Sensor and Simulation Notes
Note 179

25 May 1973

Singularity Expansion of Electromagnetic Fields and Potentials Radiated from Antennas or Scattered from Objects in Free Space

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Abstract

This note extends the formalism of the singularity expansion method (SEM) beyond the currents (and charges) induced on objects of finite size in free space to include the associated fields. These are the scattered fields if a wave is incident on the object of interest, or the total fields if the object is an antenna with appropriate sources. From the current on the object the scattered or radiated fields can be calculated and these fields can be expanded in terms of natural modes. Using retarded time concepts one can define retarded natural modes and far natural modes for the electromagnetic field quantities. From these there results what are in effect both time and frequency domain pattern functions for far fields.

Foreword

I would like to thank Dr. F. M. Tesche for several stimulating discussions related to the topics in this note.

. . .

There seemed to be no use in waiting by the little door, so she went back to the table, half hoping she might find another key on it, or at any rate a book of rules for shutting people up like telescopes: this time she found a little bottle on it ("which certainly was not here before," said Alice), and tied round the neck of the bottle was a paper label with the words "DRINK ME" beautifully printed on it in large letters.

It was all very well to say "Drink me," but the wise little Alice was not going to do *that* in a hurry. "No, I'll look first," she said, "and see whether it's marked '*poison*' or not," for she had read several nice little stories about children who had got burnt, and eaten up by wild beasts, and other unpleasant things, all because they *would* not remember the simple rules their friends had taught them, such as, that a red-hot poker will burn you if you hold it too long; and that if you cut your finger *very* deeply with a knife it usually bleeds; and she had never forgotten that if you drink much from a bottle marked "poison," it is almost certain to disagree with you sooner or later.

However, this bottle was *not* marked "poison," so Alice ventured to taste it, and finding it very nice (it had, in fact, a sort of mixed flavor of cherry tart, custard, pineapple, roast turkey, toffy, and hot buttered toast), she very soon finished it off.

"What a curious feeling!" said Alice. "I must be shutting up like a telescope."

And so it was, indeed; . . .

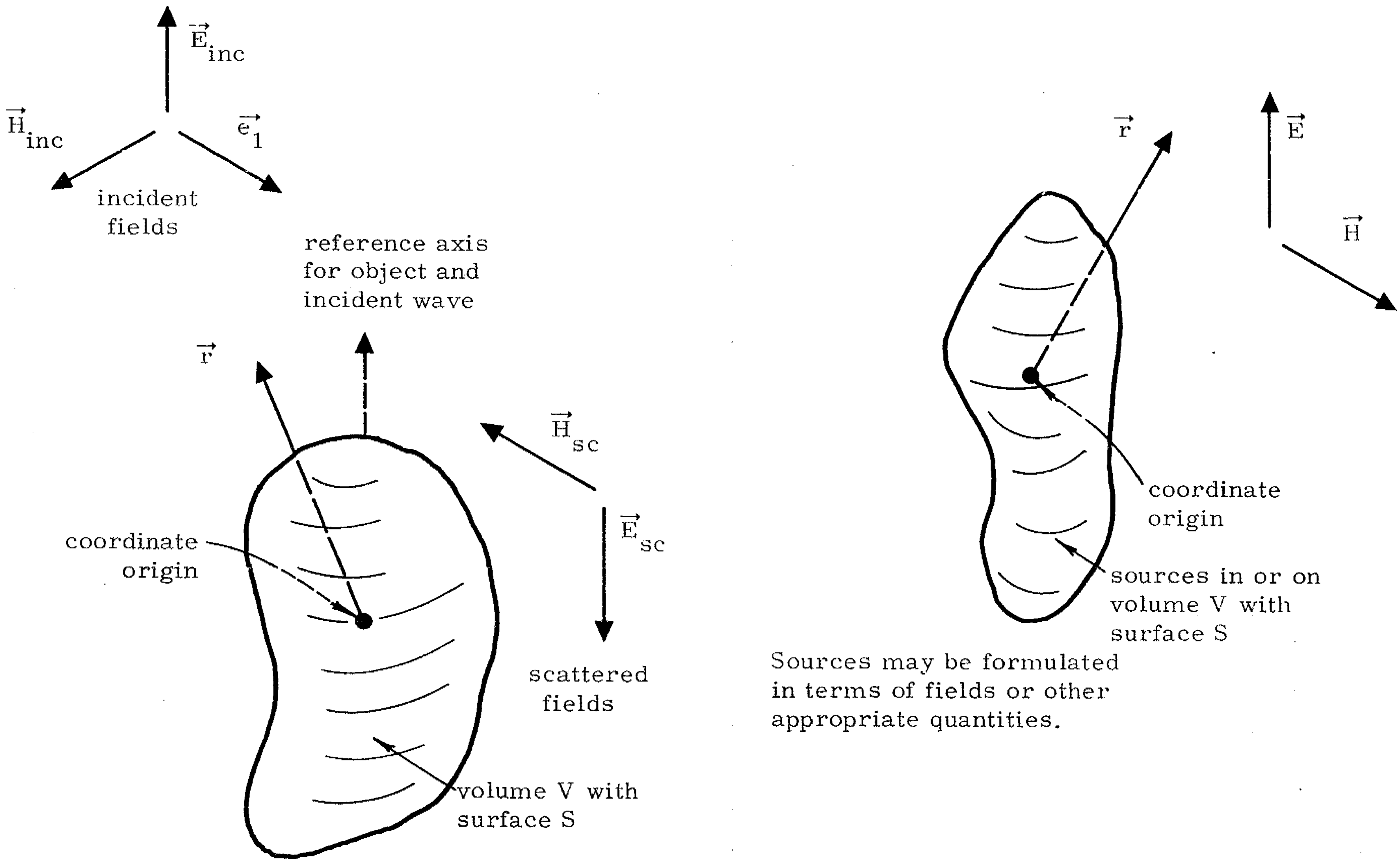
(Lewis Carroll,
Alice in Wonderland)

I. Introduction

Previous work has considered the singularity expansion of currents and charges on objects, at least as the primary emphasis. In particular there is now a modest amount of information on this method as it applies to the currents and charges on finite size objects (refs. 1 through 10). While this has primarily been stated in terms of the response of the object to an incident electromagnetic wave (the interaction or scattering problem as in figure 1A) the method applies to antenna problems (with sources on the antenna as in figure 1B) as well. The results for finite size objects carry over with the coupling coefficients describing the coupling of the poles to the antenna sources instead of the incident field. Integral equations can be written to describe the scattering and antenna problems for the currents and charges on the object. The given (or known) excitation term in the integral equation corresponds to the incident field in the scattering problem and to the sources in the antenna problem, so the formalism is very similar for the two cases.

A previous note[8] discusses the first order pole terms in the singularity expansion for current and charge densities. This note extends these considerations to radiated and scattered fields. The convenient implied integral notation of the previous note is used in this extension. We first consider some of the properties of the field natural modes from which we define retarded natural modes and far field natural modes (associated with finite size objects). Using the free space Green's function one can then calculate the various field and potential quantities from the current on the object. From these expressions one can calculate field and potential natural modes from current natural modes. The fields and potentials can then be singularity expanded in terms of these modes, although there is some flexibility in our choice of the forms of the resulting coefficients associated with entire functions of the complex frequency s.

The remainder of the note can be roughly divided into two parts. In the first part consisting of sections II through IV the natural modes are considered from a differential equation point of view for definitions and properties. In the second part consisting of sections V through VIII the natural modes are considered from the viewpoint of their integral relations to the current density natural modes; this gives explicit formulas with normalizing constants for calculating the field and potential natural modes from the current density natural modes. A recent note[16] treats the numerical calculation of the far field natural modes and the resulting far fields from a loaded linear antenna, showing a practical implementation of some of the concepts discussed in this note.



A. Scattered Fields

B. Radiated Fields

Figure 1. Radiation and Scattering of Electromagnetic Fields by an Object

II. Fields and Potentials

This section considers electromagnetic fields and potentials from a differential equation point of view as a prelude to similar considerations for the natural modes for fields and potentials. Write Maxwell's equations as

$$
\begin{aligned}
\nabla \times \tilde{\vec{E}} &= -s\mu\tilde{\vec{H}} - \tilde{\vec{J}}_m \\
\nabla \times \tilde{\vec{H}} &= s\varepsilon\tilde{\vec{E}} + \tilde{\vec{J}} \\
\nabla \cdot \tilde{\vec{B}} &= \nabla \cdot (\mu\tilde{\vec{H}}) = \tilde{\rho}_m \\
\nabla \cdot \tilde{\vec{D}} &= \nabla \cdot (\varepsilon\tilde{\vec{E}}) = \tilde{\rho} \\
\nabla \cdot \tilde{\vec{J}} &= -s\tilde{\rho} \\
\nabla \cdot \tilde{\vec{J}}_m &= -s\tilde{\rho}_m
\end{aligned}
\tag{2.1}
$$

The tilde ~ over the quantity indicates the Laplace transform (in general two sided) over time with the resulting Laplace transform variable being s, the complex frequency. In equations 2.1 the magnetic current density $\vec{J}_m$ and magnetic charge density $\tilde{\rho}_m$ are added for completeness sake and are generally taken to be zero (as being unphysical) but can be useful as fictitious quantities in certain cases. For much of our considerations we will be considering free space regions for which one has (away from the object of interest)

$$
\begin{aligned}
\vec{J} &= \vec{J}_m = \vec{0} \\
\rho &= \rho_m = \vec{0} \\
\mu &= \mu_o \\
\varepsilon &= \varepsilon_o
\end{aligned}
\tag{2.2}
$$

For finite size objects some region of space (volume V with surface S) will contain current and charge densities as well as permeability and permittivity which are possibly different from the free space values. These effects can all be lumped into the electric and magnetic current densities so that the fields and potentials can be calculated as integrals over these quantities as will be discussed later.

Combining the equations 2.1 appropriately we have wave equations (transformed) as

$$[\nabla \times \nabla \times + \gamma^2]\tilde{\vec{E}} = -s\mu_o\tilde{\vec{J}} - \nabla \times \tilde{\vec{J}}_m$$

$$[\nabla \times \nabla \times + \gamma^2]\tilde{\vec{H}} = \nabla \times \tilde{\vec{J}} - s\varepsilon_o\tilde{\vec{J}}_m \tag{2.3}$$

where μ and ε have been taken as their free space values so as to be independent of the coordinates and where the free space propagation constant, propagation speed, and wave impedance are

$$\gamma = \frac{s}{c}\ , \qquad c = \frac{1}{\sqrt{\mu_o\varepsilon_o}}\ , \qquad Z_o = \sqrt{\frac{\mu_o}{\varepsilon_o}} \tag{2.4}$$

There are alternate forms for equations 2.3 using the operator identity

$$\nabla \times \nabla \times \equiv \nabla\nabla \cdot - \nabla^2 \tag{2.5}$$

which implies

$$[\nabla^2 - \gamma^2]\tilde{\vec{E}} = s\mu_o\tilde{\vec{J}} + \frac{1}{\varepsilon_o}\nabla\tilde{\rho} + \nabla \times \tilde{\vec{J}}_m$$

$$[\nabla^2 - \gamma^2]\tilde{\vec{H}} = -\nabla \times \tilde{\vec{J}} + s\varepsilon_o\tilde{\vec{J}}_m + \frac{1}{\mu_o}\nabla\tilde{\rho}_m \tag{2.6}$$

In solving Maxwell's equations for the scattering or antenna problem for finite size objects the scattered fields or resultant fields, as appropriate, must satisfy certain boundary conditions including those at large distances from the object. This boundary condition at infinity is the radiation condition which can be stated for the electromagnetic fields as[15]

$$\lim_{r\to\infty} r\left[\nabla \times \begin{pmatrix} \tilde{\vec{E}}(\vec{r},s) \\ \\ \tilde{\vec{H}}(\vec{r},s) \end{pmatrix} + \gamma\vec{e}_r \times \begin{pmatrix} \tilde{\vec{E}}(\vec{r},s) \\ \\ \tilde{\vec{H}}(\vec{r},s) \end{pmatrix}\right] = \vec{0} \tag{2.7}$$

or

$$\lim_{r\to\infty} r\left[\begin{pmatrix} -Z_o\tilde{\vec{H}}(\vec{r},s) \\ \\ \tilde{\vec{E}}(\vec{r},s) \end{pmatrix} + \vec{e}_r \times \begin{pmatrix} \tilde{\vec{E}}(\vec{r},s) \\ \\ Z_o\tilde{\vec{H}}(\vec{r},s) \end{pmatrix}\right] = \vec{0} \tag{2.8}$$

where s is evaluated on the imaginary axis ($s = i\omega$) for the historical derivation and use of this radiation condition. Later in this note the integral form for the fields in terms of the currents on a finite size object is considered. In this integral form the radiation condition is contained explicitly in the form of the terms with a leading term of $e^{-\gamma r}/r$ for large r (distance from the object). This effectively analytically continues the radiation condition from the $i\omega$ axis into the rest of the complex frequency plane. Note that equations 2.7 and 2.8 only apply to radiated or scattered fields from finite size objects, and not to incident fields. As stated in equations 2.7 and 2.8 the radiation condition applies in the right half plane including the $i\omega$ axis, i.e., for $Re[s] \geq 0$.

