Sensor and Simulation Notes

Note 206

May 1974

MATHEMATICS OF GUIDED WAVE PROPAGATION IN OPEN STRUCTURES

Chiao-Min Chu
University of Michigan

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ABSTRACT

The propagation of electromagnetic waves along open structures whose transverse cross section are composed of two symmetrically placed circular sectors is investigated. The problem is first formulated in the form of a dual summation equation. By introducing a new set of basis functions to represent the surface field, the dispersion relation for such a structure is developed, systematic techniques of finding the approximate solutions of the dispersion relation are formulated, and numerical solutions for the first two orders of approximation are carried out to illustrate the feasibility of using the technique.

In general, the scattering and resonance problem of open structures of cylindrical, spherical or other configurations may be formulated in the form of dual summation equations. The approach introduced in this work, i.e., the use of appropriate basis functions and the numerical solution of these equations, therefore, may open a new avenue in the solution of a wide class of problems.

ACKNOWLEDGEMENT

This work has been sponsored by the Air Force Office of Scientific Research.

PL 96-1236

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wave propagation, electromagnetic waves, scattering

ACKNOWLEDGEMENT

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I

STATEMENT OF THE PROBLEM

In this study, the propagation of electromagnetic waves along a class of open structures is investigated by introducing a new set of basic functions in the representation of the surface field. To illustrate this approach, let us consider the waves guided along a curved transmission line composed of infinitely long, symmetrically located conducting circular sectors as shown in Figure 1. Assuming that the time dependence and z-variation of the field to be $e^{j\omega t - j\beta z}$, (β is the unknown propagation constant to be determined), all the field components may be generated from a single field component E_z (for TM modes) or H_z (for TE modes). Explicitly, for TM modes, we denote,

$$E_z(r, \phi) = \sum_{n=0}^{\alpha} A_n \frac{Z_{2n+1}(\gamma r)}{Z_{2n+1}(\gamma a)} \cos(2n+1)\phi \tag{1}$$

where

$$\gamma = \frac{\omega^2}{c^2} - \beta^2 \quad . \tag{2}$$

The Z_n's are Bessel functions, representing Hankel functions of the second kind $H_n^{(2)}$ for $r > a$ and the Bessel functions J_n for $r < a$. The unknown propagation constant is then determined from the boundary conditions as $r \rightarrow a$. For perfectly conducting structures, these conditions are listed below:

(a) The vanishing of the tangential component of electric field on the conductors yields:

$$E_z(a, \phi) = \sum_{n=0}^{\alpha} A_n \cos(2n+1)\phi = 0 \tag{3}$$

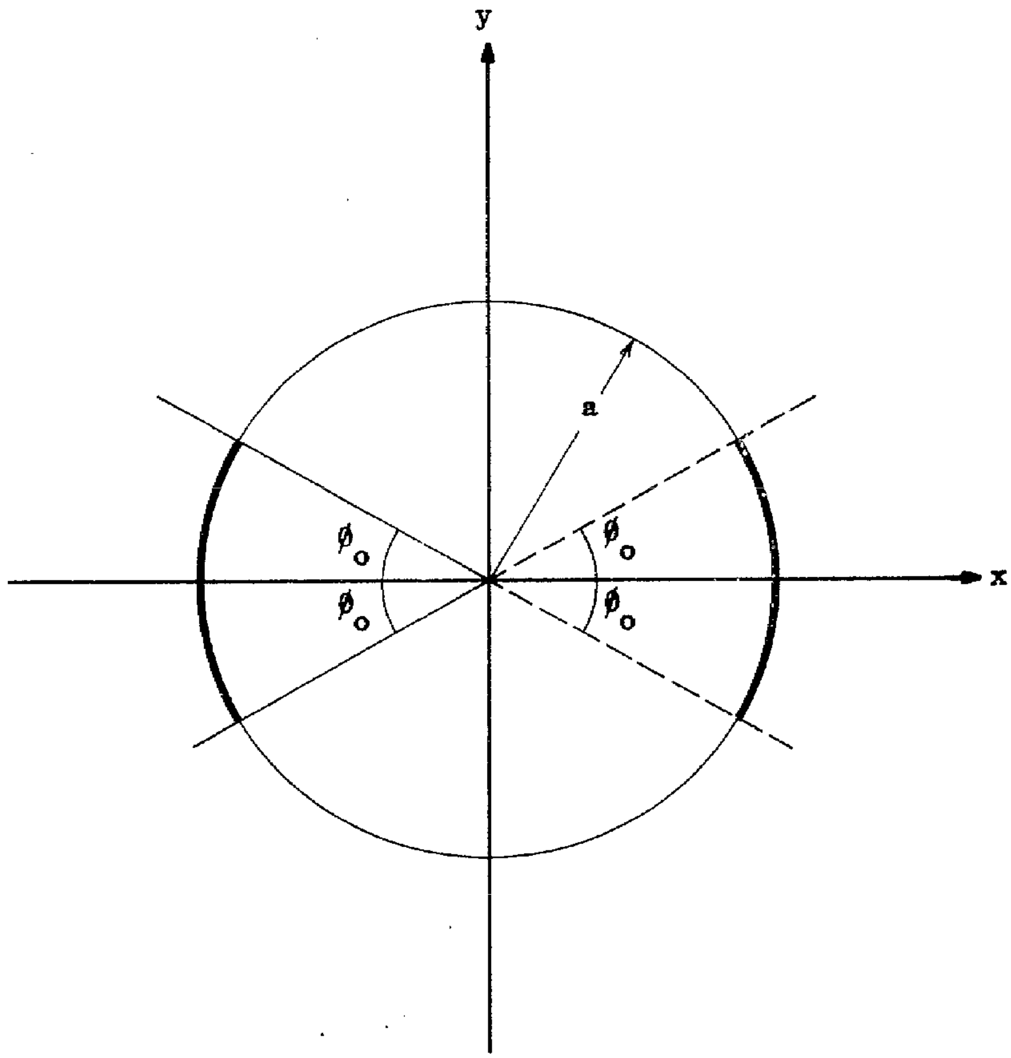

Figure 1: Curved transmission line

and

$$E_\phi(a, \phi) = -\frac{j\beta}{\gamma} \sum_{n=0}^{\alpha} (2n+1) A_n \sin(2n+1)\phi = 0 \tag{3}$$

for $\phi \epsilon L_1$, where L_1 denotes the range of ϕ such that $-\phi_0<\phi<\phi_0$, and $\pi - \phi_0 < \phi < \pi + \phi_0$,

(b) The discontinuity of the tangential component of magnetic field across the conductors yields,

$$J_z(a, \phi) = \frac{2\omega\epsilon_0}{\pi\gamma^2 a} \sum_{n=0}^{\alpha} \frac{A_n}{H_{2n+1}^{(2)}(\gamma a)\, J_{2n+1}(\gamma a)} \cos(2n+1)\phi = 0 \tag{4}$$

for $\phi \epsilon L_2$, where $\phi \epsilon L_2$ denotes the range of ϕ such that

$$\phi_0 < \phi < \pi - \phi_0 , \qquad \pi + \phi_0 < \phi < 2\pi - \phi_0 .$$

Similarly, for TE modes, we denote

$$H_z(r, \phi) = \sum_{n=0}^{\alpha} B_n \frac{Z_{2n+1}(\gamma r)}{Z'_{2n+1}(\gamma a)} \sin(2n+1)\phi \tag{5}$$

The boundary conditions are:

$$\text{(a)} \quad E_\phi(a, \phi) = \frac{-j\omega\mu_0}{\gamma} \sum_{n=0}^{\alpha} B_n \sin(2n+1)\phi = 0 \tag{6}$$

for $\phi \epsilon L_1$,

$$\text{(b)} \quad J_z(a, \phi) = \frac{-2\beta}{\pi\gamma^3 a} \sum_{n=0}^{\alpha} \frac{(2n+1) B_n}{H'^{(2)}_{2n+1}(\gamma a)\, J'_{2n+1}(\gamma a)} \cos(2n+1)\phi$$

and (7)

$$J_\phi(a, \phi) = \frac{2j}{\pi\gamma a} \sum_{n=0}^{\alpha} \frac{B_n}{H'_{2n+1}(\gamma a)\, J'_{2n+1}(\gamma a)} \sin(2n+1)\phi$$

for $\phi \epsilon L_2$.

