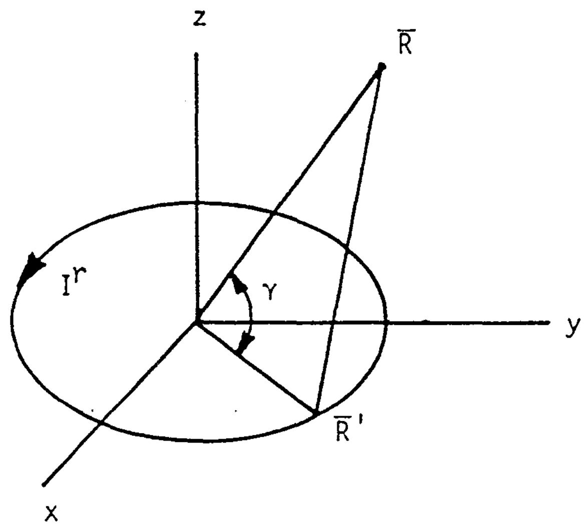

Figure 4.3 Geometry of a constant current circular loop antenna

functions the potential integral for $\overline{A}(\overline{R})$ is evaluated. For $R > a$ we have [9]

$$\overline{A}(\overline{R}) = \hat{\phi}\left(-\frac{j\mu k a I_0}{4}\right) \times$$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(4n+3)(2n)!}{2^{2n}(n+1)(n!)^2} \times$$

$$j_{2n+1}(ka)h^{(2)}_{2n+1}(kR)P^1_{2n+1}(cos\theta). \qquad (4.36)$$

Let us set

$$|\overline{R} - \overline{R}'| = \sqrt{(\nu^2 + \nu'^2 - 2\nu\nu' cos\phi')} \qquad (4.37)$$

with

$$\nu^2 + \nu'^2 = R^2 + a^2$$

$$\nu \nu' = aR sin\theta \, ,$$

comparing with:

$$|\bar{R} - \bar{R}'| = \sqrt{(R^2 + a^2 - 2aR\ cos\gamma)}$$

where $cos\gamma = sin\theta\ cos\phi'$,

We conclude that ϕ' is the angle between two position vectors $\bar{\nu}$ and $\bar{\nu}'$ with angular coordinates:

$$\theta = \frac{\pi}{2},\quad \phi = 0 \quad ; \quad \theta' = \frac{\pi}{2}, \qquad \phi = \phi' .$$

From (4.38) we have

$$\begin{matrix}\nu \\ \nu'\end{matrix} = \{\tfrac{1}{2}(R^2 + a^2 \pm \sqrt{((R^2 + a^2)^2 - 4(aRsin\theta)^2}\ )\}^{\frac{1}{2}}$$

$$\begin{matrix}\nu \\ \nu'\end{matrix} = (R^2 + a^2)^{\frac{1}{2}}\{\tfrac{1}{2}(1 \pm \sqrt{(1 - \chi^2)})\}^{\frac{1}{2}} \tag{4.39}$$

$$\chi = \frac{2aRsin\theta}{R^2+a^2} < 1 .$$

Now we can use the addition theorem for the spherical Hankel function

$$h_0^{(2)}(k|\bar{R} - \bar{R}'|) = h_0^{(2)}(k|\bar{\nu} - \bar{\nu}'|) ,$$

and the addition theorem for Legendre polynomials:

$$P_n(cos\gamma) = P_n(cos\theta)P_n(cos\theta') + 2\sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} P_n^m(cos\theta)\ \times$$

$$\times\ P_n^m(cos\theta')cosm(\phi-\phi') ,$$

where $cos\gamma = cos\theta cos\theta' + sin\theta sin\theta'\ cos(\phi - \phi') = cos\phi'$. Following analogous steps which led to (4.36) we obtain

$$\bar{A}(\bar{R}) = (\frac{-j\mu kaI_0}{4}) \sum_{n=0}^{\infty} \frac{4n+3}{(n+1)(2n+1)}[P_{2n+1}^1(0)]^2\ \times$$