Associated with the electromagnetic fields we have the vector and scalar potentials from which the field quantities can be calculated as[14]

$$\begin{aligned} \tilde{\vec{E}} &= -\nabla\tilde{\Phi} - s\tilde{\vec{A}} - \frac{1}{\epsilon_o}\nabla \times \tilde{\vec{A}}_m \\ \tilde{\vec{H}} &= \frac{1}{\mu_o}\nabla \times \tilde{\vec{A}} - \nabla\tilde{\Phi}_m - s\tilde{\vec{A}}_m \end{aligned} \tag{2.9}$$

where $\vec{A}_m$ and Φ_m are sometimes referred to as antipotentials. One might also refer to $\vec{A}$ and Φ as electric potentials and $\vec{A}_m$ and Φ_m as magnetic potentials. These potentials satisfy

$$[\nabla^2 - \gamma^2]\tilde{\vec{A}} = -\mu_o \tilde{\vec{J}}$$

$$[\nabla^2 - \gamma^2]\tilde{\Phi} = -\frac{1}{\varepsilon_o}\tilde{\rho}$$

$$[\nabla^2 - \gamma^2]\tilde{\vec{A}}_m = -\varepsilon_o \tilde{\vec{J}}_m \qquad (2.10)$$

$$[\nabla^2 - \gamma^2]\tilde{\Phi}_m = -\frac{1}{\mu_o}\tilde{\rho}_m$$

and have the Lorentz gauge relations

$$\nabla \cdot \tilde{\vec{A}} + \frac{s}{c^2}\tilde{\Phi} = 0$$

$$\nabla \cdot \tilde{\vec{A}}_m + \frac{s}{c^2}\tilde{\Phi}_m = 0 \qquad (2.11)$$

The radiation condition for the potentials (radiated or scattered) is[12,14]

$$\lim_{r\to\infty} r\left[\frac{\partial}{\partial r} + \gamma\right]\tilde{\vec{A}} = \vec{0}$$

$$\lim_{r\to\infty} r\left[\frac{\partial}{\partial r} + \gamma\right]\tilde{\Phi} = 0$$

$$\lim_{r\to\infty} r\left[\frac{\partial}{\partial r} + \gamma\right]\tilde{\vec{A}}_m = \vec{0} \qquad (2.12)$$

$$\lim_{r\to\infty} r\left[\frac{\partial}{\partial r} + \gamma\right]\tilde{\Phi}_m = 0$$

where $Re[s] > 0$. Note that the cartesian components of $\vec{A}$ (and $\vec{A}_m$) each satisfy scalar differential equations of the same form as Φ (and Φ_m). The general form for large r is $e^{-\gamma r}/r$ as with the fields (for finite size objects). This will appear in the explicit integrals for the potentials in terms of $\vec{J}$, ρ, $\vec{J}_m$, ρ_m on the object. Note that generally we will be concerned with the case where $\vec{A}_m$ and Φ_m are both zero so that often only $\vec{A}$ and Φ are used.

The electric and magnetic field vectors can be combined as[11,15]

$$\vec{F}_q \equiv \vec{E} + qiZ_o\vec{H} , \qquad Z_o \equiv \sqrt{\frac{\mu_o}{\varepsilon_o}} , \qquad q = \pm 1 \tag{2.13}$$

and the electric and magnetic current densities can be similarly combined as

$$\vec{K}_q \equiv \vec{J} + q\frac{i}{Z_o}\vec{J}_m \tag{2.14}$$

Given $\vec{F}_+$ and $\vec{F}_-$ the electric and magnetic fields can be readily reconstructed. Since we are usually concerned with only electric current densities then $\vec{K}_+$ and $\vec{K}_-$ are usually the same thing, but $\vec{J}_m$ can be also reconstructed from the two forms of $\vec{K}$. In this form the Maxwell equations for the combined field and current density vectors reduce in free space to

$$[\nabla \times - qi\gamma]\tilde{\vec{F}}_q = qiZ_o\tilde{\vec{K}}_q \tag{2.15}$$

or

$$[\nabla \times - q\,\frac{i}{c}\,\frac{\partial}{\partial t}]\vec{F}_q = qiZ_o\vec{K}_q \tag{2.16}$$

Similarly the combined charge density can be defined by

$$Q_q \equiv \rho + q\frac{i}{Z_o}\rho_m \tag{2.17}$$

so that we have the relations

$$\begin{aligned} \nabla \cdot \tilde{\vec{K}}_q &= -s\tilde{Q}_q \\ \nabla \cdot \vec{K}_q &= -\frac{\partial}{\partial t}Q_q \\ \nabla \cdot \vec{F}_q &= \frac{1}{\varepsilon_o}Q_q \end{aligned} \tag{2.18}$$

The radiation condition for this combined field vector from equations 2.7 becomes

$$\lim_{r\to\infty} r[\nabla\times + \gamma\vec{e}_r\times]\tilde{\vec{F}}_q = \vec{0} \tag{2.19}$$

and setting the currents to zero at large r gives (from equation 2.15)

$$\lim_{r\to\infty} r[qi + \vec{e}_r\times]\tilde{\vec{F}}_q = \vec{0} \tag{2.20}$$

which is a rather compact form. Again note for these last two equations we constrain Re[s] $\geq$ 0 and consider only radiated or scattered fields.

Another form for the combined field equations is found by combining equations 2.3 to give

$$[\nabla\times\nabla\times + \gamma^2]\tilde{\vec{F}}_q = [-s\mu_o + q_i Z_o \nabla\times]\tilde{\vec{K}}_q \tag{2.21}$$

This will be useful in some of the later explicit solutions involving Green's functions.

The combined field concept can be extended to combined potentials. Define

$$\begin{aligned} \vec{C}_q &= \vec{A} + qiZ_o\vec{A}_m \\ \phi_q &= \Phi + qiZ_o\Phi_m \end{aligned} \tag{2.22}$$

The combined field is then

$$\tilde{\vec{F}}_q = -\nabla\tilde{\phi}_q + [-s + qic\nabla\times]\tilde{\vec{C}}_q \tag{2.23}$$

The combined potentials satisfy

$$\begin{aligned} [\nabla^2 - \gamma^2]\tilde{\vec{C}}_q &= -\mu_o\tilde{\vec{K}}_q \\ [\nabla^2 - \gamma^2]\tilde{\phi}_q &= -\frac{1}{\varepsilon_o}\tilde{Q}_q \end{aligned} \tag{2.24}$$

with the Lorentz gauge

$$\nabla \cdot \tilde{\vec{C}}_q + \frac{s}{c^2}\tilde{\phi}_q = 0 \tag{2.25}$$

Thus the combined potentials have properties much like the electric and magnetic potentials separately. The radiation condition for the combined potentials becomes

$$\lim_{r\to\infty} r[\frac{\partial}{\partial r} + \gamma]\tilde{\vec{C}}_q = \vec{0}$$

$$\lim_{r\to\infty} r[\frac{\partial}{\partial r} + \gamma]\phi_q = 0 \tag{2.26}$$

where $Re[s] \geq 0$ for the radiation condition.

III. Natural Modes for Fields and Potentials

One of the important parts of a pole term in a singularity expansion is the natural mode. This is the spatial distribution function for the electromagnetic quantity of interest at the natural frequency s_α of interest. The complex natural frequency s_α is a frequency at which a pole occurs in the complex frequency plane. At such a natural frequency Maxwell's equations have a solution (the natural mode or modes) with no forcing function (incident field and/or sources set equal to zero). This section considers the natural modes for fields and potentials from a differential equation point of view. Later sections will give integral definitions of the various natural modes.

Previous notes[1,8] have considered the current and charge density modes and how to calculate them from general integral equation formulations. Let us now consider natural modes for various other quantities from a differential equation point of view based on the differential equations discussed in section II. One use of the combined form for the fields and potentials is to help in defining the interrelations among the various combined quantities for the natural modes for the fields and potentials. Besides making the natural modes dimensionless one can also define a set of normalizing constants (also dimensionless) which relate the various natural mode quantities and can be chosen for convenience in the problem at hand.

Let us begin with the current densities. One can write the current density in the form[1,8]

$$\begin{aligned}
\tilde{\vec{J}}(\vec{r},s) &= \tilde{\vec{J}}_2(\vec{r},s) + \tilde{\vec{J}}_3(\vec{r},s) \\
\tilde{\vec{J}}_p(\vec{r},s) &= E_o \Sigma \tilde{\vec{V}}_p^{(\vec{J})}(\vec{r},s) \qquad (3.1) \\
\tilde{\vec{V}}_p^{(\vec{J})}(\vec{r},s) &= \tilde{f}_p(s)\tilde{\vec{U}}_p^{(\vec{J})}(\vec{r},s) = \tilde{\vec{V}}_{p_o}^{(\vec{J})}(\vec{r},s) + \tilde{\vec{V}}_{p_w}^{(\vec{J})}(\vec{r},s)
\end{aligned}$$

where E_o has dimensions of volts per meter (and might be an incident field amplitude or normalized antenna source magnitude) and Σ is an appropriate normalizing constant with dimensions of conductivity (Sm^{-1}). The object part of the normalized response for the case of first order poles is

$$\tilde{\vec{V}}_{p_o}^{(\vec{J})}(\vec{r},s) = \sum_\alpha \tilde{f}_p(s_\alpha)\tilde{\eta}_\alpha(\vec{e}_1,s)\vec{\nu}_\alpha^{(\vec{J})}(\vec{r})(s - s_\alpha)^{-1} \qquad (3.2)$$

and the waveform part is

$$\tilde{\vec{V}}^{(\vec{J})}_{p_w}(\vec{r},s) = \sum_\alpha \tilde{\eta}_\alpha(\vec{e}_1,s)\vec{\nu}^{(\vec{J})}_\alpha(\vec{r})\frac{\tilde{f}_p(s) - \tilde{f}_p(s_\alpha)}{s - s_\alpha} \tag{3.3}$$

Here $\tilde{f}_p(s)$ is the Laplace transform of the waveform function for the incident wave or the antenna sources and may apply to two polarizations ($p = 2,3$) for the case of an incident wave.

For the case of higher order poles the delta function response has the general form

$$\tilde{\vec{\vec{U}}}^{(\vec{J})}_p(\vec{r},s) = \sum_\alpha \tilde{\eta}_\alpha(\vec{e}_1,s)\vec{\nu}^{(\vec{J})}_\alpha(\vec{r})(s - s_\alpha)^{-n_\alpha} + \tilde{\vec{\vec{W}}}^{(\vec{J})}_p(\vec{e}_1,\vec{r},s) \tag{3.4}$$

In general one should include the entire function $\tilde{\vec{\vec{W}}}$ in the expansion for completeness although studies have shown cases in which it is not needed. In expanding the object and waveform parts of the response higher order poles can be included by considering the behavior of the waveform function and coupling coefficients near $s = s_\alpha$. The present note is primarily concerned with the case of first order poles, $n_\alpha = 1$.

Reviewing the steps in calculating the terms in the singularity expansion for the current using the abbreviated integral notation introduced in a previous note we have[8]

$$\langle \tilde{\vec{\vec{\Gamma}}};\tilde{\vec{\vec{U}}}\rangle = \tilde{\vec{\vec{I}}} \qquad \text{Integral equation}$$

$$\tilde{\vec{\vec{U}}} = \sum_\alpha \tilde{\eta}_\alpha\vec{\nu}_\alpha(s - s_\alpha)^{-1} \qquad \text{Form of solution}$$

$$\left.\begin{aligned} \langle \vec{\vec{\Gamma}}_o;\vec{\nu}\rangle &= \vec{0} \\ \langle \vec{\mu};\vec{\vec{\Gamma}}_o\rangle &= \vec{0} \end{aligned}\right\} \qquad \text{Natural frequencies, natural modes, and coupling vectors}$$

$$\tilde{\vec{\vec{\Gamma}}} = \sum_{n=o}^{\infty} (s - s_{\alpha})^n \vec{\vec{\Gamma}}_n \qquad (3.5)$$

$$\tilde{\vec{U}} = \tilde{\eta}\vec{\nu}(s - s_{\alpha})^{-1} + \tilde{\vec{U}}' \qquad \text{Expansion near } s_{\alpha}$$

$$\tilde{\vec{I}} = \sum_{n=o}^{\infty} (s - s_{\alpha})^n \vec{I}_n$$

$$\tilde{\eta}(s_{\alpha}) = \frac{\langle \vec{\mu}; I_o \rangle}{\langle \vec{\mu}; \vec{\vec{\Gamma}}_1; \vec{\nu} \rangle} \qquad \text{Coupling coefficient at } s_{\alpha}$$

Here $\tilde{\vec{\vec{\Gamma}}}$ is the kernel of some integral equation describing the antenna or scatterer and $\tilde{\vec{I}}$ and $\tilde{\vec{U}}$ are respectively the normalized excitation and normalized current response. The natural mode $\vec{\nu}$ for the current is chosen to be dimensionless and the coupling coefficient $\tilde{\eta}$ has dimension inverse seconds.