It is interesting to note that in both cases, the boundary conditions may be expressed in the following general form:

$$J(\phi) = \sum_{n=0}^{\alpha} C_n \cos\left(n+\frac{1}{2}\right) 2\phi = 0 , \qquad \phi \in L_2 . \tag{8}$$

and

$$E(\phi) = \sum_{n=0}^{\alpha} C_n G_n(\gamma a) \sin\left(n+\frac{1}{2}\right) 2\phi = 0, \quad \phi \in L_1 . \tag{9}$$

Here, except for a constant factor $J(\phi)$ denotes $J_z(a, \phi)$, and $E(\phi)$ denotes $E_\phi(a, \phi)$. $G_n(\gamma a)$ are known functions of (γa), for the TM case, we use

$$G_n(\gamma a) = \frac{\pi}{j}(2n+1) H^{(2)}_{2n+1}(\gamma a) J_{2n+1}(\gamma a) \tag{10}$$

while for the TE case, we use*

$$G_n(\gamma a) = -\frac{\pi}{j} \frac{J'_{2n+1}(\gamma a) H^{(2)}_{2n+1}(\gamma a)}{(2n+1)} (\gamma a)^2 . \tag{11}$$

Mathematically, therefore, our problem of finding the propagation constants for the transmission line illustrated in Figure 1 is equivalent to that of finding the set of values of γ (or γa) such that nontrivial solutions exist for the homogeneous "dual summation equations" given by Eqs. (8) and (9). Although detailed analysis concerning the existence of solutions of such a problem is difficult to carry out, we assert that solutions do exist, based on the following

*These particular forms of $G_n(\gamma a)$ were chosen to simplify the expressions developed later for approximate solutions of γa.

two physical arguments:

(a) It is known that the structure illustrated in Figure 1 supports the TEM mode of propagation. In this case, $\beta = \frac{\omega}{c}$, and $\gamma = 0$, thus we know the "dual summation equation" has at least one solution - namely $\gamma a = 0$.

(b) In the published works on wave propagation along slotted cylindrical waveguide, such as the works of Harrington (1943) and Goldstone and Oliner (1961), experimental evidence of high order propagation (with imaginary part of $\beta < 0$) was reported. On physical ground, therefore, one may suspect higher order modes also exist in the present structure which may be considered as a cylindrical waveguide with two slots. Our primary objective, therefore, is to develop a systematic approach, with the help of computers, to determine numerically the solution set (γa) for the system of dual summation equations. Based on the known solutions given by the TEM mode of propagation, we first define sets of functions as the basis of approximating the field components. These sets of functions, including members that are discontinuous and unbounded, may be used to accurately represent the dominant components (from the well known edge condition) of the surface fields. Using this basis for the field representation, an infinite system of equations is obtained for the propagation constant. Since the dominant components of the fields are included in the first few terms of the field representation, truncation of the infinite system appears to be reasonable. Procedures of approximate evaluation of the higher order propagation constants by including N (arbitrary) equations of the infinite set are then developed.

II

CONSTRUCTION OF BASIC FUNCTIONS

It is well known that the structure illustrated in Figure 1 supports the TEM mode of propagation. For this mode, the field components may be represented by

$$\underline{E} = e^{-j \frac{\omega}{c} z} \nabla V(x, y) \tag{12}$$

and

$$\underline{H} = \frac{1}{\mu_0 c} e^{-j \frac{\omega}{c} z} \hat{z} \times \nabla V(x, y) \tag{13}$$

where V (x, y) satisfies the two dimensional Laplace equation and the boundary conditions. By obtaining V through the use of conformal transformation, it is easily verified that, except for multiplicative constants,

$$E_\phi(a, \phi) = \begin{cases} 0 & \phi \in L_1 \\ \dfrac{1}{\sqrt{\cos 2\phi_0 - \cos 2\phi}} & \phi \in L_2 \end{cases} \tag{14}$$

and

$$J_z(a, \phi = \begin{cases} \dfrac{1}{\sqrt{\cos 2\phi - \cos 2\phi_0}} & \phi \in L_1 \\ 0 & \phi \in L_2 \end{cases} \quad . \tag{15}$$

On the otherhand, in the dual summation equations (8) and (9), we see that for $\gamma = 0$,

$$G_n(\gamma a) = 1 \quad .$$

Thus Equation (9) and (10) are reduced to:

$$J(\phi) = \sum_{n=0}^{\alpha} C_n \cos\left(n+\frac{1}{2}\right) 2\phi = 0 \qquad \phi \in L_2 \tag{16}$$

$$E(\phi) = \sum_{n=0}^{\epsilon} C_n \sin\left(n+\frac{1}{2}\right) 2\phi = 0 \qquad \phi \in L_1 \, . \tag{17}$$

Comparison of (14), (15) with (16) and (17) reminds one of the Dirichlet-Mehler relations involving associated Legendre functions. From the Dirichlet - Mehler relations, we have (Erdélyi, et al, 1953) for $\pi > \Delta > 0$,

$$\Gamma\left(\frac{1}{2} \div m\right) \sum_{n=0}^{\alpha} P_n^{-n}(\cos\Delta)\cos\left(n+\frac{1}{2}\right)x$$

$$= \begin{cases} \sqrt{\frac{\pi}{2}} \sin^{-m}\Delta\,(\cos x - \cos\Delta)^{m-1/2} & 0 < x < \Delta \\ 0 & \Delta < x < \pi \; . \end{cases} \tag{18}$$

And, if we replace x by $\pi - x$ and Δ by $\pi - \Delta$,

$$\Gamma\left(\frac{1}{2} + m\right) \sum_{n=0}^{\alpha} P_n^{-m}(-\cos\Delta)(-1)^n \sin\left(n+\frac{1}{2}\right)x$$

$$= \begin{cases} 0 & 0 < x < \Delta \\ \frac{\pi}{2}(-1)^m \sin^{-m}\Delta(\cos\Delta - \cos x)^{m-1/2} & \Delta < x < \pi \; . \end{cases} \tag{19}$$

In the above, if we let $m = 0$, $x = 2\phi$, and $\Delta = 2\phi_0$, we have

$$\sum_{n=0}^{\alpha} P_n(\cos 2\phi_0) \cos\left(n + \frac{1}{2}\right) 2\phi$$

$$= \begin{cases} \dfrac{1}{2} \dfrac{1}{\sqrt{\cos 2\phi - \cos 2\phi_0}} & 0 < 2\phi < 2\phi_0 \\ 0 & 2\phi_0 < 2\phi < \pi \end{cases} \tag{20}$$

and

$$\sum_{n=0}^{\alpha} P_n(\cos^2 \phi_0) \sin\left(n + \frac{1}{2}\right) 2\phi$$

$$= \begin{cases} 0 & 0 < 2\phi < 2\phi_0 \\ \dfrac{1}{2} \dfrac{1}{\sqrt{\cos 2\phi_0 - \cos 2\phi}} & 2\phi_0 < 2\phi < \pi \quad . \end{cases} \tag{21}$$

From Equations (20) and (21), and their periodic extensions, it is evident that Equations (16) and (17), i.e., the dual summation equations (8) and (9) for the case of $\gamma = 0$, admit the non-trivial solution:

$$C_n = P_n(\cos 2\phi_0) \quad .$$

Moreover, this solution yield values for $E(\phi)$ and $J(\phi)$ that agree with the solution obtained by using the conformal transform techniques.