$$\times\ j_{2n+1}(k\nu')h_{2n+1}^{(2)}(k\nu) ,\ \nu' < \nu. \tag{4.40'}$$

Furthermore we have [12]

$$P_\nu^\mu(0) = 2^\mu \pi^{-1/2} cos[\tfrac{1}{2}\pi(\mu + \nu)] \quad \frac{\Gamma(\frac{1}{2} + \frac{1}{2}\nu + \frac{1}{2}\mu)}{\Gamma(1 + \frac{1}{2}\nu - \frac{1}{2}\mu)} \;;$$

Therefore:

$$P^1_{2n+1}(0) = (-1)^{n+1} \frac{(2n+1)!}{2^{2n}(n!)^2} .$$

Finally we have

$$\overline{A}(\overline{R}) = \hat{\phi}(- \frac{j\mu ka I_0}{4}) \sum_{n=0}^{\infty} \frac{(4n+3)(2n+1)}{n+1} [\frac{(2n)!}{2^{2n}(n!)^2}]^2 \times$$
$$\times\; j_{2n+1}(k\nu')h_{2n+1}(k\nu), \; \nu' < \nu. \qquad (4.40)$$

Thus electric field intensity can be obtained as:

$$\overline{E}(\overline{R}) = -j\omega(1 + \frac{1}{k^2} \nabla\nabla\cdot) \overline{A} = -j\omega\overline{A} =$$
$$= \hat{\phi}(- \frac{(ka)^2 \eta I_0}{4a}) \sum_{n=0}^{\infty} \frac{4n+3}{(n+1)(2n+1)} [P^1_{2n+1}(0)]^2 \times$$
$$j_{2n+1}(k\nu')h_{2n+1}(k\nu), \nu' < \nu \qquad (4.41)$$

Note that in obtaining the expression (4.41) we have used the following relation:

$$\nabla\cdot\overline{A} = \frac{1}{R sin\theta} \frac{\partial}{\partial\phi} A_\phi = 0.$$

Now we are in a position to obtain the mutual impedance of the antennas shown in Figure 4.2. Application of the reciprocity theorem to one of the two antennas, say antenna 2, yields [13].

$$V^r = - \frac{1}{I^t} \int_{V'} \overline{E}^i \cdot \overline{J}^t \, dv';$$

where

$V^r \equiv$ open circuit voltage of antenna 2 in receiving mode

$I^t \equiv$ current of antenna 2 in transmitting mode

$\overline{E}^i \equiv$ incident electric field when antenna 2 is removed

$\bar{J}^t \equiv$ current density of antenna 2 in transmitting mode.

Noting that

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = \frac{V_{2OC}}{I_1},$$

we can equivalently write:

$$Z_{21} = -\frac{1}{I_1 I_2}\int_{V'} \bar{E}^i \cdot \bar{J}_2(\bar{R}')dv' \tag{4.42}$$

where

$$\bar{J}_2(\bar{R}') = \hat{\phi}' \, I_2 \delta(\cos\theta') \, \frac{\delta(R'-a)}{R'} .$$

Using (4.41) in (4.42) we have

$$Z_{21} = -\int_{V'} \left(-\frac{(ka)^2\eta}{4a}\right) \sum_{n=0}^{\infty} \frac{4n+3}{(n+1)(2n+1)} [P^1_{2n+1}(0)]^2 \times$$

$$\times \, j_{2n+1}(k\nu')h^{(2)}_{2n+1}(k\nu)\delta(\cos\theta') \, \frac{\delta(R'-a)}{R'} \times R'^2 \sin\theta' dR' d\theta' d\phi'$$

$$= \frac{(ka)^2\eta}{4a} \sum_{n=0}^{\infty} \frac{4n+3}{(n+1)(2n+1)} [P^1_{2n+1}(0)]^2 \times$$

