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Source Excitation of an Open, Parallel-Plate Waveguide

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Abstract

In this work, we consider the problem of an open, finite-width, parallel-plate waveguide which is excited by a y-directed current source. The source current is assumed to be confined at $x = x_0$, have a $\sin(N\pi/2H)$ or $\cos(N\pi/2H)$ variation in the y-direction, and an $\exp(i\beta z)$ behavior along the longitudinal z-direction. Such an excitation can be interpreted as one spectral component of a transversely confined source. The solution to the longitudinally confined source problem can be subsequently constructed by an appropriate superposition of the spectral solutions derived in this paper. The important question of the excitation or non-excitation of the zero-mode in the guide is examined and the resonance condition for a leaky mode in an open, finite-width waveguide is derived.

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I. INTRODUCTION

The parallel-plate simulator [1] is an important device for EMP testing and has been investigated by a number of authors [2-8]. However, all of the previous analyses of this problem have been limited to the investigation of leaky modes in such open waveguides and the source excitation problem of such waveguides has not been previously discussed in the literature.

It is well known [9] that the leaky wave expansion of a source-excited field is an approximation to the exact, continuous spectrum representation.* Thus, an investigation of the complete solution expressed in terms of the continuous spectrum representation is useful from the point of view of evaluating the accuracy of the leaky wave representation.

The present study represents a first step toward this goal. In this work we derive the solution to the source excitation problem for an excitation function which is assumed to have an $e^{i\beta z}$ variation along the longitudinal (z) direction. Such an excitation can be interpreted as one spectral component of a transversely confined source, and the solution to the confined source problem can be subsequently constructed by an appropriate superposition of the spectral solutions derived in this paper.

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* Note that in contrast to surface wave type of structures the open parallel-plate waveguide configuration admits no discrete modes in the proper sheet.

The organization of the report is as follows: Section II presents the statement of the problem we wish to investigate. In Section III we formulate the integral equations and present the solution of these equations in Section IV. Section V is devoted to the calculation of the vector potentials which are useful for the derivation of the fields. In Section VI we investigate the special case when only the zero mode can propagate in the guide and derive the resonance condition for leaky modes. Finally, in Section VII, we discuss the case when more than one mode can propagate in the guide and present a summary of the results in Section VIII.

II. STATEMENT OF THE PROBLEM

In this work we address ourselves to the problem of source excitation of an open waveguide when the source is located inside the waveguide. The open waveguide is formed by two parallel, perfectly conducting strips: $-2L < x < 0$, $y = \pm H$ (see Figure 1).

We will investigate two types of sources, viz.,

Case A

$$\overline{J} = \hat{y}\delta(x + x_0) \sin\left(\frac{N\pi y}{2H}\right) e^{i\beta z} \quad , \quad \text{here } N = 1, 2, \ldots \tag{2.1}$$

Case B

$$\overline{J} = \hat{y}\delta(x + x_0) \cos\left(\frac{N\pi y}{2H}\right) e^{i\beta z} \quad , \text{ here } N = 0, 1, 2, \ldots \tag{2.2}$$

where the current $\overline{J}$ has only a y-component. The time factor $\exp(-i\omega t)$ is implicit throughout this report. We assume that $\mathrm{Re}\beta > 0$ and $\mathrm{Im}\beta = 0$.

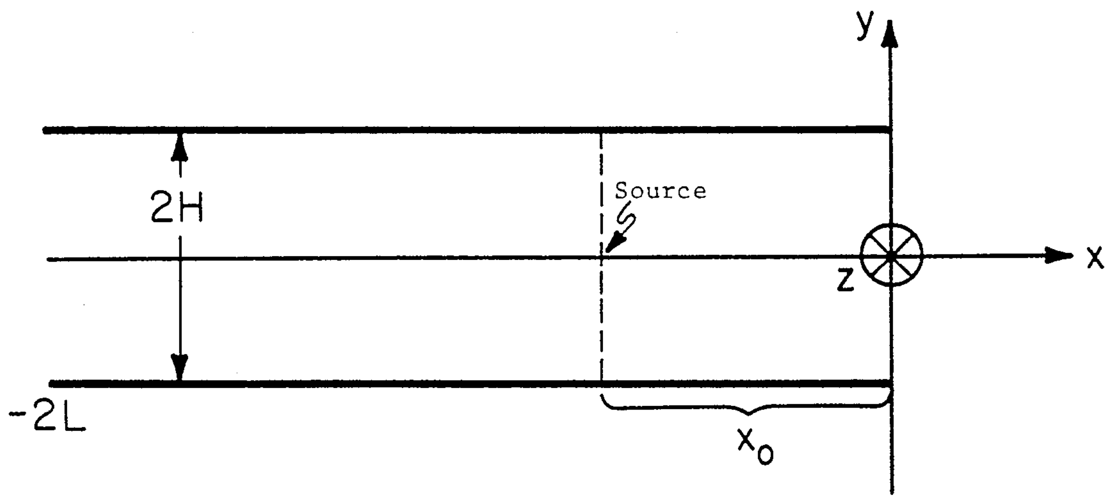

Fig. 1

Geometry of the problem of source excitation of a parallel-plate waveguide.

III. BASIC FORMULATION

We begin with Maxwell's equations:

$$\nabla_1 \times \overline{E} - i\omega\mu\overline{H} = 0 \tag{3.1}$$

$$\nabla_1 \times \overline{H} + i\omega\epsilon\overline{E} = \overline{J} \quad . \tag{3.2}$$

where

$$\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \nabla^2 + \frac{\partial^2}{\partial z^2} \quad .$$

The electromagnetic fields may be expressed in terms of a vector potential function $\overline{A}^{(1)}$ by means of the following equations:

$$\overline{H} = \frac{1}{\mu} \nabla_1 \times \overline{A}^{(1)} \tag{3.3}$$

$$\overline{E} = i\omega\overline{A}^{(1)} - \frac{1}{i\omega\epsilon\mu} \nabla_1\nabla_1 \cdot \overline{A}^{(1)} \quad , \tag{3.4}$$

where $\overline{A}^{(1)}$ is a solution to the following inhomogeneous equation:

$$\nabla_1^2\overline{A}^{(1)} + \omega^2\epsilon\mu\overline{A}^{(1)} = -\mu\overline{J} \quad . \tag{3.5}$$

Since the excitation current has only a y-component, we may let $\overline{A}^{(1)} = A_y^{(1)}\hat{y}$ and express the various field components in terms of $A_y^{(1)}$ as:

$$E_x = -\frac{1}{i\omega\epsilon\mu} \cdot \frac{\partial^2 A_y^{(1)}}{\partial x \partial y} \tag{3.6a}$$

$$E_y = i\omega A_y^{(1)} - \frac{1}{i\omega\epsilon\mu} \cdot \frac{\partial^2 A_y^{(1)}}{\partial y^2} \tag{3.6b}$$

$$E_z = -\frac{1}{i\omega\epsilon\mu} \cdot \frac{\partial^2 A_y^{(1)}}{\partial y \partial z} \tag{3.6c}$$

$$H_x = -\frac{\partial A_y^{(1)}}{\partial z} \quad ; \quad H_y = 0 \quad ; \quad H_z = \frac{\partial A_y^{(1)}}{\partial x} \quad . \tag{3.6d}$$

$A_y^{(1)}$ satisfies the inhomogeneous wave equation:

$$\nabla_1^2 A_y^{(1)} + \omega^2 \varepsilon\mu A_y^{(1)} = -\mu J_y \quad . \tag{3.7}$$

We look for the solutions of (3.7) having the form $A_y^{(1)} = Ae^{i\beta z}$. The reduced potential A satisfies

$$\nabla^2 A + k^2 A = -\mu I \quad , \tag{3.8}$$

where $k^2 = \omega^2 \varepsilon\mu - \beta^2$, and

$$I = \delta(x + x_0) \begin{bmatrix} \sin\left(\frac{N\pi y}{2H}\right) \\ \cos\left(\frac{N\pi y}{2H}\right) \end{bmatrix} \quad \begin{matrix} \text{for case (A) } N = 1, 2, \ldots \\ \text{for case (B) } N = 0, 1, 2, \ldots \end{matrix}$$

Enforcing the boundary condition on the tangential E-field at the plates, we have

$$\left.\frac{\partial A}{\partial y} = 0 \right|_{y = -H, H} \quad , -2L < x < 0 \quad . \tag{3.9}$$

For mathematical convenience we initially assume that ε has a small imaginary part, with $\text{Im}\varepsilon > 0$, intending to let $\text{Im}\varepsilon \to 0$ after the derivation of the solution is complete. We then have

$$k = \sqrt{\omega^2 \varepsilon\mu - \beta^2} = k_1 + ik_2 \quad , \tag{3.10}$$

where $k_1, k_2 > 0$.

We next proceed to derive the solution to the problem at hand using the Wiener-Hopf technique. To this end we define the transformed function Φ as

$$\Phi(\alpha, y) = \int_{-\infty}^{\infty} A(x,y) e^{i\alpha x} dx \quad , \tag{3.11}$$

where α is the Fourier transform variable. Since the region $|y| > H$ is source free, we have Φ satisfying the differential equation

$$\frac{\partial^2\Phi}{\partial y^2} - \gamma^2\Phi = 0 \quad , \; \gamma = \sqrt{\alpha^2 - k^2} \quad , \qquad (3.12)$$

which admits solutions of the form

$$\Phi(\alpha) = c_3 e^{\gamma y} \qquad \text{for } y < -H \qquad (3.13a)$$

$$\Phi(\alpha) = c_4 e^{-\gamma y} \qquad \text{for } y > H \qquad (3.13b)$$

with the requirement that

$$\mathrm{Re}\gamma = \mathrm{Re}\left(\sqrt{\alpha^2 - k^2}\right) \rightarrow +\infty \quad \text{as } \alpha \rightarrow \pm\infty \quad .$$

Figure 2 shows the branch cuts for γ in the complex α-plane. Taking the principal branch we get

$$\gamma = \sqrt{|\alpha - k||\alpha + k|}\; e^{i\frac{\phi+\psi}{2}} \quad \text{and} \qquad (3.14)$$

$$\gamma = -i\sqrt{k^2 - \alpha^2} \quad \text{for } |\alpha| < |k| \quad . \qquad (3.15)$$

In the region interior to the waveguide, i.e., for $|y| < H$, the differential equation for Φ takes the form

$$\frac{\partial^2\Phi}{\partial y^2} - \gamma^2\Phi = -\mu \begin{bmatrix} \sin\left(\frac{N\pi y}{2H}\right) \\ \cos\left(\frac{N\pi y}{2H}\right) \end{bmatrix} e^{-i\alpha x_0} \quad \begin{matrix} \text{Case (A)} \\ \text{Case (B)} \end{matrix} \qquad (3.16)$$

and the solution may be written as [see Appendix I]:

$$\Phi(y) = c_1 e^{\gamma y} + c_2 e^{-\gamma y} + \tilde{T}_N \begin{bmatrix} \sin\left(\frac{N\pi y}{2H}\right) \\ \cos\left(\frac{N\pi y}{2H}\right) \end{bmatrix} \begin{matrix} \text{Case (A) } N = 1,2,\ldots \\ \text{Case (B) } N = 0,1,2,\ldots \end{matrix} \qquad (3.17)$$

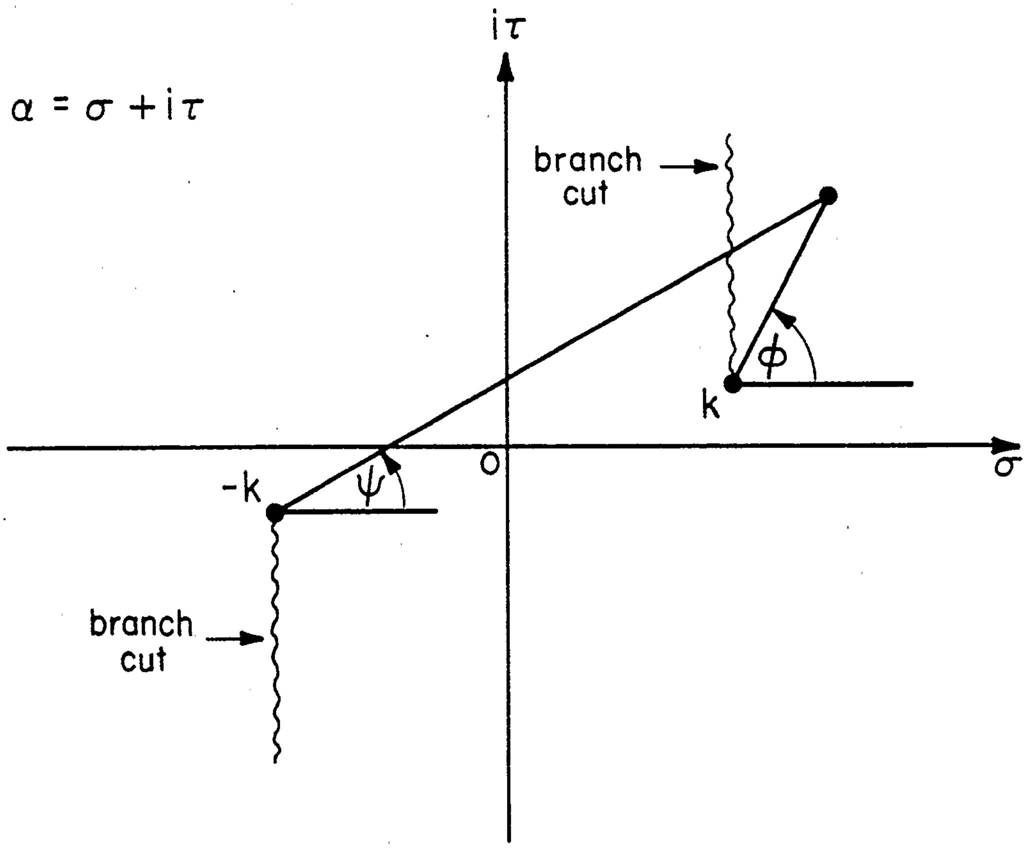

Fig. 2

Branch cuts in the complex α-plane.

with

$$\tilde{T}_N = \frac{e^{-i\alpha x_{0_\mu}}}{\gamma^2 + \left(\frac{N\pi}{2H}\right)^2} \qquad (3.18)$$

The coefficients c_1 and c_2 will, of course, be different for the cases A and B. Since

$$\frac{\partial \Phi}{\partial y} = \int_{-\infty}^{\infty} \frac{\partial A}{\partial y} e^{i\alpha x} \, dx \quad , \qquad (3.19)$$

from (3.13a) and (3.19) we have:

$$\int_{-\infty}^{\infty} \left.\frac{\partial A}{\partial y}\right|_{y = -H - 0} e^{i\alpha x} \, dx = c_3 \gamma e^{-\gamma H} \quad . \qquad (3.20)$$

Let

$$\begin{aligned} \left.\frac{\partial A}{\partial y}\right|_{y=-H} &= \psi_2 \; ; \; \left.\frac{\partial A}{\partial y}\right|_{y=H} = \psi_1 \quad \text{for } -\infty < x < -2L \\ \left.\frac{\partial A}{\partial y}\right|_{y=-H} &= \chi_2 \; ; \; \left.\frac{\partial A}{\partial y}\right|_{y=H} = \chi_1 \quad \text{for } 0 < x < \infty \; . \end{aligned} \qquad (3.21)$$