In general, since dominant components of the fields ($J(\phi)$ and $E(\phi)$) are the same as that of the TEM case, it is logical to expand the fields in a set of functions including those given by (20) and (21). We therefore introduce a new set of basis functions that may be used to represent $J(\phi)$. These are:

$$f_m(\Delta, x) = \sum_{n=0}^{\infty} \sin^m \Delta \, P_n^{-m}(\cos \Delta) \cos (n + \tfrac{1}{2}) x \quad . \tag{22}$$

For all ranges of real x, the functions $f_m(\Delta, x)$ are sketched in Figure 2. The dominant features of this set of functions are:

a) In the interval* $L_2' : \{2(k-1)\pi + \Delta < x < 2k\pi - \Delta\}$,

$$f_m(\Delta, x) = 0 \tag{23}$$

b) In the complementary interval $L_1' : \{2k\pi - \Delta < x < 2k\pi + \Delta\}$.

$$f_m(\Delta, x) = \pm \frac{\sqrt{\pi/2}\,(\cos x - \cos \Delta)^{m - \frac{1}{2}}}{\Gamma(m + \frac{1}{2})} \tag{24}$$

where the positive sign holds for even k and the negative sign for odd k.

c) f_o is discontinous, unbounded, and at the end points of L_1'

$$f_o(\Delta, x) \sim \frac{1}{\sqrt{\cos x - \cos \Delta}} \tag{25}$$

d) f_1 is continuous, but has discontinuous derivative; f_2, together with its first derivative is continuous. In general, $f_n \in C_{n-1}$.

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*Note that for $\Delta = 2\phi$, $x = 2\phi$, L_1' and L_2' are L_1 and L_2 together with their periodic extensions.

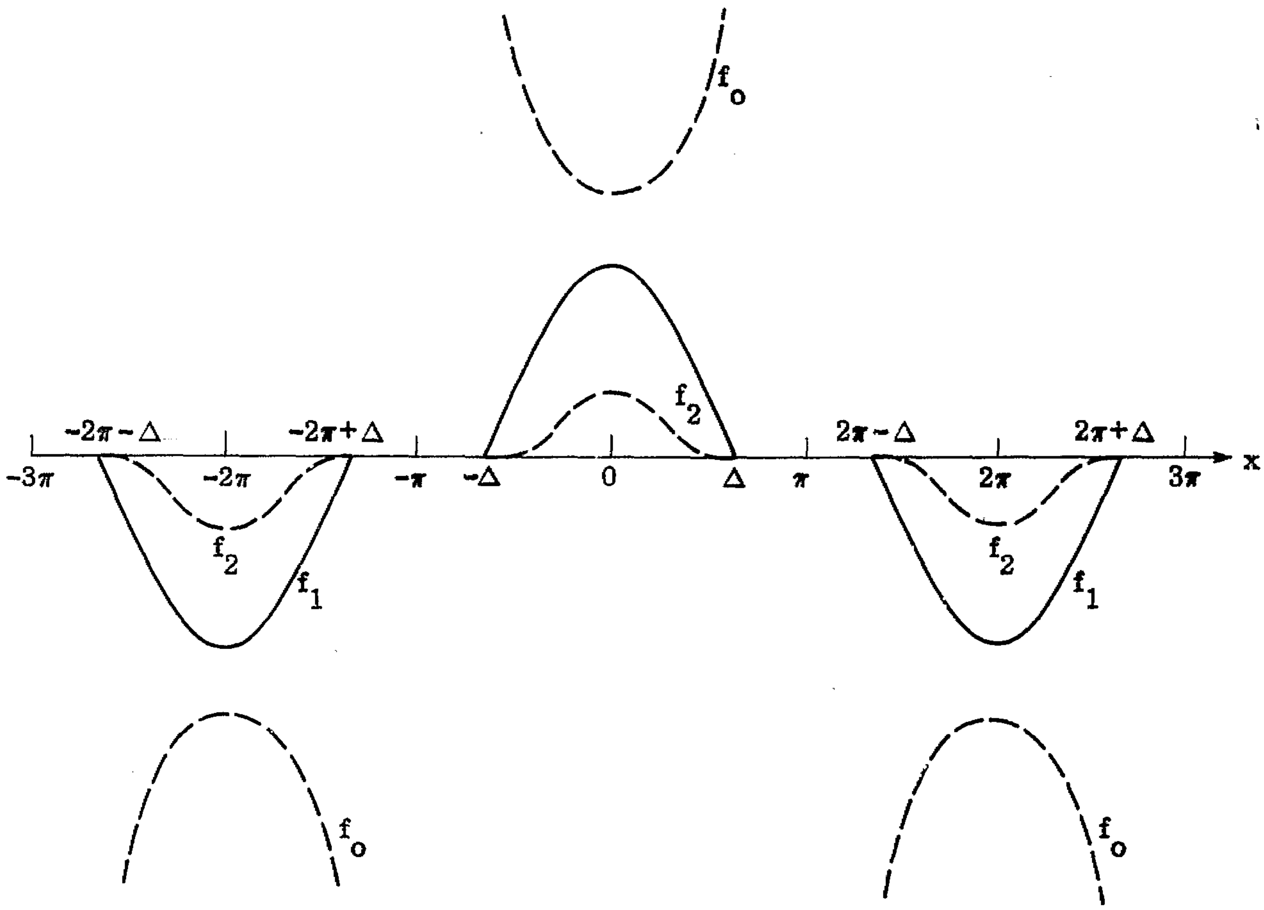

Figure 2: Sketches of f_m

These features illustrate the advantage of using the f function as the basis of expansion for $J(\phi)$.

Another set of functions, "dual" to the f_n's, can be obtained from Equation (22) by replacing Δ and x respectively by $\pi - \Delta$ and $\pi - x$. This set of functions is defined by

$$g_m(\Delta, x) = f_m(\pi - \Delta, \pi - x)$$

$$= \sum_{n=0}^{\infty} (-1)^n \sin^m \Delta P_n^{-m} (-\cos \Delta) \sin (n + \tfrac{1}{2}) x \quad . \tag{26}$$

For real values of x, the functions $g_m(\Delta, x)$ are sketched in Figure 3. Again, the following dominant features are obvious:

a) In the interval $L_1' \left\{2k\pi - \Delta < x < 2k\pi + \Delta\right\}$,

$$g_m(\Delta, x) = 0 \; . \tag{27}$$

b) In the interval $L_2' \left\{2(k-1)\pi + \Delta < x < 2k\pi - \Delta\right\}$,

$$g_m(\Delta, x) = \mp \frac{\sqrt{\pi/2}\,(\cos \Delta - \cos x)^{m - \frac{1}{2}}}{P(m + \frac{1}{2})} \tag{28}$$

c) g_o is discontinuous and unbounded. At the end points of L',

$$g_o(\Delta, x) \sim \frac{1}{\sqrt{\cos \Delta - \cos x}} \quad . \tag{29}$$

d) In general, $g_n \in C_{n-1}$.

