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Sensor and Simulation Notes

Note XXI

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Impedances and Field Distributions for Parallel Plate Transmission Line Simulators

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Abstract

The impedances and field distributions of the symmetrical three plate and two plate transmission lines are calculated. A correction is made for the replacement of the conducting plates by grids of wires (parallel to the direction of propagation). Thus we can evaluate each of these transmission lines in view of both matching various impedances and of having a certain degree of electromagnetic field uniformity over part of the cross section of the transmission line.

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I. Introduction

Certain types of EMP simulators (or calibrators as described in Sensor and Simulation Note I) make use of cylindrical transmission lines to form the electromagnetic field distribution. For these cases in which we desire a uniform field distribution we can use a parallel plate transmission line, and near the flat conductors, we have an almost uniform field over certain regions. The purpose of this note is to discuss the impedance and field distribution characteristics of such transmission lines.

We first consider the symmetrical three plate transmission line (illustrated in figure 1,, outer plates at the same potential) assuming that the outer plates are infinitely wide for our calculations. We next consider the symmetrical two plate transmission line which also covers the case of the finite width plate over an infinite (or sufficiently large) conducting plane because of the field symmetry. Finally we consider corrections needed if the conducting planes are replaced by grids of parallel wires in the direction of current flow. The finite thickness of the conducting planes is ignored.

We do not intend to derive the conformal transformations but rather to use them to obtain useful numerical results. As a matter of convention the notation for the elliptic integrals and related functions is that found in the Handbook of Mathematical Functions, AMS 55, National Bureau of Standards, 1964.

II. Symmetrical Three Plate Transmission Line

Consider the symmetrical three plate transmission line as in figure 1A. To analyze the impedance and field distribution we first study the simpler problem of a semi-infinite center plate with infinite outer plates, and then we let the center plate be finite. Finally we consider the approximation involved in assuming the outer plates infinite.

A. Semi-infinite Center Plate

First consider the case of large a/b and look at the field distribution near the edge of the center plate. Similarly we can also get an approximation for the impedance including a correction for the edge effects.

The conformal transformation for this geometry is given by[1]

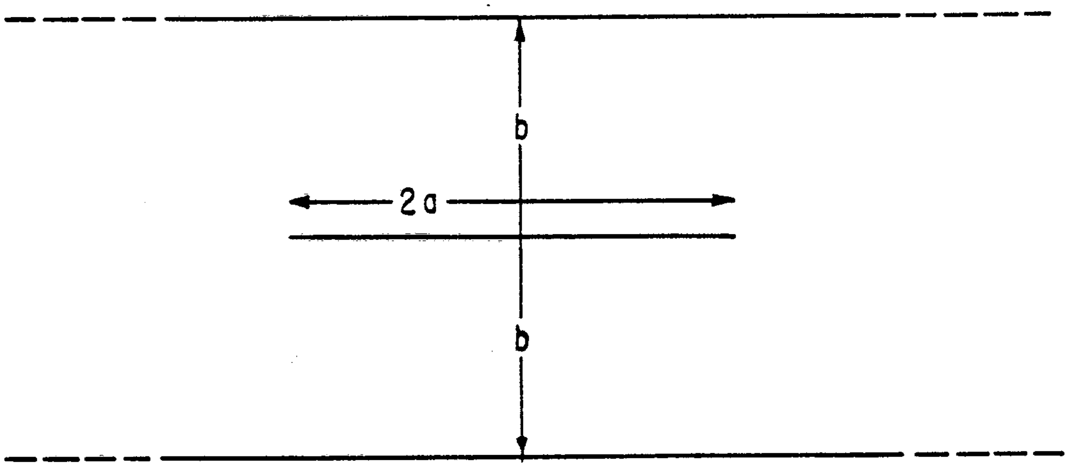
$$z = -\frac{2}{\pi} \ln\,[\cosh\,(w)] \tag{1}$$

where $z = x - j\,y$ (2)

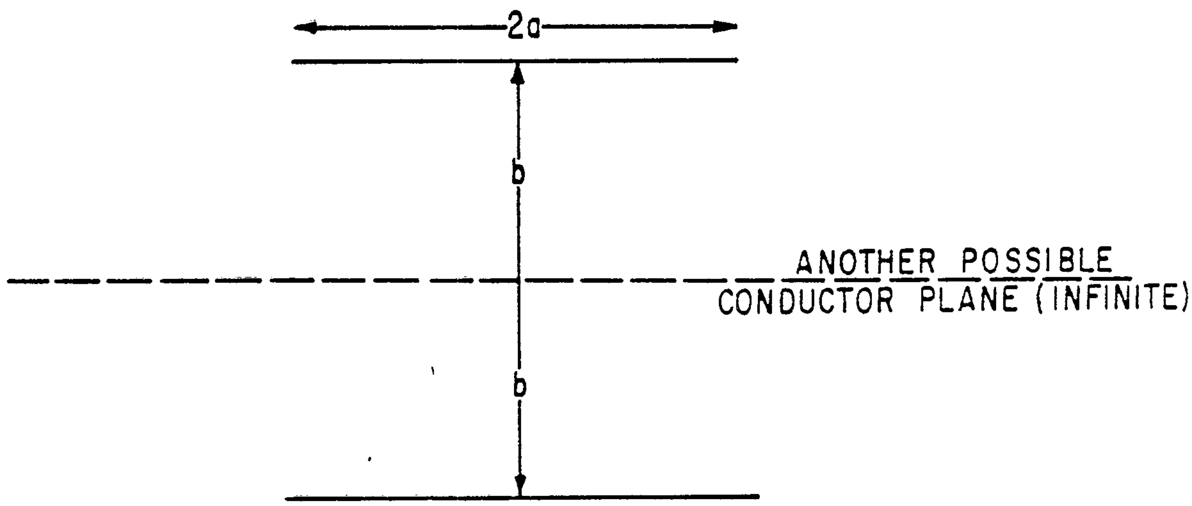
and $w = u + j\,v$ (3)

This is illustrated in figure 2. Note that we have normalized the problem by setting b to one and taking the edge of the semi-infinite center plate as the origin of the coordinate system.

1. For this and other conformal transforms see Moon and Spencer, Field Theory Handbook, 1961, except for those in Section IV.



A. INFINITE OUTER PLATES, THREE CONDUCTOR, PARALLEL PLATE TRANSMISSION LINE



B. TWO CONDUCTOR, PARALLEL PLATE TRANSMISSION LINE

FIGURE 1. SYMMETRICAL, PARALLEL PLATE TRANSMISSION LINES

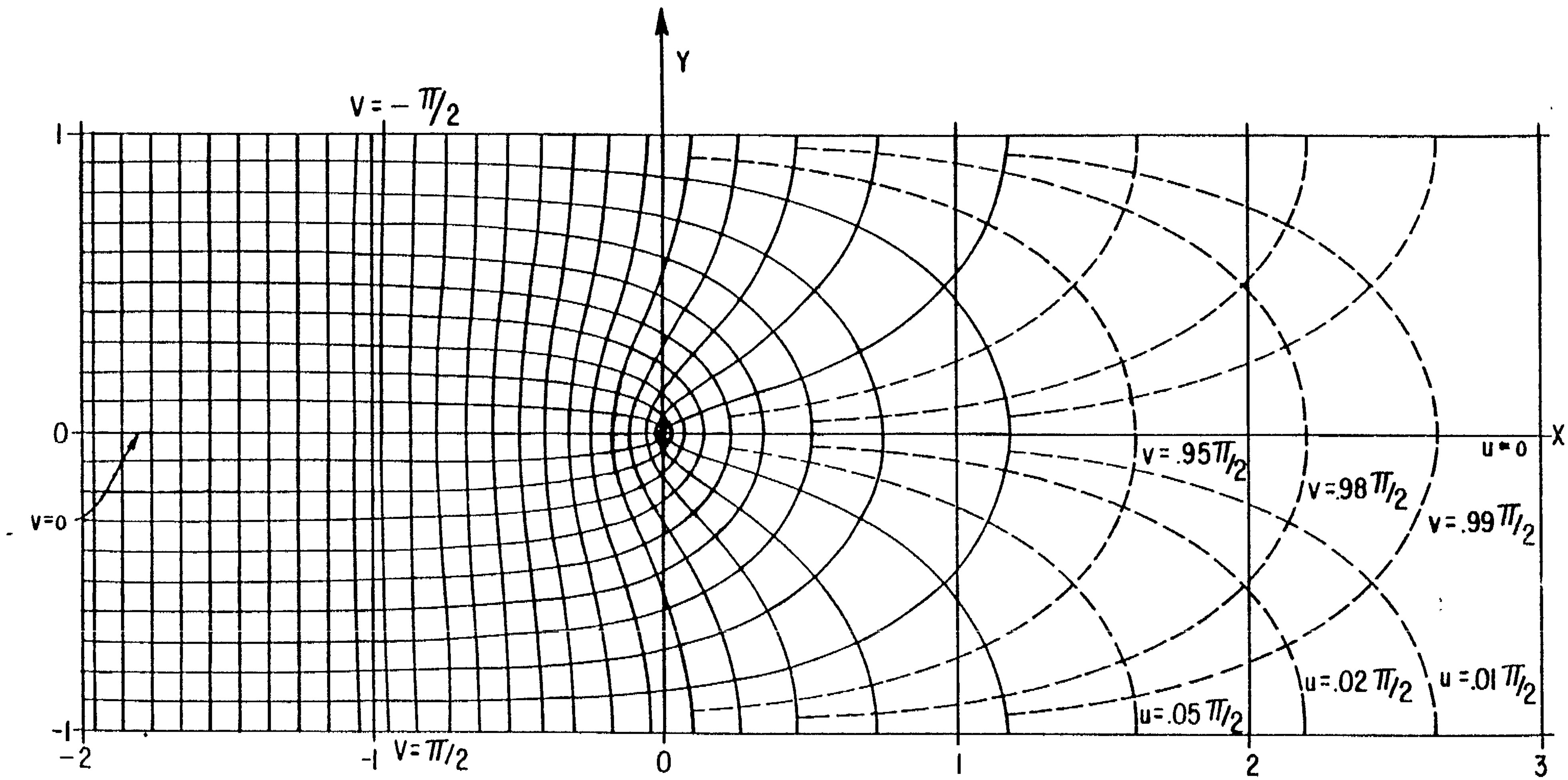


FIGURE 2. FIELD AND POTENTIAL DISTRIBUTION WITH SEMI-INFINITE CENTER CONDUCTOR IN INFINITE THREE CONDUCTOR PARALLEL-PLATE TRANSMISSION LINE

In figure 2 the equipotentials and magnetic field lines are given by constant v and the electric field lines by constant u. Rewriting the conformal transformation as

$$w = \text{arc cosh}\,(e^{-\frac{\pi}{2}z}) \tag{4}$$

we can solve for u and v as[2]

$$u = \text{arc cosh}\,(\alpha) = \ln\,[\alpha + (\alpha^2 - 1)^{1/2}] \tag{5}$$

$$v = \pm\ \text{arc cos}\,(\beta) \tag{6}$$

(+ for negative y)

where

$$\alpha = \frac{1}{2}\,[e^{-\pi x} + 2e^{-\frac{\pi}{2}x}\cos(\frac{\pi}{2}\,y)+1]^{1/2} + \frac{1}{2}\,[e^{-\pi x} - 2e^{-\frac{\pi}{2}x}\cos(\frac{\pi}{2}\,y)+1]^{1/2} \tag{7}$$

$$\beta = \frac{1}{2}\,[e^{-\pi x} + 2e^{-\frac{\pi}{2}x}\cos(\frac{\pi}{2}\,y) + 1]^{1/2} - \frac{1}{2}\,[e^{-\pi x} - 2e^{-\frac{\pi}{2}x}\cos(\frac{\pi}{2}\,y)+1]^{1/2} \tag{8}$$

Let us simplify matters by looking at the field distributions along some convenient lines. First consider the field distribution along the outer plates ($y = \pm 1$). In both cases we have from equations (5) and (7)

$$\alpha = (e^{-\pi x} + 1)^{1/2} \tag{9}$$

and

$$u = \ln\,[(e^{-\pi x} + 1)^{1/2} + e^{-\frac{\pi}{2}x}] = \text{arcsinh}\,(e^{-\frac{\pi}{2}x}) \tag{10}$$

Noting that the difference in potential function, v, between the plates is $\pi/2$ we can calculate a normalized electric field[3] (in the y direction) along the outer plates as

$$E_{y_{rel}}\Big|_{y = \pm 1} = -\frac{2}{\pi}\left|\frac{\partial u}{\partial x}\right| = (1 + e^{\pi x})^{-1/2} \tag{11}$$

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2. For these types of identities see AMS 55, Handbook of Mathematical Functions, National Bureau of Standards, 1964.