Then from (3.9), (3.20), (3.21),

$$\int_{-\infty}^{-2L} \psi_2 e^{i\alpha x} \, dx + \int_0^{\infty} \chi_2 e^{i\alpha x} \, dx = c_3 \gamma e^{-\gamma H} \qquad (3.22)$$

or

$$c_3 = \gamma^{-1} e^{\gamma H} \left[\int_{-\infty}^{-2L} \psi_2 e^{i\alpha x} dx + \int_0^{\infty} \chi_2 e^{i\alpha x} dx \right] \quad . \qquad (3.23)$$

Using the inverse Fourier transform, we finally obtain the representation of $A(x,y)$ for $y < -H$:

$$A(x,y) = \frac{1}{2\pi} \int_{-\infty+ib}^{\infty+ib} \gamma^{-1} e^{\gamma H} e^{\gamma y} e^{-i\alpha x} d\alpha \left[\int_{-\infty}^{-2L} \psi_2 e^{i\alpha\xi} d\xi + \int_0^{\infty} \chi_2 e^{i\alpha\xi} d\xi \right] . \tag{3.24}$$

Similarly for $y > H$ we have:

$$A(x,y) = -\frac{1}{2\pi} \int_{-\infty+ib}^{\infty+ib} \gamma^{-1} e^{\gamma H} e^{-\gamma y} e^{-i\alpha x} d\alpha \left[\int_{-\infty}^{-2L} \psi_1 e^{i\alpha\xi} d\xi + \int_0^{\infty} \chi_1 e^{i\alpha\xi} d\xi \right] \tag{3.25}$$

where

$$-k_2 < b < k_2 \quad .$$

Letting $y = \pm H$, we obtain from (3.9), (3.17), (3.19):

$$I_1 = \gamma c_1 e^{\gamma H} - \gamma c_2 e^{-\gamma H} + \tilde{T}_N \frac{N\pi}{2H} \begin{bmatrix} \cos\left(\frac{N\pi}{2}\right) \\ -\sin\left(\frac{N\pi}{2}\right) \end{bmatrix} \tag{3.26}$$

$$I_2 = \gamma c_1 e^{-\gamma H} - \gamma c_2 e^{\gamma H} + \tilde{T}_N \frac{N\pi}{2H} \begin{bmatrix} \cos\left(\frac{N\pi}{2}\right) \\ \sin\left(\frac{N\pi}{2}\right) \end{bmatrix} \tag{3.27}$$

where

$$I_1 = \int_{-\infty}^{-2L} \psi_1 e^{i\alpha x} dx + \int_0^{\infty} \chi_1 e^{i\alpha x} dx \tag{3.28}$$

$$I_2 = \int_{-\infty}^{-2L} \psi_2 e^{i\alpha x} dx + \int_0^{\infty} \chi_2 e^{i\alpha x} dx \tag{3.29}$$

where c_1 and c_2 are constants, given by [see Appendix II]:

$$c_1 = \frac{1}{2\sinh(2\gamma H)\gamma} \left\{ I_1 e^{\gamma H} - I_2 e^{-\gamma H} + \tilde{T}_N \frac{N\pi}{2H} 2 \begin{bmatrix} -\cos\left(\frac{N\pi}{2}\right) \sinh(\gamma H) \\ \sin\left(\frac{N\pi}{2}\right) \cosh(\gamma H) \end{bmatrix} \right\} \tag{3.30}$$

$$c_2 = \frac{1}{2\sinh(2\gamma H)\gamma}\left\{I_1 e^{-\gamma H} - I_2 e^{\gamma H} + \tilde{T}_N \frac{N\pi}{2H}\, 2\begin{bmatrix}\cos\left(\frac{N\pi}{2}\right) & \sinh(\gamma H)\\ \sin\left(\frac{N\pi}{2}\right) & \cosh(\gamma H)\end{bmatrix}\right\}. \tag{3.31}$$

In the interior region $|y| < H$ we have

$$\begin{aligned} A(x,y) &= \frac{1}{2\pi}\int_{-\infty+ib}^{\infty+ib} \Phi(\alpha,y)e^{-i\alpha x}d\alpha \\ &= \frac{1}{2\pi}\int_{-\infty+ib}^{\infty+ib}\left\{I_1\cosh[\gamma(y+H)] - I_2\cosh[(y-H)\gamma]\right\}\frac{e^{-i\alpha x}}{\gamma\sinh(2\gamma H)}d\alpha \\ &\quad + \frac{1}{2\pi}\cdot\frac{N\pi}{2H}\cdot 2\int_{-\infty+ib}^{\infty+ib}\begin{bmatrix}-\cos\frac{N\pi}{2}\sinh(\gamma H)\sinh(\gamma y)\\ \sin\frac{N\pi}{2}\cosh(\gamma H)\cosh(\gamma y)\end{bmatrix}\frac{\tilde{T}_N e^{-i\alpha x}}{\gamma\sinh(2\gamma H)}d\alpha \\ &\quad + \frac{1}{2\pi}\begin{bmatrix}\sin\left(\frac{N\pi y}{2H}\right)\\ \cos\left(\frac{N\pi y}{2H}\right)\end{bmatrix}\int_{-\infty+ib}^{\infty+ib}\tilde{T}_N e^{-i\alpha x}d\alpha \quad . \end{aligned} \tag{3.32}$$

Imposing the boundary conditions:

$$\left.\begin{aligned} A(x, +H+0) &= A(x, +H-0) \\ A(x, -H+0) &= A(x, -H-0) \end{aligned}\right. \quad \text{for} \left\{\begin{matrix} -\infty < x < -2L \\ \text{and } 0 < x < \infty \end{matrix}\right\} \tag{3.33}$$

we derive from (3.24), (3.25), (3.32) [see Appendix III] for $-\infty < x < -2L$, $0 < x < \infty$:

$$\int_{-\infty}^{-2L}\psi_1 K_1^{(*)}(k|x-\xi|)d\xi + \int_0^{\infty}\chi_1 K_1^{(*)}(k|x-\xi|)d\xi - \int_{-\infty}^{-2L}\psi_2 K_2^{(*)}(k|x-\xi|)d\xi$$

$$- \int_0^{\infty}\chi_2 K_2^{(*)}(k|x-\xi|)d\xi = f_1^{(*)}(x) \tag{3.34}$$

and

$$\int_{-\infty}^{-2L} \psi_1 K_2^{(*)}(k|x-\xi|)d\xi + \int_0^{\infty} \chi_1 K_2^{(*)}(k|x-\xi|)d\xi - \int_{-\infty}^{-2L} \psi_2 K_1^{(*)}(k|x-\xi|)d\xi$$

$$- \int_0^{\infty} \chi_2 K_1^{(*)}(k|x-\xi|)d\xi = f_2^{(*)}(x), \tag{3.35}$$

where

$$K_1^{(*)}(k|x-\xi|) = \frac{1}{2\pi} \int_{-\infty+ib}^{\infty+ib} \frac{e^{2H\gamma}}{\gamma \sinh(2H\gamma)} e^{i\alpha(\xi - x)} d\alpha \tag{3.36}$$

$$K_2^{(*)}(k|x-\xi|) = \frac{1}{2\pi} \int_{-\infty+ib}^{\infty+ib} \frac{1}{\gamma \sinh(2H\gamma)} e^{i\alpha(\xi - x)} d\alpha \tag{3.37}$$

$$f_1^{(*)}(x) = -\frac{N}{4H} \int_{-\infty+ib}^{\infty+ib} \begin{bmatrix} -\cos\left(\frac{N\pi}{2}\right) & \tanh(\gamma H) \\ \sin\left(\frac{N\pi}{2}\right) & \coth(\gamma H) \end{bmatrix} \frac{\tilde{T}_N e^{-i\alpha x}}{\gamma} d\alpha$$

$$- \frac{1}{2\pi} \begin{bmatrix} \sin\left(\frac{N\pi}{2}\right) \\ \cos\left(\frac{N\pi}{2}\right) \end{bmatrix} \int_{-\infty+ib}^{\infty+ib} \tilde{T}_N e^{-i\alpha x} d\alpha \tag{3.38}$$

$$f_2^{(*)}(x) = -\frac{N}{4H} \int_{-\infty+ib}^{\infty+ib} \begin{bmatrix} \cos\left(\frac{N\pi}{2}\right) & \tanh(\gamma H) \\ \sin\left(\frac{N\pi}{2}\right) & \coth(\gamma H) \end{bmatrix} \frac{\tilde{T}_N e^{-i\alpha x}}{\gamma} d\alpha$$

$$- \frac{1}{2\pi} \begin{bmatrix} -\sin\left(\frac{N\pi}{2}\right) \\ \cos\left(\frac{N\pi}{2}\right) \end{bmatrix} \int_{-\infty+ib}^{\infty+ib} \tilde{T}_N e^{-i\alpha x} d\alpha \quad . \tag{3.39}$$

Adding (3.34) and (3.35) we obtain

$$\int_{-\infty}^{-2L} Y_1(\xi)K_1(k|x-\xi|)d\xi + \int_{0}^{\infty} Z_1(\xi)K_1(k|x-\xi|)d\xi = f_1(x) \quad (3.40)$$

where

$$\begin{aligned} Y_1(\xi) &= \psi_1 - \psi_2 \\ Z_1(\xi) &= \chi_1 - \chi_2 \end{aligned} \quad (3.41)$$

$$K_1(k|x-\xi|) = K_1^{(*)}(k|x-\xi|) + K_2^{(*)}(k|x-\xi|) = \frac{1}{2\pi}\int_{-\infty+ib}^{\infty+ib} \frac{e^{\gamma H}}{\gamma \sinh(\gamma H)} e^{i\alpha(\xi-x)} d\alpha$$

$$(3.42)$$

$$f_1(x) = \begin{cases} 0 & \text{Case A} \\ F_1^{(N)}(x) & \text{Case B} \end{cases} \quad (3.43)$$

$$F_1^{(N)}(x) = -\frac{N}{2H}\sin\left(\frac{N\pi}{2}\right)\int_{-\infty+ib}^{\infty+ib} \tilde{T}_N \frac{\coth(\gamma H)}{\gamma} e^{-i\alpha x} d\alpha - \frac{1}{\pi}\cos\left(\frac{N\pi}{2}\right)\int_{-\infty+ib}^{\infty+ib} \tilde{T}_N e^{-i\alpha x} d\alpha \; .$$

$$(3.44)$$

Equation (3.40) represents one of the integral equations we have been seeking to derive. The unknowns in this equation are Y_1 and Z_1 and $f_1(x)$ is a known function related to the source.

Next we subtract (3.35) from (3.34) to get:

$$\int_{-\infty}^{-2L} Y_2(\xi)K_2(k|x-\xi|)d\xi + \int_{0}^{\infty} Z_2(\xi)K_2(k|x-\xi|)d\xi = f_2(x) \quad (3.45)$$

where

$$\begin{aligned} Y_2(\xi) &= \psi_1 + \psi_2 \\ Z_2(\xi) &= \chi_1 + \chi_2 \end{aligned} \quad (3.46)$$

$$K_2(k|x - \xi|) = K_1^{(*)}(k|x - \xi|) - K_2^{(*)}(k|x - \xi|) = \frac{1}{2\pi}\int_{-\infty+ib}^{\infty+ib} \frac{e^{\gamma H} e^{i\alpha(\xi - x)}}{\gamma \cosh(\gamma H)}\, d\alpha \tag{3.47}$$

$$f_2(x) = \begin{cases} F_2^{(N)}(x) & \text{Case A} \\ 0 & \text{Case B} \end{cases} \tag{3.48}$$

$$F_2^{(N)}(x) = \frac{N}{2H}\cos\left(\frac{N\pi}{2}\right)\int_{-\infty+ib}^{\infty+ib} \tilde{T}_N \frac{\tanh(\gamma H)}{\gamma} e^{-i\alpha x} d\alpha - \frac{1}{\pi}\sin\left(\frac{N\pi}{2}\right)\int_{-\infty+ib}^{\infty+ib} \tilde{T}_N e^{-i\alpha x} d\alpha \tag{3.49}$$

which is the second integral equation we have been seeking. Thus, in summary, we have reduced the original problem to that of solving a pair of integral equations (3.40) and (3.45) with four unknown functions, viz., $Y_1(\xi)$, $Z_1(\xi)$, $Y_2(\xi)$, and $Z_2(\xi)$.

IV. SOLUTION OF THE INTEGRAL EQUATIONS

As a first step we rewrite the integral equations (3.40) and (3.45) as:

$$\int_{-\infty}^{-2L} Y_1(\xi)K_1(k|x-\xi|)d\xi + \int_0^{\infty} Z_1(\xi)K_1(k|x-\xi|)d\xi = \begin{cases} f_1(x) & -\infty<x<-2L,\ 0<x<\infty \\ e_1(x) & -2L<x<0 \end{cases} \tag{4.1}$$

and

$$\int_{-\infty}^{-2L} Y_2(\xi)K_2(k|x-\xi|)d\xi + \int_0^{\infty} Z_2(\xi)K_2(k|x-\xi|)d\xi = \begin{cases} f_2(x) & -\infty<x<-2L,\ 0<x<\infty \\ e_2(x) & -2L<x<0 \end{cases} \tag{4.2}$$

where we have deliberately introduced two new unknown functions, viz., $e_1(x)$, $e_2(x)$, in order to extend the range of x from $-\infty$ to $+\infty$. This is important for the next step which is to multiply (4.1) and (4.2) throughout by $\exp(i\alpha x)$ and integrate with respect to x from $-\infty$ to ∞. This gives

$$\int_{-\infty}^{-2L} Y_1(\xi)e^{i\alpha\xi}d\xi\cdot K_3(\alpha) + \int_0^{\infty} Z_1(\xi)e^{i\alpha\xi}d\xi\cdot K_3(\alpha)$$

$$= \int_{-\infty}^{-2L} f_1(x)e^{i\alpha x}dx + \int_{-2L}^{0} e_1(x)e^{i\alpha x}dx + \int_0^{\infty} f_1(x)e^{i\alpha x}dx \tag{4.3}$$

and

$$\int_{-\infty}^{-2L} Y_2(\xi)e^{i\alpha\xi}d\xi\cdot K_4(\alpha) + \int_0^{\infty} Z_2(\xi)e^{i\alpha\xi}d\xi\cdot K_4(\alpha)$$

$$= \int_{-\infty}^{-2L} f_2(x)e^{i\alpha x}dx + \int_{-2L}^{0} e_2(x)e^{i\alpha x}dx + \int_0^{\infty} f_2(x)e^{i\alpha x}dx \tag{4.4}$$

where

$$K_3(\alpha) = \frac{e^{\gamma H}}{\gamma \sinh(\gamma H)} \quad , \quad K_4(\alpha) = \frac{e^{\gamma H}}{\gamma \cosh(\gamma H)} \tag{4.5}$$

are analytic in the strip $-k_2 < \mathrm{Im}\alpha < k_2$.