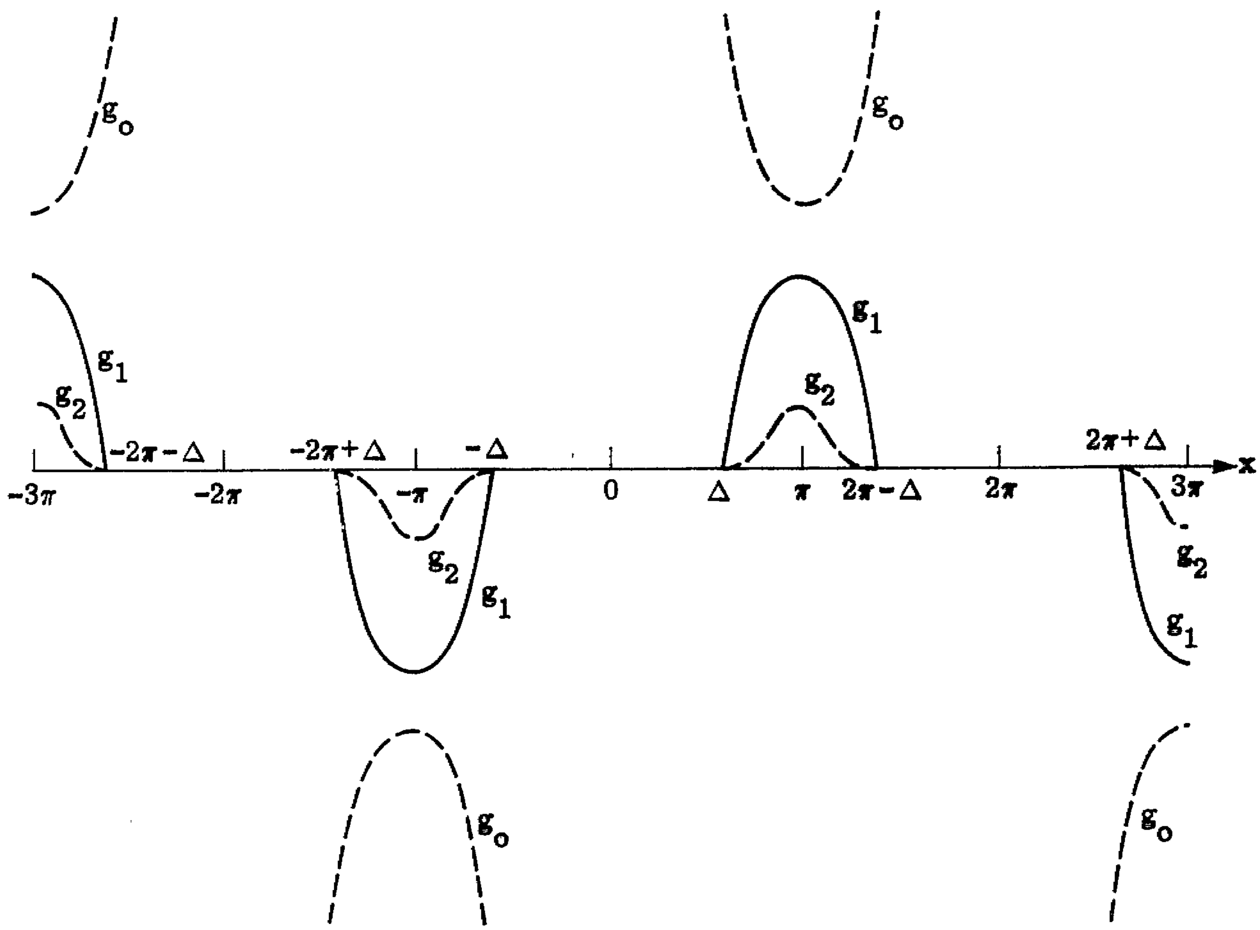

Figure 3: Sketches of g_m

This set appears to be appropriate in using as basis of expansion for E (ϕ).

A set of relations, useful in solving the type of dual summation equations of interest, are noted here. From the relations

$$(1-x^2)^{\frac{1}{2}m} P_n^{-m}(x) = \int_x^1 \int_x^1 \int_x^1 \ldots P_n(x)\, dx^m \tag{30}$$

and

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n . \tag{31}$$

We see that for $n \geq m$,

$$P_n^{-m}(-x) = (-1)^{n+m} P_n^{-m}(x) \tag{32}$$

where for $n < m$, $P_n^{-m}(x)$ and $P_n^{-m}(-x)$ are linearly independent. Using Equation (31), it is easy to deduce from Equation (22) that,

$$\sum_{n=0}^{\infty} (-1)^n \sin^m \Delta\, P_n^{-m}(-\cos\Delta) \cos\left(n+\frac{1}{2}\right) x =$$

$$=(-1)^m f_m(\Delta, x) - (-1)^m \sum_{n=0}^{m=1} \sin^m\Delta \left[P_n^{-m}(\cos\Delta) - (-1)^{m+n} P_n^{-m}(-\cos\Delta)\right] \cos\left(n+\frac{1}{2}\right) x \tag{33}$$

Similarly, from Equation (26), we have

$$\sum_{n=0}^{\infty} \sin^m \Delta\, P_n^{-m}(\cos\Delta) \sin\left(n+\frac{1}{2}\right) x =$$

$$=(-1)^m g_m(\Delta, x) + \sum_{n=0}^{m=1} \sin^m\Delta \left[P_n^{-m}(\cos\Delta) - (-1)^{n+m} P_n^{-m}(-\cos\Delta)\right] \sin\left(n+\frac{1}{2}\right) x \tag{34}$$

We shall use these new basis functions to deduce the approximate dispersion relation of the curved transission in the next section.

III

THE DISPERSION RELATION

By denoting $x = 2\phi$ $\Delta = 2\phi_0$, the problem of finding the propagation constants of the higher order modes of the open transmission line illustrated in Figure 1 may be stated as follows:

Find the set of γ such that non-trivial solutions of the following dual summation equations exist:

$$J(\phi) = \sum_{n=0}^{\infty} C_n \cos\left(n+\frac{1}{2}\right)x = 0 \qquad x \in L_2' \tag{35}$$

$$E(\phi) = \sum_{n=0}^{\infty} C_n G_n(\gamma a) \sin\left(n+\frac{1}{2}\right)x = 0 \qquad x \in L_1' \quad . \tag{36}$$

Moreover, due to the well known edge conditions, we require, as $x \longrightarrow 0$ in L'_1

$$J(\phi) \sim \frac{1}{\sqrt{\cos x - \cos\Delta}} \tag{37}$$

and as $x \longrightarrow \Delta$ in L'_2

$$E(\phi) \sim \frac{1}{\sqrt{\cos\Delta - \cos x}} \quad . \tag{38}$$

In order to derive an equation from which γa (or β) may be solved, the the following facts are to be noted.

(a) Any member of $f_m(x, \Delta)$ satisfies Equation (35) (c.f. Eq. (23)), In particular, $f_o(x, \Delta)$ also satisfies the edge condition, given by Equation (37). Thus, we may represent*

$$J(\phi) = \sum_{m=0}^{\infty} \alpha_m f_m(x, \Delta) \quad . \tag{39}$$

(b) As $n \longrightarrow \infty$,

$$n H_n^{(2)}(x) J_n(z) \longrightarrow \frac{j}{\pi}\left[1 + \frac{z^2}{4n^2}\right] \quad . \tag{40}$$

Thus, for the TM case,

$$G_n(\gamma a) = \frac{\pi}{j}(2n+1) H_{2n+1}^{(2)}(\gamma a) J_{2n+1}^{(2)}(\gamma a) \longrightarrow 1 + \frac{(\gamma a)^2}{4(2n+1)^2} \quad . \tag{41}$$

Similarly, as $n \longrightarrow \infty$,

$$\frac{H_n'^{(2)}(z) \; J_n'(z)}{n} \longrightarrow -\frac{j}{\pi}\frac{1}{z^2}\left(1 + \frac{z^2}{4n^2}\right) \quad . \tag{42}$$

Thus, for the TE case,

$$G_n(\gamma a) = -\frac{\pi}{j}\frac{H'_{2n+1}(\gamma a) \, J'_{2n+1}(\gamma a)}{(2n+1)}(\gamma a)^2 \longrightarrow 1 + \frac{(\gamma a)^2}{4(2n+1)^2}. \tag{43}$$

In either case, we may write

$$G_n(\gamma a) \triangleq 1 + S_n(\gamma a) \tag{44}$$

* The completeness of such a representation can be easily demonstrated and will not be considered here.

and note that as $n \longrightarrow \infty$,

$$S_n(\gamma a) \longrightarrow \frac{(\gamma a)^2}{4\,(2n+1)^2} \quad . \tag{45}$$

Equation (45) justifies our truncation procedure introduced later that $S_n(\gamma a)$ may be neglected for any particular γa such that

$$(\gamma a) << (2n+1)$$

(c) For $S_n(\gamma a) = 0$, $G_n(\gamma a) = 1$, we have $\gamma a = 0$, corresponding to the TEM mode of propagation.

(d) For $S_n(\gamma a) = -1$, $G_n(\gamma a) = 0$, then, γa are the zeros of $J_{2n+1}(\gamma a)$ for the TM case and the zeros of $J'_{2n+1}(\gamma a)$ for the TE case.

(e) In terms of $S_n(\gamma a)$, we may rewrite Equation (36) in the form

$$E(\phi) = \sum_{n=0}^{\infty} C_n \sin\left(n+\frac{1}{2}\right) x + \sum_{n=0}^{\infty} C_n S_n(\gamma a) \sin\left(n+\frac{1}{2}\right) x \quad . \tag{46}$$

Here, the second summation in Equation (36) is of higher order in $\frac{1}{n^2}$ in comparison to the first summation, and may be neglected (at least partially for large n) in numerical computations.