3. The normalized magnetic field is the same as the normalized electric field, but perpendicular to it.

Second consider the field distribution along the semi-infinite center plate ($y = 0$, $x \leq 0$). In this case we have

$$\alpha = e^{-\frac{\pi}{2}x} \tag{12}$$

and

$$u = \text{arc cosh } (e^{-\frac{\pi}{2}x}) \tag{13}$$

The normalized electric field (in the y direction) along the center plate is then

$$E_{y_{rel}}\Big|_{y = 0} = -\frac{2}{\pi} \quad \frac{\partial u}{\partial x} = (1 - e^{\pi x})^{-1/2} \tag{14}$$

Third we can consider the field distribution to the right of the center plate ($y = 0$, $x \geq 0$). Using the v functions we have

$$\beta = e^{-\frac{\pi}{2}x} \tag{15}$$

and

$$v = \pm \text{ arc cos } (e^{-\frac{\pi}{2}x}) \tag{16}$$

The normalized electric field (in the x direction) off the edge of the center plate is then

$$E_{x_{rel}}\Big|_{y=0} = \frac{2}{\pi} \left|\frac{\partial v}{\partial x}\right| = (e^{\pi x} - 1)^{-1/2} \tag{17}$$

These normalized fields are plotted in figures 3 and 4 on linear and logarithmic scales respectively. In figure 4 in two cases we can see how closely the normalized fields approach unity as a function of position. This shows the distance from the edge of the center plate required for a given degree of field uniformity.

The field distortion at the edge of the center plate has the effect of lowering the impedance of the transmission line. For $\frac{a}{b} >> 1$ we can approximate the impedance by calculating an effective width of the center plate. Using the potential function of equation (13) we can calculate the effective position, Δx, of the edge of the center plate required to terminate a uniform field (no fringing) as

$$\begin{aligned}
\Delta x &= \lim_{x \to -\infty} \left\{ x + \frac{2}{\pi} \ln \left[e^{-\frac{\pi}{2}x} + (e^{-\pi x} - 1)^{1/2} \right] \right\} \\
&= \lim_{x \to -\infty} \left\{ x + \frac{2}{\pi} \ln (e^{-\frac{\pi}{2}x}) + \frac{2}{\pi} \ln \left[1 + (1 - e^{\pi x})^{1/2} \right] \right\} \\
&= \lim_{x \to -\infty} \frac{2}{\pi} \ln \left[1 + (1 - e^{\pi x})^{1/2} \right]
\end{aligned} \tag{18}$$

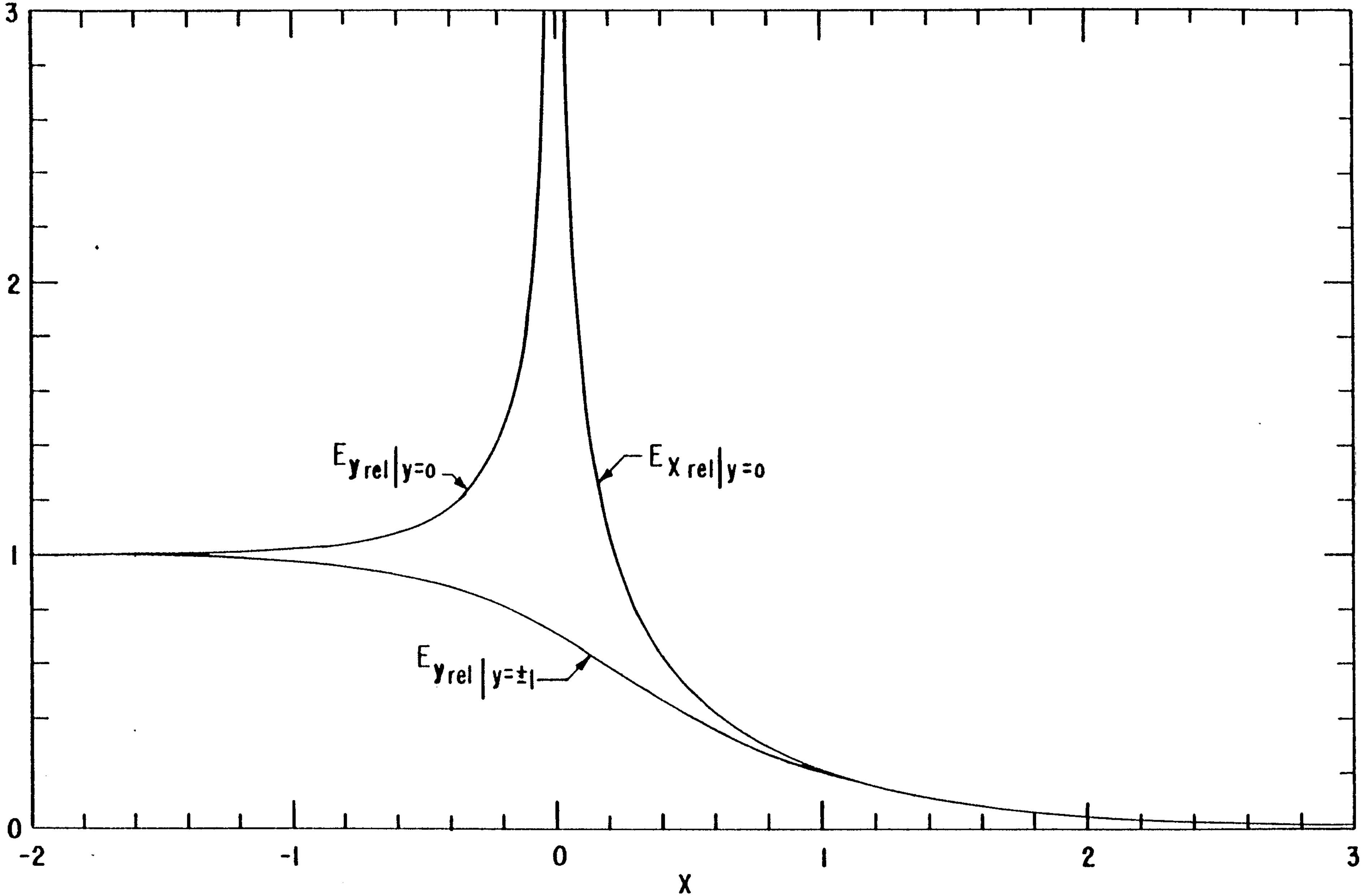

FIGURE 3. NORMALIZED FIELD DISTRIBUTION WITH SEMI-INFINITE CENTER CONDUCTOR IN INFINITE THREE CONDUCTOR PARALLEL PLATE TRANSMISSION LINE

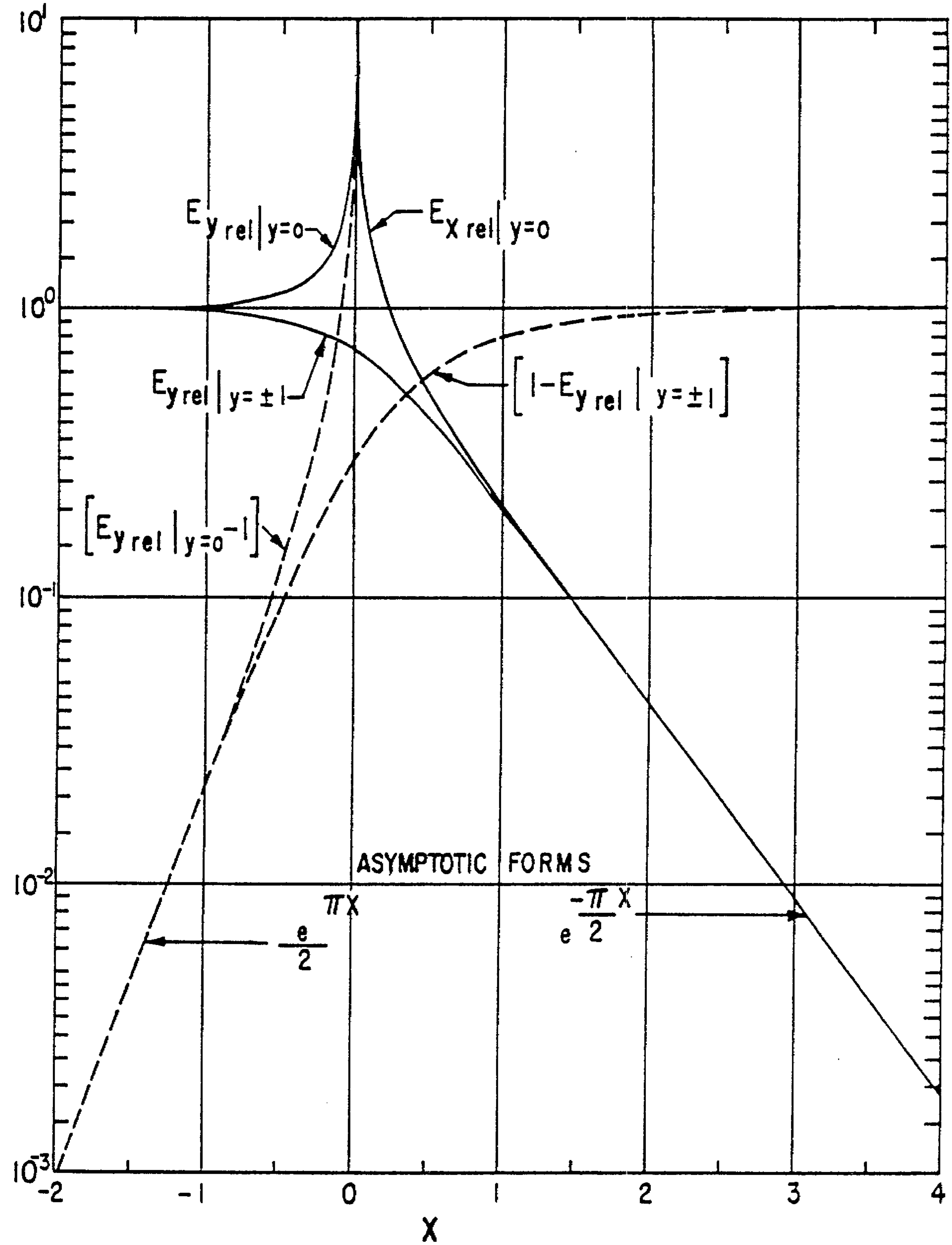

FIGURE 4. NORMALIZED FIELD DISTRIBUTION WITH SEMI-INFINITE CENTER CONDUCTOR IN INFINITE THREE CONDUCTOR PARALLEL PLATE TRANSMISSION LINE

or

$$\Delta x = \frac{2}{\pi} \ln (2) \tag{19}$$

Using the dimensions indicated in figure 1A we then have for large a/b a geometric factor, f_g, in the transmission line impedance (accounting for fringing on both edges of the center plate) of

$$f_g = \left\{ 4 \left[\frac{a}{b} + \frac{2}{\pi} \ln 2\right] \right\}^{-1} \tag{20}$$

This geometric factor relates the transmission line impedance, Z_L, to the wave impedance, Z, by

$$Z_L = f_g Z \tag{21}$$

The wave impedance in turn depends on the permeability, permittivity, and conductivity of the medium inside the transmission line.

B. Finite Center Plate

Now let the center plate be finite. The conformal transformation is[4,5]

$$\bar{z} = - \frac{1}{\pi} \ln [\sqrt{m} \; \mathrm{sn}^2 (w|m)] \tag{22}$$

where $\bar{z}$ is the complex conjugate of z. This is illustrated in figure 5 for positive x and y for the case of a 50 Ω transmission line, assuming a wave impedance equal to that of free space. Here the equipotentials and magnetic field lines are given by constant u and the electric field lines by constant v. By symmetry this figure can be extended to all four quadrants.

Instead of solving for u and v we can solve for x and y as

$$x = - \frac{1}{\pi} \ln \left[\sqrt{m} \left\{ \frac{\mathrm{sn}^2(u|m)\mathrm{dn}^2(v|m_1) + \mathrm{cn}^2(u|m)\mathrm{dn}^2(u|m)\mathrm{sn}^2(v|m_1)\mathrm{cn}^2(v|m_1)}{(1 - \mathrm{dn}^2(u|m)\mathrm{sn}^2(v|m_1))^2} \right\} \right. \tag{23}$$

4. See AMS 55 (ref. 2) for the notation regarding the elliptic integrals F (or K) and E, the Jacobian elliptic functions sn, cn, and dn, the Jacobian zeta function, Z, and related quantities.