$$\times \, j_{2n+1}(k\nu')h^{(2)}_{2n+1}(k\nu) \int_{V'} \delta(\cos\theta') \, \frac{\delta(R'-a)}{R'} R'^2 \sin\theta' dR' d\theta' d\phi',$$

so that

$$Z_{21} = \eta\frac{\pi}{2}(ka)^2 \sum_{n=0}^{\infty} \frac{4n+3}{(n+1)(2n+1)} [P^1_{2n+1}(0)]^2 \times$$

$$j_{2n+1}(k\nu')h^{(2)}_{2n+1}(k\nu). \tag{4.43}$$

For $|ka| \ll 1$ and $a \ll \rho$ we have

$$\breve{X} = \frac{2aR\sin\theta}{R^2 + a^2} = \frac{2a^2}{\rho^2 + 2a^2} \ll 1 \tag{4.39'}$$

$$\begin{matrix}\nu' \\ \nu\end{matrix} = (R^2 + a^2)^{\frac{1}{2}} \{\tfrac{1}{2}[1 \pm \surd(1 - \chi^2)]\}^{\frac{1}{2}}$$

$$= (R^2 + a^2)^{\frac{1}{2}} \{\tfrac{1}{2}[1 \pm (1 - \tfrac{1}{2}\chi^2 + O(\chi^4))]\}^{\frac{1}{2}}$$

$$\left.\begin{aligned} \nu' &= (R^2 + a^2)^{\frac{1}{2}} \frac{\chi}{2}[1 + O(\chi^2)] = \frac{a^2}{(\rho^2 + 2a^2)^{\frac{1}{2}}} [1 + O(\chi^2)], \\ \nu &= (R^2 + a^2)^{\frac{1}{2}} [1 - \tfrac{1}{8}\chi^2 + O(\chi^4)] = (\rho^2 + 2a^2)^{\frac{1}{2}} [1 + O(\chi^2)]. \end{aligned}\right\} \quad (4.44)$$

Therefore:

$$\left.\begin{aligned} k\nu' &= ka \frac{a}{(\rho^2+2a^2)^{\frac{1}{2}}} [1 + O(\chi^2)] = ka \frac{a}{(\rho^2 + 2a^2)^{\frac{1}{2}}} [1 + O(\frac{a^4}{\rho^4})], \\ k\nu &= k(\rho^2 + 2a^2)^{\frac{1}{2}} [1 + O(\chi^2)] = k(\rho^2 + 2a^2)^{\frac{1}{2}} [1 + O(\frac{a^4}{\rho^4})]. \end{aligned}\right\} (4.44')$$

Let us note that [14]:

$$j_n(z) = (\frac{\pi}{2z})^{\frac{1}{2}} J_{n+\frac{1}{2}}(z) = 2^n z^n \sum_{n=0}^{\infty} \frac{(-1)^m (n+m)!}{m!(2n+2m+1)!} z^{2m} \quad .$$

The asymptotic expansion for $h_n^{(2)}(z)$ for large argument z reads:

$$h_n^{(2)}(z) = (\frac{\pi}{2z})^{\frac{1}{2}} H_{n+\frac{1}{2}}^{(2)}(z) = \frac{1}{2} e^{-j(z-(n+1)\frac{\pi}{2})} \times$$

$$\times \{\sum_{m=0}^{p-1} \frac{(-n)_m (n+1)_m}{m!(-2jz)^m} + O(z^{-p})\} \ .$$

We conclude from the above that under the conditions imposed on a and ρ the first term of the series (4.43) will be sufficient for computation of z_{21}:

$$z_{21} = \eta\frac{3\pi}{2}(ka)^2(P_1^1(0))^2 j_1(k\nu')h_1^{\prime(2)}(k\nu) +$$

$$+ O\{(ka)^2(k\nu')^3(k\nu)^{-1}\}$$

$$= \eta\frac{3\pi}{2}(ka)^2 \ j_1(k\nu')h_1^{(2)}(k\nu) + O\{\frac{(ka)^8}{(k\rho)^4}\} \qquad (4.45)$$