Based on the above facts, let us deduce in steps the approximate dispersion relations for γa. In the zeroth approximation, let us neglect all the S_n's except S_o and take only one term of Equation (39) in representing $J(\phi)$. Thus we let,

$$J(\phi) = \alpha_o f_o(\Delta, x) \quad . \tag{47}$$

From Equation (22), we have,

$$C_n = \alpha_o P_n(\cos\Delta) \quad . \tag{48}$$

Substitute into Equation (46), and use Equation (34), we have,

$$E(\phi) = \alpha_o g_o(\Delta, x) + \alpha_o P_n(\cos\Delta) S_o(\gamma a) \sin\left(n + \frac{1}{2}\right) x = 0$$

$$x \in L'_1 \quad . \tag{49}$$

Since $g_o(\Delta, x)$ is always 0 for $x \in L'_1$, Equation (49) is satisfied only when,

$$S_o(\gamma a) = 0 \quad .$$

This is the TEM solution, i.e.,

$$\gamma a = 0$$

In the first approximation, let us assume that S_o and S_1 are non-zero, and use two terms in Equation (39). Thus,

$$J(\phi) = \alpha_o f_o(\Delta, x) + \alpha_1 f_1(\Delta, x) \quad . \tag{50}$$

This means

$$C_n = \alpha_o P_n(\cos\Delta) + \alpha_1 \sin\Delta \, P_n^{-1}(\cos\Delta) \quad . \tag{51}$$

Substituting Equation (51) into Equation (46), and using Equation (39), yields

$$E(\phi) = \alpha_o g_o(\Delta, x) - \alpha_1 g_1(\Delta, x) +$$

$$+ \sin\frac{x}{2}\left[\alpha_o S_o P_n(\cos\Delta) + \alpha_1 S_o \sin\Delta\, P_n^{-1}(\cos\Delta)\right] + \alpha_1 \sin\Delta\left[P_o^{-1}(\cos\Delta) + P_o^{-1}(-\cos\Delta)\right] +$$

$$+ \sin\frac{3}{2}x\left[\alpha_o S_1 P_n(\cos\Delta) + \alpha_1 S_1 \sin\Delta\, P_n^{-1}(\cos\Delta)\right] = 0 \qquad x \in L_1' \quad . \tag{52}$$

This is possible only if

$$\alpha_o S_o P_n(\cos\Delta) + \alpha_1\left[S_o \sin\Delta\, P_n^{-1}(\cos\Delta)\right] + \sin\Delta\left[P_o^{-1}(\cos\Delta) + P_o^{-1}(-\cos\Delta)\right] = 0 \tag{53}$$

$$\alpha_o S_1 P_n(\cos\Delta) + \alpha_1 S_1 \sin\Delta\, P_n^{-1}(\cos\Delta) = 0 \quad . \tag{54}$$

For non-trivial solution of α_o and α_1, the dispersion relation takes the form:

$$S_1\left[\frac{S_o}{2}(1-\cos\Delta)^2 - 2\cos\Delta\right] = 0 \quad . \tag{55}$$

The condition $S_1 = 0$ yields $\gamma = 0$, the TEM propagating condition, while the other condition is

$$S_o(\gamma a) = \frac{4\cos\Delta}{(1-\cos\Delta)^2} \quad . \tag{56}$$

The interpretation of Equation (56) is interesting. For $\Delta = 180^o$, $\phi_o = 90^o$, corresponding to a closed circular gyide,

$$S_o(\gamma a) = -1.$$

Thus, in the limiting case, the solution of Equation (56) yields the TM_1 or TE_1 modes of propagation. For the lowest mode, TE_{11}, if we assume that γa for the open structure is close to that of a closed circular guide, i.e., $\gamma a \cong 1.84$, then

$$S_2(\gamma a) \cong \frac{(1.84)^2}{4 \times (5)^2} = .03 << 1 \quad .$$

The approximation introduced, i.e., neglecting $S_n(\gamma a)$ for $n > 2$ appears to be reasonable. For the TM case, and for high order modes, however, higher approximation is necessary.

For the second order approximation, we assume that $S_n(\gamma a) \cong 0$ for $n > 3$, and represent

$$J(\phi) = \alpha_o f_o(x, \Delta) + \alpha_1 f_1(x, \Delta) + \alpha_a f_2(x, \Delta) \quad . \tag{57}$$

The resulting approximate dispersion relation is,

$$\begin{vmatrix} S_o P_o(\cos\Delta) & \sin\Delta \left[S_o P_o^{-1}(\cos\Delta) + P_0^{-1}(\cos\Delta) + P_o^{-1}(-\cos\Delta)\right] & \sin^2\Delta\left[S_o P_o^{-2}(\cos\Delta) + P_o^{-2}(\cos\Delta) - P_o^{-2}(-\cos\Delta)\right] \\ S_1 P_1(\cos\Delta) & \sin\Delta\, S_1 P_1^{-1}(\cos\Delta) & \sin^2\Delta\left[S_1 P_1^{-2}(\cos\Delta) + P_1^{-2}(\cos\Delta) + P_1^{-2}(-\cos\Delta)\right] \\ S_2 P_2(\cos\Delta) & \sin\Delta\, S_2 P_2^{-1}(\cos\Delta) & \sin^2\Delta\, S_2 P_2^{-2}(\cos\Delta) \end{vmatrix} = 0 \tag{58}$$

Explicitly, Equation (58) may be written as

$$S_2\left[S_oS_1 + \frac{4}{y4}\left[3y^3 - 15y^2 + 32y - 24\right] S_1 - \frac{16}{y4}(3-2y) S_o + \frac{32}{y6}(2-6y+3y^2)\right] = 0 \tag{59}$$

where, for simplicity, we denote

$$y = (1 - \cos\Delta) \quad . \tag{60}$$

In the limiting case of $\cos\Delta = -1$, $y = 2$, Equation (59) is reduced to:

$$S_2(\gamma a)\left[S_1(\gamma a) + 1\right]\left[S_o(\gamma a) + 1\right] = 0 \quad . \tag{61}$$

The three factors in Equation (26) yield the propagation constants (i..e, γa) for TEM, TM_1 (or TE_1) and TM_3 (or TE_3) modes of propagation respectively.

In general, of course, we should represent

$$J(\phi) = \sum_{m=0}^{\infty} \alpha_m f_m(x, \Delta) \tag{62}$$

and an infinite determinant is obtained. However, if we are computing γa numerically, we may truncate the determinate to $(N+1) \times (N+1)$ order, i.e., represent

$$J(\phi) = \sum_{m=0}^{N} \alpha_m f_m(x, \Delta) \quad . \tag{63}$$

Any solution of the truncated determinant satisfying the criterion that (c.f. Equation (45))

$$\frac{\gamma a^2}{4\,(2N+1)^2} << 1$$

should yield numerically acceptable propagation constants. It is to be noted that with our particular choice of basis functions in representing the surface fields $(J(\phi)$ or $E(\phi))$, the truncation appears to be reasonable since,

(a) The first few terms of the series represent the dominant components of $E(\phi)$ and $J(\phi)$ correctly,

(b) The edge conditions are automatically satisfied in this representation, and

(c) It has been shown that results of the first few orders of approximation yield exact results in the limiting case of $\Delta_o = 90^o$. This fact is easily shown to be true for any order of truncation.

For the N-th approximation when $J(\phi)$ is approximated by Equation (64) we have

$$C_n = \sum_{m=0}^{N} \alpha_m \sin^m \Delta \, P_n^{-m} (\cos \Delta) \quad . \tag{64}$$

The truncated determinant, i.e., the approximate dispersion relation, takes the form

$$\mathrm{Det} \left| K_{n,m} \right| = 0 \tag{65}$$

where

$$\begin{aligned} K_{n,m} &= S_n(\gamma a) \sin^m \Delta \, P_n^{-m} (\cos \Delta) \\ &\quad + \sin^m \Delta \, P_n^{-m} (\cos \Delta) - (-1)^{m+n} P_n^{-m} (-\cos \Delta) \end{aligned} \tag{66}$$

$n, m = 0, 1, \ldots\ldots N.$

Some reduction of the determinant, and the numerical scheme for computing γa from the determinantal equations are discussed in the next section.