Two common forms of coupling coefficients are

$$\tilde{\eta}_{\alpha}(s) = e^{(s_{\alpha}-s)t'} \frac{\langle \vec{\mu}; \vec{I}_o \rangle}{\langle \vec{\mu}; \vec{\vec{\Gamma}}_1; \vec{\nu} \rangle} \qquad \text{Class 1 coupling coefficient for turn on time } t'$$

$$(3.6)$$

$$\tilde{\eta}_{\alpha}(s) = \frac{\langle \vec{\mu}; \tilde{\vec{I}}(s) \rangle}{\langle \vec{\mu}; \vec{\vec{\Gamma}}_1; \vec{\nu} \rangle} \qquad \text{Class 2 coupling coefficient (convolution form)}$$

These are not the only forms but are useful for illustrating some of the options to be encountered later in discussing the more general coefficients for the field and potential modes.

In calculating the natural modes for the current density one usually normalizes the mode function in some convenient way, such as by making an appropriate component of it have maximum value 1. The choice of this normalization also affects the numerical values for the coupling coefficient.

What has been discussed in terms of the natural modes for $\vec{J}$ on finite size objects can be applied to the magnetic current density $\vec{J}_m$ as well, provided one has some defining equation for $\vec{J}_m$ such as an appropriate integral equation. We will not be concerned with calculational procedures involving $\vec{J}_m$ in this note. However, one can write a general form for $\vec{J}_m$ directly analogous to equation 3.1 as

$$\tilde{\vec{J}}_m(\vec{r},s) = \tilde{\vec{J}}_{m_2}(\vec{r},s) + \tilde{\vec{J}}_{m_3}(\vec{r},s)$$

$$\tilde{\vec{J}}_{m_p}(\vec{r},s) = E_o Z_o \Sigma \tilde{\vec{V}}_p^{(\vec{J}_m)}(\vec{r},s) \tag{3.7}$$

$$\tilde{\vec{V}}_p^{(\vec{J}_m)}(\vec{r},s) = \tilde{f}_p(s)\tilde{\vec{U}}_p^{(\vec{J}_m)}(\vec{r},s) = \tilde{\vec{V}}_{p_o}^{(\vec{J}_m)}(\vec{r},s) + \tilde{\vec{V}}_{p_w}(\vec{r},s)$$

Note that an additional factor Z_o is introduced to make the delta function response $\vec{U}_p$ dimensionless. This is consistent with the form of the combined current density in equation 2.14. One can then expand the magnetic current density in a form like that in equation 3.4 as

$$\tilde{\vec{U}}_p^{(\vec{J}_m)}(\vec{r},s) = \sum_\alpha \tilde{\eta}_{m_\alpha}(\vec{e}_1,s)\vec{\nu}_\alpha^{(\vec{J}_m)}(s - s_\alpha)^{-n_\alpha}$$
$$+ \tilde{\vec{W}}_p^{(\vec{J}_m)}(\vec{e}_1,\vec{r},s) \tag{3.8}$$

The same formalism carries over from current density (electric) to magnetic current density with appropriate subscripts m added to the coupling coefficients to distinguish them.

The combined current density natural modes then take the form from equation 2.14 as

$$\vec{\nu}_\alpha^{(\vec{K}_q)} = \vec{\nu}_\alpha^{(\vec{J})} + qi\eta'_\alpha \vec{\nu}_\alpha^{(\vec{J}_m)} \tag{3.9}$$

where η'_α is a constant independent of the excitation (assuming nondegenerate modes) since these natural modes are solutions of the homogeneous equations. Note that we have

$$\eta'_\alpha = \frac{\tilde{\eta}_{m_\alpha}(\vec{e}_1,s)}{\tilde{\eta}_\alpha(\vec{e}_1,s)} \tag{3.10}$$

so that for nondegenerate modes at a simple pole the coupling coefficients should have the same frequency dependence for the two types of current density provided the same form of coupling coefficient is used. Since the coupling coefficient types all reduce to the same thing at $s = s_\alpha$ then some ambiguity in calculating η'_α can be eliminated.

Within the restriction that $\rho_m = 0$ one can describe permeable object response in terms of $\vec{J}_m$, $\vec{J}$, or some combination of the two. In this context there is some flexibility in choosing how $\vec{K}_q$ is defined but the end results for the fields must be independent of this choice.

From the current density modes various other modes can be derived through the differential equations in section II. Consider the charge density modes which we define from equations 2.1 and 2.18 as

$$\begin{aligned}
\nu_\alpha^{(\rho)} &= -\frac{a_\alpha}{\gamma_\alpha}\nabla \cdot \vec{\nu}_\alpha^{(\vec{J})} \\
\nu_\alpha^{(\rho_m)} &= -\frac{a_\alpha}{\gamma_\alpha}\nabla \cdot \vec{\nu}_\alpha^{(\vec{J}_m)} \\
\nu_\alpha^{(Q_q)} &= -\frac{a_\alpha}{\gamma_\alpha}\nabla \cdot \vec{\nu}_\alpha^{(\vec{K}_q)} \\
\nu_\alpha^{(Q_q)} &= \nu_\alpha^{(\rho)} + qi\eta'_\alpha\nu_\alpha^{(\rho_m)} , \qquad \gamma_\alpha \equiv \frac{s_\alpha}{c}
\end{aligned} \tag{3.11}$$

Note that one coefficient a_α is chosen for both electric and magnetic types of modes so that the combined modes have the same form. Here a_α is chosen to be dimensionless (in contrast to its choice in two previous notes[1,8]). The resulting electric and magnetic charge density modes are dimensionless. For a given magnitude of the current density modes as one goes to higher frequencies (s_α) the spatial derivatives tend to get larger; this is offset somewhat by dividing the result by γ_α so as to keep $|a_\alpha|$ near 1 for magnitudes of the charge density modes near 1.

For some purposes it may be convenient to set $a_\alpha = 1$. This would be appropriate in considering 4-vector current densities (electric and magnetic) combining $\vec{J}$ and ρ on one hand, and $\vec{J}_m$ and ρ_m on the other. The 4-vector (and 4-tensor) form of Maxwell's equations is not considered here but may provide some useful insight into the natural mode relations. Perhaps 4-vector forms can be considered in future notes.

In converting current density expansions to charge density expansions one applies equations 2.1 to equations 3.1, 3.7, and 3.9. The expansions for ρ, ρ_m, and Q_q have additional coefficients $1/(a_\alpha c)$ in the pole terms in their respective expansions with the modes switched from current density (vector) to charge density (scalar).

Consider next the vector and scalar potential natural modes which can be related through the Lorentz gauge as

$$\nu_\alpha^{(\Phi)} = -\frac{a_\alpha}{\gamma_\alpha}\nabla \cdot \vec{\nu}_\alpha^{(\vec{A})}$$

$$\nu_\alpha^{(\Phi_m)} = -\frac{a_\alpha}{\gamma_\alpha}\nabla \cdot \vec{\nu}_\alpha^{(\vec{A}_m)}$$

$$\nu_\alpha^{(\phi_q)} = -\frac{a_\alpha}{\gamma_\alpha}\nabla \cdot \vec{\nu}_\alpha^{(\vec{C}_q)} \tag{3.12}$$

$$\vec{\nu}_\alpha^{(\vec{C}_q)} = \vec{\nu}_\alpha^{(\vec{A})} + qi\eta_\alpha'\vec{\nu}_\alpha^{(\vec{A}_m)} \quad , \qquad \nu_\alpha^{(\phi_q)} = \nu_\alpha^{(\Phi)} + qi\eta_\alpha'\nu_\alpha^{(\Phi_m)}$$

which have the same forms as equations 3.11. This occurs because the vector and scalar potentials are related to the current and charge densities through the same forms of equations as in equations 2.10 and 2.24.

In converting vector potential expansions (either electric, magnetic, or combined) to the corresponding scalar potential expansions one includes an additional factor of c/a_α when replacing the vector potential natural mode by the corresponding scalar potential natural mode. Letting $a_\alpha = 1$ would be convenient in considering 4-vector potentials.

In relating the vector and scalar potentials to the current and charge densities for natural mode definitions one can modify equations 2.24 to the form

$$[\nabla^2 - \gamma_\alpha^2]\vec{\nu}_\alpha^{(\vec{C}_q)} = -b_\alpha \gamma_\alpha^2 \vec{\nu}^{(\vec{K}_q)}$$

$$[\nabla^2 - \gamma_\alpha^2]\nu_\alpha^{(\phi_q)} = -b_\alpha \gamma_\alpha^2 \nu^{(Q_q)}$$

$$[\nabla^2 - \gamma_\alpha^2]\vec{\nu}_\alpha^{(\vec{A})} = -b_\alpha \gamma_\alpha^2 \vec{\nu}^{(\vec{J})}$$

$$[\nabla^2 - \gamma_\alpha^2]\nu_\alpha^{(\Phi)} = -b_\alpha \gamma_\alpha^2 \nu^{(\rho)} \tag{3.13}$$

$$[\nabla^2 - \gamma_\alpha^2]\vec{\nu}_\alpha^{(\vec{A}_m)} = -b_\alpha \gamma_\alpha^2 \vec{\nu}^{(\vec{J}_m)}$$

$$[\nabla^2 - \gamma_\alpha^2]\nu_\alpha^{(\rho_m)} = -b_\alpha \gamma_\alpha^2 \nu^{(\rho_m)}$$

This assures dimensionless potential modes and gives a dimensionless constant b_α for choosing potential mode normalization if desired. One can set $b_\alpha = 1$ for convenience in certain cases.

In converting electric or combined current density expansions to corresponding vector potential expansions an additional coefficient of $\mu_o/(b_\alpha \gamma_\alpha^2)$ appears; in converting electric or combined charge density expansions to scalar potential expansions an additional coefficient of $1/(\varepsilon_o b_\alpha \gamma_\alpha^2)$ appears which is a coefficient of $Z_o/(a_\alpha b_\alpha \gamma_\alpha^2)$ related to the current density. For converting magnetic current density to magnetic vector potential the additional coefficient is $\varepsilon_o/(b_\alpha \gamma_\alpha^2)$; for converting magnetic charge density to magnetic scalar potential the additional coefficient is $1/(\mu_o b_\alpha \gamma_\alpha^2)$ which becomes a coefficient of $1/(Z_o a_\alpha b_\alpha \gamma_\alpha^2)$ when related to the magnetic current density expansion.

Next we have the field natural modes. From equations 2.15 let us write

$$[\nabla\times - qi\gamma_\alpha]\vec{\nu}_\alpha^{(\vec{F}_q)} = qic_\alpha \gamma_\alpha \vec{\nu}_\alpha^{(\vec{K}_q)}$$

$$\vec{\nu}_\alpha^{(\vec{F}_q)} \equiv \vec{\nu}_\alpha^{(\vec{E})} + qi\vec{\nu}_\alpha^{(\vec{H})} \tag{3.14}$$

where the γ_α is included with the combined current natural mode to correspond to the $\nabla\times$ and γ_α operating on the combined field mode. A set of coefficients c_α is included to give some flexibility in the mode normalization; c_α may be set to 1 if desired. Note that we use this equation to define the electric and magnetic field modes as well with additional constants. This will be useful when considering far field modes.

Another set of equations for the field natural modes comes from equations 2.3 and 2.21. This gives wave equations (transformed) for the field natural modes relating them to the current density natural modes as

$$[\nabla\times\nabla\times + \gamma_\alpha^2]\vec{\nu}_\alpha^{(\vec{F}_q)} = c_\alpha[-\gamma_\alpha^2 + qi\gamma_\alpha\nabla\times]\vec{\nu}_\alpha^{(\vec{K}_q)}$$

$$[\nabla\times\nabla\times + \gamma_\alpha^2]\vec{\nu}_\alpha^{(\vec{E})} = c_\alpha\left[-\gamma_\alpha^2\vec{\nu}_\alpha^{(\vec{J})} - \gamma_\alpha\eta_\alpha'\nabla\times\vec{\nu}_\alpha^{(\vec{J}_m)}\right] \qquad (3.15)$$

$$[\nabla\times\nabla\times + \gamma_\alpha^2]\vec{\nu}_\alpha^{(\vec{H})} = c_\alpha\left[\gamma_\alpha\nabla\times\vec{\nu}_\alpha^{(\vec{J})} - \gamma_\alpha^2\eta_\alpha'\vec{\nu}_\alpha^{(\vec{J}_m)}\right]$$

where a factor of γ_α is used with each term not having a $\nabla\times$ operator for dimensional consistency. Note that the c_α coefficient must also be included so as to make the relations of the field and current density consistent with equations 3.14. The first of equations 3.15 can be derived by operating on both sides of the first of equations 3.14 with the operator $\nabla\times + qi\gamma_\alpha$.

In converting combined current density natural mode expansions to those for the combined field an additional coefficient $Z_o/(c_\alpha\gamma_\alpha)$ appears with each term. The same additional term appears when calculating the electric field ($\vec{E}$) expansion but the additional factor reduces to $1/(c_\alpha\gamma_\alpha)$ when writing the magnetic field ($\vec{H}$) expansion from the current density expansion. Note that η_α'/Z_o is already included with the magnetic current density before it is combined with the electric current density.