$$Z_{21} = -\eta \frac{\pi}{2}(ka)^2 \frac{\nu'}{\nu} e^{-jk\nu} + 0 \{\frac{(ka)^8}{(k\rho)^4}\}$$

$$= -\eta \frac{\pi}{2}(ka)^2 \frac{a^2}{\rho^2 + 2a^2} \exp\{-jk(\rho^2 + 2a^2)^{\frac{1}{2}}\} +$$

$$+ 0\{\frac{(ka)^8}{(k\rho)^4}\} \qquad (4.46)$$

Let us once more recall that the impedance which should be used in (3.1) can be obtained from (4.35) and (4.46) by means of

$$Z_{in} = Z_{11} - Z_{12} = Z_{11} - Z_{21},$$

and changing ρ to 2d.

4.4 Input impedance of two antisymmetrically driven identical coplanar circular loop antennas.

The self impedance for this configuration's of loops under assumptions imposed on the geometry of the antennas in Section 3.1 is identical to the input impedance of a single loop antenna and can be obtained from (4.35). The mutual impedance can be obtained in exactly the same manner as followed in Section 4.3, however the integrals appearing in this case are a little involved and for the purpose of the problem at hand it suffices to use an asymptotic formula for Z_{12} which is developed in [15] based on effective heights of the transmitting antennas:

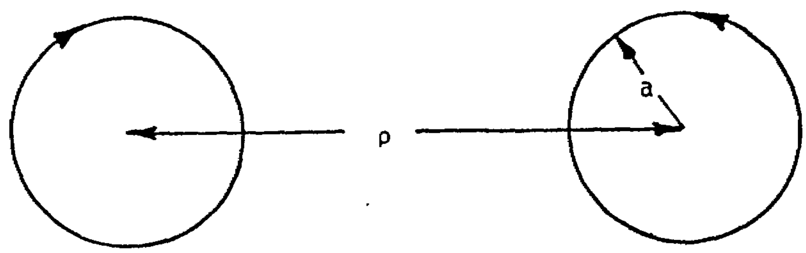

Figure 4.4 Coplanar loop antennas

$$Z_{12} = j\eta \frac{e^{-jk\rho}}{2\lambda\rho} (\overline{h}_1 \cdot \overline{h}_2) = j\eta \frac{e^{-jk\rho}}{2\lambda\rho} \times$$

$$\times \left[\frac{j}{k}\pi (ka)^2 \sin\theta_1\right]_{\theta_1=\frac{\pi}{2}} \times \left[\frac{j\pi}{k}(ka)^2 \sin\theta_2\right]_{\theta_2=\frac{\pi}{2}} (\hat{\phi}_2 \cdot \hat{\phi}_2)$$

$$z_{12} \simeq -j \frac{\pi}{4} \eta (ka)^4 \frac{e^{-jk\rho}}{k\rho} . \tag{4.47}$$

5. <u>Correlation of the unperturbed surface fields to the equivalent circuit parameters of the probes</u>.

In previous sections an attempt was made to completely describe the equivalent circuit parameters of the different probes mounted on a corner reflector. Since any physical measurement performed by the probes can be described completely in terms of the open circuit voltage and the input impedance of the sensors, we will attempt to relate the surface field quantities in the absence of the sensors to the equivalent circuit parameters of the sensors, or in other words, to the measurable quantities.

In Section 2 it was shown that for a plane wave illumination of the wedge with electric field polarized perpendicular to the wedge axis surface current and charge densities are respectively given by

$$\overline{K}(\overline{R}) = -\hat{x}\ 4H_0^i \cos(kx \sin\theta) \qquad x > 0 \quad z=0 \tag{2.3}$$

$$\rho_s(\overline{R}) = j\ 4\varepsilon\ E_0^i \sin\theta \sin(kx\sin\theta) \qquad x > 0 \quad z=0 \tag{2.5}$$