Defining the transforms of the unknown functions in (4.3) and (4.4) and indicating their domains of analyticity, we have

$$\Phi_+(\alpha) = \int_0^\infty Z_1(\xi) e^{i\alpha\xi} d\xi \quad , \tag{4.6}$$

$$\Phi_-(\alpha) = \int_{-\infty}^{-2L} Y_1(\xi) e^{i\alpha(\xi+2L)} d\xi \quad , \tag{4.7}$$

$$\psi_+(\alpha) = \int_0^\infty Z_2(\xi) e^{i\alpha\xi} d\xi \quad , \tag{4.8}$$

$$\psi_+(\alpha) = \int_{-\infty}^{-2L} Y_2(\xi) e^{i\alpha(\xi+2L)} d\xi \quad , \tag{4.9}$$

where the functions $\Phi_+(\alpha)$, $\Psi_+(\alpha)$ are analytic for $\mathrm{Im}\alpha > -k_2$ and $\Phi_-(\alpha)$, $\psi_-(\alpha)$ are analytic for $\mathrm{Im}\alpha < k_2$. We can also write the transforms of the known functions $f_1(x)$ and $f_2(x)$ in the range $-\infty < x < -2L$ as

$$H_j(\alpha) = \int_{-\infty}^{-2L} f_j(x) e^{i\alpha x} dx = e^{-i\alpha 2L} H_{j-}(\alpha) \tag{4.10}$$

where $j = 1, 2$ and

$$H_{j-}(\alpha) = \int_{-\infty}^{-2L} f_j(x) e^{i\alpha(x+2L)} dx \quad . \tag{4.11}$$

Likewise for the range $0 < x < \infty$ we have the transform

$$H_{j+}(\alpha) = \int_0^{\infty} f_j(x) e^{i\alpha x} dx \tag{4.12}$$

where

$$H_{1\pm}(\alpha) = \begin{cases} 0 & , \quad \text{Case A} \\ \tilde{H}_{1\pm}^{(N)}(\alpha) & , \quad \text{Case B} \end{cases} \tag{4.13}$$

where the superscript (N) is associated with the excitation function and is defined in (2.1) and (2.2). To obtain the expressions for $H_{1\pm}(\alpha)$, we have to calculate the functions $f_1(x)$ for two cases: (a) $-\infty < x < -2L$ and (b) $0 < x < \infty$. We also need to perform these calculations for N both even and odd. For Case (A) we need to close the contour with a semicircle in the upper half plane, whereas the closure for the second case is in the lower half plane. Substituting the results of these calculations in (4.12), we obtain the final expressions for $\tilde{H}_{1\pm}^{(2\ell)}$ and $\tilde{H}_{1\pm}^{(2\ell-1)}$, which read:

$$\tilde{H}_{1\pm}^{(2\ell)}(\alpha) = \pm (-1)^{\ell} \mu \frac{1}{\alpha_\ell'(\alpha \pm \alpha_\ell')} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{i\alpha_\ell' x_0} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{i\alpha_\ell'(2L - x_0)} \right\} \tag{4.14}$$

$\ell = 0, 1, \ldots$

$$\tilde{H}_{1\pm}^{(2\ell-1)}(\alpha) = \pm (-1)^{\ell+1} \frac{(2\ell-1)\mu}{\pi} \left\{ \frac{2}{(2\ell-1)^2} \frac{1}{k(\alpha \pm k)} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{i\alpha_\ell' x_0} \right.\right.$$

$\ell = 1, 2, \ldots$

$$\left. + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{i\alpha_\ell'(2L - x_0)} \right) - \sum_{n=1}^{n_1} \frac{1}{n^2 \alpha_n' \left[1 - \left(\frac{\ell - 1/2}{n} \right)^2 \right] (\alpha \pm \alpha_n')}$$

$$\left. \cdot \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{i\alpha_n' x_0} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{i\alpha_n'(2L - x_0)} \right) \right\} \tag{4.15}$$

Note that we have neglected the exponentially decaying terms by retaining only n_1 terms in the summation. The integer n_1 satisfies the conditions:

$$\frac{n_1\pi}{H} < k \quad , \quad \frac{(n_1+1)\pi}{H} > k \quad . \tag{4.16}$$

We can also show that

$$H_{2\pm}(\alpha) = \begin{cases} \tilde{H}^{(N)}_{2\pm}(\alpha) & , \quad \text{Case A} \\ 0 & , \quad \text{Case B} \end{cases} \tag{4.17}$$

where

$$\tilde{H}^{(2\ell)}_{2\pm}(\alpha) = \pm (-1)^{\ell}\, \ell\, \frac{2\mu}{\pi} \sum_{n=1}^{n_2} \frac{1}{\left(n-\frac{1}{2}\right)^2 \alpha'_{n-1/2}\left[1-\left(\frac{\ell}{n-1/2}\right)^2\right]\left(\alpha \pm \alpha'_{n-1/2}\right)}$$

$\ell = 1,2,\ldots$

$$\cdot\left(\begin{bmatrix}1\\0\end{bmatrix} e^{i\alpha'_{n-1/2}x_0} + \begin{bmatrix}0\\1\end{bmatrix} e^{i\alpha'_{n-1/2}(2L-x_0)}\right) \tag{4.18}$$

and n_2 is determined from the condition

$$\frac{(n_2-1/2)\pi}{H} < k \quad , \quad \frac{(n_2+1/2)\pi}{H} > k \tag{4.19}$$

and

$$\tilde{H}^{(2\ell-1)}_{2\pm}(\alpha) = \pm (-1)^{\ell+1}\, \mu \frac{1}{\alpha'_{\ell-1/2}\left(\alpha \pm \alpha'_{\ell-1/2}\right)} \left(\begin{bmatrix}1\\0\end{bmatrix} e^{i\alpha'_{\ell-1/2}x_0} + \begin{bmatrix}0\\1\end{bmatrix} e^{i\alpha'_{\ell-1/2}(2L-x_0)}\right) .$$

$\ell = 1,2,\ldots$ (4.20)

Utilizing the definition of various transforms,we rewrite (4.3) and (4.4) as:

$$e^{-i\alpha 2L}\Phi_-(\alpha) + \Phi_+(\alpha) - \gamma^2 H\, M_1(\alpha)\Phi_1(\alpha) = \left[e^{-i\alpha 2L} H_{1-}(\alpha) + H_{1+}(\alpha)\right] \gamma^2 H M_1(\alpha) \tag{4.21}$$

and

$$e^{-i\alpha 2L}\psi_-(\alpha) + \psi_+(\alpha) - \gamma M_2(\alpha)\psi_1(\alpha) = \left[e^{-i\alpha 2L} H_{2-}(\alpha) + H_{2+}(\alpha)\right] \gamma M_2(\alpha) \quad (4.22)$$

where

$$\Phi_1(\alpha) = \int_{-2L}^{0} e_1(x)e^{i\alpha x}dx \quad (4.23)$$

$$\psi_1(\alpha) = \int_{-2L}^{0} e_2(x)e^{i\alpha x}dx \quad (4.24)$$

$$M_1(\alpha) = e^{-\gamma H} \frac{\sinh(\gamma H)}{\gamma H} \quad (4.25)$$

$$M_2(\alpha) = e^{-\gamma H} \cosh(\gamma H) \quad . \quad (4.26)$$

The next step is to factorize the functions $M_1(\alpha)$ and $M_2(\alpha)$ in the form of products,

$$M_1(\alpha) = M_{1+}(\alpha)\, M_{1-}(\alpha) \quad (4.27)$$

$$M_2(\alpha) = M_{2+}(\alpha)\, M_{2-}(\alpha) \quad (4.28)$$

where $M_{1+}(\alpha)$, $M_{2+}(\alpha)$ are regular and non-zero in the upper half plane $\tau > -k_2$, whereas $M_{1-}(\alpha)$, $M_{2-}(\alpha)$ are regular and non-zero in the lower half plane $\tau < k_2$. Then, multiplication of (4.21) by

$$\frac{e^{+i\alpha 2L}}{(\alpha - k)M_{1-}(\alpha)}$$

and (4.22) by

$$\frac{e^{i\alpha 2L}}{\sqrt{\alpha-k}\; M_{2-}(\alpha)}$$

leads to the following coupled equations in the transform domain:

$$\frac{\Phi_-(\alpha)}{(\alpha-k)M_{1-}(\alpha)} + \frac{\Phi_+(\alpha)e^{i\alpha 2L}}{(\alpha-k)M_{1-}(\alpha)} - (\alpha+k)HM_{1+}(\alpha)\Phi_1(\alpha)e^{i\alpha 2L}$$

$$= (\alpha+k)HM_{1+}(\alpha)\ H_{1-}(\alpha)+(\alpha+k)HM_{1+}(\alpha)H_{1+}(\alpha)\,e^{i\alpha 2L} \tag{4.29}$$

and

$$\frac{\psi_-(\alpha)}{\sqrt{\alpha-k}\ M_{2-}(\alpha)} + \frac{\psi_+(\alpha)e^{i\alpha 2L}}{\sqrt{\alpha-k}\ M_{2-}(\alpha)} - \sqrt{\alpha+k}\ M_{2+}(\alpha)\psi_1(\alpha)e^{i\alpha 2L}$$

$$= \sqrt{\alpha+k}\ M_{2+}(\alpha)\ H_{2-}(\alpha) + \sqrt{\alpha+k}\ M_{2+}(\alpha)H_{2+}(\alpha)e^{i\alpha 2L}\quad . \tag{4.30}$$

The first terms on the left-hand side of (4.29) and (4.30) are regular in the lower half plane, whereas the third terms on the left-hand side and the second terms on the right-hand side of (4.29) and (4.30) are regular in the upper half plane.

To solve these equations we carry out the decomposition

$$e^{i\alpha 2L}\ \frac{\Phi_+(\alpha)}{(\alpha-k)M_{1-}(\alpha)} = R_+(\alpha) + R_-(\alpha) \tag{4.31}$$

$$(\alpha+k)HM_{1+}(\alpha)\ H_{1-}(\alpha) = \tilde{S}_{1+}(\alpha) + \tilde{S}_{1-}(\alpha) \tag{4.32}$$

$$e^{i\alpha 2L}\ \frac{\psi_+(\alpha)}{\sqrt{\alpha-k}\ M_{2-}(\alpha)} = Q_+(\alpha) + Q_-(\alpha) \tag{4.33}$$

$$\sqrt{\alpha+k}\ M_{2+}(\alpha)\ H_{2-}(\alpha) = \tilde{S}_{2+}(\alpha) + \tilde{S}_{2-}(\alpha)\quad . \tag{4.34}$$

Using the method of factorization we obtain the equations:

$$\frac{\Phi_-(\alpha)}{(\alpha-k)M_{1-}(\alpha)} + R_-(\alpha) - \tilde{S}_{1-}(\alpha) = 0 \tag{4.35}$$

$$\frac{\psi_-(\alpha)}{\sqrt{\alpha-k}\,M_{2-}(\alpha)} + Q_-(\alpha) - \widetilde{S}_{2-}(\alpha) = 0 \tag{4.36}$$

where

$$R_-(\alpha) = -\frac{1}{2\pi i}\int_{id-\infty}^{id+\infty} \frac{e^{i\zeta 2L}\,\Phi_+(\zeta)}{(\zeta-k)M_{1-}(\zeta)(\zeta-\alpha)}\,d\zeta \tag{4.37}$$

$$Q_-(\alpha) = -\frac{1}{2\pi i}\int_{id-\infty}^{id+\infty} \frac{e^{i\zeta 2L}\psi_+(\zeta)}{\sqrt{\zeta-k}\,M_{2-}(\zeta)(\zeta-\alpha)}\,d\zeta \tag{4.38}$$

$$\widetilde{S}_{1-}(\alpha) = -\frac{1}{2\pi i}\int_{id-\infty}^{id+\infty} \frac{(\zeta+k)HM_{1+}(\zeta)\;H_{1-}(\zeta)}{\zeta-\alpha}\,d\zeta \tag{4.39}$$

$$\widetilde{S}_{2-}(\alpha) = -\frac{1}{2\pi i}\int_{id-\infty}^{id+\infty} \frac{\sqrt{\zeta+k}\,M_{2+}(\zeta)\;H_{2-}(\zeta)}{\zeta-\alpha}\,d\zeta \tag{4.40}$$

$$\tau < d < k_2 \quad , \; \tau = \mathrm{Im}\alpha \quad .$$

After multiplying (4.21) by $1/[(\alpha+k)M_{1+}(\alpha)]$ and (4.22) by $1/[\sqrt{\alpha+k}\;M_{2+}(\alpha)]$ we obtain:

$$\frac{\Phi_+(\alpha)}{(\alpha+k)M_{1+}(\alpha)} + \frac{e^{-i\alpha 2L}\Phi_-(\alpha)}{(\alpha+k)\;M_{1+}(\alpha)} - (\alpha-k)HM_{1-}(\alpha)\Phi_1(\alpha) = e^{-i\alpha 2L}H_{1-}(\alpha)(\alpha-k)\,HM_{1-}(\alpha)$$

$$+ H_{1+}(\alpha)(\alpha-k)HM_{1-}(\alpha) \tag{4.41}$$

and

$$\frac{\psi_+(\alpha)}{\sqrt{\alpha+k}\,M_{2+}(\alpha)} + \frac{e^{-i\alpha 2L}\psi_-(\alpha)}{\sqrt{\alpha+k}\;M_{2+}(\alpha)} - \sqrt{\alpha-k}\;M_{2-}(\alpha)\psi_1(\alpha)$$

$$= e^{-i\alpha 2L}H_{2-}(\alpha)\sqrt{\alpha-k}\,M_{2-}(\alpha) + H_{2+}(\alpha)\sqrt{\alpha-k}\;M_{2-}(\alpha) \quad . \tag{4.42}$$

Note that the first terms on the left-hand side of (4.41) and (4.42) are regular in the upper half plane; the third terms on the left-hand side and the first terms on the right-hand side are regular in the lower half plane.