IV

NUMERICAL COMPUTATION OF γa

Although in principle, the solution for the set of γa satisfying the dispersion relation for any order (N) of approximation appears to be a straightforward mathematical problem, the actual numerical computation of γa is far from trivial due to the complicated functions (Hankel functions for complex arguments) involved in these equations. For the first order approximation, the dispersion relation given by Equation (57) are rewritten as:

$$F(\gamma a) = S_0(\gamma a) = \frac{4 \cos \Delta}{(1 - \cos \Delta)^2} \quad . \tag{68}$$

For a given $\Delta = 2\phi_0$, the solution of Equation (68) for complex γ a can be carried out by Newton-Raphson's iterative method. The convergence in this case appears to be good. For TM modes, some of the numerical results are tabulated in Table 1. For large ϕ_0, the solutions are close to the zero's of Bessel functions; for the smaller ϕ_0, the deviation from the zero's of Bessel functions becomes greater.

For the second order approximation, Equation (60) is again written as

$$F(\gamma a) = S_0(\gamma a)\, S_1(\gamma a) + \frac{4}{y^2}(3y^3 - 15y^2 + 32y - 24)\, S_1(\gamma a)$$

$$- \frac{16}{y^2}(3 - 2y)\, S_0(\gamma a) + \frac{32}{y6}(2 - 6y + 3y^2) = 0 \tag{69}$$

where

$$y = (1 - \cos \Delta) \quad .$$

Table 1.

γa From First Approximation TM Mode

	First Set of Roots		Second Set of Roots	
ϕ_0	Re	Im	Re	Im
87.5°	3.8316950	0.0	7.0155994	0.0
75°	3.8416062	0.0000934	7.0337080	0.0603267
60°	4.0425478	0.0474268	7.3553023	0.1318178
50°	4.3667455	0.4574768	7.6335519	0.6966593

For any given Δ, Newton - Raphson's method is again used to compute the complex roots. In this case the convergence is very poor, and no numerical acceptable roots were obtained after 10 iterations even when we start with the initial guess predicted from the first approximation. A modified conjugate gradient program was also tried in an attempt to solve

$$|F(\gamma a)| = 0 \tag{70}$$

without success. After considerable amount of numerical experimentation, we have developed a scheme combining the searching and iterating technique in computing γa. The convergence is greatly improved in using this scheme. Since the numerical problem of computing complex roots of complex equations involving transcendental equations is known to be a very difficult task, our new scheme appears to represent a significant contribution in solving such problems. A detailed description of this searching and iterating scheme is given in Appendix A. For TM modes of propagation, some numerical results are given in Table 2. It is to be noted that not all of the roots in Table 2 satisfy the criterion

$$|\gamma a|^2 << 4 \times (2N + 1)^2 = 176 \quad .$$

They are tabulated, however, to illustrate the feasibility of evaluating <u>all</u> the complex roots.

For any higher order approximation, the dispersion relation given by Equation (66) may be written in the form,

$$F(\gamma a) = \det\left[K\right] = 0 \tag{71}$$

Table 2.

(γa) From Second Approximation TM Mode

	First Set of Roots		Second Set of Roots	
ϕ_0	Re	Im	Re	Im
87.5^o	3.9685815	0.4489306	6.0104261	0.6061281
75^o	4.1029203	1.5604187	5.7068481	1.2732071
60^o	3.9995496	1.4339417	5.6891943	1.8720847
50^o	2.8350118	1.5225492	5.9726308	0.4743322

where K is a matrix, with elements

$$K_{n,m}(\gamma a) = S_n(\gamma a) \, \mathrm{Sin}^m \Delta \, P_n^m(\cos \Delta)$$

$$+ \mathrm{Sin}^m \Delta \left[P_n^{-m}(\cos \Delta) - (-1)^{n+m} P_n^{-m}(-\cos \Delta) \right]$$

$$n, \; m = 0, \; 1, \; 2, \; \ldots\ldots\ldots N. \qquad (72)$$

Explicit expression for the dispersion relation using a Laplace development of the determinant appears to be unnecessary since we are interested only in the numerical solutions. For numerical computation, however, since each of the matrix elements $K_{n,m}$ are functions of (γa) and Δ, they must be calculated for each Δ and every iteration. To simplify the computation, we have succeeded in reducing Equation (70) to the following form:

$$F(\gamma a) = \det \left([S] + [Q] \right) = 0 \qquad (73)$$

where

$$[S] = \begin{bmatrix} S_0(\gamma a) & & \\ & S_1(\gamma a) & \\ & & S_N(\gamma a) \end{bmatrix} \qquad (74)$$

is a diagonal matrix, and is independent of Δ. On the otherhand, $[Q]$ is a matrix depending only on $\Delta = 2\,\phi_0$. For any Δ, if we denote

$$y = 1 - \cos \Delta,$$

the elements of $[Q]$ may be computed by using the following equations:

$$a)\ A_{ij} = (-1)^{i+j+1} \left(\frac{2}{y}\right)^{2j} \sum_{t=0}^{i-j-1} (-1)^t \left(\frac{y}{2}\right)^t \frac{(2j-t-1)!}{(j-t-i-1)!\ t!\ (j+i-t)!} \qquad i > j \tag{75}$$

$$A_{ij} = 0 \qquad i \leq j$$

$$b)\ C_{n,k} = \sum_{n=0}^{k} \frac{(n+k+r)!}{(2k+r)!} \frac{1}{(n-k-r)!} \frac{(-1)^r}{r!} \left(\frac{y}{2}\right)^r \tag{76}$$

$$c)\ Q_{i,N} = A_{i,N} \tag{77}$$

$$d)\ Q_{i,N-1} = A_{i,N-1} - Q_{i,N} C_{N,N-1} \tag{78}$$

$$e)\ Q_{i,k} = A_{i,k} - \sum_{j=k+1}^{N} Q_{ij} C_{j,k} \tag{79}$$

For a fixed N and Δ, the matrix elements $Q_{i,k}$ can be computed first, and in the iteration solution for γa, only the elements of S matrix need to be computed in each iteration. The detailed derivation of Equation (73) is given in Appendix B. Computation of γa, using this scheme appears to be feasible if N is not too large. Actual computation to date, however, has been completed for $N = 2$ only.

V

CONCLUSIONS AND RECOMMENDATIONS

The modal analysis of the guided wave propagation along an open structure such as illustrated in Figure 1 yields a set of "dual summation equations" for the propagation constants. By introducing new sets of basis functions in the expansions of surface fields, the approximate dispersion relation (for any order N of approximation) have been developed. Numerical scheme for obtaining the complex roots of the dispersion relation were developed. Although the actual computation was carried out only for $N = 2$, there appears to be no doubt that the scheme is applicable for moderate values of N. Future work concerning this problem should probably include (a) more calculations for higher values of N to investigate numerically the effect of truncation, and (b) more detailed analysis to determine the behavior of the matrix elements Q_{ij} which, physically may be interpreted as "coupling coefficients" between different modes.

The basic mathematical scheme developed in this study may in principle be modified and extended to include the solutions of the following problems:

(a) The propagation of waves along a circular waveguide with one longitudinal slot. Although others have performed a theoretical analysis of propagation along slotted cylinders previously, for example, in the works of Goldstone and Oliner (1961), Harrington (1959) and Chen (1973), theoretical analysis and numerical computation of the propagation constants for higher order modes are still lacking.

(b) The complex resonant frequency of spherical resonators. The modal analysis of such an open structure yield also a set of dual summation equations involving associated Legendre functions $P_n^m(\cos\theta)$. The solution of the dual summation equations by introducing the proper basis function to represent the surface field should be worth investigating.

(c) The modal analysis of scattering problems involving open structures such as slotted cylinder and slotted spheres yields a set of inhomogeneous dual summation equations. A systematic approach for solving such scattering problems by introducing proper basis functions for the representation of the surface fields should also be tried. The scattering problem of the sources by slotted cylinders has been investigated by Hayashi (1966) by using singular integral equations. It appears that the basic advantage of using singular integral equations is to obtain the dominant component of the surface fields which is the first term of our representation. The computation of other higher order terms, however, is more involved in the singular integral formulation.