5. In Moon and Spencer (ref. 1) this transform is interchanged with another one in the figures (pp. 74 and 75).

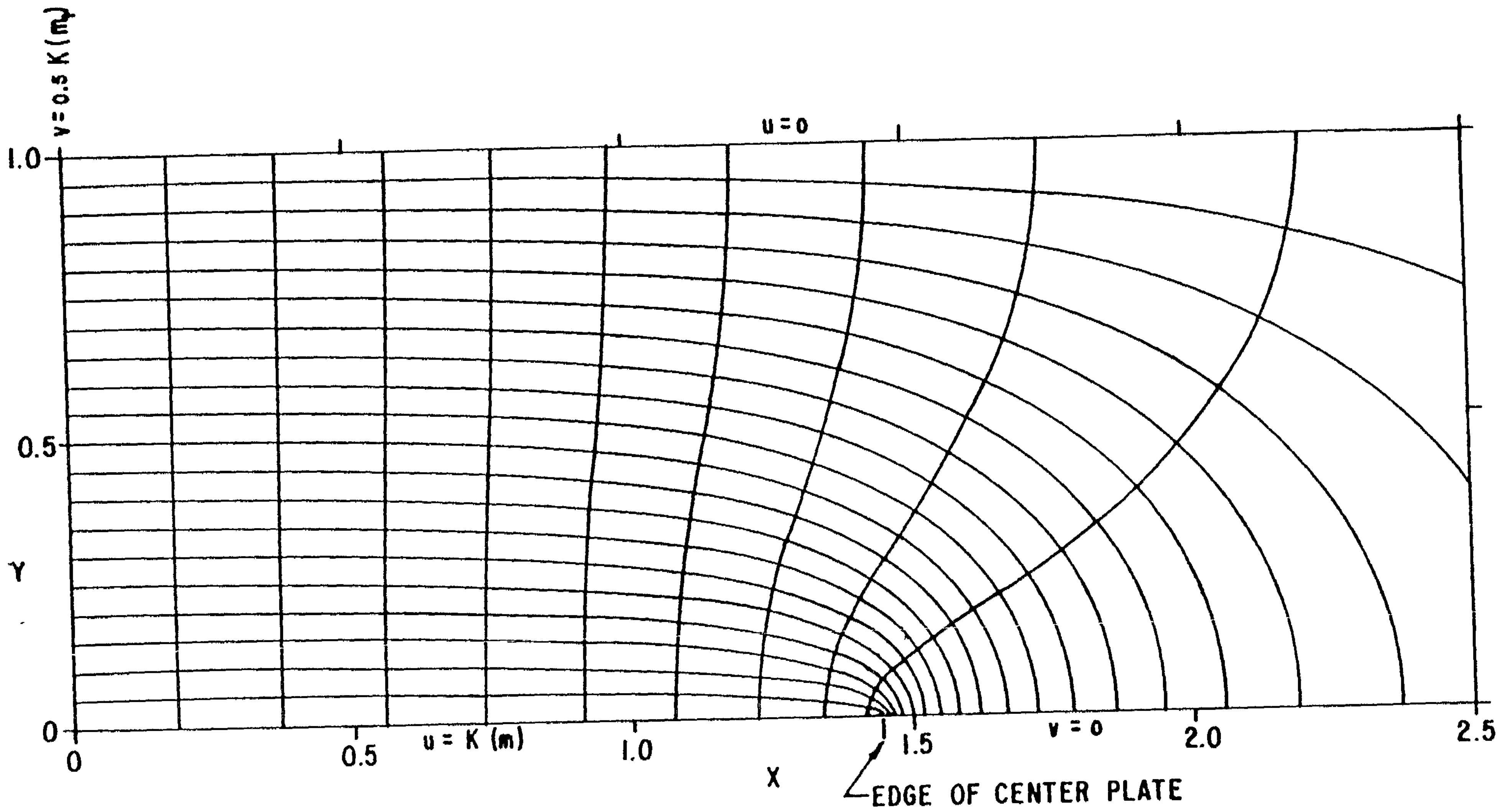

FIGURE 5. FIELD AND POTENTIAL DISTRIBUTION FOR FINITE CENTER CONDUCTOR IN INFINITE THREE CONDUCTOR PARALLEL-PLATE TRANSMISSION LINE: 50Ω

and

$$y = \frac{2}{\pi} \arctan \left[\frac{cn(u|m)dn(u|m)sn(v|m_1)cn(v|m_1)}{sn(u|m)dn(v|m_1)} \right] \tag{24}$$

In this normalized geometry where the outer plates are at $y = \pm 1$ we can relate the width of the center plate to the parameter, m, (or its complement, $m_1 = 1-m$) in the elliptic functions. At y = 0 for $|x| < a/b$ we have

$$u = K(m) \tag{25}$$

and

$$x = \frac{1}{\pi} \ln \left[\frac{dn^2(v|m_1)}{\sqrt{m}} \right] \tag{26}$$

At v = 0 we then have

$$x \Big|_{v=0} = -\frac{1}{2\pi} \ln(m) \tag{27}$$

and at the other end

$$x \Big|_{v=K(m_1)} = \frac{1}{2\pi} \ln(m) \tag{28}$$

Thus the origin of the coordinates is in the center of the middle plate and we have the important result that

$$\frac{a}{b} = -\frac{1}{2\pi} \ln(m) \tag{29}$$

We also have the geometric factor in the impedance as

$$f_g = \frac{K(m)}{2K(m_1)} \tag{30}$$

Since

$$m = 1-m_1 = e^{-2\pi\frac{a}{b}} \tag{31}$$

we can calculate the impedance as graphed in figure 6.

In equation (20) we have an approximation to f_g for small b/a. For large b/a we can take limiting forms of K(m) and K (m_1) to obtain

$$f_g \approx \frac{1}{2\pi} \ln\left(\frac{16}{1-m}\right) \approx \frac{1}{2\pi} \ln\left(\frac{8}{\pi} \frac{b}{a}\right) \tag{32}$$

Thus in figure 6 we also plot approximations to f_g for both large and small b/a as well as the difference of these approximations from f_g.

For various applications we may desire certain specific transmission line impedances (generally a convenient number times 50 Ω). Taking the definition of the permeability of free space

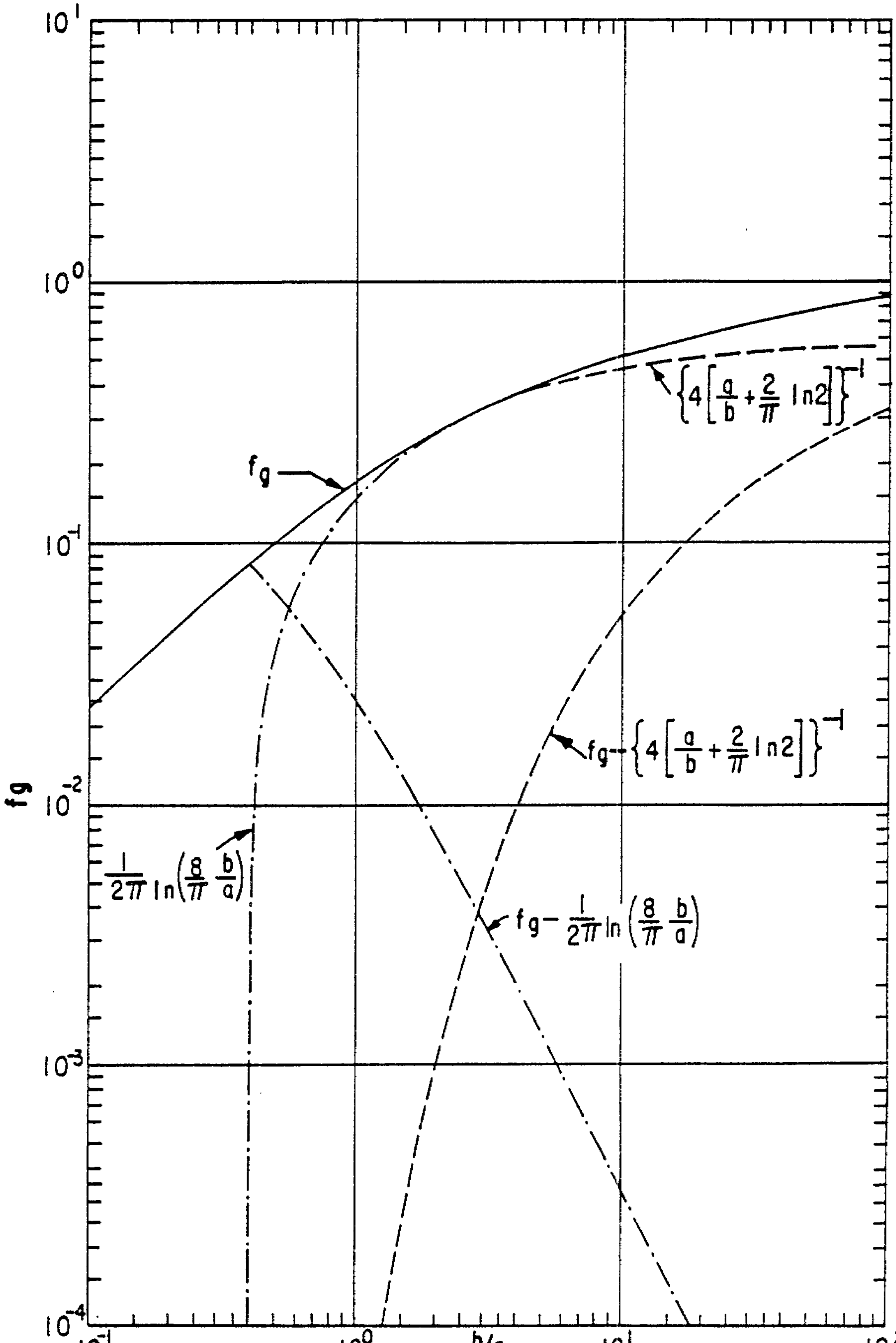

FIGURE 6. GEOMETRIC IMPEDANCE FACTOR FOR FINITE CENTER CONDUCTOR IN INFINITE THREE CONDUCTOR, PARALLEL PLATE TRANSMISSION LINE

$$\mu_0 = 4 \pi \times 10^{-7} \ \frac{\text{henries}}{\text{meter}} \tag{33}$$

and the measured value[6] of the speed of light in vacuo

$$c = 2.99793 \ (\pm 10^{-5}) \times 10^{8} \ \frac{\text{meters}}{\text{sec}} \tag{34}$$

we can construct a table of b/a for desired values of Z_L(equation (21)).

Z_L (ohms)	f_g	b/a
25	.06627	.30018
50	.13254	.6920
100	.26508	1.9793
200	.5302	10.964

Table I. Impedances for Three Plate Transmission Line

As before we can calculate the field distribution at convenient locations. First along the outer plates ($y = \pm 1$, $u = 0$) we have from equation (23)

$$x = \frac{1}{\pi} \ \ln \left[\frac{\mathrm{cn}^2 (v|m_1)}{\sqrt{m} \ \mathrm{sn}^2 (v|m_1)} \right] \tag{35}$$

The normalized electric field (in the y direction) along the outer plates is then

$$E_{y_{rel}} \Big|_{y = \pm 1} = - \frac{1}{K(m)} \ \frac{\partial v}{\partial x} \tag{36}$$

which after some manipulation becomes

$$E_{y_{rel}} \Big|_{y = \pm 1} = \frac{\pi}{2K(m)} [1 + m + 2\sqrt{m} \cosh (\pi x)]^{-1/2} \tag{37}$$

For the special case of x = 0 we have

$$E_{y_{rel}} \Big|_{\substack{x = 0 \\ y = \pm 1}} = \frac{\pi}{2 \ K(m) \ [1+\sqrt{m}]} \tag{38}$$

With the use of equation (31) we can relate these quantities to b/a. We can note that in the limit of small b/a the normalized field (equation (37)) behaves like that for the semi infinite center plate near the edge of the center plate. Using equation (38) we can see how small we must make b/a to approach a normalized field of one in the center of the outer plates. This latter parameter is plotted in figure 7.

Second let us consider the field distribution along the center plate ($y=0$, $|x| \le a/b$, $u = K(m)$) where we have

$$x = \frac{1}{\pi} \ln \left[\frac{\mathrm{dn}^2 (v|m_1)}{\sqrt{m}} \right] \tag{39}$$

6. American Institute of Physics Handbook, Second Edition, 1963.

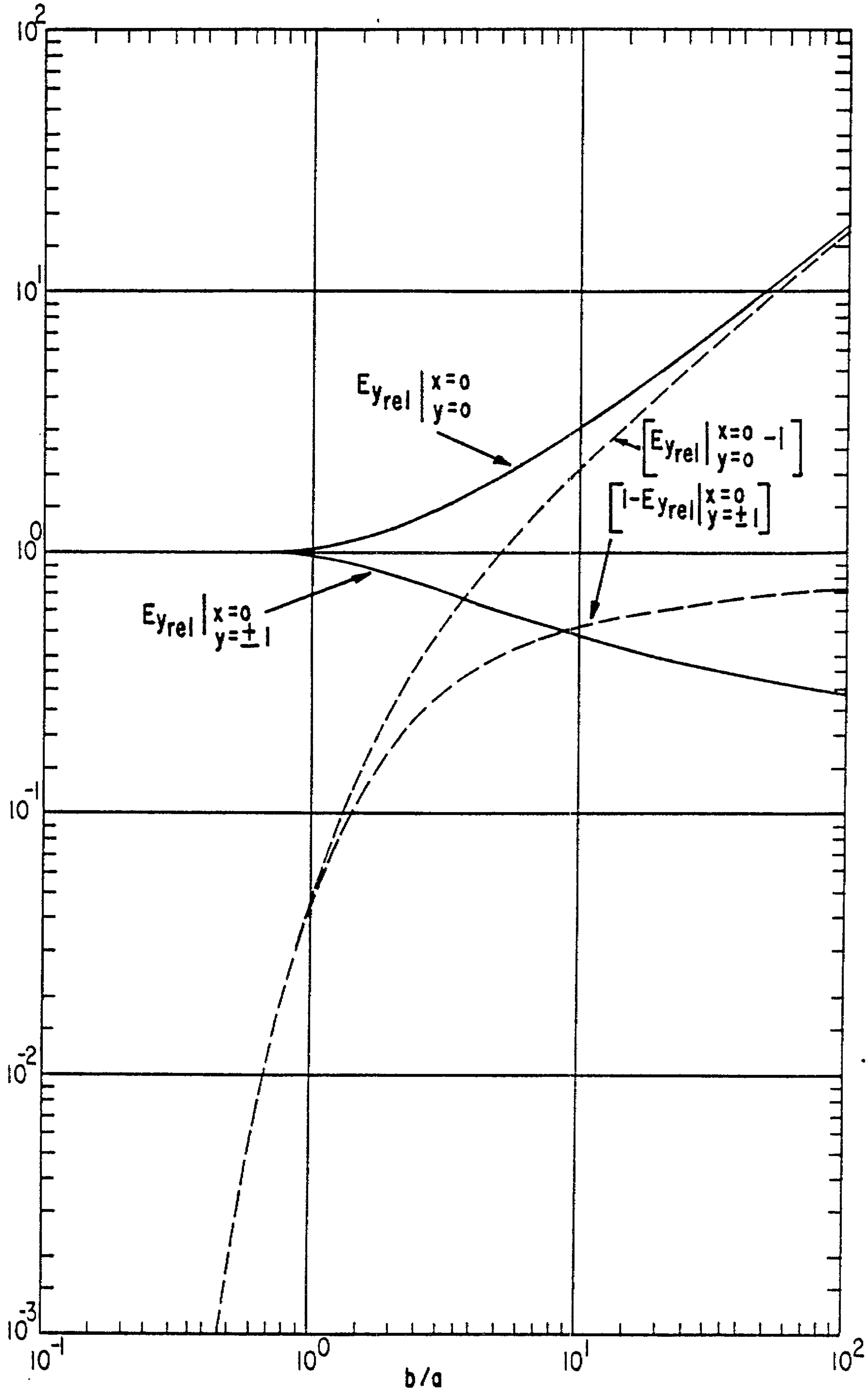

FIGURE 7. NORMALIZED FIELD VARIATION FOR FINITE CENTER CONDUCTOR IN INFINITE THREE CONDUCTOR, PARALLEL PLATE TRANSMISSION LINE