We now use the decompositions:

$$\frac{e^{-i\alpha 2L}\Phi_-(\alpha)}{(\alpha + k)\, M_{1+}(\alpha)} = U_+(\alpha) + U_-(\alpha) \tag{4.43a}$$

$$\frac{e^{-i\alpha 2L}\psi_-(\alpha)}{\sqrt{\alpha + k}\, M_{2+}(\alpha)} = \theta_+(\alpha) + \theta_-(\alpha) \tag{4.43b}$$

$$H_{1+}(\alpha)(\alpha - k)\, HM_{1-}(\alpha) = V_{1+}(\alpha) + V_{1-}(\alpha) \tag{4.43c}$$

$$H_{2+}(\alpha)\sqrt{\alpha - k}\, M_{2-}(\alpha) = V_{2+}(\alpha) + V_{2-}(\alpha) \quad . \tag{4.43d}$$

Substituting (4.43) into (4.41) and (4.42) and using the Wiener-Hopf technique [10,11] result in the equations:

$$\frac{\Phi_+(\alpha)}{(\alpha + k)M_{1+}(\alpha)} + U_+(\alpha) - V_{1+}(\alpha) = 0 \tag{4.44}$$

$$\frac{\psi_+(\alpha)}{\sqrt{\alpha + k}\, M_{2+}(\alpha)} + \theta_+(\alpha) - V_{2+}(\alpha) = 0 \tag{4.45}$$

where

$$U_+(\alpha) = \frac{1}{2\pi i}\int_{ic-\infty}^{ic+\infty} \frac{e^{-i\zeta 2L}\Phi_-(\zeta)}{(\zeta + k)M_{1+}(\zeta)(\zeta - \alpha)}\, d\zeta \quad , \tag{4.46}$$

$$\theta_+(\alpha) = \frac{1}{2\pi i}\int_{ic-\infty}^{ic+\infty} \frac{e^{-i\zeta 2L}\psi_-(\zeta)}{\sqrt{\zeta + k}\, M_{2+}(\zeta)(\zeta - \alpha)}\, d\zeta \quad , \tag{4.47}$$

$$V_{1+}(\alpha) = \frac{1}{2\pi i} \int_{ic-\infty}^{ic+\infty} \frac{H_{1+}(\zeta)(\zeta - k)HM_{1-}(\zeta)}{\zeta - \alpha} d\zeta \quad , \tag{4.48}$$

$$V_{2+}(\alpha) = \frac{1}{2\pi i} \int_{ic-\infty}^{ic+\infty} \frac{H_{2+}(\zeta)\sqrt{\zeta - k}\, M_{2-}(\zeta)}{\zeta - \alpha} d\zeta \quad , \tag{4.49}$$

$$-k_2 < c < \tau \quad .$$

The above manipulations have resulted in four coupled integral equations (4.35), (4.36), (4.44) and (4.45) each of which contains two unknowns. We now proceed to derive a set of new equations each with only a single unknown. To this end, we replace α by $-\alpha$ in (4.35) and (4.36) and ζ by $(-\zeta)$ in (4.46) and (4.47). Using the representations $\sqrt{-\alpha + k} = i\sqrt{\alpha - k}$, $\sqrt{-\alpha - k} = -i\sqrt{\alpha + k}$, adding and subtracting the resulting equations and defining

$$\begin{bmatrix} S_{1+}(\alpha) \\ D_{1+}(\alpha) \end{bmatrix} = \Phi_+(\alpha) \pm \Phi_-(-\alpha) \tag{4.50a}$$

$$\begin{bmatrix} S_{2+}(\alpha) \\ D_{2+}(\alpha) \end{bmatrix} = \psi_+(\alpha) \pm \psi_-(-\alpha) \tag{4.50b}$$

We obtain the two sets of coupled equations

$$\begin{bmatrix} S_{1+}(\alpha) \\ D_{1+}(\alpha) \end{bmatrix} \cdot \frac{1}{(\alpha + k)M_{1+}(\alpha)} \pm \frac{1}{2\pi i} \int_{id-\infty}^{id+\infty} \begin{bmatrix} S_{1+}(\zeta) \\ D_{1+}(\zeta) \end{bmatrix} \frac{e^{i\zeta 2L} d\zeta}{(\zeta - k)M_{1-}(\zeta)(\zeta + \alpha)}$$
$$- \left[V_{1+}(\alpha) \mp \widetilde{S}_{1-}(-\alpha) \right] = 0 \tag{4.51}$$

$$\begin{bmatrix} S_{2+}(\alpha) \\ D_{2+}(\alpha) \end{bmatrix} \cdot \frac{i}{\sqrt{\alpha + k}\, M_{2+}(\alpha)} \mp \frac{1}{2\pi i} \int_{id-\infty}^{id+\infty} \begin{bmatrix} S_{2+}(\zeta) \\ D_{2+}(\zeta) \end{bmatrix} \frac{e^{i\zeta 2L} d\zeta}{\sqrt{\zeta - k}\, M_{2-}(\zeta)(\zeta + \alpha)}$$
$$- \left[iV_{2+}(\alpha) \pm \widetilde{S}_{2-}(-\alpha) \right] = 0 \quad . \tag{4.52}$$

It is shown in Appendix IV that for $k2L \gg 1$ the integrals appearing in (4.51) and (4.52) can be evaluated in a series form as follows:

$$\frac{1}{2\pi i} \int_{id-\infty}^{id+\infty} \begin{bmatrix} S_{1+}(\zeta) \\ D_{1+}(\zeta) \end{bmatrix} \frac{e^{i\zeta 2L} d\zeta}{(\zeta - k) M_{1-}(\zeta)(\zeta + \alpha)}$$

$$= \begin{bmatrix} S_{1+}(k) \\ D_{1+}(k) \end{bmatrix} \frac{e^{ik2L} M_{1+}(k)}{(\alpha + k)} \left\{ 1 + \frac{H}{\pi} e^{-i(3/4)\pi} \sqrt{2k}\,\sqrt{2L}\,(\alpha + k) W_{-1}\left[-i2L(\alpha + k)\right] \right\}$$

$$+ \sum_{n=1}^{n_1} \begin{bmatrix} S_{1+}(\alpha_n') \\ D_{1+}(\alpha_n') \end{bmatrix} \frac{e^{i\alpha_n' 2L} M_{1+}(\alpha_n')(\alpha_n' + k)}{\alpha_n'(\alpha + \alpha_n')} \qquad (4.53)$$

and

$$\frac{1}{2\pi i} \int_{id-\infty}^{id+\infty} \begin{bmatrix} S_{2+}(S) \\ D_{2+}(S) \end{bmatrix} \frac{e^{i\zeta 2L} d\zeta}{\sqrt{\zeta - k}\, M_{2-}(\zeta)(\zeta + \alpha)}$$

$$= \frac{1}{\pi} e^{-i(3/4)\pi} \sqrt{2L}\, W_{-1}\left[-i2L(\alpha + k)\right] e^{ik2L} M_{2+}(k) \cdot \begin{bmatrix} S_{2+}(k) \\ D_{2+}(k) \end{bmatrix}$$

$$+ \frac{1}{H} \sum_{n=1}^{n_2} \begin{bmatrix} S_{2+}(\alpha_{n-1/2}') \\ D_{2+}(\alpha_{n-1/2}') \end{bmatrix} \frac{e^{i\alpha_{n-1/2}' 2L} M_{2+}(\alpha_{n-1/2}')\sqrt{\alpha_{n-1/2}' + k}}{(\alpha + \alpha_{n-1/2}')\alpha_{n-1/2}'} \quad . \qquad (4.54)$$

Substituting the various series expressions given in (4.53), and (4.54) into (4.51) and (4.52) gives:

$$\begin{bmatrix} S_{1+}(\alpha) \\ D_{1+}(\alpha) \end{bmatrix} = \mp (\alpha + k) M_{1+}(\alpha) \left\{ \begin{bmatrix} S_{1+}(k) \\ D_{1+}(k) \end{bmatrix} \frac{e^{ik2L} M_{1+}(k)}{\alpha + k} \right.$$

$$\cdot \left(1 + \frac{H}{\pi} e^{-i3/4\pi} \sqrt{2k}\,\sqrt{2L}\,(\alpha + k)\, W_{-1}\left[-i2L(\alpha + k)\right] \right)$$

$$+\sum_{n=1}^{n_1}\begin{bmatrix}S_{1+}(\alpha_n')\\ D_{1+}(\alpha_n')\end{bmatrix}\frac{e^{i\alpha_n' 2L}M_{1+}(\alpha_n')(\alpha_n'+k)}{\alpha_n'(\alpha+\alpha_n')}+\left[\mp V_{1+}(\alpha)+\tilde{S}_{1-}(-\alpha)\right]\Bigg\} \tag{4.55}$$

and

$$\begin{bmatrix}S_{2+}(\alpha)\\ D_{2+}(\alpha)\end{bmatrix}=\mp\ i\sqrt{\alpha+k}\ M_{2+}(\alpha)\Bigg\{\frac{1}{\pi}e^{-i3/4\pi}\sqrt{2L}\ W_{-1}\ [-i2L(\alpha+k)]$$

$$\cdot\ e^{ik2L}M_{2+}(k)\cdot\begin{bmatrix}S_{2+}(k)\\ D_{2+}(k)\end{bmatrix}+\frac{1}{H}\sum_{n=1}^{n_2}\begin{bmatrix}S_{2+}(\alpha_{n-1/2}')\\ D_{2+}(\alpha_{n-1/2}')\end{bmatrix}$$

$$\cdot\ \frac{e^{i\alpha_{n-1/2}' 2L}M_{2+}(\alpha_{n-1/2}')\sqrt{\alpha_{n-1/2}'+k}}{\alpha_{n-1/2}'(\alpha+\alpha_{n-1/2}')}+[\pm\ iV_{2+}(\alpha)+\tilde{S}_{2-}(-\alpha)]\Bigg\}\ . \tag{4.56}$$

Our next step is to obtain the expressions for the vector potentials which depend on the unknown functions $\Phi_-(\alpha)$, $\Phi_+(\alpha)$, $\psi_+(\alpha)$, $\psi_-(\alpha)$, and which are in turn expressed in terms of the functions $S_{1+}(\alpha)$, $D_{1+}(\alpha)$, $S_{2+}(\alpha)$, $D_{2+}(\alpha)$.

From (4.50) we have

$$\Phi_\pm(\pm\alpha)=\tfrac{1}{2}[S_{1+}(\alpha)\pm D_{1+}(\alpha)] \tag{4.57a}$$

$$\psi_\pm(\pm\alpha)=\tfrac{1}{2}[S_{2+}(\alpha)\pm D_{2+}(\alpha)]\quad . \tag{4.57b}$$

Changing $\alpha\to-\alpha$ in (4.57) yields

$$\Phi_-(\alpha)=\tfrac{1}{2}[S_{1+}(-\alpha)-D_{1+}(-\alpha)] \tag{4.58a}$$

$$\psi_-(\alpha)=\tfrac{1}{2}[S_{2+}(-\alpha)-D_{2+}(-\alpha)]\quad . \tag{4.58b}$$

We can rewrite (4.57) and (4.58) as:

$$\Phi_\pm(\alpha)=\tfrac{1}{2}[S_{1+}(\pm\alpha)\pm D_{1+}(\pm\alpha)] \tag{4.59a}$$

$$\psi_\pm(\alpha)=\tfrac{1}{2}[S_{2+}(\pm\alpha)\pm D_{2+}(\pm\alpha)]\quad . \tag{4.59b}$$

To obtain the expressions for $\Phi_+(\alpha)$ and $\psi_+(\alpha)$, we need to substitute (4.55) and (4.56) into (4.59). Likewise $\Phi_-(\alpha)$ and $\psi_-(\alpha)$ are obtained by changing $\alpha \to -\alpha$ in (4.55) and (4.56) and substituting the results in (4.59). Following these steps we derive the equations:

$$\begin{aligned}\Phi_\pm(\alpha) = \pm\frac{1}{2}(\alpha \pm k)M_{1\pm}(\alpha) \Bigg\{ & \frac{\left[D_{1+}(k) \mp S_{1+}(k)\right] M_{1+}(k)e^{ik2L}}{\alpha \pm k} \\ & \cdot \left(1 \pm \frac{H}{\pi} e^{-i3/4\pi} \sqrt{2k}\,\sqrt{2L}\,(\alpha \pm k)\, W_{-1}[\mp i2L(\alpha \pm k)]\right) \\ & + \sum_{n=1}^{n_1} \frac{[D_{1+}(\alpha_n') \mp S_{1+}(\alpha_n')]e^{i\alpha_n' 2L} M_{1+}(\alpha_n')(\alpha_n' + k)}{\alpha_n'(\alpha \pm \alpha_n')} \\ & + 2\begin{bmatrix} V_{1+}(\alpha) \\ -\tilde{S}_{1-}(\alpha)\end{bmatrix} \Bigg\} .\end{aligned} \qquad (4.60)$$

and

$$\begin{aligned}\psi_\pm(\alpha) = \begin{bmatrix} i \\ 1 \end{bmatrix} \frac{\sqrt{\alpha \pm k}\, M_{2\pm}(\alpha)}{2} \Bigg\{ & \left[D_{2+}(k) \mp S_{2+}(k)\right] \\ & \cdot \frac{1}{\pi} e^{-i3/4\pi} \sqrt{2L}\, W_{-1}[\mp i2L(\alpha \pm k)]e^{ik2L}M_{2+}(k) \\ & - \frac{1}{H}\sum_{n=1}^{n_2} \frac{\left[\mp D_{2+}(\alpha_{n-1/2}') + S_{2+}(\alpha_{n-1/2}')\right] e^{i\alpha_{n-1/2}' 2L} M_{2+}(\alpha_{n-1/2}')\sqrt{\alpha_{n-1/2}' + k}}{\alpha_{n-1/2}'(\alpha \pm \alpha_{n-1/2}')} \\ & + 2\begin{bmatrix} -iV_{2+}(\alpha) \\ \tilde{S}_{2-}(\alpha)\end{bmatrix} \Bigg\} .\end{aligned} \qquad (4.61)$$

The last two equations provide us with the expressions for the functions $\Phi_\pm(\alpha)$ and $\psi_\pm(\alpha)$. They depend on the group of constants $[S_{1+}(k), D_{1+}(k), S_{2+}(k), D_{2+}(k)]$ and also on $[S_{1+}(\alpha_n'), D_{1+}(\alpha_n'), S_2(\alpha_{n-1/2}'), D_{2+}(\alpha_{n-1/2}')]$.