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APPENDIX A

SEARCHING AND ITERATING PROCEDURE

Let us consider the problem of finding a complex root $z = x + j y$ satisying the equation

$$f(z) = 0 \tag{A.1}$$

where $f(z)$ is a complex function, involving transcendental functions such as Hankel functions. When the standard Newton-Raphson's method of finding roots of Equation (A.1) is not successful, a searching and iterating procedure may be used in improving the convergence. In illustrating this procedure, we assume that near any zero of $f(z)$, the function is analytic and the derivative of $f(z)$ may be computed. We shall denote

$$f(z) = f(x + jy) = U(x, y) + j V(x, y) \tag{A.2}$$

$$\begin{aligned} f'(z) \cong \frac{\Delta f(z)}{\Delta z} &= U_x(x, y) + i V_x(x, y) \\ &= + V_y(x, y) - i U_y(x, y) \end{aligned} \tag{A.3}$$

and assume that given x and y, U, V and their partial derivitives may be evaluated. Our suggested procedure for finding the complex roots are illustrated schematically in Figure A-1. The procedure may be described in the following steps:

(a) Given any initial y_o, compute $U(x, y_o)$ and $V(x, y_o)$, and scan x coarsely over a chosen range. As illustrated in Figure A-1, there exists points $A(x_o^a, y_o)$ such that $U(x_o^a, y_o) = 0$, and $B(x_o^b, y_o)$ such that $V(x_o^a, y_o) = 0$. From the coarse searching, in general for simple roots, U changes sign at x_o^a and V changes sign at x_o^b. Values of x_o^a and x_o^b may be computed more exactly, from the results of coarse searching and Newton-Raphson iterative procedure. To improve convergence, if the difference between x_o^a and x_o^b are too large, another value of y_o may be chosen and the coarse searching repeated.

(b) From x_o^a, y_o, and x_o^b, y_o, we determine y_1, corresponding to the y coordinate of C, which is the intersection of the two tangent lines AC and BC. To determine y_1, we first computed

$$U_x^a = U_x(x_o^a, y_o)$$

$$U_y^a = U_y(x_o^a, y_o)$$

$$V_x^b = V_x(x_o^b, y_o)$$

$$V_y^b = V_y(x_o^b, y_o)$$

From these partial derivatives, it is easily seen that

$$y_1 = (x_o^a - x_o^b) \Bigg/ \left(\frac{U_y^a}{U_x^a} - \frac{V_y^b}{V_x^b} \right) \tag{A.4}$$

$$x_1 = \left(\frac{x_o^a U_x^a}{U_y^a} - \frac{x_o^b V_x^b}{V_y^b} \right) \Bigg/ \left(\frac{U_x^a}{U_y^a} - \frac{V_x^b}{U_y^b} \right) \tag{A.5}$$

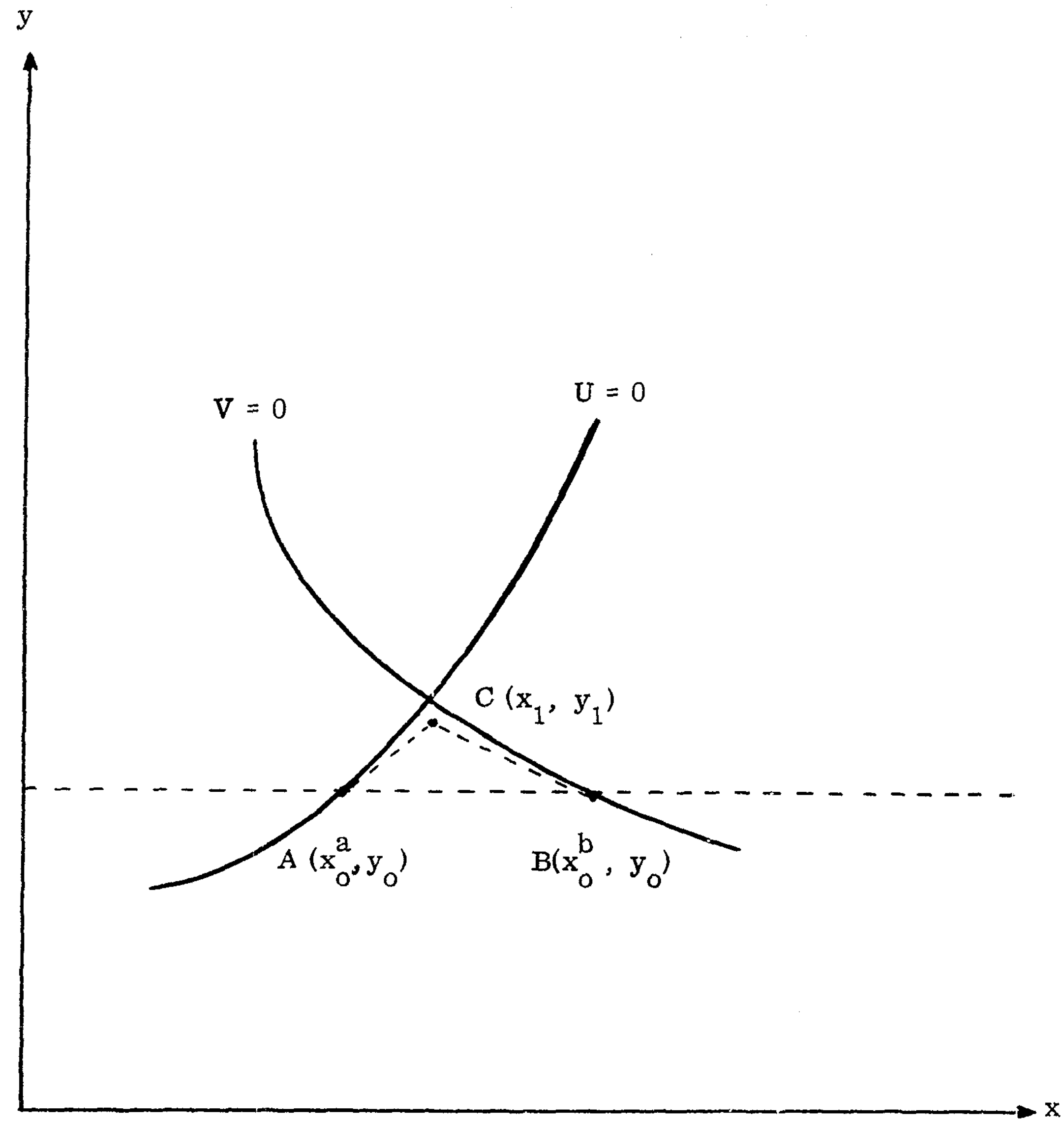

Figure A-1: Illustration of Searching and Iterating Procedure.

(c) Start from (x_1, y_1), i.e., the point C. The process may be repeated, until for any y_m,

$$|x_m^a - x_m| , \quad |x_m^b - x_m|$$

are less than some preset criterion of convergence.

(d) In some cases, this procedure may fail. For these cases in general

$$|x_m^a - x_m^b| > |x_{m-1}^a - x_{m-1}^b|.$$

If this happens, the role of x and y should be interchanged in order to obtain convergence.

APPENDIX B

REDUCTION OF THE DISPERSION RELATION

The dispersion relation for the N-th approximation, given by Equation (66), indicates that the determinant of matrix $\left[K\right]$ should be zero. If we denote

$$y = (1 - \cos\Delta) \tag{B.1}$$

$$p_{n,m}(y) = \sin^m \Delta \; P_n^{-m}(\cos\Delta) \, , \tag{B.2}$$

$$H_{n,m} = \mathrm{Sin}^m \Delta \left[P_n^{-m}(\cos\Delta) - (-1)^{m+n} P_n^{-m}(-\cos\Delta)\right]$$

$$= p_{n,m}(y) - (-1)^{m+n} p_{n,m}(2-y) \tag{B.3}$$

the elements of $\left[K\right]$ matrix are given by

$$K_{n,m} = S_n(\gamma a)\, p_{n,m}(y) + H_{n,m}(y) \; .$$

Thus, the $\left[K\right]$ matrix may be written as

$$\left[K\right] = \left[S\right]\left[p\right] + \left[H\right] \tag{B.4}$$

where

$$\left[S\right] = \begin{bmatrix} S_o & & & & & \\ & S_1 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & S_N \end{bmatrix} , \tag{B.5}$$

and the matrices $[p]$ and $[H]$ have elements $p_{n,m}$, and $H_{n,m}$ respectively.