The normalized electric field (in the y direction) along the center plate is then

$$E_{y_{rel}}\Big|_{y=0} = -\frac{1}{K(m)} \quad \frac{\partial v}{\partial x} \tag{40}$$

which after some manipulation becomes

$$E_{y_{rel}}\Big|_{y=0} = \frac{\pi}{2K(m)} [1 + m - 2\sqrt{m} \cosh (\pi x)]^{-1/2} \tag{41}$$

and which is similar in form to equation (37). For the special case of x = 0 we have

$$E_{y_{rel}}\Bigg|_{\substack{x = 0 \\ y = 0}} = \frac{\pi}{2 K(m) [1 - \sqrt{m}]} \tag{42}$$

which is also plotted in figure 7.

Thus we can determine the impedances and field distributions in this type of parallel plate transmission line. In addition we can determine the degree of uniformity of the fields over a given part of the transmission line.

C. Effect of Finite Outer Plates

In practice the outer plates must be of finite width, introducing an error into the previous calculations. Since the case of small b/a is of interest (for uniform fields) and a comparatively simple case, we can look at figure 4 to get an idea of how far beyond the edge of the center plate we need extend the outer plates until the fringing field is below a specified level (relative to the field at x = 0). If a' represents the half width of either of the outer plates, then the one percent level is, for example, at

$$\frac{a' - a}{b} \approx 3 \tag{43}$$

Instead of looking at the fringing field level we might look at that part of the impedance attributable to field lines terminating on the outer plates for $|x| >> a/b$. From equation (37) we have for $|x| > a/b$,

$$E_{y_{rel}}\Bigg|_{y = \pm 1} \approx \frac{\pi}{2 K (m)} [\sqrt{m} \quad e^{\pi x}]^{-1/2} \tag{44}$$

Using equation (30) for f_g we can then say for a relative part of $1/f_g$ due to field lines beyond a' (for all four outer plate edges)

$$f_g \quad \Delta \left(\frac{1}{f_g}\right) \approx - \frac{K (m)}{2 K(m)} \quad x \quad 4 \quad \frac{\pi}{2 K(m) m^{1/4}} \int_{\frac{a'}{b}}^{\infty} e^{-\frac{\pi}{2}x} \, dx \tag{45}$$

or

$$\frac{\Delta f_g}{f_g} \cong - f_g \, \Delta\left(\frac{1}{f_g}\right) \approx \frac{2}{K(m_1)} \frac{e^{-\frac{\pi}{2}\frac{a'}{b}}}{m^{1/4}} \tag{46}$$

For small b/a then

$$\frac{\Delta f_g}{f_g} \approx \frac{2}{\pi}\frac{b}{a} e^{-\frac{\pi}{2}\frac{a'-a}{b}} \tag{47}$$

showing that both small b/a and large (a'-a)/b contribute to a small uncertainty in the impedance.

Thus we have two criteria with which to judge the effect of the outer plates, one based on the size of the fringing field at the edge of the outer plates and one based on that part of the impedance due to field lines beyond the edge of the outer plates. However these criteria are very approximate in that if the outer plates are of finite extent the field distribution will be rearranged in the vicinity of the edges of these outer plates. Thus equation (47) is not a correction to the impedance but an indication of the size of the error.

III. Symmetrical Two Plate Transmission Line

Now consider the symmetric two-plate transmission line as in figure 1B. We consider first the case of semi-infinite plates and second the case of finite plates. Our results also apply for a semi-infinite or finite plate parallel to an infinite plate because of symmetry in the field lines.

A. Semi-infinite Plates

Let a/b be large and look at the field distribution near the edges of the plates. The conformal transformation is

$$z = \frac{1}{\pi} \left[w + 1 + e^{w}\right] \tag{48}$$

which is illustrated in figure 8 in normalized form for positive y. The plot for negative y is a mirror image.

The equipotentials and magnetic field lines are given by constant v and the electric field lines by constant u. We can solve for x and y as

$$x = \frac{1}{\pi} \left[u + 1 + e^{u} \cos (v)\right] \tag{49}$$

and

$$y = \frac{1}{\pi} \left[v + e^{u} \sin (v)\right] \tag{50}$$

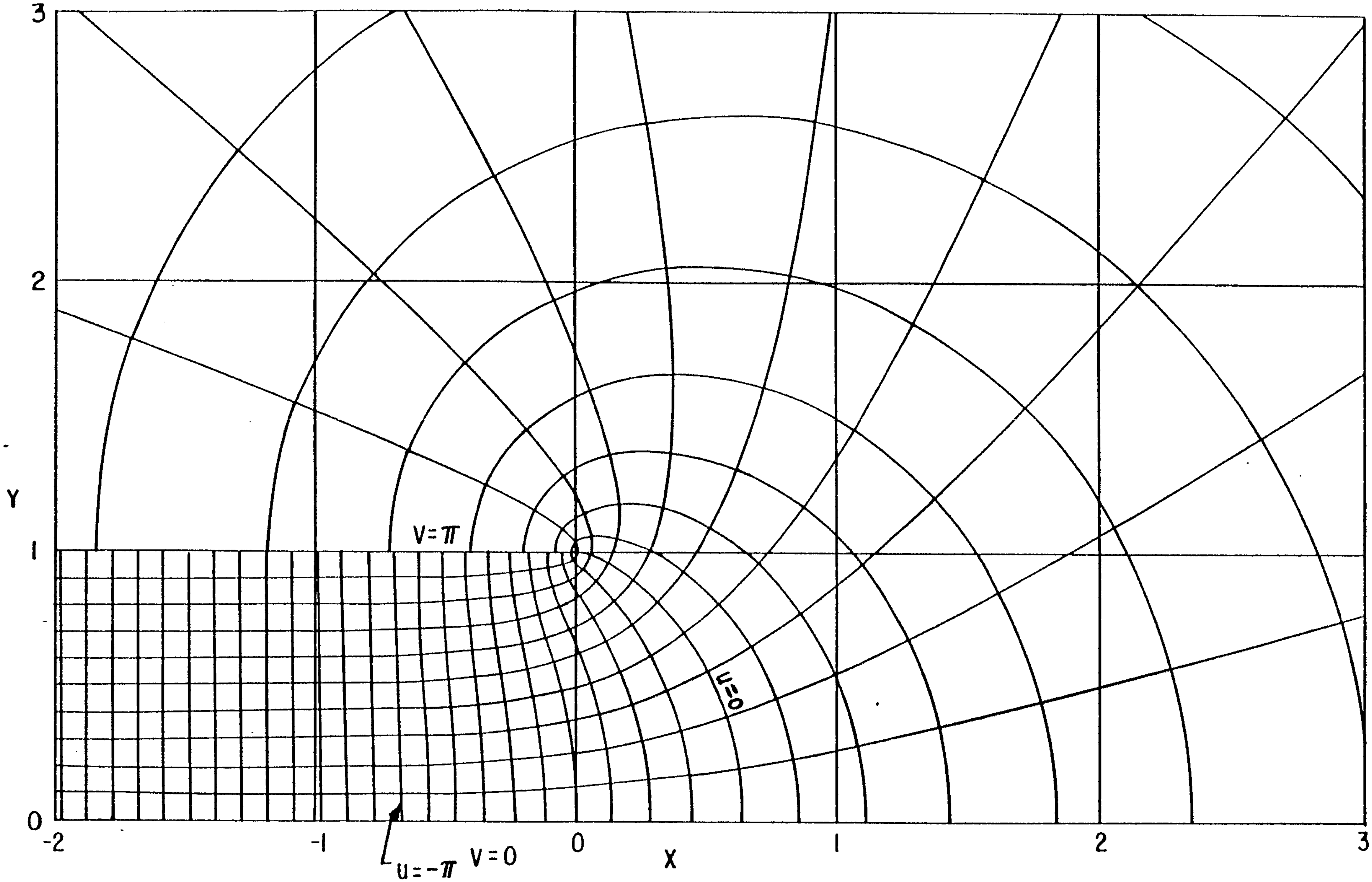

FIGURE 8. FIELD AND POTENTIAL DISTRIBUTION FOR SEMI-INFINITE, PARALLEL, TWO-PLATE TRANSMISSION LINE (UPPER HALF)

First calculate the field distribution along the plane of symmetry $(y = 0, v = 0)$. Here we have

$$x = \frac{1}{\pi} [u + 1 + e^u] \tag{51}$$

Thus we have a normalized field distribution

$$E_{y_{rel}}\Big|_{y = 0} = \frac{1}{\pi} \frac{\partial u}{\partial x} = [1 + e^u]^{-1} \tag{52}$$

obtaining u from equation (51).

Since in a conformal solution we can place a conductor along any equipotential line then equation (52) also describes the relative field over an infinite conducting plate (y=0) from a semi-infinite conducting plate. Thus we are solving for two interesting transmission-line geometries.

Second calculate the field distribution along the semi-infinite plates $(y = \pm 1, v = \pm \pi)$. Here we have

$$x = \frac{1}{\pi} [u + 1 - e^u] \tag{53}$$

However we have two field distributions: one outside of the plates $(u > 0)$ and one inside the plates $(u < 0)$. Thus

$$E_{y_{rel}}\Big|_{y=\pm 1} = \frac{1}{\pi} \left|\frac{\partial u}{\partial x}\right| = \begin{cases} [1 - e^u]^{-1} & \text{inside } (u < 0) \\ [e^u - 1]^{-1} & \text{outside } (u > 0) \end{cases} \tag{54}$$

obtaining u from equation (53).

These normalized fields are plotted in figures 9 and 10 on linear and logarithmic scales respectively. Again in figure 10 the magnitude of the difference of the normalized fields from unity is also plotted so that we can determine the distance from the edge required for a given degree of field uniformity.

Unfortunately we cannot calculate an approximation for f_g for small b/a in the same manner as for the three plate transmission line (equation (20)). If we try to integrate the "excess" field (the difference from a uniform field) over one of the semi-infinite plates we shall have a divergent answer from the contribution of the field on the outside of the plates.