We can express the first set of four constants in terms of the ones appearing inside the second bracket. This is accomplished by returning to (4.55) and (4.56) and substituting $\alpha \to k$. The resulting relationships can be further simplified for $2kL >> 1$, i.e., for wide plates, by introducing the asymptotic forms

$$W_{-1}(z) = \frac{\sqrt{\pi}}{z} \quad \text{for} \quad z \to \infty \ , \quad -\pi < \arg z < \pi \quad . \tag{4.62}$$

We then obtain the following desired equation relating $S_{1+}(k)$, and $D_{1+}(k)$ with $S_{1+}(\alpha_n')$, $D_{1+}(\alpha_n')$, etc.,

$$\begin{bmatrix} S_{1+}(k) \\ D_{1+}(k) \end{bmatrix} = \frac{2kM_{1+}(k)}{1 \pm T_1} \left\{ \mp \sum_{n=1}^{n_1} \begin{bmatrix} S_{1+}(\alpha_n') \\ D_{1+}(\alpha_n') \end{bmatrix} \frac{e^{i\alpha_n' 2L} M_{1+}(\alpha_n')}{\alpha_n'} \right.$$
$$\left. + [V_{1+}(k) \mp \tilde{S}_{1-}(-k)] \right\} \tag{4.63}$$

and

$$\begin{bmatrix} S_{2+}(k) \\ D_{2+}(k) \end{bmatrix} = \mp \frac{i\sqrt{2k}M_{2+}(k)}{1 \pm T_2} \left\{ \frac{1}{H} \sum_{n=1}^{n_2} \begin{bmatrix} S_{2+}(\alpha_{n-1/2}') \\ D_{2+}(\alpha_{n-1/2}') \end{bmatrix} \frac{e^{i\alpha_{n-1/2}' 2L} M_{2+}(\alpha_{n-1/2}')}{\alpha_{n-1/2}' \sqrt{k + \alpha_{n-1/2}'}} \right.$$
$$\left. + [\tilde{S}_{2-}(-k) \pm iV_{2+}(k)] \right\} \quad , \tag{4.64}$$

where

$$T_1 = [M_{1+}(k)]^2 e^{ik2L} \left[1 + \frac{H}{\sqrt{\pi}} e^{-i\pi/4} \sqrt{\frac{2k}{2L}} \right] \quad , \tag{4.65}$$

$$T_2 = [M_{2+}(k)]^2 e^{ik2L} \frac{e^{i\pi/4}}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{2k2L}} \quad . \tag{4.66}$$

The constants $S_{1+}(\alpha_n')$, $D_{1+}(\alpha_n')$, $S_{2+}(\alpha_{n-1/2}')$ and $D_{2+}(\alpha_{n-1/2}')$ satisfy a set of algebraic equations, which is derived by substituting $\alpha = \alpha_m'$ into (4.55), $\alpha = \alpha_{m-1/2}'$ into (4.56) and using the asymptotic representation $W_{-1}(z)$ and

(4.63) through (4.66) [see Appendix V]. These equations take the form:

For $S_{1+}(\alpha_n')$, $D_{1+}(\alpha_n')$:

$$\sum_{n=1}^{n_1} \begin{bmatrix} S_{1+}(\alpha_n') \\ D_{1+}(\alpha_n') \end{bmatrix} \begin{bmatrix} G_{mn}^{(1)\pm} - \delta_m^n \end{bmatrix} = P_m^{(1)\pm} \quad \text{where } m = 1,2,\ldots,n_1 \tag{4.67}$$

where

$$G_{mn}^{(1)\pm} = \frac{e^{i\alpha_n' 2L} M_{1+}(\alpha_m') M_{1+}(\alpha_n')}{\alpha_n'} \left[\frac{2kT_1}{1 \pm T_1} \mp \frac{(\alpha_m' + k)(\alpha_n' + k)}{\alpha_m' + \alpha_n'} \right] \tag{4.68}$$

$$\delta_m^n = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases} \tag{4.69}$$

$$P_m^{(1)\pm} = \begin{cases} 0 & \text{Case A} \\ \tilde{P}_{m,N}^{(1)\pm} & \text{Case B} \end{cases} \tag{4.70}$$

(i) for N-even:

$$\tilde{P}_{m,2\ell}^{(1)\pm} = (-1)^{\ell} H\mu \frac{M_{1+}(\alpha_m') M_{1+}(\alpha_\ell')}{\alpha_\ell'} \left[\mp \frac{2kT_1}{1 \pm T_1} + \frac{(\alpha_m' + k)(\alpha_\ell' + k)}{(\alpha_m' + \alpha_\ell')} \right]$$
$$\left[e^{i\alpha_\ell' x_0} \pm e^{i\alpha_\ell'(2L - x_0)} \right] \quad \ell = 0,1,2,\ldots \tag{4.71}$$

(ii) for N-odd

$$\tilde{P}_{m,2\ell-1}^{(1)\pm} = (-1)^{\ell+1} H\mu \frac{M_{1+}(\alpha_m')(2\ell - 1)}{\pi} \left\{ \frac{1}{1 \pm T_1} \frac{4\, M_{1+}(k)}{(2\ell - 1)^2} \right.$$
$$\cdot \left[e^{ikx_0} \pm e^{ik(2L - x_0)} \right] - \sum_{n=1}^{n_1} \left[\mp \frac{2kT_1}{1 \pm T_1} + \frac{(\alpha_m' + k)(\alpha_n' + k)}{(\alpha_m' + \alpha_n')} \right]$$
$$\left. \cdot \frac{M_{1+}(\alpha_n')}{n^2 \alpha_n' [1 - ((\ell - 1/2)/n)^2]} \left[e^{i\alpha_n' x_0} \pm e^{i\alpha_n'(2L - x_0)} \right] \right\}. \tag{4.72}$$

$\ell = 1,2,\ldots$

For $S_{2+}(\alpha'_{n-1/2})$; $D_{2+}(\alpha'_{n-1/2})$;

$$\sum_{n=1}^{n_2} \begin{bmatrix} S_{2+}(\alpha'_{n-1/2}) \\ D_{2+}(\alpha'_{n-1/2}) \end{bmatrix} \cdot \left[G_{mn}^{(2)\pm} - \delta_m^n \right] = P_m^{(2)\pm} \quad , \tag{4.73}$$

where

$$G_{mn}^{(2)\pm} = \frac{i\sqrt{\alpha'_{m-1/2} + k}\; M_{2+}(\alpha'_{m-1/2}) M_{2+}(\alpha'_{n-1/2}) e^{i\alpha'_{n-1/2} 2L}}{H\alpha'_{n-1/2}}$$

$$\cdot \left[\frac{2kT_2}{(1 \pm T_2)} \cdot \frac{1}{(\alpha'_{m-1/2} + k)\sqrt{\alpha'_{n-1/2} + k}} \mp \frac{\sqrt{\alpha'_{n-1/2} + k}}{(\alpha'_{m-1/2} + \alpha'_{n-1/2})} \right] , \tag{4.74}$$

$$P_m^{(2)\pm} = \begin{cases} \tilde{P}_{m,N}^{(2)\pm} & \text{Case A} \\ 0 & \text{Case B} \end{cases} \quad , \tag{4.75}$$

(i) for N-even:

$$\tilde{P}_{m,2e}^{(2)\pm} = i(-1)^{\ell} \frac{2\mu\ell}{\pi} M_{2+}(\alpha'_{m-1/2}) \cdot \sum_{n=1}^{n_2} \frac{M_{2+}(\alpha'_{n-1/2})}{(n-1/2)^2 \alpha'_{n-1/2}[1 - (\ell/(n-1/2))^2]}$$

$\ell = 1, 2, \ldots$

$$\cdot \left[\frac{\sqrt{\alpha'_{m-1/2} + k}\;\sqrt{\alpha'_{n-1/2} + k}}{(\alpha'_{m-1/2} + \alpha'_{n-1/2})} \mp \frac{2kT_2}{1 \pm T_2} \cdot \frac{1}{\sqrt{\alpha'_{m-1/2} + k}\;\sqrt{\alpha'_{n-1/2} + k}} \right]$$

$$\cdot \left[e^{i\alpha'_{n-1/2} x_0} \pm e^{i\alpha'_{n-1/2}(2L - x_0)} \right] \quad , \tag{4.76}$$

(ii) for N-odd:

$$\tilde{P}_{m,2\ell-1}^{(2)\pm} = i(-1)^{\ell+1} \mu \frac{M_{2+}(\alpha'_{m-1/2}) M_{2+}(\alpha'_{\ell-1/2})}{\alpha'_{\ell-1/2}} \left[\frac{\sqrt{\alpha'_{m-1/2} + k}\;\sqrt{\alpha'_{\ell-1/2} + k}}{(\alpha'_{m-1/2} + \alpha'_{\ell-1/2})} \right.$$

$\ell = 1, 2, \ldots$

$$\left. \mp \frac{2kT_2}{1 \pm T_2} \cdot \frac{1}{\sqrt{\alpha'_{m-1/2} + k}\;\sqrt{\alpha'_{\ell-1/2} + k}} \right] \left[e^{i\alpha'_{\ell-1/2} x_0} \pm e^{i\alpha'_{\ell-1/2}(2L - x_0)} \right] . \tag{4.77}$$

The M_{1+} and M_{2+} functions appearing in the last few equations will now be written explicitly. To this end, we return to the definitions of M_1 and M_2 :

$$M_1(\alpha) = e^{-\gamma H} \frac{\sinh(\gamma H)}{\gamma H} = M_{1+}(\alpha) M_{1-}(\alpha) \tag{4.78}$$

$$M_2(\alpha) = e^{-\gamma H} \cosh(\gamma H) = M_{2+}(\alpha) M_{2-}(\alpha) \tag{4.79}$$

where [see [10] pages 131 and 175]

$$M_{1+}(\alpha) = \sqrt{\frac{\sin kH}{kH}} \exp\left\{\frac{iH\alpha}{\pi}\left[1 - C + \ell n\left(\frac{2\pi}{kH}\right) + i\frac{\pi}{2}\right]\right\}$$

$$\cdot \exp\left[\frac{iH\gamma}{\pi} \ell n\left(\frac{\alpha - \gamma}{k}\right)\right] \prod_{n=1}^{\infty} \left(1 + \frac{\alpha}{\alpha'_n}\right) e^{i\frac{\alpha H}{n\pi}} \quad , \tag{4.80}$$

$$M_{1-}(\alpha) = M_{1+}(-\alpha) \quad , \tag{4.81}$$

$$M_{2+}(\alpha) = \sqrt{\cos (kH)} \exp\left\{\frac{i\alpha H}{\pi}\left[1 - C + \ell n\left(\frac{\pi}{2kH}\right) + i\frac{\pi}{2}\right]\right\}$$

$$\cdot \exp\left[\frac{iH\gamma}{\pi} \ell n \frac{\alpha - \gamma}{k}\right] \prod_{n=1}^{\infty} \left(1 + \frac{\alpha}{\alpha'_{n-1/2}}\right) e^{i\frac{\alpha H}{(n-1/2)\pi}} \tag{4.82}$$

$$M_{2-}(\alpha) = M_{2+}(-\alpha) \quad . \tag{4.83}$$

Substituting $\alpha = k$ and $\alpha = \alpha'_m$ into (141) and $\alpha = \alpha'_{m-1/2}$ into (143), we obtain:

$$M_{1+}(k) = \sqrt{\frac{\sin(kH)}{kH}} \exp\left\{\frac{iHK}{\pi}\left[1 - C + \ell n\left(\frac{2\pi}{kH}\right) + i\frac{\pi}{2}\right]\right\}$$

$$\cdot \prod_{n=1}^{\infty} \left(1 + \frac{k}{\alpha'_n}\right) e^{i\frac{kH}{n\pi}} \tag{4.84}$$

$$M_{1+}(\alpha_m') = \sqrt{\frac{\sin(kH)}{kH}} \exp\left\{\frac{iH\alpha_m'}{\pi}\left[1 - C + \ln\left(\frac{2\pi}{kH}\right) + i\,\frac{\pi}{2}\right]\right\}$$

$$\cdot \exp\left[m\ln \frac{\alpha_m' + i\,m\pi/H}{k}\right] \prod_{n=1}^{\infty}\left(1 + \frac{\alpha_m'}{\alpha_n'}\right) e^{i\frac{\alpha_m' H}{n\pi}} \quad . \tag{4.85}$$

$$M_{2+}(\alpha_{m-1/2}') = \sqrt{\cos(kH)} \exp\left\{i\,\frac{\alpha_{m-1/2}'H}{\pi}\left[1 - C + \ln\left(\frac{\pi}{2kH}\right) + i\,\frac{\pi}{2}\right]\right\}$$

$$\cdot \exp\left[(m - 1/2)\ln\left(\frac{\alpha_{m-1/2}' + i\,\frac{(m-1/2)\pi}{H}}{k}\right)\right]$$

$$\cdot \prod_{n=1}^{\infty}\left(1 + \frac{\alpha_{m-1/2}'}{\alpha_{n-1/2}'}\right) e^{\frac{i\alpha_{m-1/2}'H}{(n-1/2)\pi}} \tag{4.86}$$

which are the desired expressions for $M_{1+}(\alpha_m)$ and $M_{2+}(\alpha_{m-1/2}')$ we were seeking.

V. CALCULATION OF THE VECTOR POTENTIALS

From (3.41) and (3.46) we have:

$$\left\{\begin{aligned} \psi_1 &= \tfrac{1}{2}[Y_2(\xi) + Y_1(\xi)] && (5.1a)\\ \psi_2 &= \tfrac{1}{2}[Y_2(\xi) - Y_1(\xi)] && (5.1b)\\ \chi_1 &= \tfrac{1}{2}[Z_2(\xi) + Z_1(\xi)] && (5.1c)\\ \chi_2 &= \tfrac{1}{2}[Z_2(\xi) - Z_1(\xi)] && (5.1d)\end{aligned}\right\}.$$

Substituting (5.1) into (3.28) and (3.29) and using (4.6) through (4.9) we obtain:

$$I_1 = \frac{1}{2}\, e^{-i\alpha 2L}\, [\psi_-(\alpha) + \phi_-(\alpha)] + \frac{1}{2}\, [\psi_+(\alpha) + \phi_+(\alpha)] \tag{5.2a}$$

$$I_2 = \frac{1}{2}\, e^{-i\alpha 2L}\, [\psi_-(\alpha) - \phi_-(\alpha)] + \frac{1}{2}\, [\psi_+(\alpha) - \phi_+(\alpha)]\ . \tag{5.2b}$$

It is known from the theory of the integral equations that if an integral equation $g(y) = \int_a^b g(x)\, K(y,x)\, dx + f(y)$ has a unique solution for every $f(y)$ then the integral equation $g(y) = \int_a^b g(x)\, K(y,x)\, dx$ has only a trivial solution $g \equiv 0$. In view of this, we obtain:

For Case A:

$S_{1+}(\alpha) = 0$, $D_{1+}(\alpha) = 0$ and from (4.50) $\phi_+(\alpha) = 0$, $\phi_-(-\alpha) = 0$.

Furthermore, by substituting $\alpha = -\alpha$ we have $\phi_-(\alpha) = 0$.

For Case B:

$S_{2+}(\alpha) = 0$, $D_{2+}(\alpha) = 0$ and from (4.50) $\psi_+(\alpha) = 0$, $\psi_-(-\alpha) = 0$.

Again, substituting $\alpha = -\alpha$ we obtain $\psi_-(\alpha) = 0$.

and from (5.2)

$$I_1 = \begin{cases} \frac{1}{2} e^{-i\alpha 2L} \psi_-(\alpha) + \frac{1}{2} \psi_+(\alpha) \quad , \quad \text{Case A} & (5.3a) \\ \frac{1}{2} e^{-i\alpha 2L} \phi_-(\alpha) + \frac{1}{2} \phi_+(\alpha) \quad , \quad \text{Case B} & (5.3b) \end{cases}$$

$$I_2 = \begin{cases} I_1 \quad , \quad \text{Case A} & (5.4a) \\ -I_1 \quad , \quad \text{Case B} & (5.4b) \end{cases}$$

Next, inserting (5.4) into (3.32), we obtain the expression for the vector potential

$$A(x,y) = \underbrace{\frac{1}{2\pi} \int_{-\infty+ib}^{\infty+ib} \begin{bmatrix} \frac{\sinh(\gamma y)}{\cosh(\gamma H)} \\ \frac{\cosh(\gamma y)}{\sinh(\gamma H)} \end{bmatrix} \cdot I_1 \frac{e^{-i\alpha x}}{\gamma} d\alpha}_{J_1} +$$

$$+ \underbrace{\frac{N}{4H} \int_{-\infty+ib}^{\infty+ib} \begin{bmatrix} -\cos(\frac{N\pi}{2}) \frac{\sinh(\gamma y)}{\cosh(\gamma H)} \\ \sin(\frac{N\pi}{2}) \frac{\cosh(\gamma y)}{\sinh(\gamma H)} \end{bmatrix} \cdot \frac{\tilde{T}_N e^{-i\alpha x}}{\gamma} d\alpha}_{J_2}$$

$$+ \underbrace{\frac{1}{2\pi} \begin{bmatrix} \sin(\frac{N\pi y}{2H}) \\ \cos(\frac{N\pi y}{2H}) \end{bmatrix} \int_{-\infty+ib}^{\infty+ib} \tilde{T}_N e^{-i\alpha x} d\alpha}_{J_3} \quad . \qquad (5.5)$$

To calculate the integrals in (5.5) we complete the contours by semicircles in the proper half planes. Integrands connected with Case A have poles in $\alpha = \pm \alpha'_{m-1/2}$, and integrands connected with Case B have poles in $\alpha = \pm k, \pm \alpha'_m$.