Now relate the field and potential natural modes. From equations 2.23 together with the various coefficients to be included with the various modes we have

$$\vec{\nu}_\alpha^{(\vec{F}_q)} = \frac{c_\alpha}{b_\alpha}\left\{-\frac{1}{a_\alpha\gamma_\alpha}\nabla\nu_\alpha^{(\phi_q)} + \left[-1 + \frac{qi}{\gamma_\alpha}\nabla\times\right]\vec{\nu}_\alpha^{(\vec{C}_q)}\right\}$$

$$\vec{\nu}_\alpha^{(\vec{E})} = \frac{c_\alpha}{b_\alpha}\left\{-\frac{1}{a_\alpha\gamma_\alpha}\nabla\nu_\alpha^{(\Phi)} - \vec{\nu}_\alpha^{(\vec{A})} - \frac{\eta'_\alpha}{\gamma_\alpha}\nabla\times\vec{\nu}_\alpha^{(\vec{A}_m)}\right\} \tag{3.16}$$

$$\vec{\nu}_\alpha^{(\vec{H})} = \frac{c_\alpha}{b_\alpha}\left\{-\frac{\eta'_\alpha}{a_\alpha\gamma_\alpha}\nabla\nu_\alpha^{(\Phi_m)} - \eta'_\alpha\vec{\nu}^{(\vec{A}_m)} + \frac{1}{\gamma_\alpha}\nabla\times\vec{\nu}_\alpha^{(\vec{A})}\right\}$$

The Maxwell field equations can be written for the natural modes as

$$\nabla\times\vec{\nu}_\alpha^{(\vec{E})} = -\gamma_\alpha\vec{\nu}_\alpha^{(\vec{H})} - c_\alpha\gamma_\alpha\eta'_\alpha\vec{\nu}_\alpha^{(\vec{J}_m)}$$

$$\nabla\times\vec{\nu}_\alpha^{(\vec{H})} = \gamma_\alpha\vec{\nu}_\alpha^{(\vec{E})} + c_\alpha\gamma_\alpha\vec{\nu}_\alpha^{(\vec{J})}$$

$$\nabla\cdot\vec{\nu}_\alpha^{(\vec{E})} = -c_\alpha\nabla\cdot\vec{\nu}_\alpha^{(\vec{J})} = \frac{c_\alpha}{a_\alpha}\gamma_\alpha\nu_\alpha^{(\rho)} \tag{3.17}$$

$$\nabla\cdot\vec{\nu}_\alpha^{(\vec{H})} = -c_\alpha\eta'_\alpha\nabla\cdot\vec{\nu}_\alpha^{(\vec{J}_m)} = \frac{c_\alpha}{a_\alpha}\gamma_\alpha\eta'_\alpha\nu_\alpha^{(\rho_m)}$$

In combined form we have

$$[\nabla\times - qi\gamma_\alpha]\vec{\nu}_\alpha^{(\vec{F}_q)} = qic_\alpha\gamma_\alpha\vec{\nu}_\alpha^{(\vec{K}_q)}$$

$$\nabla\cdot\vec{\nu}_\alpha^{(\vec{F}_q)} = -c_\alpha\nabla\cdot\vec{\nu}_\alpha^{(\vec{K}_q)} = \frac{c_\alpha}{a_\alpha}\gamma_\alpha\nu_\alpha^{(Q_q)} \tag{3.18}$$

Let us now try to summarize the normalizing conventions for the natural modes. The various quantities have an expansion for the delta function response in the form (for simple poles)

$$\tilde{\vec{U}}_p^{(\vec{X})}(\vec{r},s) = \sum_\alpha \tilde{\eta}_\alpha(\vec{e}_1,s) Y \vec{\nu}_\alpha^{(\vec{X})}(\vec{r})(s - s_\alpha)^{-1}$$

$$+ \tilde{\vec{W}}_p^{(\vec{X})}(\vec{e}_1,\vec{r},s) \qquad (3.19)$$

where $\vec{X}$ is the quantity to be expanded and where it might also be a scalar in which case ν is also a scalar. Note the entire function $\vec{W}$ added for completeness. As it will turn out from later portions of this note there are other convolution forms involving the natural modes which change the form for $s \neq s_\alpha$; the difference can also be included with the entire function. It need not concern us here as it will not affect the additional coefficient Y.

Choosing Y = 1 for the case that $\vec{X}$ is the current density $\vec{J}$ or the combined current density $\vec{K}_q$ then equation 3.19 is in a form consistent with the form of the current density expansions in equations 3.1 through 3.6. Table 1 lists the additional coefficient Y associated with various electromagnetic quantities. Note that the factor $E_o \Sigma \tilde{f}_p$ is needed to multiply by the delta function response as in equations 3.1 so as to obtain the desired electromagnetic quantity.

	Electromagnetic quantity	Additional coefficient
	$\vec{X}$ (or X)	Y
current densities	$\vec{K}_q$, $\vec{J}$	1
	$\vec{J}_m$	η'_α
charge densities	Q_q , ρ	$\frac{1}{a_\alpha c}$
	ρ_m	$\frac{\eta'_\alpha}{a_\alpha c}$
vector potentials	$\vec{C}_q$, $\vec{A}$	$\frac{\mu_o}{b_\alpha \gamma_\alpha^2}$
	$\vec{A}_m$	$\frac{\varepsilon_o Z_o \eta'_\alpha}{b_\alpha \gamma_\alpha^2}$
scalar potentials	ϕ_q , Φ	$\frac{Z_o}{a_\alpha b_\alpha \gamma_\alpha^2}$
	Φ_m	$\frac{\eta'_\alpha}{a_\alpha b_\alpha \gamma_\alpha^2}$
fields	$\vec{F}_q$, $\vec{E}$	$\frac{Z_o}{c_\alpha \gamma_\alpha}$
	$\vec{H}$	$\frac{1}{c_\alpha \gamma_\alpha}$

Table 1. Additional Coefficients Needed in Expansion of Electromagnetic Quantities

IV. Retarded and Far-Field Forms for Fields, Potentials, and Associated Natural Modes

When dealing with radiated or scattered fields, particularly at large distances from the object, it is convenient to define a retarded time as

$$t^* = t - \frac{r}{c} \tag{4.1}$$

where $r = |\vec{r}|$ is the distance from the coordinate origin as indicated in figure 1. In the complex frequency domain this time shift corresponds to multiplication by $e^{\gamma r}$. Thus we define retarded electromagnetic quantities from

$$\tilde{\vec{X}}_{ret}(\vec{r},s) \equiv e^{\gamma r}\tilde{\vec{X}}(\vec{r},s) \tag{4.2}$$

where $\vec{X}$ here is a vector or scalar quantity, or even a matrix quantity if desired.

Now the fields, potentials, Green's functions, etc. behave as $e^{-\gamma r}/r$ for large r. The retarded quantities then fall off as 1/r. This suggests that far field quantities be defined as

$$\begin{aligned} r\tilde{\vec{X}}_f(\vec{r},s) &\equiv \lim_{r\to\infty} r\tilde{\vec{X}}_{ret}(\vec{r},s) \\ &= \lim_{r\to\infty} re^{\gamma r}\tilde{\vec{X}}(\vec{r},s) \end{aligned} \tag{4.3}$$

Here $r\tilde{\vec{X}}_f$ is independent of r and is written in this combination to point out that it is an r times field (potential, etc.) product that is being calculated.

The retarded and far field definitions have equivalent time domain forms as

$$\begin{aligned} \vec{X}_{ret}(\vec{r},t^*) &= \vec{X}(\vec{r},t) \\ r\vec{X}_f(\vec{r},t^*) &= \lim_{r\to\infty} r\vec{X}_{ret}(\vec{r},t^*) \\ &= \lim_{r\to\infty} r\vec{X}(\vec{r},t) \quad \text{with } t^* \text{ constant} \end{aligned} \tag{4.4}$$

The choice of coordinate origin $\vec{r} = \vec{0}$ is arbitrary as long as it is within a finite distance of the finite size object of interest. However there are advantages to choosing $\vec{r} = \vec{0}$ to be at the "center" of the object in some sense so as to equalize as much as possible the transit times of radiated or scattered waves out to a given r in all directions from the object. In particular $\vec{r} = \vec{0}$ should be chosen to lie on points, lines, and planes of symmetry to the extent possible. This will retain the object symmetry properties in the retarded and far-field quantities, thereby simplifying their computation and the understanding of their properties. This symmetry will also carry over into the retarded and far-field natural modes.

Next we define retarded natural modes from the general equation

$$\nu_{ret_\alpha}^{(\vec{X})}(\vec{r}) \equiv e^{\gamma_\alpha r} \vec{\nu}_\alpha^{(\vec{X})}(\vec{r}) \tag{4.5}$$

where this also applies to scalar quantities etc. The far field modes are defined through

$$\begin{aligned} \vec{\nu}_{f_\alpha}^{(\vec{X})}(\vec{e}_r) &\equiv \lim_{r\to\infty} \frac{r}{\ell_\alpha} \vec{\nu}_{ret_\alpha}^{(\vec{X})}(\vec{r}) \\ &= \lim_{r\to\infty} \frac{r}{\ell_\alpha} e^{\gamma_\alpha r} \nu_\alpha^{(\vec{X})}(\vec{r}) \end{aligned} \tag{4.6}$$

In this definition the retarded and far-field natural modes are still dimensionless. A characteristic length ℓ_α (dimension meters) is introduced for giving some flexibility in normalizing the far-field modes. This length ℓ_α might be chosen as some dimension of the object independent of α. It could also be chosen as $1/\gamma_\alpha$ or $1/(d_\alpha \gamma_\alpha)$ where d_α would be another dimensionless constant.

Having chosen the form for the far-field modes there are various relations among them. Since the fields and potentials fall off as $e^{-\gamma r}/r$ then the radiation condition relations in section II can be used to find far field relations by multiplying by $e^{\gamma r}$ (the factor r being already included). Equation 2.20 can then be written for the far field natural modes as

$$[qi + \vec{e}_r \times]\vec{\nu}_{f_\alpha}^{(\vec{F}_q)} = \vec{0}$$

$$\vec{e}_r \times \vec{\nu}_{f_\alpha}^{(\vec{E})} = \vec{\nu}_{f_\alpha}^{(\vec{H})} \qquad (4.7)$$

$$\vec{e}_r \times \vec{\nu}_{f_\alpha}^{(\vec{H})} = -\vec{\nu}_{f_\alpha}^{(\vec{E})}$$

so that the electric and magnetic field natural modes are transverse in the far field and form an orthogonal set of vectors including the radial unit vector $\vec{e}_r$.

In spherical coordinates (r,θ,ϕ) we have the differential operations

$$\nabla X = \vec{e}_r \frac{\partial}{\partial r} X + \vec{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} X + \vec{e}_\phi \frac{1}{r \sin(\theta)} \frac{\partial}{\partial \phi} X$$

$$\nabla \cdot \vec{X} = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 X_r) + \frac{1}{r \sin(\theta)} \frac{\partial}{\partial \theta}(\sin(\theta) X_\theta) + \frac{1}{r \sin(\theta)} \frac{\partial}{\partial \phi} X_\phi$$

$$\nabla \times \vec{X} = \vec{e}_r\left[\frac{1}{r \sin(\theta)} \frac{\partial}{\partial \theta}(\sin(\theta) X_\phi) - \frac{1}{r \sin(\theta)} \frac{\partial}{\partial \phi} X_\theta\right] \qquad (4.8)$$

$$+ \vec{e}_\theta\left[\frac{1}{r \sin(\theta)} \frac{\partial}{\partial \phi} X_r - \frac{1}{r} \frac{\partial}{\partial r}(rX\phi)\right]$$

$$+ \vec{e}_\phi\left[\frac{1}{r} \frac{\partial}{\partial r}(rX_\theta) - \frac{1}{r} \frac{\partial}{\partial \theta} X_r\right]$$

For quantities with dependence at large r of the form $\vec{X}'(\theta,\phi)e^{-\gamma r}/r + O(e^{-\gamma r}/r^2)$ we can write these differential operators so as to include only the resulting terms of order $e^{-\gamma r}/r$, giving

$$\nabla X = -\vec{e}_r \gamma X + O(e^{-\gamma r}/r^2)$$

$$\nabla \cdot \vec{X} = -\gamma X_r + O(e^{-\gamma r}/r^2)$$

$$\nabla \times \vec{X} = \vec{e}_\theta \gamma X_\phi - \vec{e}_\phi \gamma X_\theta + O(e^{-\gamma r}/r^2) \qquad (4.9)$$

$$= -\gamma \vec{e}_r \times \vec{X} + O(e^{-\gamma r}/r^2)$$

The potential functions then have the far-field form of the Lorentz gauge condition (from equation 2.25) as

$$\lim_{r\to\infty} r e^{\gamma r}\left[-\gamma \tilde{C}_{q_r} + \frac{\gamma}{c}\tilde{\phi}_q\right] = 0$$

$$\lim_{r\to\infty} r e^{\gamma r}\left[-\gamma \tilde{A}_r + \frac{\gamma}{c}\tilde{\Phi}\right] = 0 \qquad (4.10)$$

$$\lim_{r\to\infty} r e^{\gamma r}\left[-\gamma \tilde{A}_{m_r} + \frac{\gamma}{c}\tilde{\Phi}_m\right] = 0$$