B. Finite Plates

For finite plates (as in figure 1B) we have the conformal transformation

$$\bar{z} = \frac{2\,K(m)}{\pi} \quad Z(w + j\,K(m)|m) + j \tag{55}$$

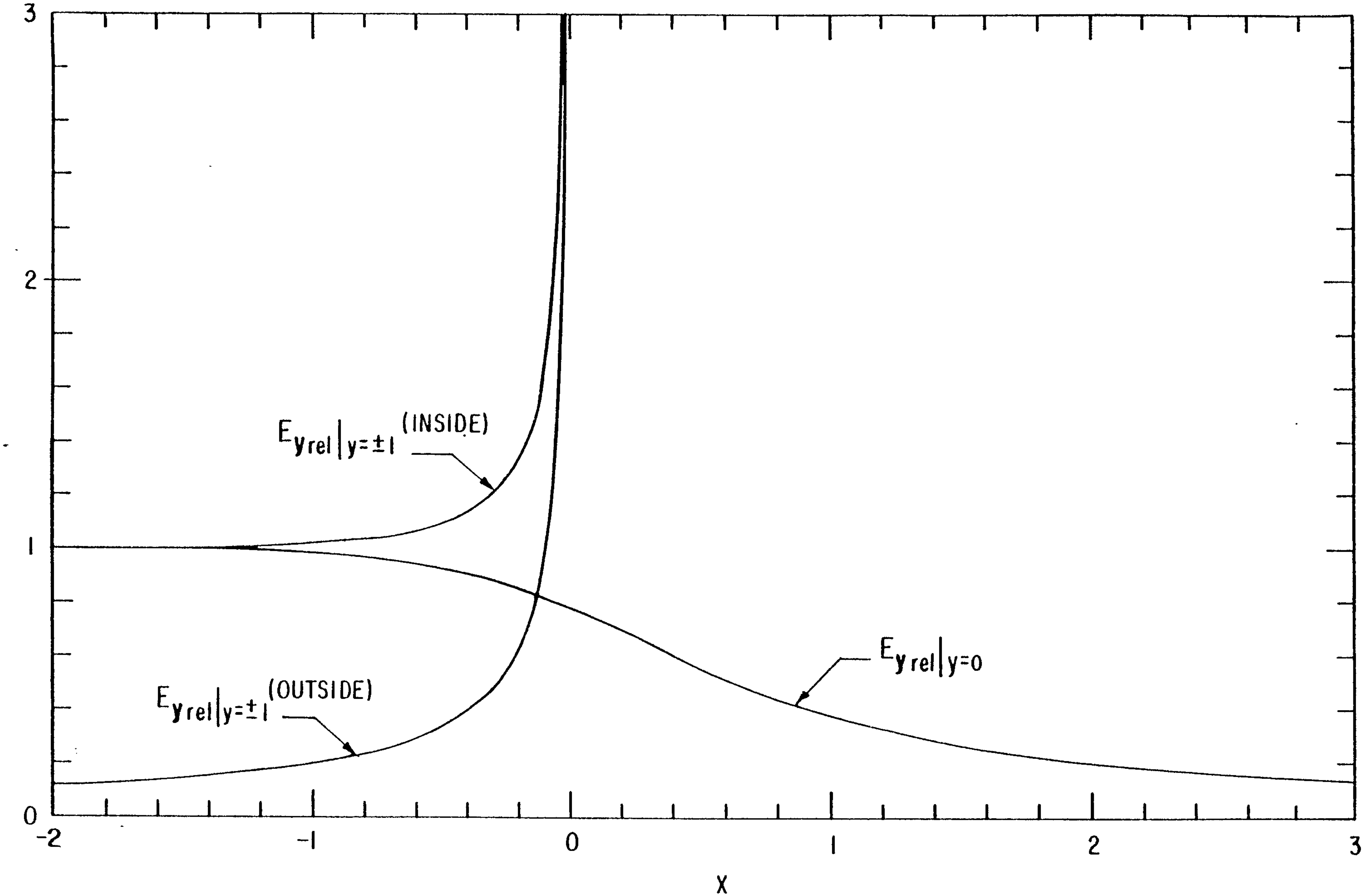

FIGURE 9. NORMALIZED FIELD DISTRIBUTION FOR SEMI-INFINITE, PARALLEL, TWO-PLATE TRANSMISSION LINE

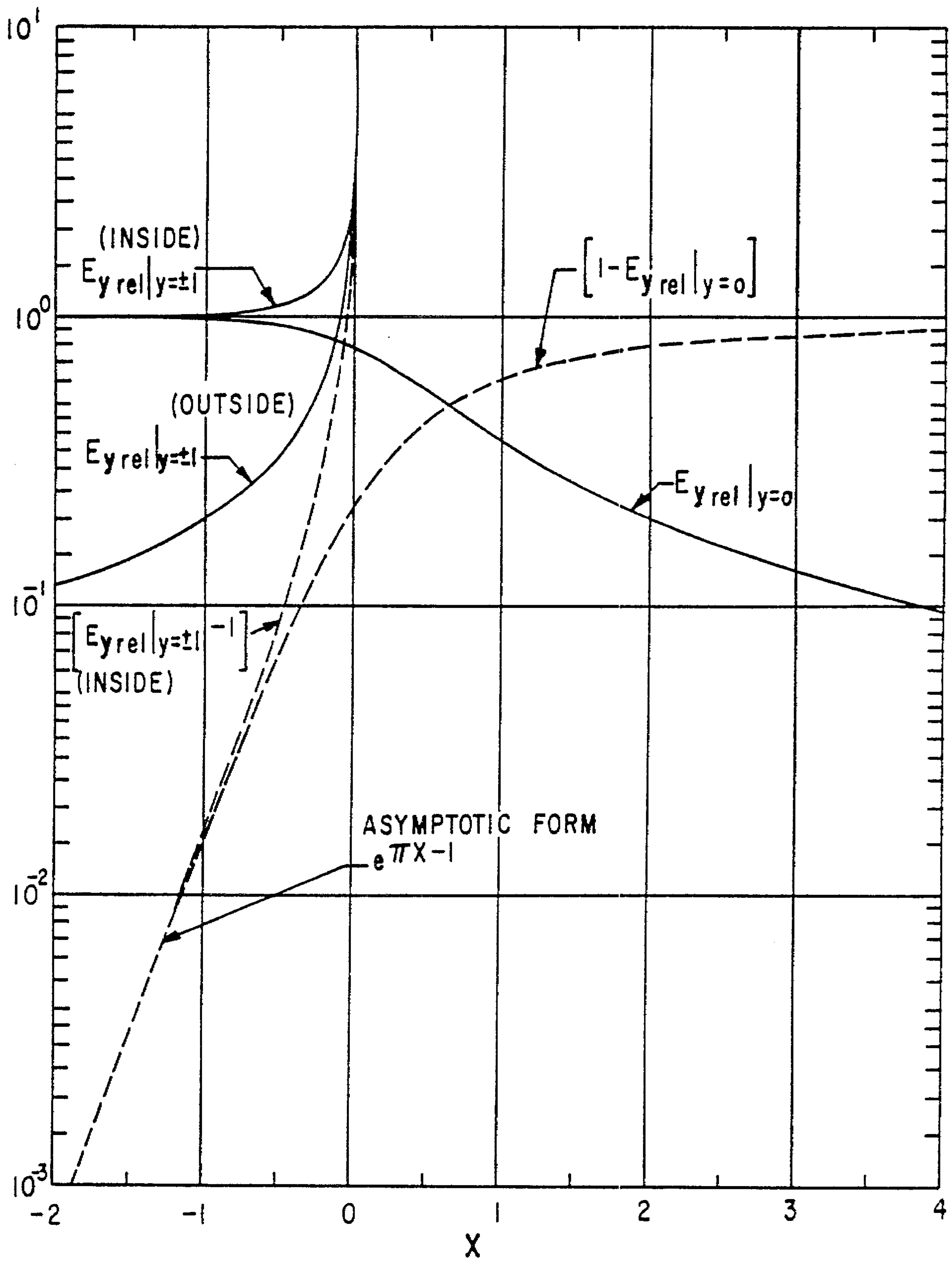

FIGURE 10. NORMALIZED FIELD DISTRIBUTION FOR SEMI-INFINITE, PARALLEL, TWO-PLATE TRANSMISSION LINE

This is illustrated in figure 11 for positive x and y for the case of a 100 Ω transmission line (assuming a wave impedance equal to that of free space). If we place a conducting plane at y=0 we also have the field distribution for a 50 Ω transmission line consisting of a finite width conducting plate parallel to an infinite width conducting plate. The equipotentials and magnetic field lines are given by constant v and the electric field lines by constant u. By symmetry this figure can be extended to all four quadrants.

Expanding the transformation we have

$$x = \frac{2\ K(m)}{\pi}\left\{E(u|m)-u\frac{E(m)}{K(m)} + m\ \frac{sn(u|m)\ cn(u|m)dn(u|m)sn^2(v'|m_1)}{1-dn^2(u|m)sn^2(v'|m_1)}\right\} \tag{56}$$

and

$$y = \frac{2\ K(m)}{\pi}\left\{E(v'|m_1)-\frac{v'\ E(m_1)}{K(m_1)}+\frac{\pi v}{2K(m)K(m_1)} - \frac{dn^2(u|m)sn(v'|m_1)cn(v'|m_1)dn(v'|m_1)}{1 - dn^2(u|m)sn^2(v'|m_1)}\right\} \tag{57}$$

where

$$v' = v + K(m_1) \tag{58}$$

In this normalized geometry (outer plates at $y = \pm 1$) let us next relate a/b to the parameter, m, in the elliptic functions. At $y = \pm 1$ for $|x| < \frac{a}{b}$ we have

$$v = \pm K(m_1) \tag{59}$$

and

$$x = \frac{2\ K(m)}{\pi}\left\{E(u|m) - u\ \frac{E(m)}{K(m)}\right\} = \frac{2\ K(m)}{\pi}\ Z(u|m) \tag{60}$$

Varying u between 0 and K(m) corresponds to moving from x = 0 to x = a/b on the outside of one of the plates and then from x = a/b back to x = 0 on the inside of the plate. (See figure 11.) Thus we have

$$\frac{a}{b} = \frac{2\ K(m)}{\pi}\ Z_{max} \tag{61}$$

where Z_{max} is the maximum value of $Z(u|m)$ for a fixed m.

For convenience we can rewrite equation (60) as

$$x = \frac{2}{\pi}\left\{K(m)\ E(\phi|m) - E(m)\ F(\phi|m)\right\} \tag{62}$$

where we use the amplitude, ϕ, (instead of u). Then let

$$\left.\frac{\partial x}{\partial \phi}\right|_{\phi = \phi_0} = 0 = \frac{2}{\pi}\left\{K(m)\ [1-m\sin^2(\phi_0)]^{1/2} - E(m)\ [1-m\sin^2(\phi_0)]^{-1/2}\right\} \tag{63}$$

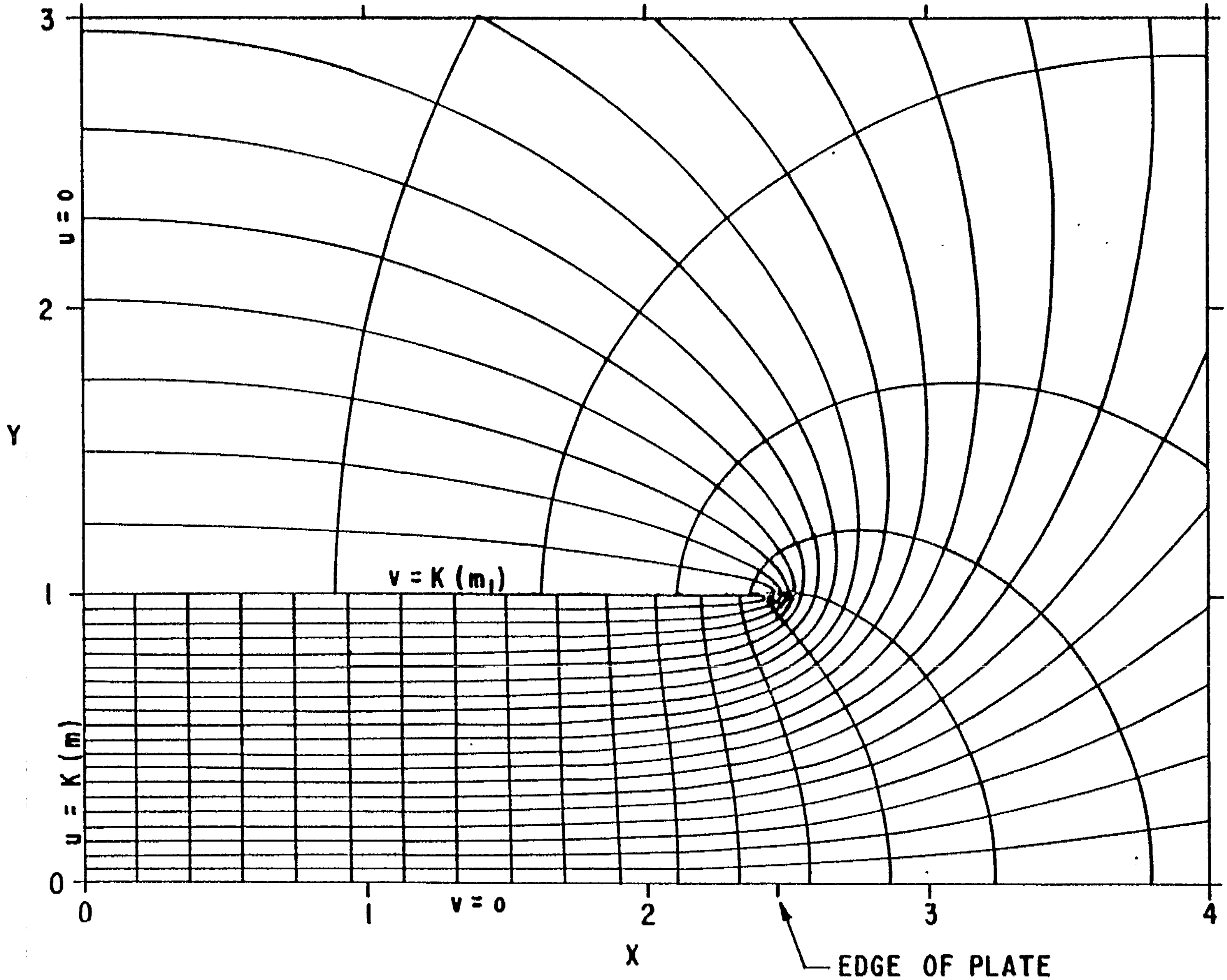

FIGURE 11. FIELD AND POTENTIAL DISTRIBUTION FOR PARALLEL, TWO-PLATE TRANSMISSION LINE: 100 Ω

Thus

$$1 - m\sin^2(\phi_o) = \frac{E(m)}{K(m)} \tag{64}$$

or

$$\sin(\phi_o) = \left[\frac{1}{m}\left(1 - \frac{E(m)}{K(m)}\right)\right]^{1/2} \tag{65}$$

and finally

$$\frac{a}{b} = \frac{2}{\pi}\left\{K(m)\,E(\phi_c|m) - E(m)F(\phi_o|m)\right\} \tag{66}$$

The geometric factor in the impedance is

$$f_g = \frac{K(m)}{K(m_1)} \tag{67}$$

Thus relating m to $\frac{a}{b}$ from equations (65) and (66) we can calculate f_g as graphed in figure 12.