The result:

$$A(x,y) = \begin{cases} A_N^{(A)}(x,y) & \text{Case A} \\ A_N^{(B)}(x,y) & \text{Case B} \end{cases} \tag{5.6}$$

where

$$A_N^{(A)}(x,y) = \frac{i}{2H} \sum_{n=1}^{n_2} (-1)^{m+1} \frac{\sin[(2m-1)\pi y/2H]}{\alpha'_{m-1/2}}$$

$N = 1,2,\ldots$

$$\cdot \left[\psi_+^{(N)}(\alpha'_{m-1/2}) e^{-i\alpha'_{m-1/2}x} + \psi_-^{(N)}(-\alpha'_{m-1/2})\, e^{i\alpha'_{m-1/2}(x+2L)}\right]$$

$$+ Q_N^{(1)}(x,y) \tag{5.7}$$

$$Q_{2\ell}^{(1)}(x,y) = i\mu(-1)^{\ell+1} \frac{\ell}{\pi} \sum_{m=1}^{n_2} (-1)^m \frac{\sin[(2m-1)\pi y/2H] e^{i\alpha'_{m-1/2}|x+x_0|}}{\alpha'_{m-1/2}(m-1/2)^2[1-(\ell/(m-1/2))^2]} \tag{5.8a}$$

$$Q_{2\ell-1}^{(1)}(x,y) = \frac{i\mu}{2\alpha'_{\ell-1/2}} \sin[(2\ell-1)\pi y/2H] e^{i\alpha'_{\ell-1/2}|x+x_0|} \tag{5.8b}$$

and

$$A_N^{(B)}(x,y) = \frac{i}{4kH}\left[\phi_+^{(N)}(k) e^{-ikx} + \phi_-^{(N)}(-k) e^{ik(x+2L)}\right]$$

$N=0,1,\ldots$

$$+ \frac{i}{2H} \sum_{m=1}^{n_1} (-1)^m \frac{\cos((m\pi/H)y)}{\alpha'_m} \left[\phi_+^{(N)}(\alpha'_m) e^{-i\alpha'_m x}\right.$$

$$\left. + \phi_-^{(N)}(-\alpha'_m) e^{i\alpha'_m(x+2L)}\right] + Q_N^{(2)}(x,y) \tag{5.9}$$

where

$$Q_{2\ell}^{(2)}(x,y) = \frac{i\mu}{2\alpha'_\ell} \cos(\ell\pi \frac{y}{H})\, e^{i\alpha'_\ell|x+x_0|} \tag{5.10a}$$

$$Q_{2\ell-1}^{(2)}(x,y) = i\mu(-1)^{\ell+1} \frac{1}{\pi} \left\{\frac{1}{2\ell-1} \frac{e^{ik|x+x_0|}}{k} - \frac{(2\ell-1)}{2}\right.$$

$$\left. \cdot \sum_{m=1}^{n_1} (-1)^m \frac{\cos(\frac{m\pi}{H}y) e^{i\alpha'_m|x+x_0|}}{m^2[1-((\ell-1/2)/m)^2]\alpha'_m}\right\} . \tag{5.10b}$$

The expressions for $\psi_{\pm}^{(N)}(\alpha)\Big|_{\alpha = \pm\alpha'_{m-1/2}}$ and $\phi_{\pm}^{(N)}(\alpha)\Big|_{\alpha=\pm k,\ \pm\alpha'_m}$, are obtained from (4.60) and (4.61) by substituting $S_{i+}(k)$ and $D_{i\pm}(k)$ from (4.63) and (4.64), $\alpha = \pm k, \pm\alpha'_m, \pm\alpha'_{m-1/2}$ and using (4.62).

We obtain:

$$\psi_{\pm}^{(N)}(\pm\alpha'_{m-1/2}) = \psi_{\pm 1}^{(N)}(\pm\alpha'_{m-1/2}) + \psi_{\pm 2}^{(N)}(\pm\alpha'_{m-1/2}) \tag{5.11}$$

where

$$\begin{aligned}
\psi_{\pm 1}^{(N)}(\pm\alpha'_{m-1/2}) = \pm\frac{i}{2}\sqrt{\alpha'_{m-1/2}+k}\, M_{2+}(\alpha'_{m-1/2}) \Bigg\{ & \frac{2kT_2}{(\alpha'_{m-1/2}+k)H} \\
& \cdot \Bigg[\frac{1}{1-T_2}\sum_{n=1}^{n_2} \frac{e^{i\alpha'_{n-1/2}2L} D_{2+}^{(N)}(\alpha'_{n-1/2}) M_{2+}(\alpha'_{n-1/2})}{\alpha'_{n-1/2}\sqrt{k+\alpha'_{n-1/2}}} \\
& \pm \frac{1}{(1+T_2)}\sum_{n=1}^{n_2} \frac{e^{i\alpha'_{n-1/2}2L} S_{+}^{(N)}(\alpha'_{n-1/2}) M_{2+}(\alpha'_{n-1/2})}{\alpha'_{n-1/2}\sqrt{k+\alpha'_{n-1/2}}}\Bigg] \\
& + \frac{1}{H}\sum_{n=1}^{n_2} \frac{e^{i\alpha'_{n-1/2}2L}\left[D_{2+}^{(N)}(\alpha'_{n-1/2}) \mp S_{2+}^{(N)}(\alpha'_{n-1/2})\right] M_{2+}(\alpha'_{n-1/2})\sqrt{\alpha'_{n-1/2}+k}}{\alpha'_{n-1/2}(\alpha'_{m-1/2}+\alpha'_{n-1/2})}\Bigg\}
\end{aligned} \tag{5.12}$$

$$\begin{aligned}
\psi_{\pm 2}^{(2\ell)}(\pm\alpha'_{m-1/2}) = +\frac{i}{\pi}\sqrt{\alpha'_{m-1/2}+k}\, M_{2+}(\alpha'_{m-1/2})(-1)^{\ell+1}\, 2\mu\ell
\end{aligned}$$

$\ell = 1,2,\ldots$

$$\begin{aligned}
& \cdot \Bigg\{\frac{2kT_2}{(\alpha'_{m-1/2}+k)(1-T_2^2)}\sum_{n=1}^{n_2}\frac{M_{2+}(\alpha'_{n-1/2})}{(n-1/2)^2\alpha'_{n-1/2}[1-(\ell/(n-1/2))^2]\sqrt{\alpha'_n+k}} \\
& \cdot \left(\begin{bmatrix} T_2 \\ -1 \end{bmatrix} e^{i\alpha'_{n-1/2}x_0} + \begin{bmatrix} -1 \\ +T_2 \end{bmatrix} e^{i\alpha'_{n-1/2}(2L-x_0)}\right) \\
& + \sum_{n=1}^{n_2}\frac{\sqrt{\alpha'_{n-1/2}+k}\, M_{2+}(\alpha'_{n-1/2})}{(n-1/2)^2[1-(\ell/(n-1/2))^2](\alpha'_{m-1/2}+\alpha'_{n-1/2})\,\alpha'_{n-1/2}} \\
& \cdot \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{i\alpha'_{n-1/2}x_0} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{i\alpha'_{n-1/2}(2L-x_0)}\right)\Bigg\}
\end{aligned} \tag{5.13}$$

$$\psi_{\pm 2}^{(2\ell-1)}(\pm\alpha'_{m-1/2}) = i\sqrt{\alpha'_{m-1/2}+k}\; M_{2+}(\alpha'_{m-1/2})(-1)^{\ell}\mu \frac{M_{2+}(\alpha'_{\ell-1/2})}{\alpha'_{\ell-1/2}}$$

$\ell = 1,2,\ldots$

$$\cdot\left\{ \frac{2kT_2}{(\alpha'_{m-1/2}+k)(1-T_2^2)\sqrt{\alpha'_{\ell-1/2}+k}}\left(\begin{bmatrix}T_2\\-1\end{bmatrix} e^{i\alpha'_{1-1/2}x_0}\right.\right.$$

$$\left. + \begin{bmatrix}-1\\T_2\end{bmatrix} e^{i\alpha'_{\ell-1/2}(2L-x_0)}\right) + \frac{\sqrt{\alpha'_{\ell-1/2}+k}}{\alpha'_{m-1/2}+\alpha'_{\ell-1/2}}$$

$$\left.\cdot\left(\begin{bmatrix}1\\0\end{bmatrix} e^{i\alpha'_{\ell-1/2}x_0} + \begin{bmatrix}0\\1\end{bmatrix} e^{i\alpha'_{\ell-1/2}(2L-x_0)}\right)\right\} \tag{5.14}$$

$$\Phi_{\pm}^{(N)}(\pm k) = \Phi_{\pm 1}^{(N)}(\pm k) + \Phi_{\pm 2}^{(N)}(\pm k) \quad , \tag{5.15}$$

where

$$\Phi_{\pm 1}^{(N)}(\pm k) = kM_{1+}(k)\left\{\pm T_1\left[\frac{1}{1-T_1}\sum_{n=1}^{n_1}\frac{e^{i\alpha'_n 2L} D_{1+}^{(N)}(\alpha'_n)M_{1+}(\alpha'_n)}{\alpha'_n} \pm \frac{1}{1+T_1}\right.\right.$$

$$\left.\left.\cdot\sum_{n=1}^{n_1}\frac{e^{i\alpha'_n 2L} S_{1+}^{(N)}(\alpha'_n)M_{1+}(\alpha'_n)}{\alpha'_n}\right] - \sum_{n=1}^{n_1}\frac{e^{i\alpha'_n 2L}\left[S_{1+}^{(N)}(\alpha'_n) \mp D_{1+}^{(N)}(\alpha'_n)\right]M_{1+}(\alpha'_n)}{\alpha'_n}\right\} , \tag{5.16}$$

$$\Phi_{\pm 2}^{(2\ell)}(\pm k) = \frac{2k}{(1-T_1^2)}(-1)^{\ell+1} H\mu \frac{M_{1+}(k)M_{1+}(\alpha'_\ell)}{\alpha'_\ell}\left(\begin{bmatrix}1\\-T_1\end{bmatrix} e^{i\alpha'_\ell x_0}\right.$$

$\ell = 0,1,2,\ldots$

$$\left. + \begin{bmatrix}-T_1\\1\end{bmatrix} e^{i\alpha'_\ell(2L-x_0)}\right) , \tag{5.17}$$

$$\Phi_{\pm 2}^{(2\ell-1)}(\pm k) = \frac{2k}{(1-T_1^2)}(-1)^{\ell} H\mu\frac{(2\ell-1)}{\pi} M_{1+}(k)\left\{\frac{2}{(2\ell-1)^2}\frac{M_{1+}(k)}{k}\right.$$

$\ell = 1,2,\ldots$

$$\cdot\left(\begin{bmatrix}1\\-T_1\end{bmatrix}\cdot e^{ikx_0} + \begin{bmatrix}-T_1\\1\end{bmatrix}\cdot e^{ik(2L-x_0)}\right) - \sum_{n=1}^{n_1}\frac{M_{1+}(\alpha'_n)}{n^2\alpha'_n[1-((\ell-1/2)/n)^2]}$$

$$\left.\cdot\left(\begin{bmatrix}1\\-T_1\end{bmatrix}\cdot e^{i\alpha'_n x_0} + \begin{bmatrix}-T_1\\1\end{bmatrix}\cdot e^{i\alpha'_n(2L-x_0)}\right)\right\} \tag{5.18}$$

$$\Phi_{\pm}^{(N)}(\pm\,\alpha_m') = \Phi_{\pm 1}^{(N)}(\pm\alpha_m') + \Phi_{\pm 2}^{(N)}(\pm\alpha_m') \quad , \tag{5.19}$$

where

$$\begin{aligned}\Phi_{\pm 1}^{(N)}(\pm\,\alpha_m') = \frac{1}{2}(\alpha_m' + k)\, M_{1+}(\alpha_m')\Bigg\{ &\pm \frac{2kT_1}{\alpha_m' + k}\Bigg[\frac{1}{1 - T_1}\sum_{n=1}^{n_1}\frac{e^{i\alpha_n' 2L} D_{1+}^{(N)}(\alpha_n') M_{1+}(\alpha_n')}{\alpha_n'} \\ &\pm \frac{1}{1 + T_1}\sum_{n=1}^{n_1}\frac{e^{i\alpha_n' 2L} S_{1+}^{(N)}(\alpha_n') M_{1+}(\alpha_n')}{\alpha_n'}\Bigg] \\ &- \sum_{n=1}^{n_1}\frac{e^{i\alpha_n' 2L}[S_{1+}^{(N)}(\alpha_n') \mp D_{1+}^{(N)}(\alpha_n')] M_{1+}(\alpha_n')(\alpha_n' + k)}{\alpha_n'(\alpha_m' + \alpha_n')}\Bigg\} ,\end{aligned} \tag{5.20}$$

$$\begin{aligned}\Phi_{\pm 2}^{(2\ell)}(\pm\,\alpha_m') = (-1)^{\ell+1}\mu H\, \frac{M_{1+}(\alpha_\ell')}{\alpha_\ell'}\Bigg\{ &\frac{2kT_1}{(\alpha_m' + k)(1 - T_1^2)}\left(\begin{bmatrix} T_1 \\ -1 \end{bmatrix} e^{i\alpha_\ell' x_0}\right. \\ \ell = 0,1,\ldots \qquad &\left. + \begin{bmatrix} -1 \\ T_1 \end{bmatrix} e^{i\alpha_\ell'(2L - x_0)}\right) + \frac{\alpha_\ell' + k}{(\alpha_m' + \alpha_\ell')}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{i\alpha_\ell' x_0}\right. \\ &\left. + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{i\alpha_\ell'(2L - x_0)}\right)\Bigg\} \cdot (\alpha_m' + k) M_{1+}(\alpha_m') \quad ,\end{aligned} \tag{5.21}$$

$$\begin{aligned}\Phi_{\pm 2}^{(2\ell-1)}(\pm\,\alpha_m') = (-1)^{\ell}\, H\mu\, \frac{(2\ell - 1)}{\pi}(\alpha_m' + k)\, M_{1+}(\alpha_m')\Bigg\{ &\frac{2kT_1}{(\alpha_m' + k)(1 - T_1^2)} \\ \ell = 1,2,\ldots \qquad \cdot\Bigg[&\frac{2}{(2\ell - 1)^2}\,\frac{M_{1+}(k)}{k}\left(\begin{bmatrix} T_1 \\ -1 \end{bmatrix} e^{ikx_0} + \begin{bmatrix} -1 \\ T_1 \end{bmatrix} e^{ik(2L - x_0)}\right) \\ &- \sum_{n=1}^{n_1}\frac{M_{1+}(\alpha_n')}{n^2\alpha_n'[1 - ((\ell - 1/2)/n)^2]}\left(\begin{bmatrix} T_1 \\ -1 \end{bmatrix} e^{i\alpha_n' x_0} + \begin{bmatrix} -1 \\ T_1 \end{bmatrix} e^{i\alpha_n'(2L - x_0)}\right)\Bigg]\end{aligned}$$

$$+ \frac{4}{(2\ell-1)^2} \frac{M_{1+}(k)}{\left(\alpha_m' + k\right)} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{ikx_0} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{ik(2L-x_0)} \right)$$

$$- \sum_{n=1}^{n_1} \frac{(\alpha_n' + k)M_{1+}(\alpha_n')}{n^2\alpha_n'[1 - ((\ell - 1/2)/n)^2](\alpha_m' + \alpha_n')} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{i\alpha_n' x_0} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{i\alpha_n'(2L-x_o)} \right) \Bigg\} \tag{5.22}$$