This has a natural mode form from equations 3.12 as

$$\nu_{f_\alpha}^{(\phi_q)} = a_\alpha \nu_{f_{\alpha_r}}^{(\vec{C}_q)}$$

$$\nu_{f_\alpha}^{(\Phi)} = a_\alpha \nu_{f_{\alpha_r}}^{(\vec{A})} \qquad (4.11)$$

$$\nu_{f_\alpha}^{(\Phi_m)} = a_\alpha \nu_{f_{\alpha_r}}^{(\vec{A}_m)}$$

The natural modes for the fields and potentials can be related in the far field from equations 3.16 giving

$$\vec{\nu}_{f_\alpha}^{(\vec{F}_q)} = \frac{c_\alpha}{b_\alpha}\left\{\frac{1}{a_\alpha}\,\nu_{f_\alpha}^{(\phi_q)} + [-1 - qi\vec{e}_r\times]\vec{\nu}_{f_\alpha}^{(\vec{C}_q)}\right\}$$

$$= \frac{c_\alpha}{b_\alpha}[-[\vec{\vec{I}} - \vec{e}_r\vec{e}_r] \cdot -qi\vec{e}_r\times]\vec{\nu}_{f_\alpha}^{(\vec{C}_q)} \tag{4.12}$$

and similarly

$$\vec{\nu}_{f_\alpha}^{(\vec{E})} = \frac{c_\alpha}{b_\alpha}\left\{\vec{e}_r\times\vec{e}_r\times\vec{\nu}_{f_\alpha}^{(\vec{A})} + \eta_\alpha'\vec{e}_r\times\vec{\nu}_{f_\alpha}^{(\vec{A}_m)}\right\}$$

$$\vec{\nu}_{f_\alpha}^{(\vec{H})} = \frac{c_\alpha}{b_\alpha}\left\{\eta_\alpha'\vec{e}_r\times\vec{e}_r\times\vec{\nu}_{f_\alpha}^{(\vec{A}_m)} - \vec{e}_r\times\vec{\nu}_{f_\alpha}^{(\vec{A})}\right\} \tag{4.13}$$

Thus the far field natural modes are simply related to the transverse part of the far potential natural modes.

Having defined the far fields and far potentials by multiplying by $re^{\gamma r}$ and letting $r \to \infty$ then we can write a general equation for the delta function response in the far field for the case of first order poles as

$$\tilde{\vec{U}}_{f_p}^{(\vec{X})}(\vec{r},s) = \lim_{r\to\infty} re^{\gamma r}\tilde{\vec{U}}_p^{(\vec{X})}(\vec{r},s)$$

$$= \sum_\alpha \tilde{\eta}_\alpha(\vec{e}_1,s)Y_f\vec{\nu}_{f_\alpha}^{(\vec{X})}(\vec{e}_r)(s - s_\alpha)^{-1}$$

$$+ \tilde{\vec{W}}_{f_p}^{(\vec{X})}(\vec{e}_1,\vec{r},s) \tag{4.14}$$

$$Y_f = \ell_\alpha Y = \frac{1}{d_\alpha\gamma_\alpha}Y$$

The Y coefficients for the expansions of the various electromagnetic quantities in equation 3.19 and table 1 can be carried over directly to far field and potential expansions as in equations 4.14. The length coefficient ℓ_α or the dimensionless coefficient d_α enters directly. Note that an entire function is included in equations 4.14. There are also other forms for the expansion in equation 4.14 involving operators on $\vec{v}_{f_\alpha}$ which keep it as the far natural mode for $s = s_\alpha$ but alter it for $s \neq s_\alpha$. Even with such changes the Y_f coefficients will still be the same.

V. Free Space Green's Function

In calculating the fields and potentials as integrals over the currents one uses scalar and dyadic Green's functions appropriate to free space. These are introduced in various books.[13,14,15] Note that no boundary conditions or effects of media are included. The radiation condition at infinity is included, however. This section considers some of the characteristics and representations of the free space Green's functions.

For the scalar wave equation we have the scalar Green's function which satisfies

$$\begin{gathered}[\nabla^2 - \gamma^2]\tilde{G}_o(\vec{r},\vec{r}\,';s) = -\delta(\vec{r} - \vec{r}\,') \\ \lim_{r\to\infty} r\left[\frac{\partial}{\partial r} + \gamma\right]\tilde{G}_o(\vec{r},\vec{r}\,';s) = 0\end{gathered} \tag{5.1}$$

It has the explicit representation

$$\begin{gathered}\tilde{G}_o(\vec{r},\vec{r}\,';s) = \frac{e^{-\gamma|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|} = \gamma\,\frac{e^{-\zeta}}{4\pi\zeta} = \frac{e^{-\zeta}}{4\pi|\vec{r}-\vec{r}\,'|} \\ \zeta \equiv \gamma|\vec{r} - \vec{r}\,'|\end{gathered} \tag{5.2}$$

Note that

$$\int_V \delta(\vec{r} - \vec{r}\,') = \begin{cases} 1 & \text{for } \vec{r}\,' \in V \\ 0 & \text{for } \vec{r}\,' \notin V \end{cases} \tag{5.3}$$

In terms of a spherical Bessel function $\tilde{G}_o$ can be written as[1]

$$\tilde{G}_o(\vec{r},\vec{r}\,';s) = \frac{\gamma}{4\pi}\,k_o(\zeta) = \frac{\zeta k_o(\zeta)}{4\pi|\vec{r}-\vec{r}\,'|} \tag{5.4}$$

A differential equation for $\tilde{G}_o$ in terms of ζ can then be obtained from

$$\zeta^2[\zeta k_n(\zeta)]'' - [\zeta^2 + n(n+1)]\zeta k_n(\zeta) = 0$$

$$\zeta^2 k_n''(\zeta) + 2\zeta k_n'(\zeta) - [\zeta^2 + n(n+1)]k_n(\zeta) = 0 \tag{5.5}$$

where a prime is used to indicate differentiation with respect to the argument of the Bessel function, ζ. In terms of the complex frequency s we then have

$$\left\{\left[\frac{c}{|\vec{r} - \vec{r}\,'|}\right]^2 \frac{\partial^2}{\partial s^2} - 1\right\}\tilde{G}_o(\vec{r},\vec{r}\,';s) = 0$$

$$\left[\frac{\partial^2}{\partial \zeta^2} - 1\right]\tilde{G}_o(\vec{r},\vec{r}\,';s) = 0 \tag{5.6}$$

A first order differential equation with respect to the complex frequency s can be obtained by inspection as

$$\left[\frac{c}{|\vec{r} - \vec{r}\,'|} \frac{\partial}{\partial s} + 1\right]\tilde{G}_o(\vec{r},\vec{r}\,';s) = 0$$

$$\left[\frac{\partial}{\partial \zeta} + 1\right]\tilde{G}_o(\vec{r},\vec{r}\,';s) = 0 \tag{5.7}$$

(with $\vec{r} - \vec{r}\,'$ assumed constant). Note that $\tilde{G}_o$ and its derivatives with respect to s are never zero in the finite s plane, excluding s = 0, for $|\vec{r} - \vec{r}\,'|$ nonzero but finite.

For the vector wave equation we have the dyadic Green's function for free space which satisfies

$$[\nabla\times\nabla\times + \gamma^2]\overset{\leftrightarrow}{G}_o(\vec{r},\vec{r}\,';s) = \delta(\vec{r} - \vec{r}\,')\overset{\leftrightarrow}{I}$$

$$\lim_{r\to\infty} r[\nabla\times + \gamma\vec{e}_r\times]\overset{\leftrightarrow}{G}_o(\vec{r},\vec{r}\,';s) = \overset{\leftrightarrow}{0} \tag{5.8}$$

where $\overset{\leftrightarrow}{I}$ is the identity dyadic. It has a representation in terms of the scalar Green's function as

$$\tilde{\overleftrightarrow{G}}_o(\vec{r},\vec{r}\,';s) = \left[1 - \frac{1}{\gamma^2}\nabla\nabla\cdot\right][\tilde{G}_o(\vec{r},\vec{r}\,';s)\overleftrightarrow{I}]$$

$$= \left[\overleftrightarrow{I} - \frac{1}{\gamma^2}\nabla\nabla\right]\tilde{G}_o(\vec{r},\vec{r}\,';s) \qquad (5.9)$$

Both the scalar and dyadic free space Green's functions are symmetric in the forms

$$\tilde{G}_o(\vec{r},\vec{r}\,';s) = \tilde{G}_o(\vec{r}\,',\vec{r};s)$$

$$\tilde{\overleftrightarrow{G}}_o(\vec{r},\vec{r}\,';s) = \tilde{\overleftrightarrow{G}}_o(\vec{r}\,',\vec{r};s) \qquad (5.10)$$

$$\tilde{\overleftrightarrow{G}}_o(\vec{r},\vec{r}\,';s) = \tilde{\overleftrightarrow{G}}_o^T(\vec{r},\vec{r}\,';s)$$

where the superscript T indicates the dyadic or matrix transpose.

In obtaining a more explicit form for the dyadic Green's function introduce a spherical coordinate system R, θ_R, ϕ_R centered on r = r' with

$$R \equiv |\vec{r} - \vec{r}\,'| = \frac{\zeta}{\gamma} \qquad (5.11)$$

The polar and azimuthal angles θ_R and ϕ_R are given from some convenient choice of an axis through the origin R = 0 and a convenient choice of the direction $\phi_R = 0$ perpendicular to this axis. Using the differential operators for spherical coordinates from equations 4.8 we have

$$\nabla\tilde{G}_o(\vec{r},\vec{r}\,';s) = \vec{e}_R\frac{\partial}{\partial R}\tilde{G}_o(\vec{r},\vec{r}\,';s)$$

$$= \vec{e}_R\frac{\gamma^2}{4\pi}\frac{\partial}{\partial\zeta}\frac{e^{-\zeta}}{\zeta} = \vec{e}_R\frac{\gamma^2}{4\pi}k_o'(\zeta)$$

$$= -\vec{e}_R\frac{\gamma^2}{4\pi}k_1(\zeta) = \vec{e}_R\frac{\gamma^2}{4\pi}[-\zeta^{-2} - \zeta^{-1}]e^{-\zeta}$$

$$= \vec{e}_R\frac{1}{4\pi}[-R^{-2} - \gamma R^{-1}]e^{-\gamma R} \qquad (5.12)$$

The gradient of the scalar Green's function itself is important for some integrals as indicated elsewhere. There are alternate forms in which this result is found such as

$$\nabla \times \tilde{\vec{\vec{G}}}_o(\vec{r},\vec{r}\,';s) = \nabla \times [\tilde{G}_o(\vec{r},\vec{r}\,';s)\vec{\vec{I}}]$$

$$= \nabla\tilde{G}_o(\vec{r},\vec{r}\,';s) \times \vec{\vec{I}}$$

$$= -\frac{\gamma^2}{4\pi}k_1(\zeta)\vec{e}_R \times \vec{\vec{I}} \tag{5.13}$$

where

$$\vec{\vec{I}} = \vec{e}_R\vec{e}_R + \vec{e}_{\theta_R}\vec{e}_{\theta_R} + \vec{e}_{\phi_R}\vec{e}_{\phi_R}$$

$$\vec{e}_R \times \vec{\vec{I}} = -\vec{\vec{I}} \times \vec{e}_R \tag{5.14}$$

$$= \vec{e}_{\phi_R}\vec{e}_{\theta_R} - \vec{e}_{\theta_R}\vec{e}_{\phi_R}$$

By inspection a first order differential equation for $\nabla\tilde{G}_o$ and its other forms as above is

$$\left\{\left[\frac{c}{R} + \frac{c^2}{sR^2}\right]\frac{\partial}{\partial s} + 1\right\}\nabla\tilde{G}_o(\vec{r},\vec{r}\,';s) = \vec{0}$$

$$\left\{\left[1 + \frac{1}{\zeta}\right]\frac{\partial}{\partial \zeta} + 1\right\}\nabla\tilde{G}_o(\vec{r},\vec{r}\,';s) = \vec{0} \tag{5.15}$$

where $\vec{e}_R$ and R (and thus $\vec{r} - \vec{r}\,'$) are assumed constant. Similarly a second order differential equation can be obtained as

$$\left\{c^2\frac{\partial^2}{\partial s^2} + 2cR\frac{\partial}{\partial s} + R^2\right\}\nabla\tilde{G}_o(\vec{r},\vec{r}\,';s) = \vec{0}$$

$$\left\{\frac{\partial^2}{\partial \zeta^2} + 2\frac{\partial}{\partial \zeta} + 1\right\}\nabla\tilde{G}_o(\vec{r},\vec{r}\,';s) = \vec{0} \tag{5.16}$$

Using equation 5.5 (with R constant) another second order equation is

$$\left\{\zeta \frac{\partial^2}{\partial\zeta^2} - 2\frac{\partial}{\partial\zeta} - \zeta\right\}\nabla\tilde{G}_o(\vec{r},\vec{r}\,';s) = \vec{0} \tag{5.17}$$