For both large and small b/a we can obtain approximate expressions for f_g. For small b/a we have the approximation[7]

$$f_g \approx \frac{b}{a}\left\{1 + \frac{b}{\pi a}\left[1 + \ln\left(\frac{2\pi a}{b}\right)\right]\right\}^{-1} \tag{68}$$

This is also plotted in figure 12 together with the difference from the more exact calculation. The latter could only be calculated down to a b/a close to 0.1 because of computer accuracy problems associated with m very close to one. Fortunately the approximation of equation (68) is within about a percent at this point as can be seen in figure 12. For large b/a (corresponding to small m) we can take first order expansions, in m, of equations (65), (66), and (67), obtaining

$$\phi_o \approx \frac{\pi}{4} \tag{69}$$

and

$$\frac{a}{b} \approx \frac{m}{4} \tag{70}$$

and finally

$$f_g \approx \frac{1}{\pi}\ln\left(\frac{16}{m}\right) \quad \frac{1}{\pi}\ln\left(4\frac{b}{a}\right) \tag{71}$$

which is also plotted in figure 12.

7. A.E.H. Love, Proc. London Math. Soc. 22, 337-369, 1923

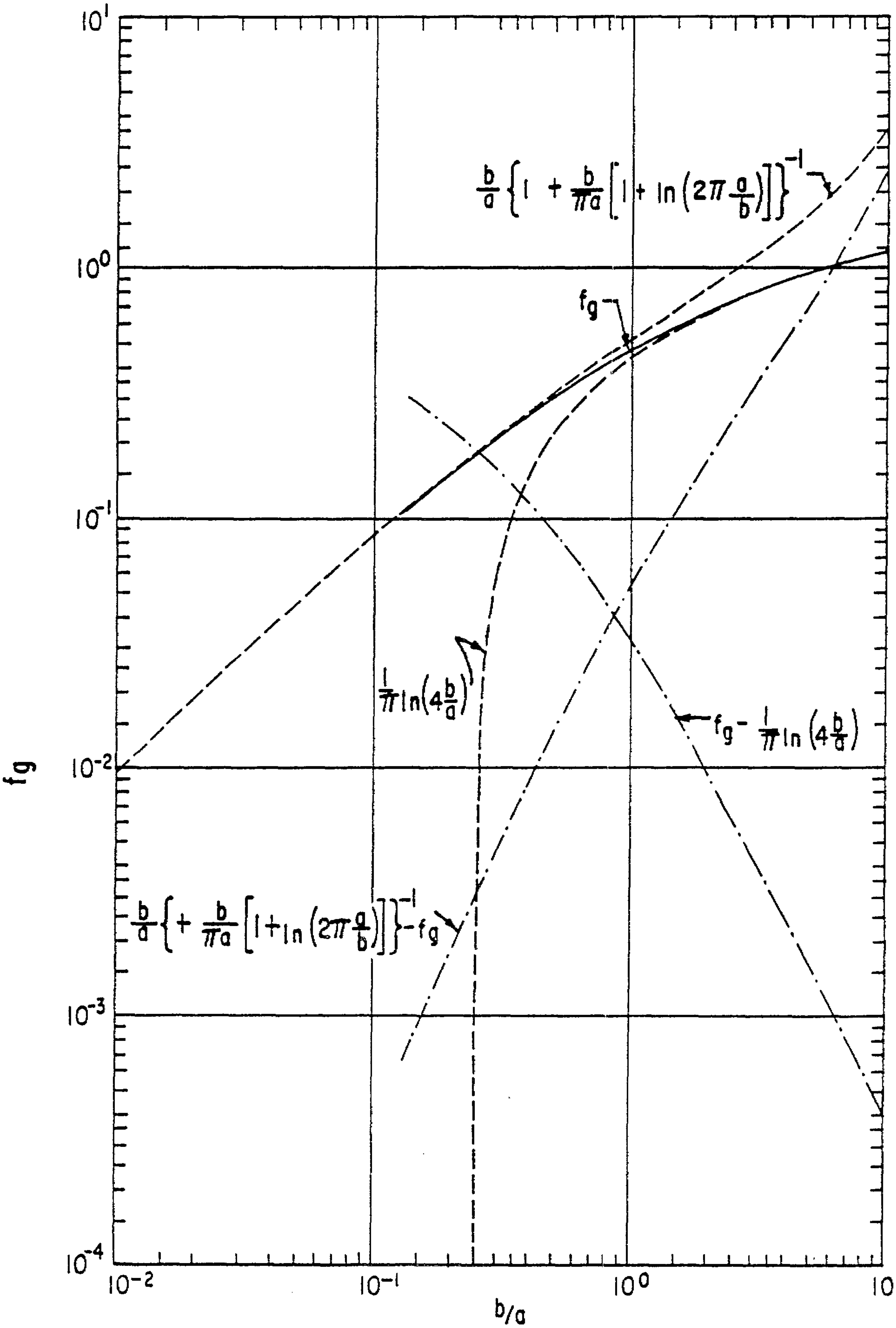

FIGURE 12. GEOMETRIC IMPEDANCE FACTOR FOR FINITE, SYMMETRICAL, TWO PLATE TRANSMISSION LINE

For convenience as in Section II B we can tabulate b/a for a few impedances (for a wave impedance equal to that of free space).

Z_L (ohms)	f_g	b/a
25	.06627	.07484
50	.13254	.16652
100	.26508	.40641
200	.5302	1.2335

Table II. Impedances for Two Plate Transmission Line

Now let us consider the field distribution. First along the plates ($y=\pm 1$, $v=\pm K(m_1)$) we have

$$x = \frac{2}{\pi}\left\{K(m)\ E(u|m) - u\ E(m)\right\} \tag{72}$$

The normalized electric field along the outer plates is then

$$E_{y_{rel}}\Big|_{y=\pm 1} = \left|\frac{1}{K(m_1)} \quad \frac{\partial u}{\partial x}\right| \tag{73}$$

or

$$E_{y_{rel}}\Big|_{y=\pm 1} = \left|\frac{\pi}{2\ K(m_1)} \cdot \left\{K(m)\ dn^2\ (u|m) - E(m)\right\}^{-1}\right| \tag{74}$$

The absolute value is indicated because the expression changes sign depending on the value of u which governs whether the expression pertains to fields on the outside ($|y| = 1+$) or on the inside ($|y| = 1-$) of the plates.

For x = 0 we then have two special cases. At the center inside the plates ($u = K(m)$) we have

$$E_{y_{rel}}\Big|_{\substack{x=0 \\ y=\pm 1}} (\text{inside}) = \frac{\pi}{2\ K(m_1)\ [E(m) - m_1 K(m)]} \tag{75}$$

and at the center outside the plates ($u=0$) we have

$$E_{y_{rel}}\Big|_{\substack{x=0 \\ y=\pm 1}} (\text{outside}) = \frac{\pi}{2\ K(m_1)\ [K(m) - E(m)]} \tag{76}$$

With the use of equation (66) we can plot these last two quantities against b/a as in figure 13.

Second along the plane of symmetry which is also an equipotential ($y=0$, $v=0$) we have

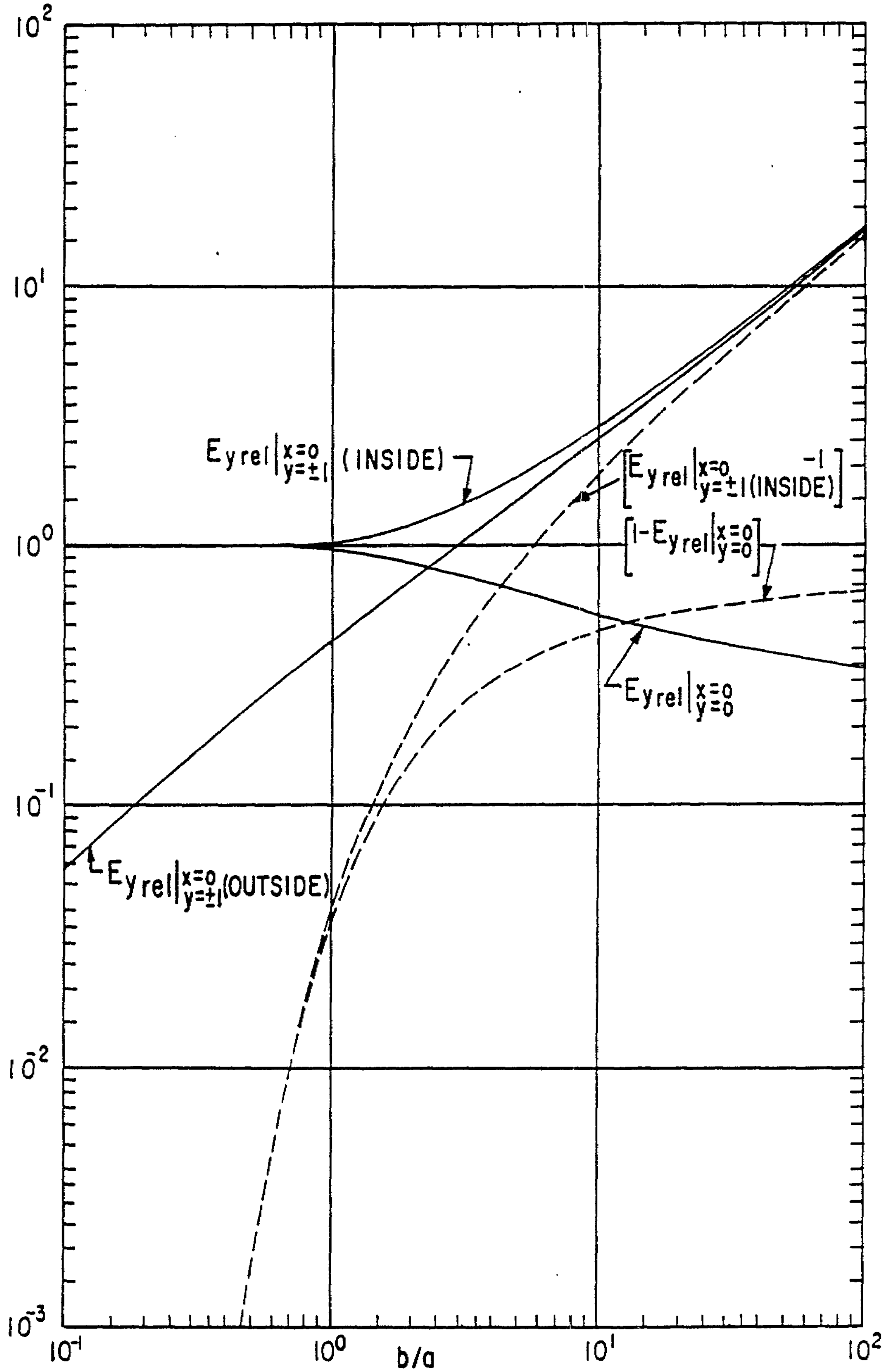

FIGURE 13. NORMALIZED FIELD VARIATION FOR FINITE, SYMMETRICAL, TWO PLATE TRANSMISSION LINE

$$x = \frac{2\ K(m)}{\pi} \left\{ E(u|m) - u\ \frac{E(m)}{K(m)} + \frac{cn(u|m)\ dn(u|m)}{sn(u|m)} \right\} \tag{77}$$

This gives a normalized field

$$E_{y_{rel}}\Big|_{y=0} = -\ \frac{1}{K(m_1)}\ \frac{\partial u}{\partial x} \tag{78}$$

or

$$E_{y_{rel}}\Big|_{y=0} = \frac{\pi}{2\ K(m_1)\ K(m)} \left\{ \frac{E(m)}{K(m)} + \frac{cn^2(u|m)}{sn^2(u|m)} \right\}^{-1} \tag{79}$$

For x=0 (and u = K(m)) we have

$$E_{y_{rel}}\Big|_{\substack{x=0\\y=0}} = \frac{\pi}{2\ K(m_1)\ E(m)} \tag{80}$$

This is also plotted in figure 13.

Thus we can determine the impedance and field distribution for this two plate transmission line. Likewise using equations (75) and (80) we can determine the field uniformity near the middle (x=0) of the transmission line. These results also apply to a transmission line consisting of a conducting plate of width 2a at a distance b from an infinite conducting plate if we divide f_g by two. However if we look at figures 9 and 10 we can see how slowly the field falls off with increasing distance from the finite plate, at least compared with figure 3 and 4 for the three plate transmission line. Thus the conducting plate at y=0 should extend significantly beyond the edges of the other plate to intercept most of the field for the calculations to be valid.

IV. Effect of Replacing Conducting Plates with Wires

In some cases it is desirable to replace some of the continuous conducting plates with grids of parallel wires, each wire being parallel to the direction of propagation of the wave on the transmission line. Let us then consider the field distribution around a grid of parallel wires, each of radius, c , and spacing between centers, 2d, all lying in a common plane as illustrated in figure 14. To simplify the calculations let the grid extend to infinity on both sides, i.e., let the structure be periodic. Thus we can consider a repetitive cell of width, 2d, which we can solve by conformal transformation. Also consider only the case in which the wire diameter is much less then the spacing between wires.