This completes the derivation of the vector potentials. We now write them explicitly for the two different excitations, viz., Case A and Case B, and for the zero mode as Case A:

$$A_{N,0}^{(A)}(x,y) = 0 \tag{5.23}$$

Case B:

1) N-even

$$A_{2\ell,0}^{(B)}(x,y) = \frac{i\mu}{2k} e^{ik|x+x_0|} \delta_\ell^0 + \frac{i}{4kH} \left[\phi_+^{(2\ell)}(k)e^{-ikx} + \phi_-^{(2\ell)}(-k)e^{ik(x+2L)} \right]$$
$$\ell = 0,1,2,... \tag{5.24}$$

2) N-odd

$$A_{2\ell-1,0}^{(B)}(x,y) = i\mu(-1)^{\ell+1} \frac{1}{\pi(2\ell-1)} \frac{e^{ik|x+x_0|}}{k}$$
$$\ell = 1,2,...$$
$$+ \frac{i}{4kH} \left[\phi_+^{(2\ell-1)}(k)e^{-ikx} + \phi_-^{(2\ell-1)}(-k)e^{ik(x+2L)} \right] \tag{5.25}$$

where

$$\delta_\ell^0 = \begin{cases} 1 & \ell = 0 \\ 0 & \ell \neq 0 \end{cases} \tag{5.26}$$

VI. INVESTIGATION OF THE SPECIAL CASE WHEN ONLY THE ZERO MODE CAN PROPAGATE IN THE GUIDE AND DERIVATION OF RESONANCE CONDITION

From (5.15) through (5.18) we have

$$A_{0,0}^{(B)}(x,y) = \frac{i\mu}{2k} e^{ik|x+x_0|} - \frac{i\mu}{2k} [M_{1+}(k)]^2 e^{ikx_0} L(T_1,x_0,x) \tag{6.1}$$

$$A_{2\ell,0}^{(B)}(x,y) = 0 \qquad \ell = 1,2,\ldots \tag{6.2}$$

$$A_{2\ell-1,0}^{(B)}(x,y) = \frac{i\mu(-1)^{\ell+1}}{\pi(2\ell-1)k} e^{ik|x+x_0|} + \frac{i(-1)^{\ell}\mu}{\pi(2\ell-1)k} [M_{1+}(k)]^2 e^{ikx_0} \cdot L(T_1,x_0,x) \qquad \ell = 1,2,\ldots \tag{6.3}$$

where

$$L(T_1,x_0,x) = \frac{1}{(1-T_1^2)} \left\{ \left[1 - T_1 e^{ik2(L-x_0)}\right] e^{-ikx} + \left[1 - T_1 e^{-ik2(L-x_0)}\right] e^{ik2(L-x_0)} e^{ik(x+2L)} \right\} . \tag{6.4}$$

If $k(L - x_0) = n\pi$ where $n = 0, \pm 1, \pm 2,\ldots, \pm n_3$; n_3 satisfies inequalities

$$\begin{aligned} n_3 \frac{\pi}{k} &< L \\ (n_3 + 1) \frac{\pi}{k} &> L \end{aligned} \tag{6.5}$$

or

$$x_0 = L - \frac{n\pi}{k} .$$

Then

$$L(T_1 x_0,x)\Big|_{x_0 = L - \frac{n\pi}{k}} = \frac{1}{(1 + T_1)} \left[e^{-ikx} + e^{ik(x+2L)}\right] \tag{6.6}$$

and the resonance condition is

$$T_1 = -1 \tag{6.7}$$

if $2k\,(L-x_0) = (2n+1)\pi$

or (6.8)

$$x_0 = L - \frac{(2n+1)\pi}{2k}$$

where $n = 0, \pm 1, \ldots, \pm n_4$; n_4 satisfies inequalities:

$$\frac{(2n_4+1)\pi}{2k} < L$$

(6.9)

$$\frac{(2n_4+3)\pi}{2k} > L$$

$$\text{then } L(T,x_0,x)\Big|_{x_0 = L - \frac{(2n+1)\pi}{2k}} = \frac{1}{(1-T_1)}\left[e^{-ikx} - e^{ik(x+2L)}\right] \tag{6.10}$$

and the resonance condition is

$$T_1 = 1 \quad . \tag{6.11}$$

Thus, in general, the resonance conditions are given by

$$T_1 = \pm 1 \quad . \tag{6.12}$$

We can rewrite (6.12) as

$$[M_{1+}(k)]^2 \, e^{ik2L} \cdot f = \pm 1 \tag{6.13}$$

where

$$f = 1 + \frac{e^{-i\pi/4}}{\sqrt{\pi}} \frac{Hk}{\sqrt{kL}} \quad . \tag{6.14}$$

The expression $M_{1+}(k)$, as given in (4.84), can be simplified for the case when only the zero mode can propagate in the guide. For this case we have

$$M_{1+}(k) = \exp\left\{ i \frac{Hk}{\pi} \left[1 - C + \ln(\frac{2\pi}{kH}) + i \frac{\pi}{2}\right] + i \sum_{\ell=1}^{\infty} \left[\frac{kH}{\ell\pi} - \arcsin(\frac{kH}{\ell\pi})\right] \right\} \tag{6.15}$$

Let us investigate the case

$$(\frac{Hk}{\sqrt{\pi}})^2 \lessapprox \frac{1}{kL} << 1 \tag{6.16}$$

returning only the terms with an accuracy $0[\frac{1}{kL}]$. We then have

$$f = 1 + 0(\frac{1}{kL}) \tag{6.17}$$

$$[M_+(k)]^2 = \exp\left\{ i2 \frac{Hk}{\pi} \left[1 - C + \ln\ (\frac{2\pi}{kH}) + i \frac{\pi}{2}\right]\right\} \left[1 + 0\ (\frac{1}{kL})\right] . \tag{6.18}$$

The resonance conditions for this case reduce to

$$\exp \left\{ i2 \frac{Hk}{\pi} \left[1 - C + \ln\ (\frac{2\pi}{kH}) + i \frac{\pi}{2}\right] + i2kL \right\} = \pm 1 \quad . \tag{6.19}$$

For the choice of a positive sign in the r·h·s· of (6.19) we get

$$\exp\left\{i2\,\frac{Hk}{\pi}\left[1 - C + \ln\left(\frac{2\pi}{kH}\right) + i\,\frac{\pi}{2}\right] + i2kL\right\} = +1 = \exp(i2m\pi) \tag{6.20}$$

or

$$2\,\frac{Hk}{\pi}\left[1 - C + \ln\left(\frac{2\pi}{kH}\right) + i\,\frac{\pi}{2}\right] + 2kL = 2m\pi \tag{6.21}$$

where m is an integer. Since (6.21) implies that $m >> 1$, we can rewrite it as

$$\xi = m\pi - \frac{H\xi}{\pi L}\left[\ln\left(\frac{\pi L}{H\xi}\right) + 1 - C + \ln 2 + i\,\frac{\pi}{2}\right] \tag{6.22}$$

where

$$\xi = kL \quad . \tag{6.23}$$

Solving (6.22) by the iteration method we obtain:

$$\xi_{(+)} = m\pi - \frac{Hm}{L}\ln\left(\frac{L}{Hm}\right) - \frac{Hm}{L}\left(1 - C + \ln 2 + i\,\frac{\pi}{2}\right) + 0\left[\frac{H^2 m}{\pi L^2}\ln^2\left(\frac{L}{Hm}\right)\right] \tag{6.24}$$

and

$$k_{(+)} = k_{1+} + ik_{2+} \tag{6.25}$$

where

$$k_{1+} = \frac{m\pi}{L} - \frac{Hm}{L^2}\ln\left(\frac{L}{Hm}\right) - \frac{Hm}{L^2}\left(1 - C + \ln 2\right) \tag{6.26a}$$

$$k_{2+} = -\frac{Hm}{L^2}\,\frac{\pi}{2} \quad . \tag{6.26b}$$

In a similar manner for the choice of negative sign in (6.19) we obtain

$$\xi_{(-)} = \left(m + \frac{1}{2}\right)\pi - \frac{Hm}{L}\ln\frac{L}{Hm} - \frac{Hm}{L}\left(1 - C + \ln 2 + i\,\frac{\pi}{2}\right) + 0\left[\frac{H}{L}\ln\left(\frac{Hm}{L}\right)\right] \tag{6.27}$$

and

$$k_{(-)} = k_{1-} + ik_{2-} \tag{6.28}$$

where

$$k_{1-} = \frac{(m+0,5)}{L}\,\pi - \frac{Hm}{L^2}\ln\left(\frac{L}{Hm}\right) - \frac{Hm}{L^2}\,(1 - C + \ln 2) \tag{6.29a}$$

$$k_2 = -\frac{Hm}{L^2}\,\frac{\pi}{2} \quad . \tag{6.29b}$$

VII. INVESTIGATION OF THE GENERAL CASE WHEN MORE THAN ONE MODE CAN PROPAGATE

As a first step we show that the resonance condition is no longer given by the conventional formulas $T_1 = \pm 1$ when more than one mode can propagate in the guide.

Let us consider the case $T_1 \simeq -1$. Then from (4.67) we have

$$\sum_{n=1}^{n_1} \frac{\exp(i\alpha_n' 2L)\, \eta_{1n}\, M_{1+}(\alpha_n')}{\alpha_n'} = -\,[V_{1+}(k) - \tilde{S}_{1-}(-k)] \qquad (7.1)$$

where we have used

$$S_{1+}(\alpha_m') = \eta_{1m} + \eta_{2m}(1 + T_1) + \ldots \quad . \qquad (7.2)$$

Inserting (7.1) in the equation for $\Phi_+(k)$, we obtain

$$\Phi_+(k) = 0[(1 + T_1)^0] \quad . \qquad (7.3)$$

It is possible to show in the same manner that

$$\Phi_-(-k) = 0[(1 - T_1)^0] \quad . \qquad (7.4)$$

The resonance conditions are given by

$$\text{Case A:} \quad \left| G_{mn}^{(2)\pm} - \delta_m^n \right| = 0 \qquad (7.5)$$

$$\text{Case B:} \quad \left| G_{mn}^{(1)\pm} - \delta_m^n \right| = 0 \quad . \qquad (7.6)$$

VIII. SUMMARY OF RESULTS

In this work we have addressed ourselves to the problem of a finite-width, parallel-plate waveguide excited by a source located in the interior of the guide. Two types of sources have been investigated, viz.:

Case A:

$$\overline{J} = \hat{y}\, \delta(x+x_o) \sin\left(\frac{N\pi y}{2H}\right) e^{i\beta z}, \quad N = 1,2\ldots$$

Case B:

$$\overline{J} = \hat{y}\, \delta(x+x_o) \cos\left(\frac{N\pi y}{2H}\right) e^{i\beta z}, \quad N = 0,1,2,\ldots$$

We have assumed that the current has only a y-component and that β is real and greater than zero. Using the vector potential approach, we have reduced the original problem to that of solving the inhomogeneous wave equation (3.8) together with the boundary condition stated in (3.9). Next, two coupled equations for four unknowns (Y_1, Y_2, Z_1 and Z_2) have been derived where these unknowns are related to the vector potential at the extensions of the parallel plates. These equations read [same as (3.40) and (3.45)].

$$\int_{-\infty}^{2L} Y_i(\xi) K_i\,(k|x-\xi|)\,d\xi + \int_{0}^{\infty} Z_i(\xi) K_i\,(k|x-\xi|)\,d\xi = f_i(x)$$

where the functions $f_i(x)$ appearing in the r·h·s are related to the prescribed source and are given in (3.43), (3.44), (3.48) and (3.49). The kernel functions K_i appearing in the integral equation may be found in (3.42) and (3.47).

Our next step was to solve the integral equations using Fourier transforms and the Wiener-Hopf technique. The results for the vector potential constructed in this manner are given in (5.6) through (5.22) for both cases considered. We have shown that there is no zero mode excited in Case A, and in Case B this mode is excited only when $N < 2Hk/\pi$, if N is even. For N odd, the zero mode is always excited in Case B.

An important result of the analysis presented here is the expression for the resonance condition. We have shown that this is given by

$$T = (M_{1+}(k)]^2 \, e^{i2kl} \, f = \pm 1$$

where $f = 1 + e^{-i\pi/4}(\pi)^{-1/2} \, Hk/\sqrt{kL}$

and $M_{1+}(k)$ is given in (4.84).

For $(\frac{Hk}{\sqrt{\pi}})^2 \lesssim \frac{1}{kL} << 1$ the function f above can be replaced by unity and the resonance condition is correspondingly simplified. It is interesting to note that for the source located at $x_0 = L - n\pi/k$ the resonance condition is reduced to $T_1 = -1$ whereas for $x_0 = L - \frac{(n+\frac{1}{2})\pi}{k}$ the same condition becomes $T_1 = +1$. In general both the plus sign and the minus sign are admissible for the resonance condition. Equations (6.25) and (6.29) state the resonance condition under the constraint that only the TEM mode can propagate in the infinite, parallel-plate guide. For the more general case, the condition for resonance is given by (7.5) and (7.6) and an examination of this reveals that $T = \pm 1$ no longer represents the resonance condition for this general case. We point out that the resonance condition is useful for solving the complex (leaky) modes in such open waveguides.

We also draw the attention of the reader to the fact that the resonance condition derived herein is in general more accurate than that given by previous workers. We show, however, in Section 6 that when the condition on the waveguide parameters as expressed by (6.16) applies, the resonance equation reduces to that obtainable by multiple reflection method applied to semi-infinite parallel-plate waveguides, a technique that has been used in the past by other workers [7,8].