Next we need[13]

$$\begin{aligned} \nabla\vec{e}_R &= \frac{1}{R}[\vec{e}_{\theta_R}\vec{e}_{\theta_R} + \vec{e}_{\phi_R}\vec{e}_{\phi_R}] = \frac{1}{R}[\vec{\vec{I}} - \vec{e}_R\vec{e}_R] \\ \nabla(k_1(\zeta)\vec{e}_R) &= [\nabla k_1(\zeta)]\vec{e}_R + k_1(\zeta)\nabla\vec{e}_R \end{aligned} \tag{5.18}$$

From these we find

$$\begin{aligned} \nabla\nabla\tilde{G}_o(\vec{r},\vec{r}\,';s) &= -\frac{\gamma^3}{4\pi}\frac{\partial}{\partial\zeta}k_1(\zeta)\vec{e}_R\vec{e}_R - \frac{\gamma^3}{4\pi}\frac{k_1(\zeta)}{\zeta}[\vec{\vec{I}} - \vec{e}_R\vec{e}_R] \\ &= \frac{\gamma^3}{4\pi}\Big\{[2\zeta^{-3} + 2\zeta^{-2} + \zeta^{-1}]e^{-\zeta}\vec{e}_R\vec{e}_R \\ &\quad + [-\zeta^{-3} - \zeta^{-2}]e^{-\zeta}[\vec{\vec{I}} - \vec{e}_R\vec{e}_R]\Big\} \\ &= \frac{\gamma^3}{4\pi}\Big\{[-\zeta^{-3} - \zeta^{-2}]e^{-\zeta}\vec{\vec{I}} + [3\zeta^{-3} + 3\zeta^{-2} + \zeta^{-1}]e^{-\zeta}\vec{e}_R\vec{e}_R\Big\} \end{aligned} \tag{5.19}$$

This gives the dyadic Green's function as

$$\begin{aligned} \vec{\vec{G}}_o(\vec{r},\vec{r}\,';s) &= \frac{\gamma}{4\pi}\Big\{[\zeta^{-3} + \zeta^{-2} + \zeta^{-1}]e^{-\zeta}\vec{\vec{I}} + [-3\zeta^{-3} - 3\zeta^{-2} - \zeta^{-1}]e^{-\zeta}\vec{e}_R\vec{e}_R\Big\} \\ &= \frac{\gamma}{4\pi}\Big\{[-2\zeta^{-3} - 2\zeta^{-2}]e^{-\zeta}\vec{e}_R\vec{e}_R + [\zeta^{-3} + \zeta^{-2} + \zeta^{-1}]e^{-\zeta}[\vec{\vec{I}} - \vec{e}_R\vec{e}_R]\Big\} \\ &= \frac{\gamma}{4\pi}\Big\{-\frac{2}{\zeta}k_1(\zeta)\vec{e}_R\vec{e}_R + \left[k_o(\zeta) + \frac{k_1(\zeta)}{\zeta}\right][\vec{\vec{I}} - \vec{e}_R\vec{e}_R]\Big\} \\ &= \frac{\gamma}{4\pi}\Big\{-\frac{2}{\zeta}k_1(\zeta)\vec{e}_R\vec{e}_R - \frac{[\zeta k_1(\zeta)]'}{\zeta}[\vec{\vec{I}} - \vec{e}_R\vec{e}_R]\Big\} \end{aligned} \tag{5.20}$$

Considering the dyadic Green's function as the sum of a radial and a transverse part we have

$$\tilde{\overleftrightarrow{G}}_{o_R}(\vec{r},\vec{r}';s) = -\frac{\gamma}{2\pi}\frac{k_1(\zeta)}{\zeta}\vec{e}_R\vec{e}_R$$

$$\tilde{\overleftrightarrow{G}}_{o_T}(\vec{r},\vec{r}';s) = -\frac{\gamma}{4\pi}\frac{[\zeta k_1(\zeta)]'}{\zeta}[\overleftrightarrow{I} - \vec{e}_R\vec{e}_R] \tag{5.21}$$

For constant R we can find first order differential equations with respect to the complex frequency (with R constant) as

$$\left\{\zeta\frac{\partial}{\partial\zeta} - \frac{\zeta^2 + 2\zeta + 2}{\zeta + 1}\right\}\tilde{\overleftrightarrow{G}}_{o_R}(\vec{r},\vec{r}';s) = \overleftrightarrow{0}$$

$$\left\{\zeta\frac{\partial}{\partial\zeta} - \frac{\zeta^3 - \zeta^2 - 2\zeta - 1}{\zeta^2 + \zeta + 1}\right\}\tilde{\overleftrightarrow{G}}_{o_T}(\vec{r},\vec{r}';s) = \overleftrightarrow{0} \tag{5.22}$$

These have rather complicated coefficients. Second order differential equations (for constant R) can be derived from the defining differential equations for spherical Bessel functions as

$$\left\{\zeta^2\frac{\partial^2}{\partial\zeta^2} + 2\zeta\frac{\partial}{\partial\zeta} - \zeta^2 + 2\right\}\tilde{\overleftrightarrow{G}}_{o_R}(\vec{r},\vec{r}';s) = \overleftrightarrow{0}$$

$$\left\{\zeta^2[\zeta^2 + 2]\frac{\partial^2}{\partial\zeta^2} + 4\zeta\frac{\partial}{\partial\zeta} - [\zeta^2 + 2]\right\}\tilde{\overleftrightarrow{G}}_{o_T}(\vec{r},\vec{r}';s) = \overleftrightarrow{0} \tag{5.23}$$

For use in calculating the potentials and fields at large distances we need the far Green's functions. Considering first the scalar Green's function we define the far scalar Green's function following the procedures in section IV as

$$\tilde{g}_{o_f}(\vec{e}_r,\vec{r}';s) \equiv r\tilde{G}_{o_f} = \lim_{r\to\infty} re^{\gamma r}\tilde{G}_o(\vec{r},\vec{r}';s)$$

$$= \frac{e^{\gamma\vec{e}_r\cdot\vec{r}'}}{4\pi} \tag{5.24}$$

For the gradient of the scalar Green's function in the far field we have

$$r(\nabla\tilde{G}_o(\vec{r},\vec{r}\,';s))_f \equiv \lim_{r\to\infty} re^{\gamma r}\nabla\tilde{G}_o(\vec{r},\vec{r}\,';s)$$

$$= -\vec{e}_r\gamma\tilde{g}_{o_f}(\vec{e}_r,\vec{r}\,';s)$$

$$r(\nabla\tilde{G}_o(\vec{r},\vec{r}\,';s)\times\vec{\vec{I}})_f = r(\nabla\times\tilde{\vec{\vec{G}}}_o(\vec{r},\vec{r}\,';s))_f \tag{5.25}$$

$$= -\gamma\tilde{g}_{o_f}(\vec{e}_r,\vec{r}\,';s)\vec{e}_r\times\vec{\vec{I}}$$

For the far dyadic Green's function we have

$$\tilde{\vec{\vec{g}}}_{o_f}(\vec{e}_r,\vec{r}\,';s) \equiv r\tilde{\vec{\vec{G}}}_{o_f}(\vec{r},\vec{r}\,';s) = \lim_{r\to\infty} re^{\gamma r}\tilde{\vec{\vec{G}}}_o(\vec{r},\vec{r}\,';s)$$

$$= \tilde{g}_{o_f}(\vec{e}_r,\vec{r}\,';s)[\vec{\vec{I}} - \vec{e}_R\vec{e}_R] \tag{5.26}$$

Note that as defined here the various far Green's functions are conveniently dimensionless.

As in the case of the fields and potentials in section IV the retarded forms of the free space Green's functions are obtained by multiplying by $e^{\gamma r}$ to give

$$\tilde{G}_{o_{ret}}(\vec{r},\vec{r}\,';s) \equiv e^{\gamma r}\tilde{G}_o(\vec{r},\vec{r}\,';s)$$

$$(\nabla\tilde{G}_o(\vec{r},\vec{r}\,';s))_{ret} \equiv e^{\gamma r}\nabla\tilde{G}_o(\vec{r},\vec{r}\,';s)$$

$$(\nabla\tilde{G}_o(\vec{r},\vec{r}\,';s)\times\vec{\vec{I}})_{ret} = (\nabla\times\tilde{\vec{\vec{G}}}_o(\vec{r},\vec{r}\,';s))_{ret}$$

$$\equiv e^{\gamma r}\nabla\tilde{G}_o(\vec{r},\vec{r}\,';s) \times \overset{\leftrightarrow}{I} = e^{\gamma r}\nabla \times \tilde{\overset{\leftrightarrow}{G}}_o(\vec{r},\vec{r}\,';s) \tag{5.27}$$

$$\tilde{\overset{\leftrightarrow}{G}}_{o_{ret}}(\vec{r},\vec{r}\,';s) \equiv e^{\gamma r}\tilde{\overset{\leftrightarrow}{G}}_o(\vec{r},\vec{r}\,';s)$$

$$r\tilde{\vec{A}}_f = \mu_o \langle \tilde{g}_{o_f}, \tilde{\vec{J}} \rangle$$

$$r\tilde{\Phi}_f = \frac{1}{\varepsilon_o} \langle \tilde{g}_{o_f}, \tilde{\rho} \rangle$$

$$r\tilde{\vec{A}}_{m_f} = \varepsilon_o \langle \tilde{g}_{o_f}, \tilde{\vec{J}}_m \rangle$$

$$r\tilde{\Phi}_{m_f} = \frac{1}{\mu_o} \langle \tilde{g}_{o_f}, \tilde{\rho}_m \rangle \qquad (6.3)$$

$$r\tilde{\vec{C}}_{q_f} = \mu_o \langle \tilde{g}_{o_f}, \tilde{\vec{K}}_q \rangle$$

$$r\tilde{\phi}_{q_f} = \frac{1}{\varepsilon_o} \langle \tilde{g}_{o_f}, \tilde{Q}_q \rangle$$

The far fields are

$$r\tilde{\vec{E}}_f = -s\mu_o \langle \tilde{\vec{\vec{g}}}_{o_f} ; \tilde{\vec{J}} \rangle + \frac{s}{c} \langle \tilde{g}_{o_f} \vec{e}_r \overset{\times}{,} \tilde{\vec{J}}_m \rangle$$

$$r\tilde{\vec{H}}_f = -\frac{s}{c} \langle \tilde{g}_{o_f} \vec{e}_r \overset{\times}{,} \tilde{\vec{J}} \rangle - s\varepsilon_o \langle \tilde{\vec{\vec{g}}}_{o_f} ; \tilde{\vec{J}}_m \rangle \qquad (6.4)$$

$$r\tilde{\vec{F}}_{q_f} = -s\mu_o [\vec{\vec{I}} - \vec{e}_r \vec{e}_r + qi\vec{e}_r \times \vec{\vec{I}}] \cdot \langle \tilde{g}_{o_f}, \tilde{\vec{K}}_q \rangle$$

VII. Field and Potential Natural Modes from Current Natural Modes

Now we are in a position to express the field and potential natural modes as integrals over the current or charge density natural modes. Including the constants introduced in the differential equations for the natural modes in section III for normalization purposes the potential and field natural modes are

$$\vec{\nu}_\alpha^{(\vec{A})} = b_\alpha \gamma_\alpha^2 \left\langle \tilde{G}_{o_\alpha}, \vec{\nu}_\alpha^{(\vec{J})} \right\rangle$$

$$\nu_\alpha^{(\Phi)} = b_\alpha \gamma_\alpha^2 \left\langle \tilde{G}_{o_\alpha}, \nu_\alpha^{(\rho)} \right\rangle$$

$$\vec{\nu}_\alpha^{(\vec{A}_m)} = b_\alpha \gamma_\alpha^2 \left\langle \tilde{G}_{o_\alpha}, \vec{\nu}_\alpha^{(\vec{J}_m)} \right\rangle$$

$$\nu_\alpha^{(\Phi_m)} = b_\alpha \gamma_\alpha^2 \left\langle \tilde{G}_{o_\alpha}, \nu_\alpha^{(\rho_m)} \right\rangle$$

$$\vec{\nu}_\alpha^{(\vec{C}_q)} = b_\alpha \gamma_\alpha^2 \left\langle \tilde{G}_{o_\alpha}, \vec{\nu}_\alpha^{(\vec{K}_q)} \right\rangle \qquad (7.1)$$

$$\nu_\alpha^{(\phi_q)} = b_\alpha \gamma_\alpha^2 \left\langle \tilde{G}_{o_\alpha}, \nu_\alpha^{(Q_q)} \right\rangle$$

$$\vec{\nu}_\alpha^{(\vec{E})} = -c_\alpha \gamma_\alpha^2 \left\langle \tilde{\vec{\vec{G}}}_{o_\alpha} ; \vec{\nu}_\alpha^{(\vec{J})} \right\rangle - c_\alpha \eta_\alpha' \gamma_\alpha \left\langle \nabla \tilde{G}_{o_\alpha} \overset{\times}{,} \vec{\nu}_\alpha^{(\vec{J}_m)} \right\rangle$$

$$\vec{\nu}_\alpha^{(\vec{H})} = c_\alpha \gamma_\alpha \left\langle \nabla \tilde{G}_{o_\alpha} \overset{\times}{,} \vec{\nu}_\alpha^{(\vec{J})} \right\rangle - c_\alpha \eta_\alpha' \gamma_\alpha^2 \left\langle \tilde{\vec{\vec{G}}}_{o_\alpha} ; \vec{\nu}_\alpha^{(\vec{J}_m)} \right\rangle$$

$$\vec{\nu}_\alpha^{(\vec{F}_q)} = -c_\alpha \gamma_\alpha^2 \left\langle \tilde{\vec{\vec{G}}}_{o_\alpha} ; \vec{\nu}_\alpha^{(\vec{K}_q)} \right\rangle + qic_\alpha \gamma_\alpha \left\langle \nabla \tilde{G}_{o_\alpha} \overset{\times}{,} \vec{\nu}_\alpha^{(\vec{K}_q)} \right\rangle$$

where

$$\tilde{G}_{o_\alpha} \equiv \tilde{G}_o(\vec{r},\vec{r}\,';s_\alpha)$$

$$\nabla\tilde{G}_{o_\alpha} \equiv \nabla\tilde{G}_o(\vec{r},\vec{r}\,';s_\alpha) \tag{7.2}$$

$$\tilde{\vec{\vec{G}}}_{o_\alpha} \equiv \tilde{\vec{\vec{G}}}_o(\vec{r},\vec{r}\,';s_\alpha)$$