Since S depends only on (γa) and p and H depends only on y, the roots (γa) of the dispersion relation are the same as that of the equation,

$$\det \left([S] + [H][p]^{-1} \right) = 0 \quad . \tag{B.6}$$

Therefore, the matrix $[Q]$ in equation (73) is given by

$$[Q] = [H][p]^{-1} \tag{B.7}$$

or

$$[Q][p] = [H] \quad . \tag{B.8}$$

The scheme for evaluating the elements Q_{ij} of the matrix $[Q]$ given in Equation (75) through Equation (79) can be derived from the expression of $p_{n,m}$ (c.f. Hobson, 1955).

$$p_{n,m}(y) = y^m \sum_{n=0}^{n} \frac{(n+r)!}{(n-r)!} \frac{1}{r!} \frac{(-1)^r}{(m+r)!} \left(\frac{y}{2}\right)^r \quad . \tag{B.9}$$

The derivation makes use of a discontinuous summation formula which we shall state now. This relation is given by

$$\sum_{r=0}^{n} \frac{n!}{(n-r)!\ r!} (-1)^r \frac{(a+r)!}{(b+r)!}$$

$$= \begin{cases} \dfrac{(a-b)!}{(a-b-n)!} (-1)^n \dfrac{a!}{(b+n)!} & a \geq (a-b) \geq n \\[2ex] \dfrac{a!}{(b-a-1)!} \quad \dfrac{(b-a+n-1)!}{(b+n)!} & b > a \\[2ex] 0 & \text{otherwise.} \end{cases} \tag{b.10}$$

The proof of this relation is straightforward. One starts from the binomial expansion

$$(1 - U)^n = \sum_{r=0}^{n} \frac{n!}{(n-r)!} \frac{(-1)^r}{r!} U^r \tag{B.11}$$

Multiply the above by U^a, differentiate both sides of the produce (a - b) times and let $U \to 1$ thus obtaining the first part of Equation (B.10). Similarly, if we integrate both sides of the product (b - a) times, and let $U \to 1$, the second part of Equation (B.10) is obtained. The third part of Equation (B.10) becomes clear after the first two parts have been determined. Based on Equation (B.10), one finds that

$$\begin{aligned} p_{n,m}(2-y) &= 2^m \sum_{r=0}^{n} \frac{(n+r)!}{(n-r)!} \frac{1}{r!} \frac{(-1)^r}{(m+r)} \left(1 - \frac{y}{2}\right) r + m \\ &= 2^m \sum_{s=0}^{m-1} \frac{(-1)^s}{s!} \left(\frac{y}{2}\right)^s \left[\sum_{r=0}^{n} \frac{(n+r)!}{(n-r)!} \frac{(-1)^r}{r!} \frac{1}{(m-s+r)!}\right] \\ &+ 2^m \sum_{s=m}^{n+m} \frac{(-1)^s}{s!} \left(\frac{y}{2}\right)^s \left[\sum_{r=s-m}^{n} \frac{(n+r)!}{(n-r)!} \frac{(-1)^r}{r!} \frac{1}{(m-s+r)!}\right]. \end{aligned} \tag{B.12}$$

In Equation (B.12) the series in the square brackets may be expressed in closed form by using Equation (B.11). Therefore, we have

$$\begin{aligned} p_{n,m}(2-y) &= 2^m (-1)^{m+n} \sum_{r=0}^{x} \frac{(-1)^r}{(r+m)!} \frac{(n+r)!}{(n-r)!\,r!} \left(\frac{y}{2}\right)^{r+m} \\ &+ 2^m \sum_{s=0}^{n-m-1} \frac{(-1)^s}{s!} \left(\frac{y}{2}\right)^s \frac{(m-s-1)!}{(m-n-s-1)!} \frac{1}{(n+m-s)!} \end{aligned} \tag{B.13}$$

where the second series is nonvanishing only if $m > n$. Thus, we have

$$H_{n,m}(y) = \begin{cases} 2^m (-1)^{m+n+1} \sum_{s=0}^{m-n-1} \frac{(-1)^s}{s!} \left(\frac{y}{2}\right)^s \frac{(m-s-1)!}{(m-n-s-1)!\,(m+n-s)!} & m > n \\ 0 & m \leq n \end{cases} \quad . \qquad (B.14)$$

In other words, the matrix $[H]$ is an upper triangular matrix with diagonal terms zero.

In order to find $[Q]$, we would like to find the inverse matrix $[p]^{-1}$. However, since the order of matrix N is kept arbitrary, one finds that a relation independent of N, which reduces the $[p]$ matrix into triangular form may be more suitable for computational purposes. This relation is given by:

$$[p]\,[\alpha] = [C] \qquad (B.15)$$

where $[\alpha]$ is a lower triangular matrix while $[C]$ is an upper triangular matrix. The diagonal elements of $[C]$ are unity, and the element of $[\alpha]$ are

$$\frac{\alpha_{ij}}{y^{\,i+j}}$$

where α_{ij} are constants (independent of y) and satisfying

$$\alpha_{ij} = 0 \qquad \alpha > j \quad . \qquad (B.16)$$

The quantities α_{ij} and C_{ij} can again be deduced by using Equation (B.10). By matrix multiplication, we see that the elements of the $[C]$ matrix is given by,

$$C_{rn} = \sum_{s=0}^{n} \frac{p_{r,s}\,\alpha_{sn}}{y\ s+n}$$

$$= \frac{1}{y^n} \sum_{t=0}^{r} \frac{(r+t)!}{(r-t)} (-1)^t \left(\frac{y}{2}\right) \sum_{s=0}^{n} \frac{\alpha_{sn}}{(s+t)!} \tag{B.17}$$

Since $C_{rn} = 1$, $r = n$, and $C_{rn} = 0$ $r < n$, the set α_{sn} satisfy

$$\sum_{s=0}^{n} \frac{\alpha_{sn}}{(s+t)!} = 0 \qquad t < n$$

$$= \frac{(-1)^n\, n!}{(2n)!}\, 2^n \qquad t = n \quad . \tag{B.18}$$

Using Equation (B.10), it is evident that

$$\alpha_{n,m} = (-1)^{n+m} \frac{m!}{(m-1)!} \frac{2^m}{n!} \frac{(n+m-1)!}{(m-n)!} \qquad n \geq m \tag{B.19}$$

and

$$C_{n,s} = \sum_{r=0}^{n-s} \frac{(n+r+s)!}{(n-r-s)!} (-1)^r \frac{1}{r!} \left(\frac{y}{2}\right)^r \frac{1}{(2s+r)!} \tag{B.20}$$

Now, if we multiply both sides of Equation (B.8), by $[\alpha]$ we have

$$[Q][C] = [H][\alpha] \triangleq [A] \tag{B.21}$$

The elements of the A matrix is easily shown to be:

$$A_{ij} = \sum_{k=i+1}^{j} H_{ik} \frac{\alpha_{kj}}{y\, k + j}$$

$$= \frac{(j)!}{(j-1)!} (-1)^{i+1} \left(\frac{2}{y}\right)^{2j}$$

$$\sum_{t=0}^{j-i-1} (-1)^t \left(\frac{y}{2}\right)^t \frac{(j-t-1)!}{(j-t-i-1)!\,(j-t+i)!} \sum_{k=j-t}^{j} (-1)^k \frac{(k+j-1)!}{(k+t-j)}\frac{1}{(j-k)!}\frac{1}{k!}$$

Again, use Equation (B.10), we have

$$A_{ij} = (-1)^{i+j+1} \left(\frac{2}{y}\right)^{2j} \sum_{t=0}^{j-i-1} (-1)^t \left(\frac{y}{2}\right)^t \frac{(2j-t-1)!}{(j-t-i-1)!} \frac{1}{(t)!\,(j-t+i)!} \qquad (B.22)$$

From Equation (B.21) and (B.22), the expressions for the elements Q_{ij} are then easily deduced.