APPENDIX I

The general solution of the equation

$$\frac{\partial^2\Phi}{\partial y^2} - \gamma^2\Phi(y) = f(y) \tag{I.1}$$

is

$$\Phi(y) = \tilde{C}_1 e^{\gamma y} + \tilde{C}_2 e^{-\gamma y} + \frac{1}{\gamma}\int_0^y f(\xi)\,\sinh[\gamma(y-\xi)]\,d\xi \quad . \tag{I.2}$$

For

$$f(\xi) = -\mu \begin{bmatrix} \sin(\frac{N\pi\xi}{2H}) \\ \cos(\frac{N\pi\xi}{2H}) \end{bmatrix} e^{-i\alpha x_0} \tag{I.3}$$

we have

$$\Phi(y) = C_1 e^{\gamma y} + C_2 e^{-\gamma y} + \tilde{T}_N \begin{bmatrix} \sin(\frac{N\pi y}{2H}) \\ \cos(\frac{N\pi y}{2H}) \end{bmatrix} \tag{I.4}$$

where

$$\tilde{T}_N = \frac{e^{-i\alpha x_0}\mu}{\gamma^2 + [N\pi/(2H)]^2} \quad . \tag{I.5}$$

APPENDIX II

Write (3.40) as

$$C_1\gamma e^{\gamma H} - C_2\gamma e^{-\gamma H} = I_1 - F_1 \tag{II.1a}$$

$$C_1\gamma e^{-\gamma H} - C_2\gamma e^{\gamma H} = I_2 - F_2 \tag{II.1b}$$

where

$$F_{1,2} = \tilde{T}_N \frac{N\pi}{2H} \begin{bmatrix} \cos(\frac{N\pi}{2}) \\ \mp \sin(\frac{N\pi}{2}) \end{bmatrix} . \tag{II.2}$$

The solution of the system equations (II.1) is

$$C_1 = \frac{1}{2\sinh(2\gamma H)\gamma} \left\{ I_1 e^{\gamma H} - I_2 e^{-\gamma H} - I_2 e^{-\gamma H} + \tilde{T}_N \frac{N\pi}{H} \begin{bmatrix} -\cos(\frac{N\pi}{2})\sinh(\gamma H) \\ \sin(\frac{N\pi}{2})\cosh(\gamma H) \end{bmatrix} \right\} \tag{II.3}$$

and

$$C_2 = \frac{1}{2\sinh(2\gamma H)\gamma} \left\{ I_1 e^{-\gamma H} - I_2 e^{\gamma H} + \tilde{T}_N \frac{N\pi}{H} \begin{bmatrix} \cos(\frac{N\pi}{2})\sinh(\gamma H) \\ \sin(\frac{N\pi}{2})\cosh(\gamma H) \end{bmatrix} \right\} . \tag{II.4}$$

The continuity condition of the vector potential across the boundaries $-\infty < x < -2L$, $0 < x < \infty$ for $y = +H$ and $y = -H$ is given by (3.44). Substituting the expressions for the vector potential for all three fields from (3.38), (3.39) and (3.43) into (3.44) and changing the order of the integrations gives

$$\int_{-\infty}^{-2L} \psi_1 K_{1,2}^{(*)}(k|x - \xi|)d\xi + \int_{0}^{\infty} h_1 K_{1,2}^{(*)}(k|x - \xi|)d\xi$$

$$- \int_{-\infty}^{-2L} \psi_2 K_{2,1}^{(*)}(k|x - \xi|d\xi - \int_{0}^{\infty} h_2 K_{2,1}^{(*)}(k|x - \xi|)d\xi = \tilde{f}_{1,2}(x) \qquad \text{(III.1)}$$

where

$$K_{1,2}^{*}(k|x - \xi|) = \frac{1}{2\pi} \int_{-\infty+ib}^{\infty+ib} \begin{bmatrix} \exp(2H\gamma) \\ 1 \end{bmatrix} \frac{e^{i\alpha(\xi-x)}}{\gamma \sinh(2H\gamma)} d\alpha \quad . \qquad \text{(III.2)}$$

$$\tilde{f}_{1,2}(x) = -\frac{N}{4H} \int_{-\infty+ib}^{\infty+ib} \begin{bmatrix} \mp \cos(\frac{N\pi}{2}) \tanh(\gamma H) \\ \sin(\frac{N\pi}{2}) \coth(\gamma H) \end{bmatrix} \frac{\tilde{T}_N e^{-i\alpha x}}{\gamma} d\alpha$$

$$- \frac{1}{2\pi} \begin{bmatrix} \pm \sin(\frac{N\pi}{2}) \\ \cos(\frac{N\pi}{2}) \end{bmatrix} \int_{-\infty+ib}^{\infty+ib} \tilde{T}_N e^{-i\alpha x} d\alpha \qquad \text{(III.3)}$$

APPENDIX IV

We are interested in calculating the integrals:

$$I = \frac{1}{2\pi i} \int_{id-\infty}^{id+\infty} \begin{bmatrix} S_{1+}(\zeta) \\ D_{1+}(\zeta) \end{bmatrix} \cdot \frac{e^{i\zeta 2L}\, d\zeta}{(\zeta - k) M_{1-}(\zeta)(\zeta + \alpha)} .$$

Multiplying the numerator and denominator by $M_{1+}(\zeta)$, we obtain

$$= \frac{1}{2\pi i} \int_{id-\infty}^{id+\infty} \begin{bmatrix} S_{1+}(\zeta) \\ D_{1+}(\zeta) \end{bmatrix} \frac{e^{i\zeta 2L}\, e^{\gamma H}\, M_{1+}(\zeta)}{(\zeta - k)(\zeta + \alpha)\sinh(\gamma H)/\gamma H}\, d\zeta \tag{IV.1}$$

We have branch cuts from k to Rek + i∞ and from -k to -Rek - i∞.

Closing the contour with a semicircle in the upper half and using the theory or residues, we get

$$\frac{1}{2\pi i} \int_C \begin{bmatrix} S_{1+}(\zeta) \\ D_{1+}(\zeta) \end{bmatrix} \frac{e^{i\zeta 2L}\, e^{\gamma H}\, M_{1+}(\zeta)\, d\zeta}{(\zeta - k)(\zeta + \alpha)\sinh(\gamma H)/\gamma H} = \sum_{n=1}^{n_1} \begin{bmatrix} S_{1+}(\alpha_n') \\ D_{1+}(\alpha_n') \end{bmatrix} \frac{e^{i\alpha_n' 2L} M_{1+}(\alpha_n')(\alpha_n' + k)}{\alpha_n'(\alpha + \alpha_n')} \tag{IV.2}$$

where

$$\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_\tau} + \int_{C_R} . \tag{IV.3}$$

It can be shown that $\int_{C_R} \to 0$, when $R \to \infty$. Calculating $\int_{C_\tau}$ when $\tau \to 0$ gives

$$\lim_{\tau \to 0} \frac{1}{2\pi i} \int_{C\tau} = - \begin{bmatrix} S_{1+}(k) \\ D_{1+}(k) \end{bmatrix} \frac{e^{ik2L}\, M_{1+}(k)}{\alpha + k} . \tag{IV.4}$$

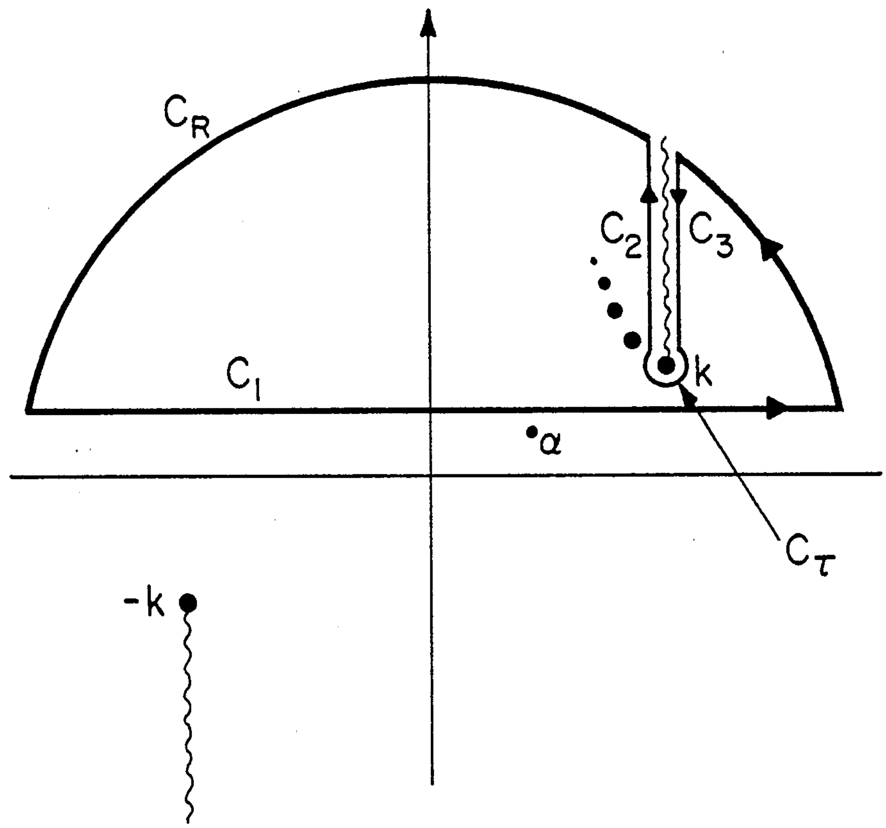

Fig. 3

Contour for integration of the integral I in (IV.1)

For $\int_{c_2} + \int_{c_3}$, taking note of the fact that $\sqrt{\zeta^2 - k^2}$ has the + sign on the right-hand side and the - sign on the left-hand side of the cut, respectively, we get

$$\frac{1}{2\pi i}\left[\int_{c_2} + \int_{c_3}\right] = \frac{H}{\pi i}\int_{\mathrm{Re}k+i\infty}^{k}\begin{bmatrix} S_{1+}(\zeta) \\ D_{1+}(\zeta)\end{bmatrix} \frac{e^{i\zeta 2L} M_{1+}(\zeta)\sqrt{\zeta + k}}{\sqrt{\zeta - k}\ (\zeta+\alpha)}\, d\zeta \quad . \qquad \text{(IV.5)}$$

To calculate the integral in the r·h·s· of (IV.5) for large guide width, i.e., $k2L \gg 1$ we note that the integrand decreases exponentially along the path of integration from k to k + i∞. This allows us to expand

$$\begin{bmatrix} S_{1+}(\zeta) \\ D_{1+}(\zeta)\end{bmatrix} M_{1+}(\zeta)\sqrt{\zeta + k}$$

in a Taylor's series and retain only the first term for asymptotic evaluation. This gives the r·h·s· of (IV.5) = I

$$\approx \frac{H}{\pi i}\begin{bmatrix} S_{1+}(k) \\ D_{1+}(k)\end{bmatrix} M_{1+}(k)\sqrt{2k}\int_{\mathrm{Re}k+i\infty}^{k} \frac{e^{i\zeta 2L}}{(\zeta + \alpha)\sqrt{\zeta - k}}\, d\zeta \quad .$$

Introducing a new variable u

$$(\zeta - k)2L = iu \quad , \quad \sqrt{\zeta - k} = \sqrt{\frac{u}{2L}}\, e^{i\frac{\pi}{4}}$$

$$\zeta 2L = iu + 2kL \quad , \quad d\zeta = \frac{i}{2L}\, du$$

we obtain

$$I = -\frac{H}{\pi i}\begin{bmatrix} S_{1+}(k) \\ D_{1+}(k)\end{bmatrix} M_{1+}(k)\sqrt{2k}\ \sqrt{2L}\ e^{-i\frac{\pi}{4}}\ e^{ik2L}\ W_{-1}\left[-i2L(\alpha + k)\right] \qquad \text{(IV.6)}$$

where

$$W_{-1}[-i2L(\alpha + k)] = \int_0^{\infty} \frac{e^{-u}}{\sqrt{u}[u - i2L(\alpha + k)]} \, du \quad . \tag{IV.7}$$

From (IV.2) through (IV.6) we get

$$\begin{aligned} I = & \begin{bmatrix} S_{1+}(k) \\ D_{1+}(k) \end{bmatrix} \frac{e^{ik2L} M_{1+}(k)}{\alpha + k} \\ & \cdot \left\{ 1 + \frac{H}{\pi} e^{-i\frac{3}{4}\pi} \sqrt{2k} \sqrt{2L} (\alpha + k) W_{-1}[-i2L(\alpha + k)] \right\} \\ & + \sum_{n=1}^{n_1} \begin{bmatrix} S_{1+}(\alpha_n') \\ D_{1+}(\alpha_n') \end{bmatrix} \frac{e^{i\alpha_n' 2L} M_{1+}(\alpha_n')}{\alpha_n'(\alpha + \alpha_n')} \quad . \end{aligned} \tag{IV.8}$$

Note that in (IV.8) we have neglected the exponentially decreasing terms in the summation.

In the same manner we can show:

$$\begin{aligned} & \frac{1}{2\pi i} \int_{id-\infty}^{id+\infty} \begin{bmatrix} S_{2+}(\zeta) \\ D_{2+}(\zeta) \end{bmatrix} \frac{e^{i\zeta 2L}}{\sqrt{\zeta - k}(\zeta + \alpha) M_{2-}(\zeta)} \, d\zeta \\ & = \frac{1}{\pi} e^{-i\frac{3}{4}\pi} \sqrt{2L} \, W_{-1}[-2L(\alpha + k)] e^{ik2L} M_{2+}(k) \begin{bmatrix} S_{2+}(k) \\ D_{2+}(k) \end{bmatrix} \\ & + \frac{1}{H} \sum_{n=1}^{n_2} \begin{bmatrix} S_{2+}(\alpha_{n-1/2}') \\ D_{2+}(\alpha_{n-1/2}') \end{bmatrix} \frac{e^{i\alpha_{n-1/2}' 2L} M_{2+}(\alpha_{n-1/2}') \sqrt{\alpha_{n-1/2}' + k}}{\alpha_{n-1/2}'(\alpha + \alpha_{n-1/2}')} \end{aligned} \tag{IV.9}$$

APPENDIX V

In this appendix we discuss the problem of deriving the systems of the equations that are satisfied by the constants $S_{1+}(\alpha_n')$, $D_{1+}(\alpha_n')$, $S_{2+}(\alpha_{n-1/2}')$, $D_{2+}(\alpha_{n-1/2}')$. We present only the calculations for $S_{1+}(\alpha_n')$ and $D_{1+}(\alpha')$ because the same procedure can be followed to solve for the other constants.

After substituting $\alpha = \alpha_m'$ into (4.55), and using the expression (4.63) together with the asymptotic form of the function $W_{-1}[-2L(\alpha + k)]$, we have

$$\sum_{n=1}^{n_1} S_{1+}(\alpha_n') \left[G_{mn}^{(1)} - \delta_m^n\right] = P_m^{(1)\pm} \qquad m = 1,2,\ldots \tag{V.1}$$

where

$$G_{mn}^{(1)} = \frac{e^{i\alpha_n' 2L} M_{1+}(\alpha_m')M_{1+}(\alpha_n')}{\alpha_n'} \left[\frac{2kT_1}{1 \pm T_1} \mp \frac{(\alpha_m' + k)(\alpha_n' + k)}{\alpha_m' + \alpha_n'}\right]. \tag{V.2}$$

$$P_m^{(1)\pm} = \pm M_{1+}(\alpha_m') \left\{\frac{2kT_1}{1 \pm T_1} [V_{1+}(k) \mp \hat{S}_{1-}(-k)] \mp (\alpha_m' + k)[V_{1+}(\alpha_m') \mp S_{1-}(-\alpha_m')]\right\}. \tag{V.3}$$

To calculate the constants $V_{1+}(\alpha)|_{\alpha=k,\alpha_m'}$ and $\tilde{S}_{1-}(-\alpha)|_{\alpha=k,\alpha_m'}$ we insert the expressions for $H_{1\pm}(\zeta)$ from (4.13) into (4.39), (4.48) and substitute $\alpha = k$ and $\alpha = \alpha_m'$. This yields certain integrals which we can calculate using the theory of residues. We are interested in examining two cases a) N-even; b) N-odd. Substituting results into (V.3) we obtain the expressions for $P_m^{(1)\pm}$. In the same manner we have obtained results for Case A.

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