The far potential and field natural modes are found by multiplying both sides of equations 7.1 by $(r/\ell_\alpha)e^{\gamma_\alpha r}$ and letting $r \to \infty$. This gives

$$\vec{\nu}_{f_\alpha}^{(\vec{A})} = \frac{b_\alpha\gamma_\alpha^2}{\ell_\alpha}\left\langle\tilde{g}_{o_{f_\alpha}},\nu_\alpha^{(J)}\right\rangle$$

$$\nu_{f_\alpha}^{(\Phi)} = \frac{b_\alpha\gamma_\alpha^2}{\ell_\alpha}\left\langle\tilde{g}_{o_{f_\alpha}},\nu_\alpha^{(\rho)}\right\rangle$$

$$\vec{\nu}_{f_\alpha}^{(\vec{A}_m)} = \frac{b_\alpha\gamma_\alpha^2}{\ell_\alpha}\left\langle\tilde{g}_{o_{f_\alpha}},\vec{\nu}_\alpha^{(\vec{J}_m)}\right\rangle$$

$$\nu_{f_\alpha}^{(\Phi_m)} = \frac{b_\alpha\gamma_\alpha^2}{\ell_\alpha}\left\langle\tilde{g}_{o_{f_\alpha}},\nu_\alpha^{(\rho_m)}\right\rangle$$

$$\vec{\nu}_{f_\alpha}^{(\vec{C}_q)} = \frac{b_\alpha\gamma_\alpha^2}{\ell_\alpha}\left\langle\tilde{g}_{o_{f_\alpha}},\vec{\nu}_\alpha^{(\vec{K}_q)}\right\rangle$$

$$\nu_{f_\alpha}^{(\phi_q)} = \frac{b_\alpha\gamma_\alpha^2}{\ell_\alpha}\left\langle\tilde{g}_{o_{f_\alpha}},\nu_\alpha^{(Q_q)}\right\rangle$$

$$\vec{\nu}_{f_\alpha}^{(\vec{E})} = -\frac{c_\alpha\gamma_\alpha^2}{\ell_\alpha}\left\langle\tilde{\vec{\vec{g}}}_{o_{f_\alpha}};\vec{\nu}_\alpha^{(\vec{J})}\right\rangle + \frac{c_\alpha\eta_\alpha'\gamma_\alpha^2}{\ell_\alpha}\left\langle\tilde{g}_{o_{f_\alpha}}\vec{e}_r \times ; \vec{\nu}_\alpha^{(\vec{J}_m)}\right\rangle$$

$$\vec{\nu}_{f_\alpha}^{(\vec{H})} = -\frac{c_\alpha \gamma_\alpha^2}{\ell_\alpha}\left\langle \tilde{g}_{o_{f_\alpha}} \vec{e}_r \times ; \vec{\nu}_\alpha^{(\vec{J})} \right\rangle - \frac{c_\alpha \eta'_\alpha \gamma_\alpha^2}{\ell_\alpha}\left\langle \tilde{\vec{\vec{g}}}_{o_{f_\alpha}} ; \vec{\nu}_\alpha^{(\vec{J}_m)} \right\rangle \tag{7.3}$$

$$\vec{\nu}_{f_\alpha}^{(\vec{F}_q)} = -\frac{c_\alpha \gamma_\alpha^2}{\ell_\alpha}[\vec{\vec{I}} - \vec{e}_r\vec{e}_r + qi\vec{e}_r \times \vec{\vec{I}}] \cdot \left\langle \tilde{g}_{o_{f_\alpha}}, \vec{\nu}_\alpha^{(\vec{K}_q)} \right\rangle$$

where

$$\tilde{g}_{o_{f_\alpha}} \equiv \tilde{g}_{o_f}(\vec{e}_r, \vec{r}'; s_\alpha)$$

$$\tilde{\vec{\vec{g}}}_{o_{f_\alpha}} \equiv \tilde{\vec{\vec{g}}}_{o_f}(\vec{e}_r, \vec{r}'; s_\alpha) \tag{7.4}$$

To round out the integral formulas for the natural modes for fields and potentials consider some common idealizations for current and charge densities. Surface current and charge densities can be defined through

$$\tilde{\vec{K}}_q(\vec{r}, s) = \tilde{\vec{K}}_{s_q}(\vec{r}_s, s)\,\delta(\vec{n}_s \cdot (\vec{r} - \vec{r}_s))$$

$$\vec{n}_s(\vec{r}_s) \cdot \tilde{\vec{K}}_{s_q}(\vec{r}_s, s) = 0$$

$$\tilde{Q}_q(\vec{r}, s) = \tilde{Q}_{s_q}(\vec{r}_s, s)\,\delta(\vec{n}_s \cdot (\vec{r} - \vec{r}_s)) \tag{7.5}$$

$$\nabla_s \cdot \tilde{\vec{K}}_{s_q}(\vec{r}_s, s) + s\tilde{Q}_{s_q}(\vec{r}_s, s) = 0$$

where the subscript s refers to the coordinates of the surface of interest or the surface electromagnetic quantities and where $\vec{n}_s$ is the unit vector normal to the surface. Similarly if the current is confined to a line path we have

$$\tilde{\vec{K}}_q(\vec{r},s) = \tilde{\vec{K}}_{\ell_q}(\vec{r}_\ell,s)\,\delta(\vec{n}_\ell \cdot (\vec{r}-\vec{r}_\ell))\,\delta((\vec{\tau}_\ell \times \vec{n}_\ell)\cdot(\vec{r}-\vec{r}_\ell))$$

$$\vec{n}_\ell(\vec{r}_\ell) \cdot \tilde{\vec{K}}_{\ell_q}(\vec{r}_\ell,s) = 0 \;, \qquad (\vec{\tau}_\ell \times \vec{n}_\ell) \cdot \tilde{\vec{K}}_{\ell_q}(\vec{r}_\ell,s) = 0$$

$$\tilde{Q}_q(\vec{r},s) = \tilde{Q}_{\ell_q}(\vec{r}_\ell,s)\,\delta(\vec{n}_\ell \cdot (\vec{r}-\vec{r}_\ell))\,\delta((\vec{\tau}_\ell \times \vec{n}_\ell)\cdot(\vec{r}-\vec{r}_\ell)) \tag{7.6}$$

$$\nabla_\ell \cdot \tilde{\vec{K}}_{\ell_q}(\vec{r}_\ell,s) + s\tilde{Q}_{\ell_q}(\vec{r}_\ell,s) = 0$$

where the subscript ℓ refers to the coordinates of the line of interest or the line electromagnetic quantities (current and charge per unit length), where $\vec{n}_\ell$ is a convenient unit normal to the line (path) and $\vec{\tau}_\ell$ is the unit tangent vector of the path. The delta functions should be interpreted such that $\delta(\vec{r} - \vec{r}\,')$ is one (or zero) when integrated over a volume (three dimensions) while each of the delta functions in equations 7.5 and 7.6 has a scalar argument and as such integrate to one (or zero) when integrated in the direction (one dimension) of the vector which dot (scalar) multiplies $\vec{r} - \vec{r}\,'$, i.e., $\vec{n}_s$, $\vec{n}_\ell$, and $\vec{\tau}_\ell \times \vec{n}_\ell$.

The natural modes for current and charge densities can be similarly modified to include idealized surface and line current and charge densities. In order to keep the modes dimensionless, factors of γ_α and γ_α^2 are introduced as

$$\vec{\nu}_\alpha^{(\vec{K}_q)}(\vec{r}) = \vec{\nu}_\alpha^{(\vec{K}_{s_q})}(\vec{r}_s)\,\gamma_\alpha^{-1}\,\delta(\vec{n}_s \cdot (\vec{r}-\vec{r}_s))$$

$$\vec{n}_s(\vec{r}_s) \cdot \vec{\nu}_\alpha^{(\vec{K}_{s_q})}(\vec{r}_s) = 0$$

$$\nu_\alpha^{(Q_q)}(\vec{r}) = \nu_\alpha^{(Q_{s_q})}(\vec{r}_s)\,\gamma_\alpha^{-1}\,\delta(\vec{n}_s \cdot (\vec{r}-\vec{r}_s)) \tag{7.7}$$

$$\frac{a_\alpha}{\gamma_\alpha}\nabla_s \cdot \vec{\nu}_\alpha^{(\vec{K}_{s_q})}(\vec{r}_s) + \nu_\alpha^{(Q_{s_q})}(\vec{r}_s) = 0$$

and

$$\vec{\nu}_\alpha^{(\vec{K}_q)}(\vec{r}) = \vec{\nu}_\alpha^{(\vec{K}_{\ell_q})}(\vec{r}_\ell)\gamma_\alpha^{-2}\delta(\vec{n}_\ell \cdot (\vec{r} - \vec{r}_\ell))\delta((\vec{\tau}_\ell \times \vec{n}_\ell) \cdot (\vec{r} - \vec{r}_\ell))$$

$$\vec{n}_\ell(\vec{r}_\ell) \cdot \vec{\nu}_\alpha^{(\vec{K}_{\ell_q})}(\vec{r}_\ell) = 0 \;, \quad (\vec{\tau}_\ell \times \vec{n}_\ell) \cdot \vec{\nu}_\alpha^{(\vec{K}_{\ell_q})}(\vec{r}_\ell) = 0$$

$$\nu_\alpha^{(Q_q)}(\vec{r}) = \nu_\alpha^{(Q_{s_q})}(\vec{r}_s)\gamma_\alpha^{-2}\delta(\vec{n}_\ell \cdot (\vec{r} - \vec{r}_\ell))\delta((\vec{\tau}_\ell \times \vec{n}_\ell) \cdot (\vec{r} - \vec{r}_\ell)) \tag{7.8}$$

$$\frac{a_\alpha}{\gamma_\alpha} \nabla_\ell \cdot \vec{\nu}_\alpha^{(\vec{K}_{\ell_q})}(\vec{r}_\ell) + \nu_\alpha^{(Q_{\ell_q})}(\vec{r}_\ell) = 0$$

Equations 7.5 through 7.8 are written in terms of the combined current and charge densities but also apply separately to the electric and magnetic current and charge densities as well due to the way they related through equations 2.14 and 2.17.