On the other hand the open circuit voltage of a short monopole mounted on the wedge was found to be

$$V_{oc}^r = j2\ell E_0^i \sin\theta \sin(kx\sin\theta). \tag{3.3}$$

Comparing (2.5) with (3.3) we establish that

$$\rho_s = \frac{2\varepsilon}{\ell} V_{oc}^r \tag{5.1}$$

The unperturbed surface charge density is therefore related to the open circuit voltage of the probe by an equivalent capacitance per unit area

$$C_{eq} \triangleq \frac{2\epsilon}{\ell} \qquad \text{farad/m}^2 \ . \tag{5.2}$$

Equation (5.1) is the manifestation of the electric coupling of the monopole probes and further justifies the name of 'charge probes' given to this kind of sensor.

Similarly the V^r_{oc} for a semiloop whose axis is parallel to the wedge axis was found to be

$$V^r_{oc} = -j2\omega(\mu\pi a^2)H^i_o \cos(kx\sin\theta). \tag{3.4}$$

Comparing (3.4) with (2.3) we have

$$V^r_{oc} = j\omega \frac{\mu\pi a^2}{2} k_x \ . \tag{5.3}$$

Thus, the equivalent inductance relating V^r_{oc} to unperturbed current density is given by

$$L_{eq} = \frac{1}{2}\mu(\pi a^2) \qquad \text{henry-m.} \tag{5.4}$$

It is obvious that the coupling of the probe to the electromagnetic field in this case is of magnetic type. Let us finally note that for the plane wave whose electric field is polarized parallel to the wedge axis the surface currents and the corresponding surface magnetic field can be detected by a semiloop sensor whose plane is parallel to the current lines. From (2.7) and (3.5) we obtain for this case

$$V^r_{oc} = j\omega \frac{\mu\pi a^2}{2} k_y \ . \tag{5.5}$$

That is to say the equivalent inductance for this case is also given by (5.4).

Let us conclude from the above results that as long as our sensors are electrically small, low frequency elements relating the open circuit voltage of the probes to the surface fields only depend on the geometrical characteristics of the probes and are independent of the characteristics of the source.

1. Baum, C. E., Sensor and Simulation Notes, Note 212, 8 December 1975.

2. Dyson, J. D., IEEE Trans. AP., VOL. AP21, July 1973, p. 446.

3. Jordan, E. C. and Balmain, K. G., Electromagnetic Waves and Radiating Systems, second edition, Prentice-Hall, N.J. 1968, p. 336.

4. King, R. W. P. and Chang, V. W. H., IEEE Trans. AP., Vol. AP16, May 1968, p. 309.

5. Ramo, S., Whinnery, Jr. R. and Wan Duzer, T., Fields and Waves in Communication Electronics, John Wiley, New York, 1965, p. 656.

6. Ramo, S., Whinnery, J. R. and Wan Duzer, T., loc cit, p. 309.

7. Collin, R. E. and Zucker, F. J., Antenna Theory Part 1, McGraw-Hill Book Company, New York, 1969, Chap. 11.

8. Tai, C. T., Technical Report No. 32, Project No. 591, Stanford Research Institute, Stanford, California, Dec. 1952.

9. Martin, Jr., E. J., Proc. IEEE, Vol. 51, July 1963, p. 1042.

10. Green, F. M., J. Res. Nat. Bur. Stand., Sec. 3, Vol. 71C, Oct. - Dec. 1967, p. 319.

11. Huerta, M. A. and Gonzalez, G., J. Appl. Phys., Vol. 43, No. 10, Oct. 1972, p. 3975.

12. Magnus, W., Oberhettinger, F. and Soni, R. P., Formulas and Theorems for the Special Functions of Mathematical Physics, third edition, Springer-Verlag, New York, 1966, p. 171.

13. Jordan, E. C. and Balmain, K. G., loc cit, p. 485.

14. Watson, G. N., Theory of Bessel Functions, second edition, Cambridge University Press, London, 1952, p. 40 and 198.

15. Tai, C. T., Antenna Theory Notes, Department of Electrical and Computer Engineering, The University of Michigan, Ann Arbor, Michigan.