All the wires are assumed to be at the same potential and two types of field distributions are considered. First is the case in which the grid supports a uniform electric field on one side and no field on the other. In the second

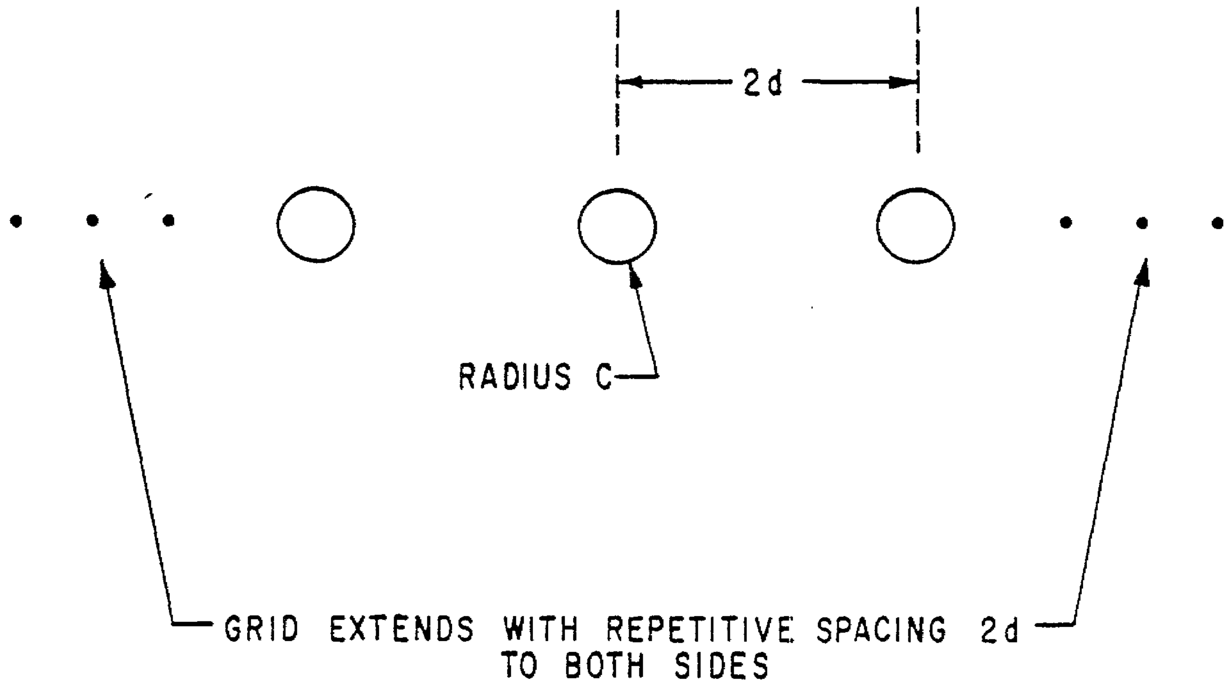

FIGURE 14. UNIFORM GRID OF PARALLEL WIRES

case the grid supports equal (but opposite in direction) uniform electric fields on both sides of the grid. Since these grids can be used to replace the conducting plates of the parallel plate transmission lines (as in figure 1), it is necessary in such a case that the wire spacing be much less than the characteristic dimensions (b and a) of the transmission line if these calculations are to apply. Thus significant distortions will be confined to the vicinity of the grids over a distance on the order of the wire spacing. This field distortion is also reflected in an increase in the transmission line impedance which we can look upon as an increase in the "effective" plate spacing.

A. Uniform Field on One Side of Grid

Consider first the case in which the grid supports a uniform field on one side. The conformal transformation is

$$z = \frac{j}{\pi} \ln[e^{w} + 1] \tag{81}$$

one cell of which is illustrated in figure 15. The equipotentials and magnetic field lines are given by constant u and the electric field lines by constant v. We have normalized the problem by setting d to one.

Rewriting equation (81) as

$$w = \ln [e^{-j\pi z} - 1] \tag{82}$$

we have

$$u = \frac{1}{2} \ln [e^{2\pi y} - 2 e^{\pi y} \cos (\pi x) + 1] \tag{83}$$

and

$$v = - \arctan \left[\frac{e^{\pi y} \sin(\pi x)}{e^{\pi y} \cos(\pi x) - 1}\right] \tag{84}$$

For small $|z|$ (or large negative u) the equipotentials approach circles, approximating the wire shape. For convenience define the wire potential, u_o, by setting $x = 0$ and $y = \frac{c}{d}$.
Thus

$$u_o = \ln [e^{\pi\frac{c}{d}} - 1] \tag{85}$$

For large positive y we have

$$u \approx \pi y \tag{86}$$

giving an approximately uniform field distribution. Let us then determine that y for a uniform field distribution, y_o, which would be at a potential, u_o, and match the potential of equation (86). Thus

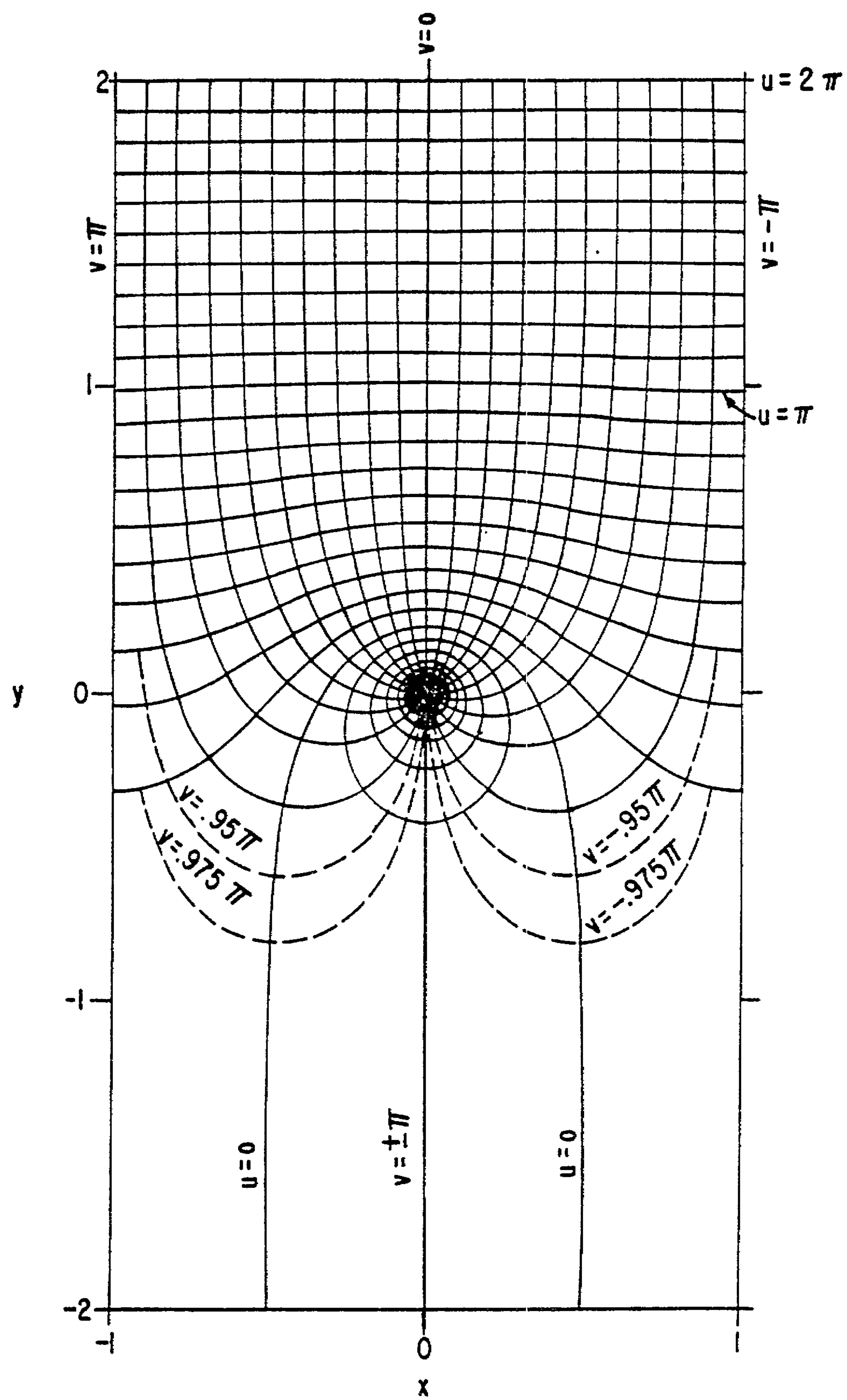

FIGURE 15. WIRE GRID (ONE CELL) TERMINATING UNIFORM FIELD ON ONE SIDE OF GRID

$$y_0 \approx \frac{1}{\pi} u_0 \approx \frac{1}{\pi} \ln \left[e^{\pi \frac{c}{d}} - 1 \right] \tag{87}$$

This represents an "effective" position of the grid for use in impedance calculations, i.e., this is the position of the flat conducting surface equivalent to the grid. Removing the normalization of the coordinates we have a shift, Δy, of the grid to an equivalent electrical position (as a conducting plane)

$$\Delta y \approx -\frac{d}{\pi} \ln \left[e^{\pi \frac{c}{d}} - 1 \right] \approx \frac{d}{\pi} \ln \left(\frac{d}{\pi c} \right) \tag{88}$$

We can use this approximate formula to calculate an increase in transmission line impedance due to an increase, Δy, in effective plate spacing.

Near the wires the field distribution will be significantly distorted. In the normalized geometry consider first the potential distribution along a plane midway between two adjacent wires ($x=\pm 1$, $v=\mp\pi$). Thus from equation (83)

$$u = \ln \left[e^{\pi y} + 1 \right] \tag{89}$$

which gives a relative field distribution

$$E_{y_{rel}} \Big|_{x=\pm 1} = \frac{1}{\pi} \frac{\partial u}{\partial y} = \left[1 + e^{-\pi y} \right]^{-1} \tag{90}$$

Second along the plane, $x=0$, we have

$$u = \ln \left[\left| e^{\pi y} - 1 \right| \right] \tag{91}$$

which gives a relative field distribution

$$E_{y_{rel}} \Big|_{x=0} = \left| \frac{1}{\pi} \frac{\partial u}{\partial y} \right| = \left| \left[1 - e^{-\pi y} \right]^{-1} \right| \tag{92}$$

Note that for negative y the field along the plane, $x=0$, is opposite in direction to that along the planes, $x=\pm 1$. These field distributions are plotted in figure 16 showing the extent of field distortion due to the wire grid.

B. Equal and Opposite Uniform Fields on Both Sides of Grid

For the case that the grid supports equal and opposite uniform fields on both sides we have the conformal transformation

$$z = \frac{2}{\pi} \arcsin \left[e^{w} \right] \tag{93}$$

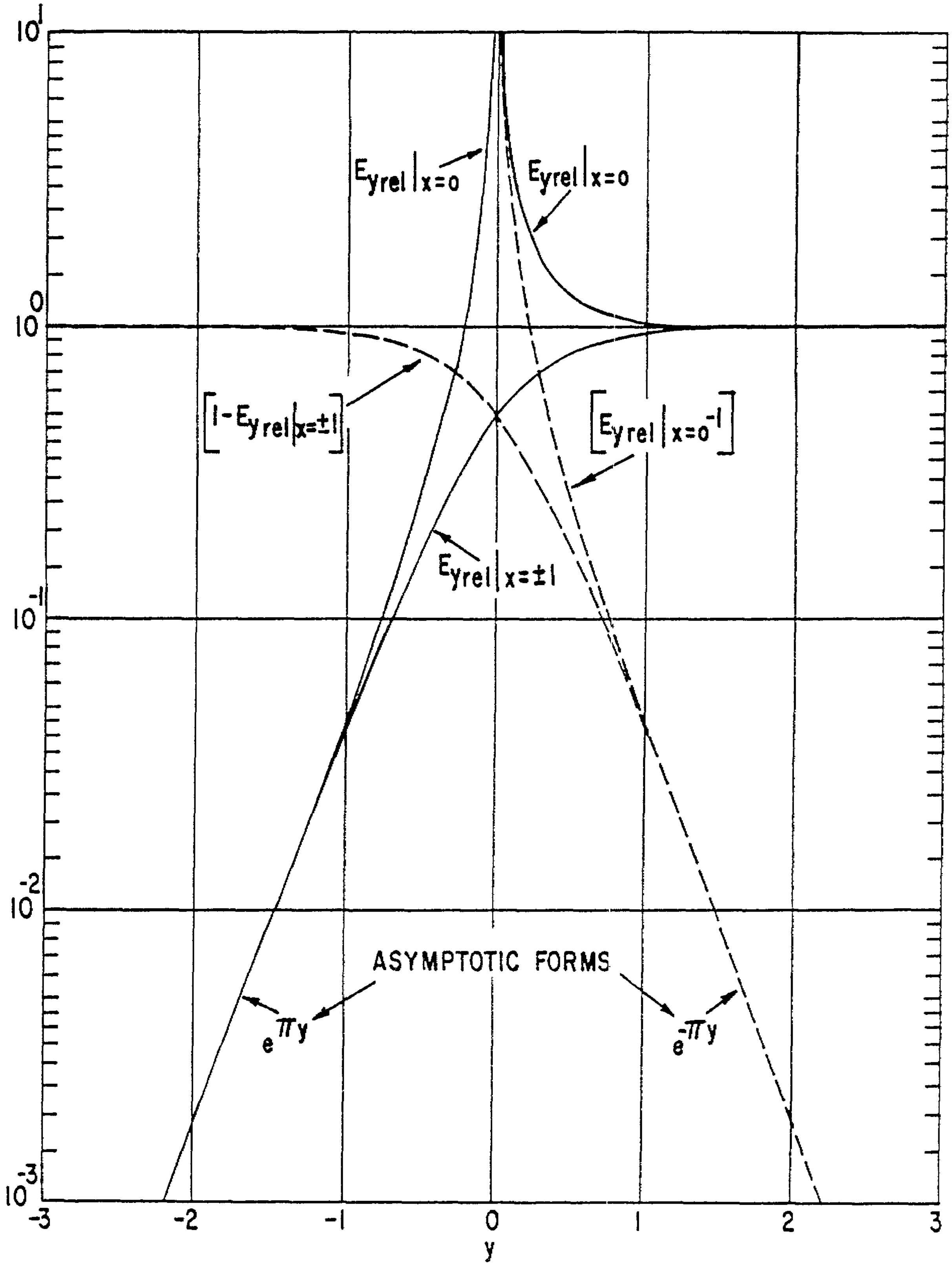

FIGURE 16. NORMALIZED FIELD DISTRIBUTION FOR WIRE GRID TERMINATING UNIFORM FIELD ON ONE SIDE OF GRID

one cell of which is illustrated in figure 17. The equipotentials and magnetic field lines are given by constant u and the electric field lines by constant v. Again we have set d to one for the calculations