In calculating the field and potential natural modes as integrals over surface or line current density natural modes the integrals reduce from volume to surface or line integrals respectively, and factors of γ_α^{-1} and γ_α^{-2} respectively are included with the coefficients. For the case of surface current and charge densities then the modes for the combined potentials and fields have the representations

$$\vec{\nu}_\alpha^{(\vec{C}_q)} = b_\alpha \gamma_\alpha \left\langle \tilde{G}_{o_\alpha}, \vec{\nu}_\alpha^{(\vec{K}_{s_q})} \right\rangle$$

$$\nu_\alpha^{(\phi_q)} = b_\alpha \gamma_\alpha \left\langle \tilde{G}_{o_\alpha}, \nu_\alpha^{(Q_{s_q})} \right\rangle$$

$$\vec{\nu}_\alpha^{(\vec{F}_q)} = -c_\alpha \gamma_\alpha \left\langle \tilde{\vec{\vec{G}}}_{o_\alpha} ; \vec{\nu}_\alpha^{(\vec{K}_{s_q})} \right\rangle + qic_\alpha \left\langle \nabla \tilde{G}_{o_\alpha} \overset{\times}{,} \vec{\nu}_\alpha^{(\vec{K}_{s_q})} \right\rangle$$

$$\vec{\nu}_{f_\alpha}^{(\vec{C}_q)} = \frac{b_\alpha \gamma_\alpha}{\ell_\alpha} \left\langle \tilde{g}_{o_{f_\alpha}}, \vec{\nu}_\alpha^{(\vec{K}_{s_q})} \right\rangle$$

$$\nu_{f_\alpha}^{(\phi_q)} = \frac{b_\alpha \gamma_\alpha}{\ell_\alpha} \left\langle \tilde{g}_{o_{f_\alpha}}, \nu_\alpha^{(Q_{s_q})} \right\rangle \tag{7.9}$$

$$\vec{\nu}_{f_\alpha}^{(\vec{F}_q)} = -\frac{c_\alpha \gamma_\alpha}{\ell_\alpha} [\vec{\vec{I}} - \vec{e}_r \vec{e}_r + qi\vec{e}_r \times \vec{\vec{I}}] \cdot \left\langle \tilde{g}_{o_{f_\alpha}}, \vec{\nu}_\alpha^{(\vec{K}_{s_q})} \right\rangle$$

For the case of line current and charge densities the combined potentials and fields have the representations

$$\vec{\nu}_\alpha^{(\vec{C}_q)} = b_\alpha \left\langle \tilde{G}_{o_\alpha}, \vec{\nu}_\alpha^{(\vec{K}_{\ell_q})} \right\rangle$$

$$\nu_\alpha^{(\phi_q)} = b_\alpha \left\langle \tilde{G}_{o_\alpha}, \nu_\alpha^{(Q_{\ell_q})} \right\rangle$$

$$\vec{\nu}_\alpha^{(\vec{F}_q)} = -c_\alpha \left\langle \tilde{\vec{\vec{G}}}_{o_\alpha} ; \vec{\nu}_\alpha^{(\vec{K}_{\ell_q})} \right\rangle + qic_\alpha \gamma_\alpha^{-1} \left\langle \nabla \tilde{G}_{o_\alpha} \times \vec{\nu}_\alpha^{(\vec{K}_{\ell_q})} \right\rangle$$

$$\vec{\nu}_{f_\alpha}^{(\vec{C}_q)} = \frac{b_\alpha}{\ell_\alpha} \left\langle \tilde{g}_{o_{f_\alpha}}, \vec{\nu}_\alpha^{(\vec{K}_{\ell_q})} \right\rangle \tag{7.10}$$

$$\nu_{f_\alpha}^{(\phi_q)} = \frac{b_\alpha}{\ell_\alpha} \left\langle \tilde{g}_{o_{f_\alpha}}, \nu_\alpha^{(Q_{\ell_q})} \right\rangle$$

$$\vec{\nu}_{f_\alpha}^{(\vec{F}_q)} = -\frac{c_\alpha}{\ell_\alpha} [\vec{\vec{I}} - \vec{e}_r \vec{e}_r + qi\vec{e}_r \times \vec{\vec{I}}] \cdot \left\langle \tilde{g}_{o_{f_\alpha}}, \vec{\nu}_\alpha^{(\vec{K}_{\ell_q})} \right\rangle$$

These formulas can be split to give the electric and magnetic quantities separately or one can simply multiply the right side of equations 7.1 and 7.3 by γ_α^{-1} and use surface modes and integrals, or multiply the right side of the same equations by γ_α^{-2} and use line modes and integrals in place of the volume modes and integrals. Note that with the constants a_α, b_α, c_α, and ℓ_α (or d_α) to be chosen at convenience one can still include any γ_α dependence in these coefficients as one wishes.

Including surface and line current densities with the definitions in this section the results for the additional coefficients for the pole terms for the various electromagnetic quantities as in table 1 can still be used. However the coefficient (Σ) used in the expansion of the current density as in equations 3.1 in defining the normalized response needs to be generalized. Consider then the three forms

$$\tilde{\vec{J}}_p(\vec{r},s) = E_o \Sigma \tilde{\vec{V}}_p^{(\vec{J})}(\vec{r},s) \quad \text{volume density}$$

$$\tilde{\vec{J}}_{s_p}(\vec{r},s) = E_o Z_o^{-1} \tilde{\vec{V}}_p^{(\vec{J}_s)}(\vec{r},s) \quad \text{surface density} \tag{7.11}$$

$$\tilde{\vec{J}}_{\ell_p}(\vec{r},s) = E_o \ell Z_o^{-1} \tilde{\vec{V}}_p^{(\vec{J}_\ell)}(\vec{r},s) \quad \text{line density}$$

where Σ (perhaps equal to $(\ell Z_o)^{-1}$) has dimensions Sm^{-1}, Z_o (impedance of free space) has dimensions S, and ℓZ_o^{-1} has dimensions Sm. The characteristic length ℓ (dimension m) may be some specific size of the object (radius, overall length, etc.).

One can then summarize the coefficient relations for the various types of current densities and natural modes as in table 2.

		Normalizing ceofficient	Additional coefficient for natural modes (see table 1)	Additional coefficient for far natural modes
current densities	$\vec{K}_q$	Σ (or $(\ell Z_o)^{-1}$)	Y	$Y_f = \ell_\alpha Y$
surface current densities	$\vec{K}_{s_q}$	Z_o^{-1}	Y	$Y_f = \ell_\alpha Y$
line current densities	$\vec{K}_{\ell_q}$	ℓZ_o^{-1}	Y	$Y_f = \ell_\alpha Y$

Table 2. Coefficients for Use with Various Forms of Current Densities and Natural Modes

VIII. Alternate Forms for Pole Terms for Fields and Potentials

In the previous sections we have taken the pole terms in the SEM expansion of the current density (equations 3.1) and extended the result to include fields and potentials by replacing the current density natural modes by field and potential (and far field and far potential) natural modes with appropriate additional coefficients. In this process the coupling coefficients $\tilde{\eta}_\alpha$ have not been changed in form and various previously developed forms for $\tilde{\eta}_\alpha$ (such as in equations 3.6: classes 1 and 2) can all be used in the formulas in this note.

In a form analogous to the class 2 (convolution) coupling coefficients one can start with the expansion of the current density normalized delta function response (equation 3.4). Operate on this response term by term in the sum by the appropriate Green's function which is left as a function of s (not just s_α). The spatial mode that results is the natural mode for $s = s_\alpha$, but not in general for $s \neq s_\alpha$. One could rewrite equation 3.19 in the form

$$\tilde{\vec{U}}_p^{(\vec{X})}(\vec{r},s) = \sum_\alpha \tilde{\eta}_\alpha(\vec{e}_1,s) Y \tilde{\vec{\nu}}_\alpha^{(\vec{X})}(\vec{r},s)(s-s_\alpha)^{-1}$$

$$+ \text{ possible entire function} \tag{8.1}$$

The coefficient Y stays the same as in tables 1 and 2 but the mode function changes. At $s = s_\alpha$ we have the natural mode

$$\tilde{\vec{\nu}}_\alpha^{(\vec{X})}(\vec{r},s_\alpha) = \vec{\nu}_\alpha^{(\vec{X})}(\vec{r}) \tag{8.2}$$

The convoluted mode is found by using $\tilde{G}_o$ instead of $\tilde{G}_{o_\alpha}$ in the natural mode formulas (as well as far natural mode formulas) in section VII. There is the possible exception of the combined charge and combined scalar potential modes since the charge is derived from the current and one might use

$$\tilde{\nu}_\alpha^{(Q_q)}(\vec{r},s) = -\frac{a_\alpha c}{s}\nabla \cdot \vec{\nu}_\alpha^{(\vec{K}_q)}(\vec{r}) \tag{8.3}$$

instead of changing s to s_α in equations 3.11.

The class 2 or convoluted modes can then be defined for the combined quantities as

$$\tilde{\vec{\nu}}_\alpha^{(C_q)}(\vec{r},s) = b_\alpha \gamma_\alpha^2 \left\langle \tilde{G}_o, \vec{\nu}_\alpha^{(\vec{K}_q)} \right\rangle$$

$$\tilde{\nu}_\alpha^{(\phi_q)}(\vec{r},s) = b_\alpha \gamma_\alpha^2 \begin{cases} \left\langle \tilde{G}_o, \nu_\alpha^{(Q_q)}(\vec{r}) \right\rangle \\ \text{or} \\ \left\langle \tilde{G}_o, \tilde{\nu}_\alpha^{(Q_q)}(\vec{r},s) \right\rangle \end{cases} \tag{8.4}$$

$$\tilde{\vec{\nu}}_\alpha^{(\vec{F}_q)}(\vec{r},s) = -c_\alpha \gamma_\alpha^2 \left\langle \tilde{\vec{\vec{G}}}_o ; \vec{\nu}_\alpha^{(\vec{K}_q)} \right\rangle + qic_\alpha \gamma_\alpha \left\langle \nabla \tilde{G}_o \overset{\times}{,} \vec{\nu}_\alpha^{(\vec{K}_q)} \right\rangle$$

The corresponding far modes are

$$\tilde{\vec{\nu}}_{f_\alpha}^{(\vec{C}_q)}(\vec{e}_r,s) = \frac{b_\alpha \gamma_\alpha^2}{\ell_\alpha} \left\langle \tilde{g}_{o_f}, \vec{\nu}_\alpha^{(\vec{K}_q)} \right\rangle$$

$$\tilde{\nu}_{f_\alpha}^{(\phi_q)}(\vec{e}_r,s) = \frac{b_\alpha \gamma_\alpha^2}{\ell_\alpha} \begin{cases} \left\langle \tilde{g}_{o_f}, \nu_\alpha^{(Q_q)}(\vec{r}) \right\rangle \\ \text{or} \\ \left\langle \tilde{g}_{o_f}, \tilde{\nu}_\alpha^{(Q_q)}(\vec{r},s) \right\rangle \end{cases} \tag{8.5}$$

$$\tilde{\vec{\nu}}_{f_\alpha}^{(\vec{F}_q)}(\vec{e}_r,s) = -\frac{c_\alpha \gamma_\alpha^2}{\ell_\alpha} [\vec{\vec{I}} - \vec{e}_r \vec{e}_r + qi\vec{e}_r \times \vec{\vec{I}}] \cdot \left\langle \tilde{g}_{o_f}, \vec{\nu}_\alpha^{(\vec{K}_q)} \right\rangle$$

These combined modes can be readily split into the corresponding electric and magnetic modes. The normalized delta function response for the far quantities is written by modifying the first of equations 4.14 as

$$\tilde{\vec{U}}_{f_p}^{(\vec{X})} = \sum_\alpha \tilde{\eta}_\alpha(\vec{e}_1,s) Y_f \tilde{\vec{\nu}}_{f_\alpha}^{(\vec{X})}(\vec{e}_r)(s - s_\alpha)^{-1}$$

$$+ \text{ possible entire function} \tag{8.6}$$

Note that various other changes of s_α to s can be used to change the form of the expansion for possible advantages in various regimes of frequency and/or time.

Thus we can define class 1 and class 2 modes (and perhaps other classes as well). The class 1 modes are the natural modes (or far natural modes) while the class 2 modes involve a convolution in the time domain. This type of class division corresponds directly to the classes of coupling coefficients. One can combine the class of coupling coefficient with the class of mode to give what one might term the expansion form which might be expressed as

$$\vec{\Sigma} = (\text{class of coupling coefficient, class of mode}) \quad (8.7)$$

This symbolic vector (or matrix, or whatever else it is expanded to for flexibility) can be used to label the form of the singularity expansion used in a particular study. The simplest expansion form is (1,1) for pole terms but this is not necessarily the most useful for certain purposes, such as convergence at early times or high frequencies.

IX. Summary

This note has considered the singularity expansion of potentials and fields for the case of first order pole terms. These potentials and fields can be either scattered or radiated from an antenna, but in both cases the object is assumed to be of finite size. The potential and field natural modes can be related to the current and charge density modes through differential equations. Using the appropriate free space Green's functions evaluated at the natural frequencies the potential and field natural modes can be expressed as integrals over the current and charge density natural modes.

The potential and field natural modes can be modified to give retarded and far natural modes. The far natural modes have various properties that are somewhat simpler than the more general natural modes for all distances from the object of interest. The far natural modes for the fields give antenna and scattering patterns applicable to both frequency and time domains. Thus we seem to have at least one form of answer to the question of what is a time domain antenna pattern. Note, however, that the definition of retarded and far natural modes relies on a definition of $\vec{r} = \vec{0}$ which might be considered the center of the object. For objects with sufficient symmetry the choice for $\vec{r} = \vec{0}$ is clear, but for more general object shapes this choice is not so clear but needed nonetheless.

There is some flexibility in the choice of the form of the singularity expansion for potentials and fields. Not only can one combine various types of frequency dependence with the coupling coefficients, but also with the natural modes (and retarded and far natural modes) as well. These various expansion forms are associated with the presence or absence of a separate entire function in the expansion. The various forms have their own convergence properties for the series so that different forms may be most appropriate for different portions of the time and/or frequency domains.

There are many interesting and practical questions associated with the singularity expansion of potentials and fields that require further investigation. These include how to handle second order poles, convergence rates of the series under various conditions, relations to other types of modes, and efficient numerical procedures. In any event various problems involving specific types of antennas or scatterers can be solved. This will help give insight into some of these more general theoretical questions while at the same time generating useful response data.

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Advice From a Caterpillar

The Caterpillar and Alice looked at each other for some time in silence: at last the Caterpillar took the hookah out of its mouth, and addressed her in a languid, sleepy voice.

"Who are *you*?" said the Caterpillar.

This was not an encouraging opening for a conversation. Alice replied, rather shyly, "I—I hardly know, sir, just at present—at least I know who I *was* when I got up this morning, but I think I must have been changed several times since then."

"What do you mean by that?" said the Caterpillar sternly. "Explain yourself!"

"I can't explain *myself*, I'm afraid, sir," said Alice, "because I'm not myself, you see."

(Lewis Carroll,
Alice in Wonderland)

X. References

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