Rewriting equation (93) as

$$w = \ln\left[\sin\left(\frac{\pi}{2}z\right)\right] \tag{94}$$

we have

$$u = \frac{1}{2}\ln\left[\sin^2\left(\frac{\pi}{2}x\right)\cosh^2\left(\frac{\pi}{2}y\right) + \cos^2\left(\frac{\pi}{2}x\right)\sinh^2\left(\frac{\pi}{2}y\right)\right] \tag{95}$$

and

$$v = \arctan\left[\cot\left(\frac{\pi}{2}x\right)\tanh\left(\frac{\pi}{2}y\right)\right] \tag{96}$$

Again the equipotentials approach circles for small $|z|$, approximating the wire shape. Taking the position x=0 and $y = \pm\frac{c}{d}$ to define the wire potential, u_o, we have

$$u_o \approx \ln\left[\sinh\left(\frac{\pi}{2}\frac{c}{d}\right)\right] \tag{97}$$

For large positive y we have

$$u \approx \frac{\pi}{2}y - \ln(2) \tag{98}$$

That y for a uniform field distribution, y_o, which would be at a potential, u_o, and match the potential of equation (98) is

$$y_o \approx \frac{2}{\pi}\left[u_o + \ln(2)\right] \approx \frac{2}{\pi}\ln\left[2\sinh\left(\frac{\pi}{2}\frac{c}{d}\right)\right] \tag{99}$$

Removing the normalization of the coordinates we have a shift, Δy, of the wire grid to an equivalent electrical position (as a conducting plane)

$$\Delta y \approx -\frac{2d}{\pi}\ln\left[2\sinh\left(\frac{\pi}{2}\frac{c}{d}\right)\right] \approx \frac{2d}{\pi}\ln\left(\frac{d}{\pi c}\right) \tag{100}$$

Note that this is twice as large as the effective grid displacement for a uniform field on one side of the grid (equation (88)). We should also note that the effective displacement of equation (100) contributes to an impedance increase on both sides of the grid. For example, in the case of the three plate transmission line (figure 1A), if the center plate were replaced with a wire grid we would use equation (100) to calculate an effective increase in the plate spacing, b, on each

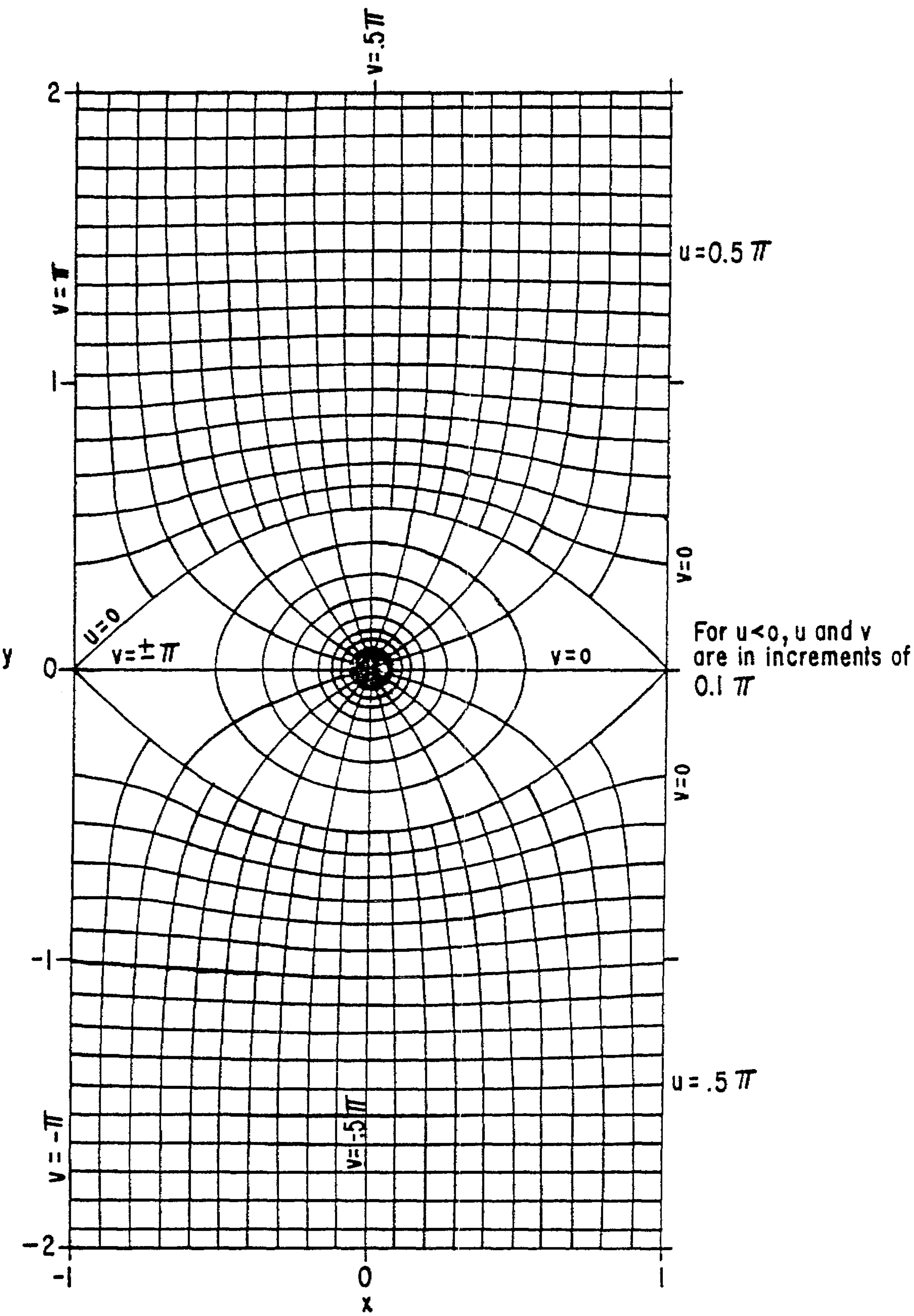

FIGURE 17. WIRE GRID (ONE CELL) TERMINATING EQUAL AND OPPOSITE UNIFORM FIELDS ON BOTH SIDES OF GRID

side of the center plate. Equation (88) would apply to replacing one of the outer plates with a wire grid.

Looking at the field distribution near the wires first consider the potential distribution along a plane midway between adjacent wires ($x=\pm 1$, $v=0,\pm\pi$). Thus from equation (95)

$$u = \ln\left[\cosh\left(\frac{\pi}{2}\, y\right)\right] \tag{101}$$

which gives a relative field distribution

$$E_{y_{rel}}\Big|_{x=\pm 1} = \left|\frac{2}{\pi}\ \frac{\partial u}{\partial y}\right| = \left|\tanh\left(\frac{\pi}{2}\, y\right)\right| \tag{102}$$

Second along the plane, x=0, we have

$$u = \ln\left[\left|\sinh\left(\frac{\pi}{2}\, y\right)\right|\right] \tag{103}$$

which gives a relative field distribution

$$E_{y_{rel}}\Big|_{x=0} = \left|\frac{2}{\pi}\ \frac{\partial u}{\partial y}\right| = \left|\coth\left(\frac{\pi}{2}\, y\right)\right| \tag{104}$$

Both of these field distributions are symmetrical about the plane of the grid (y=0) and are plotted for positive y in figure 18.

Thus we can make a first order correction to our impedance calculations for the use of periodic wire grids (parallel to the current flow) in place of conducting plates. Likewise we can calculate the minimum distance away from the grid we must maintain for a given field uniformity.

V. Summary

We can calculate the impedances of both the symmetrical three plate transmission line with sufficiently large outer plates (figure 6) and the symmetrical two plate transmission line (figure 12). The latter solution also applies to the transmission line consisting of a finite width conducting plate parallel to a sufficiently large conducting plane if we halve the impedance. If the conducting plates are replaced by grids of parallel wires we can make a first order correction to the impedance, increasing the effective plate spacing.

For use in design of electromagnetic field simulators we can also determine the field distribution and degree of field uniformity over certain regions in the transmission line structures. However we should be careful in our application of these results because there may be some other effects which influence the impedances and field distributions in certain cases.

We would like to thank Mrs. Linda Crosby and Mr. Robert Mercer who programmed the computer solutions for the plots and tables contained in this note.

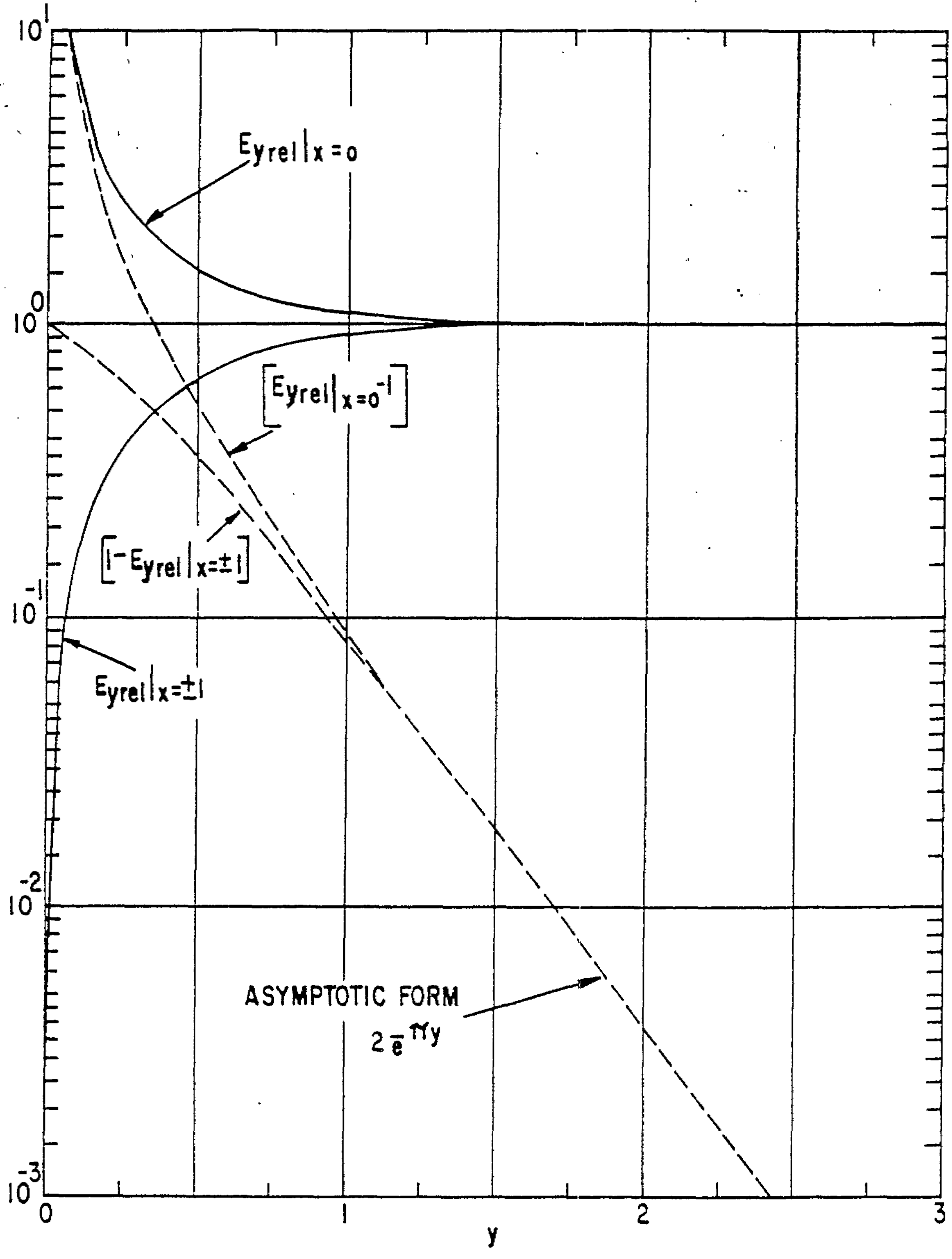

FIGURE 18. NORMALIZED FIELD DISTRIBUTION FOR WIRE GRID TERMINATING EQUAL AND OPPOSITE UNIFORM FIELDS ON BOTH SIDES OF GRID